The Giant Component: The Golden Anniversary

Joel Spencer

1 Paul Erdős and Alfréd Rényi

Oftentimes the beginnings of a mathematical area are obscure or disputed. The subject of *Random Graphs* had, however, a clear beginning and it occurred fifty years ago. Alfréd Rényi (1921-1970) was head of the Hungarian Mathematical Institute which today bears his name. Rényi had a great love of literature and philosophy, a true Renaissance man. His life, in the words of Paul Turán, was one of intense and creative involvement in the exchange of ideas and in public affairs. His mathematics centered on Probability Theory. Paul Erdős (1913-1996) was a central figure in twentieth century mathematics. This author recalls well the Memorial Conference organized by his long-time friend and collaborator Vera Sós in 1999, at which fifteen plenary lecturers discussed his contributions. What was so surprising to us all was the sheer breadth of his work, which spanned so many vital areas of mathematics. To this very prejudiced author, it is his work in Discrete Mathematics that is having the greatest lasting impact. In this area, Erdős tended toward asymptotic questions, a style very much relevant to today's world of o, O, Θ and Ω. For both Erdős and Rényi, mathematics was a collaborative enterprise. They both had numerous coauthors, and they wrote 32 joint papers.

In 1960 they produced [8] their masterwork, *On the Evolution of Random Graphs* They considered the following process. Begin with n vertices and no edges and add edges randomly (that is, uniformly from among the potential edges) one by one. Let $G[n, e]$ be the state when there are e edges. Of course, $G[n, e]$ could be any graph with n vertices and e edges and technically is the uniform probability distribution over all such graphs. Erdős and Rényi analyzed the typical behavior of $G[n, e]$ as e "evolved" from 0 to $\binom{n}{2}$. Their interest, and ours, lies in the asymptotic behavior of this process as n goes to infinity. When e approaches and passes $\frac{n}{2}$ the random graph undergoes a phase transition, as described in Theorem 1.1 below. A typical computer run on a million vertices illustrates this rapid change. At $e = 400000$ edges the size of the largest component is only 168. By $e = 600000$ edges the size of the largest component has exploded to 309433.

Let $G(n, p)$ denote the probability space over graphs on n vertices where each pair is adjacent with independent probability p. (We may make a "thought experiment" in which each pair of vertices flips a coin to decide if the pair is adjacent.) Such graphs have very close to a proportion p of the edges. The behaviors of $G[n,e]$ and $G(n,p)$, where $e = p\binom{n}{2}$ are asymptotically the same for all the topics we discuss here. While Erdős and Rényi analyzed $G[n,e]$, most modern work gives results in the $G(n,p)$ format and we shall describe results in that format.

We shall parameterize $p = \frac{c}{n}$. The graph with $\frac{n}{2}$ edges then corresponds to $c = 1$. Let C_1, C_2 denote the largest and second largest components in the graph with $|C_i|$ denoting their number of vertices. We define the *complexity* of a component with V vertices and E edges as $E - V + 1$. Trees and unicyclic graphs have complexity 0 and 1 respectively and are called *simple*.

Theorem 1.1 *[Erdős-Rényi] The behavior of $G(n,p)$ with $p = \frac{c}{n}$ can be broken into three regions.*
`Subcritical` $c < 1$*: All components are simple and very small, $|C_1| = O(\ln n)$.*
`Critical` $c = 1$*: $|C_1| = \Theta(n^{2/3})$ A delicate situation!*
`Supercritical` $c > 1$*: $|C_1| \sim yn$ where $y = y(c)$ is the positive solution to the equation*

$$e^{-cy} = 1 - y \tag{1}$$

C_1 has high complexity. All other components are simple and very small, $|C_2| = O(\ln n)$.

