of its children the i-th and $(i+1)$-st coordinates are at least $.48x_i^*$ and so, by induction, also large. $\square$

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$y = (y_0, \ldots, y_k)$. We need define a move $y^* = (y_0^*, \ldots, y_k^*)$ for the new tree. If coordinate i is small in the new tree then $x_i = x_i^*$ and in that case we set $y_i^* = y_i$. Let L denote the set of i for which the i-th coordinate is large in P^*. The requirement that the weights of the children be equal in the two trees may be written

$$\sum_{i=0}^{n}(2y_i - x_i)\binom{q-t}{k-i} = \sum_{i=0}^{n}(2y_i^* - x_i^*)\binom{q-t}{k-i} \quad (**)$$

The left hand side is bounded in absolute value by $O(q^k)$. It suffices to restrict to the sum over $i \in L$ since the terms are identical for the small coordinates. Considered as an equation in the y_i^*, $i \in L$, the equation (**) has the integer solution

$$y_i^* = y_i + \frac{x_i^* - x_i}{2}$$

which, by the induction hypothesis, would make y_i, y_i^* congruent modulo 2^{k-t-1}. Now consider real solutions to (**), again with only y_i^*, $i \in L$ as variables. If, say, we try $y_i^* = .51x_i^*$, then each $i \in L$ contributes at least $+.02qA$ to the right hand side sum while the $i \notin L$ can contribute (the extreme case being $y_i^* = 0$) $-w_i = O(A)$ negatively and the right hand side sum is at least $+\Omega(Aq)$ while the left hand side is $O(q^k) = o(Aq)$. Thus these y_i^* are too big. Similarly, $y_i^* = .49x_i^*$ would be too small. Thus there is an $\alpha \in [.49, .51]$ so that setting $y_i^* = \alpha x_i^*$ for $i \in L$ gives a real solution to (**). Now consider the integer solution to (**) with $y_i^* \equiv y_i$ modulo 2^{k-t} (we know there is such a solution) which minimizes $\sum_{i \in L}|y_i^* - \alpha x_i^*|$. Given any solution we get another by replacing any y_i^*, y_j^* by $y_i^* + 2^{k-t}\binom{q-t-1}{k-j}$ and $y_j^* - 2^{k-t}\binom{q-t-1}{k-i}$ respectively. Thus in the minimum solution we would not have i, j with $y_i^* > \alpha x_i^* + cq^k$ and $y_j^* < \alpha x_j^* - cq^k$, where c is a large constant. But

$$\sum_{i \in L}(y_i^* - \alpha x_i^*)2\binom{q-t-1}{k-i} = 0$$

so that if, say, the negative values are all $O(q^k)$ then the positive values are all $O(q^{2k})$ so that there is a solution with all

$$|y_i^* - \alpha x_i^*| = O(q^{2k})$$

Since $A > q^{5k}$ (which leaves room to spare) all $i \