`Remark:` The notation above allows the statements to be a little more brief than they would be otherwise. For example, $|C_1| \sim yn$ means that for all $\epsilon > 0$ the limit as $n \to \infty$ of the probability that $(y-\epsilon)n < |C_1| < (y+\epsilon)n$ is one.
`Remark:` The average degree of a vertex is $p(n-1) \sim c$. The critical behavior takes place when the average degree reaches one.
`Remark:` Elementary, albeit challenging, calculus gives that (1) has a unique positive solution $y = y(c)$ when $c > 1$. Further, parametrizing $c = 1 + \epsilon$ and letting ϵ approach zero from above:

$$y(1+\epsilon) \sim 2\epsilon \tag{2}$$

When $c > 1$, Erdős and Rényi called C_1 the *giant component*. There are two salient features of the giant component: its existence and its uniqueness. Their system does not have a Jupiter and an also huge Saturn, it is more like Jupiter and Ceres. Several books [3],[4], [16] give very full discussions.

2 Francis Galton and Henry Watson

In 1873 Francis Galton initiated the modern theory of branching processes, posing the following problem in the Educational Times:

> A large nation, of whom we will only concern ourselves with adult males, N in number, and who each bear separate surnames colonise a district. Their law of population is such that, in each generation, a_0 percent of the adult males have no male children

who reach adult life; a_1 have one such male child; a_2 have two; and so on up to a_5 who have five. Find (1) what proportion of their surnames will have become extinct after r generations; and (2) how many instances there will be of the surname being held by m persons.

Peter Jagers comments [12]

> Rarely does a mathematical problem convey so much of the flavour of its time, colonialism and male supremacy hand in hand, as well as the underlying concern for a diminished fertility of noble families, paving the way for the crowds from the genetically dubious lower classes.

The challenge was taken up by Henry Watson. Watson was a clergyman who, not unlike his more illustrious contemporary Charles Dodgson [1], had keen mathematical abilities. He considers a process beginning with a single node which we will, for balance, call Eve. Eve has k children with probability a_k. These children then have children independently with the same distribution and the process continues through the generations. These processes are now called Galton-Watson processes. We shall here assume that the number of Eve's children is Poisson with mean c. That is, Eve has k children with probability $e^{-c}c^k/k!$. Let T_c be the total population generated. There is a precise formula

$$\Pr[T_c = k] = \frac{e^{-ck}(ck)^{k-1}}{k!} \tag{3}$$

However, it is also possible the T_c is infinite.

Theorem 2.1 *The Galton-Watson process has three regions.*
`Subcritical` $c < 1$*: T_c is finite with probability one. and* $E[T_c] = \sum_{k=0}^{\infty} c^k = \frac{1}{1-c}$
`Critical` $c = 1$*: T_c is finite with probability one but has infinite expectation.* [2]
`Supercritical` $c > 1$*. T_c is infinite with probability $y = y(c)$ given by (1).*

With $c > 1$ let $z = 1 - y$ be the probability Eve generates an finite tree. If Eve has k children the full tree will be finite if and only if all of the children generate finite trees, which has probability z^k. Thus

$$z = \sum_{k=0}^{\infty} e^{-c}\frac{c^k}{k!}z^k \tag{4}$$

and some manipulation gives that $y = 1-z$ satisfies (1). Galton and Watson [10] noted that $z = 1, y = 0$ is a solution to (4) and deduced, incorrectly, that the Galton-Watson process dies with probability one. But when $c > 1$ there is another solution $z < 1$ and that, we now know, gives the correct probability.

[1] Better known under his pen name Lewis Carroll

[2] This author recalls in undergraduate days first seeing a finite random variable with infinite expectation and thinking it was a very funny and totally anomalous creation. Wrong! Such variables occur frequently at critical points in percolation processes.

2.1 Erdős meets Galton

Fix a vertex v in $G(n,p)$ with $p = \frac{c}{n}$. The component of v may be found by what is known as a Breadth First Search (BFS) algorithm. One first finds the neighbors of v. Then for each such neighbor w we find its neighbors. Then for each such second generation u we find its neighbors and so on. What occurs on this random graph? v has Binomial Distribution $B[n-1,p]$ neighbors, asymptotically Poisson with mean c. The neighbors w of v then have Poisson c new neighbors, and so on. The component of v is approximated by the Galton-Watson process. Sometimes. For $c < 1$ this approximation works well. But for $c > 1$ the Galton-Watson process may go on forever while the component of v can have at most n vertices. What goes wrong with the approximation? BFS requires *new* vertices. After δn vertices have been found the new distribution is Binomial $B[n(1-\delta),p]$ which is Poisson with mean $c(1-\delta)$. The success of BFS causes δ to rise which makes it harder to find new vertices leading the process to eventually die. This has the colorful term *ecological limitation.* The effect of the ecological limitation is only felt after a positive proportion δn of vertices have been found. Consider BFS from each vertex v. With probability $1-y$ the process will die early, giving a small component. But for $\sim yn$ the process will not die early. All of these vertices have their components merge into the giant component.

2.2 Jupiter without Saturn

Why can we not have Jupiter and Saturn, two components both of size bigger than δn? This would be highly unstable. Each additional edge would have probability at least $(\delta n)^2/\binom{n}{2} \sim 2\delta^2$ of merging them. High instability and nonexistence are not the same. Indeed, while there are many proofs of the uniqueness of the giant component we do not know one that is both simple and rigorous.

3 The Critical Window

Erdős and Rényi normally repressed their enthusiasm in their formal writings. But not now!

> This double "jump" in the size of the largest component when $\frac{e}{n}$ passes the value $\frac{1}{2}$ is one of the most striking facts concerning random graphs. [8]

In the 1980s, spearheaded by the work of Béla Bollobás [5] and Tomasz Łuczak [13], the value $c = 1$ was stretched out and a Critical Window was found. The stretching was done by adding a second order term. The correct parameterization is

$$p = \frac{1}{n} + \frac{\lambda}{n^{4/3}} \tag{5}$$

Now there are three regions:

`Barely Subcritical` $pn \sim 1$ and λ is a function of n which approaches

$-\infty$: All components are simple. $|C_1| \sim |C_2|$, their sizes increasing with λ.
`The Critical Window` $pn \sim 1$ and λ is a constant: Here (and only here!) we have chaotic behavior, distributions instead of almost sure behavior. Parametrizing $|C_i| = X_i n^{2/3}$ the X_i ($i = 1, 2$ and beyond) have a nontrivial distribution and a nontrivial joint distribution. The complexity Y_i of C_i also has a nontrivial distribution.
`Barely Supercritical` $pn \sim 1$ and λ is a function of n which approaches $+\infty$: $|C_1| \gg n^{2/3} \gg |C_2|$. C_1 is the *dominant component*, much bigger than C_2 but still small. C_1 has high complexity but all other components are simple.

Stirling's Formula applied to (3) with $c = 1$ gives $\Pr[T_1 = k] \sim (2\pi)^{-1/2} k^{-3/2}$ and $\Pr[T_1 \geq k] \sim 2(2\pi)^{-1/2} k^{-1/2}$. Now consider $G(n, p)$ with $pn = 1$ and let $C(v)$ be the component containing v. Call $C(v)$ large if its size is at least $Kn^{2/3}$ and let Z be the number of v with $C(v)$ large. Estimating $|C(v)|$ by T_1 would give that $C(v)$ is large with probability $2(2\pi)^{-1/2} K^{-1/2} n^{-1/3}$ and so Z would have expectation $2(2\pi)^{-1/2} K^{-1/2} n^{2/3}$. The actual value of $E[Z]$ is somewhat smaller due to the ecological limitation, but let us assume it as a heuristic. If any large component exists every vertex of it would be in a large component so that Z would be at least $Kn^{2/3}$. When K is large $E[Z]$ is much lower than $Kn^{2/3}$ so that with high probability there would be no large component. Conversely, when K is a small positive constant we expect many components of size bigger than $Kn^{2/3}$.

3.1 A Strange Physics

To understand evolution inside the Critical Window we set $p = n^{-1} + \lambda n^{-4/3}$ and consider λ (ranging over all real numbers) as a parametrized time. Let $c_i n^{2/3}$ be the size of the i-th largest component at $p = n^{-1} + \lambda n^{-4/3}$. Let $\Delta\lambda$ be an "infinitesimal" and increase time λ by $\Delta\lambda$. There are $(c_i n^{2/3})(c_j n^{2/3})$ potential edges that would merge C_i, C_j and each is added with probability $(\Delta\lambda) n^{-4/3}$. The n factors cancel: C_i, C_j merge to form a component of size $(c_i + c_j) n^{2/3}$ with probability $c_i c_j (\Delta\lambda)$. This gravitational attraction merges the large components and forms the dominant component. We can include the complexity in this model. When C_i, C_j with complexities r_i, r_j merge, the new component has complexity $r_i + r_j$. Further, each C_i has $\sim \frac{1}{2} c_i^2 n^{4/3}$ potential internal edges. In the infinitesimal time $\Delta\lambda$ with probability $c_i^2 (\Delta\lambda)/2$ such an edge is added and the complexity of C_i is incremented by one. Over time, the complexities get larger and larger. The limiting process, called the multiplicative coalescent process, has interesting connections to Brownian motion. [2]

3.2 A Computer Exercise

·	-4	N	-3	N	-2	N	-1	N	0	N	+1	N	+2	N	+3	N	+4
0	0.14	1	0.18	0	0.24	1	0.28	0	0.37	0	0.82	0	1.16	0	4.21	0	5.88
1	0.10	2	0.16	1	0.19	2	0.26	1	0.36	1	0.39	1	0.86	0	0.22	0	0.24
2	0.10	3	0.13	3	0.13	4	0.19	3	0.26	0	0.28	3	0.49	0	0.13	0	0.12
3	0.09	4	0.12	4	0.13	0	0.16	2	0.21	2	0.23	4	0.46	0	0.12	0	0.10
4	0.07	0	0.09	5	0.13	3	0.14	5	0.16	0	0.20	2	0.32	0	0.11	2	0.10
5	0.07	0	0.08	7	0.09	5	0.12	4	0.15	4	0.14	5	0.16	1	0.10	1	0.10
6	0.07	5	0.06	6	0.09	7	0.10	8	0.12	5	0.12	2	0.12	3	0.09	1	0.10
7	0.06	8	0.06	2	0.08	6	0.09	-	0.12	3	0.10	1	0.11	4	0.09	4	0.10
8	0.05	-	0.06	8	0.06	9	0.09	-	0.10	3	0.10	7	0.09	2	0.09	0	0.08
9	0.05	9	0.05	-	0.06	0	0.07	-	0.10	9	0.10	0	0.08	5	0.08	6	0.07

Computer experimentation vividly shows the rapid development in the critical window. In the above run [3] we begin with $n = 10^5$ vertices and no edges. At each step a random edge is added and a Union-Find algorithm is used to keep track of component sizes. We parameterize the number of edges as $e = \binom{n}{2}(n^{-1} + \lambda n^{-4/3})$ and take "snapshots" at $\lambda = -4, -3, \ldots, +4$. The ten largest components sizes (listed $0, \ldots, 9$ here, and divided by $n^{2/3}$) are given for each λ. At $\lambda = +2$ there is a 1.16 Jupiter and 0.86 Saturn. The next digit, under N, gives the new ranking ($-$ if not in the top ten) for that component for the next λ. Components $0, 1, 2, 3, 4$ have $N = 0$ meaning they have all merged by $\lambda = +3$. At $\lambda = 3$ Jupiter has blown up to 4.21. (Smaller components have also joined Jupiter, explaining the discrepancy in the sum.) The size of the second largest component has *decreased* (it is the component formerly ranked 5) to a 0.22 Ceres.

3.3 Inside the Critical Window

At $\lambda = -4$ there is a "jostling for position" among the top components, while by $\lambda = +4$ a dominant component has emerged. The last time the largest component loses that distinction occurs [13] during the critical window. At $\lambda = -4$ all components are simple while by $\lambda = +4$ the dominant component has high complexity. Complexity at least 4 is necessary for nonplanarity. Planarity is lost [19] in the critical window. In a masterful work [15] the development of complex components is studied. One exceptionally striking result: the probability that the evolution ever simultaneously has two complex components is, asymptotically, $\frac{5}{18}\pi$.

3.4 Classical Bond Percolation

Mathematical physicists examine Z^d as a lattice, the pairs $\vec{v}, \vec{w}$ that are one apart are called *bonds*. They imagine that each bond is *occupied* with independent probability p. (For graph theorists, the occupied bonds form a random subgraph of Z^d.) The occupied components then form clusters, or components. There is a critical probability, denoted p_c (dependent on d) so

[3] Thanks to Juliana Freire

that:
`Subcritical` $p < p_c$: All components are finite.
`Critical` $p = p_c$: A delicate situation!
`Supercritical` $p > p_c$: There is precisely one infinite component.

There are natural analogies between this infinite model and the asymptotic Erdős-Rényi model. Infinite size corresponds to $\Omega(n)$ while finite size corresponds to $O(\ln n)$. There is particular interest in p being very close to p_c. Let $f(p)$ be the probability that $\vec{0}$ (or, by symmetry, any particular $\vec{v}$) lies in an infinite component. The *critical exponent* β is that [4] real number such $f(p_c + x) \sim x^{\beta+o(1)}$ as $x \to 0^+$. $p_c + x$ corresponds to $pn = 1 + x$ and f to the probability that a given vertex v lies in the giant component or, equivalently, the proportion $y(1 + x)$ of vertices in the giant component. As (2) $y(1+x) \sim 2x = x^{1+o(1)}$ the β value for the Erdős-Rényi model is considered 1. This is known to be the β value in Z^d for all $d \geq 19$. Grimmett [11] gives many other critical exponents and in all cases the analogous value for the Erdős-Rényi model matches the known value in high dimensional space. Mathematical physicists loosely use the term *mean field behavior* to describe percolation phenomenon in high dimensions and the Erdős-Rényi model has this mean field behavior.

4 Recent Results

Today it is recognized that percolation and the critical window appear in many guises. Here is a highly subjective description of recent work. We generally give simplified versions.

4.1 Random 2-SAT

We generate m random clauses $C_1, \ldots, C_m$ on Boolean variables $x_1, \ldots, x_n$. That is, each clause $C = y \vee z$ with y, z drawn randomly from $\{x_1, \overline{x_1}, \ldots, x_n, \overline{x_n}\}$. We ask if all C_i can be simultaneously satisfied. The answer changes from yes to no in the critical window $m = n + \lambda n^{2/3}$. [6]

4.2 d-regular graph

Let G_n be a sequence of transitive d-regular graphs. Under reasonable conditions $p_c = \frac{1}{d-1}$ acts as the critical probability for a random subgraph of G_n. For $p < p_c$ the components are small while for $p > p_c$ there is a giant component. More delicately, at $p = p_c$ the largest component has size $\Theta(n^{2/3})$. The scaling $p(d-1) = 1 + \lambda n^{-1/3}$ acts as the critical window. [21]

4.3 An Improving Walk

Consider an infinite walk starting at $W_0 = 1$ with $W_t = W_{t-1} + X_t - 1$ where X_t is Poisson with mean $\frac{t}{n}$. When $W_t = 0$ (crashes) it is reset to $W_t = 1$. When $t = \frac{1-\epsilon}{n}$ the walk has negative drift and crashes repeatedly. When

[4] β might not exist, but all mathematical physicists assume it does.

$t = \frac{1+\epsilon}{n}$ the walk has positive drift and goes to infinity. The walk will crash for the last time in the critical window $t = n + \lambda n^{2/3}$. [9]

4.4 A First Order Phase Transition

Modify the Erdős-Rényi evolution as follows. Each round an edge is added to G, initially empty. *Two* random pairs $\{u, v\}$, $\{w, x\}$ are given. Add that pair for which the product of the component sizes of the two vertices is smaller. This provides a powerful anti-gravity that deters large components from joining. Parameterizing $e = t\frac{n}{2}$ edges chosen (so that $t = 1$ is the critical value in the Erdős-Rényi evolution) the giant component occurs at $t \sim 1.77$. More interesting, extensive computer simulation (but no mathematical proof!) indicates strongly that when the critical value is reached there is a first order phase transition. That is, let $t = t_c$ be the critical probability and let $f(t)$ be the proportion of vertices in the largest component at "time" t. Then the limit of $f(t)$ as t approaches t_c from above appears not to be zero, but rather something like 0.6. [1]

4.5 General Critical Points

Let G be a graph on n vertices. Let d_v denote the degree of vertex v and set $d^* = (\sum_v d_v^2)(\sum_v d_v)$. Note that d^* gives the average degree of a vertex if one first selects an edge uniformly and then one of its vertices uniformly. Let G_p denote the random subgraph of G, accepting each edge with independent probability p. Then, under certain mild conditions on G, $p = \frac{1}{d^*}$ is the critical point in the evolution of G_p. When $p = \frac{1-\epsilon}{d^*}$, G_p contains no giant component while when $p = \frac{1+\epsilon}{d^*}$, G_p does contains a unique giant component. [7]

4.6 Degree Sequence

For given $d_1, \ldots, d_n$ we consider the graph on n vertices chosen uniformly amongst all with that degree sequence, that is, v having precisely d_v neighbors. Suppose for each n we have a degree sequence, $\lambda_i(n) \sim \lambda_i n$ vertices having degree i. Then (with d^* from above), $d^* \sim [\sum i^2\lambda_i]/[\sum i\lambda_i]$. Set $Q := \sum_i i(i-2)\lambda_i$ so that $Q > 0$ if and only if $d^* > 2$. In analyzing BFS a new edge will have a new vertex with expected degree d^* and an expected $d^* - 1$ edges to new vertices. With $d^* < 2$ the process will die while for $d^* > 2$ it might continue. When $Q < 0$ the random graph with this degree sequence has no giant component while when when $Q > 0$ it does. [20]

Set $Q_n := \sum_i i(i-2)\lambda_i(n)$ and assume $Q_n \to 0$. Under moderate assumptions, $Q_n = \lambda n^{-1/3}$ provides a critical window. For $\lambda \to -\infty$ the random graph is subcritical and all components have size $o(n^{2/3})$. When $\lambda \to +\infty$ the random graph is supercritical, there is a dominant component of size $\gg n^{2/3}$ and all other components has size $o(n^{2/3})$. In the power law random graphs, thought by many to model the web graph and other phenomenon, it is assummed that $\lambda_i \sim i^{-\gamma}$ for a constant γ. For certain γ the above critical window does not work, and work in progress indicates that there is a critical window whose exponent depends on γ. [17] [14]

4.7 A Potts Model

In the Potts Model, the distribution of graphs is biased toward having more components. There are three parameters, $p \in [0,1]$, $q \geq 1$, and the number of vertices n. A graph G with e edges, $s := \binom{n}{2} - e$ nonedges, and c components has probability $p^e(1-p)^s q^c/Z$ where Z is a normalizing constant chosen so that the sum of the probabilities is one. For $q = 2$ this is called the Ising model, for $q \geq 3$ and integral, this is the Potts model. For $2 < q < 3$ the critical value is $pn = c_q := 2\frac{q-1}{q-2}\ln(q-1)$. At $pn = c_q + \epsilon$ there is a giant component, while at $pn = c_q - \epsilon$ the largest component has logarithmic size. The critical window has parameterization is $pn = c_q + \frac{\lambda}{n}$. There the graph has two different personalities. Either it has a giant component *or* the largest component has logarithmic size. Both occur with positive limiting probability and these limiting probabilities sum to one. At no p is there a middle ground with the size of the largest component being bigger than logarithmic but sublinear. [18]

5 In conclusion

The mathematical landscape at the time of Paul Erdős's birth, nearly one hundred years ago, was very different than what it is today. Discrete Mathematics was disparaged as "the slums of topology." Probability was useful for gambling, but not proper work for a serious mathematician. Today both areas are thriving. It is the fecund intersection of Discrete Mathematics and Probability that has seen the most spectacular growth. A wide variety of random processes on large discrete structures are studied. These processes, to use Erdős and Rényi's well chosen word, undergo an *evolution*. At a critical moment they undergo a phase transition, from water to ice, from satisfiable to not satifiable, from freeflow to gridlock, from small components to a giant component. To understand a process we need to understand these critical moments. The Erdős-Rényi process provides a bedrock, to which all other processes may be compared.

References

[1] D. Achlioptas, R. M. D'Souza and J. Spencer, Explosive Percolation in Random Networks, Science, 323 (5920) 1453-1455, 2009.

[2] David Aldous, Brownian Excursions, Critical Random Graphs, and the Multiplicative Coalescent, Annals of Probability, 25 (1997), 812-854.

[3] N. Alon and J. Spencer, The Probabilistic Method, 3nd ed., John Wiley, 2009.

[4] Béla Bollobás, Random Graphs, 2nd Edition, Cambridge University Press. 2001.

[5] Béla Bollobás, The evolution of random graphs, Trans. Amer. Math. Soc., 286 (1984), 257-274.

[6] Béla Bollobás, Christian Borgs, Jennifer Chayes, Jeong Han Kim, David B. Wilson, The Scaling Window of the 2-SAT Transition, Random Structures and Algorithms, 21 (2002), 182-195.

[7] Fan Chung, Paul Horn and Lincoln Lu, The Giant component in a Random Subgraph of a Given Graph, Proceedings of the 6th Workshop on Algorithms and Models for the Web-Graph (WAW2009), Lecture Notes in Computer Science 5427, 38–49.

[8] Paul Erdős and Alfréd Rényi, On the Evolution of Random Graphs, Magyar Tud. Akad. Mat. Kutató Int. Közl., 5 (1960), 17-61.

[9] Juliana Freire, Analysis Techniques for Nudged Random Graphs, Ph.D. Thesis, Courant Institute. 2009.

[10] Francis Galton and Henry Watson, On the probability of extinction of families. J. Anthropol. Inst., 4 (1874), 138-144.

[11] Geoffrey Grimmett, Percolation. 2nd Edition. Springer-Verlag. 1999

[12] Peter Jagers, Some Notes on the History of Branching Processes, from my Perspective. Oberwolfach 2009. Unpublished manuscript.

[13] T.Łuczak, Components behavior near the critical point of the random graph process, Random Structures and Algorithms 1 (1990) 287-310.

[14] Svante Janson and Malwina Luczak, A new approach to the giant component problem, Random Structures and Algorithms. 34 (2008), 197-216.

[15] Svante Janson, Tomasz Łuczak, Donald Knuth, Boris Pittel, The Birth of the Giant Component, Random Structures and Algorithms, 3 (1993), 233-358.

[16] Svante Janson, Tomasz Łuczak and Andrzej Rucinski, Random Graphs, John Wiley, 2000.

[17] Milyun Kang and Taral Seierstad, The critical phase for random graphs with a given degree sequence, Combin. Probab. Comput. 17 (2008), 67-86.

[18] Malwina Luczak and Tomasz Łuczak, The phase transition in the cluster-scaled model of a random graph, Random Structures and Algorithms, 28 (2006), 215-246.

[19] Tomasz Łuczak and John Wierman, The chromatic number of random graphs at the double jump threshold, Combinatorica, 9 (1989), 39-49.

[20] Mike Molloy and Bruce Reed, The size of the largest component of a random graph on a fixed degree sequence, Combin. Probab. Comput. 7 (1998), 295-306.

[21] Asaf Nachmias, Mean-field conditions for percolation on finite graphs, GAFA (to appear).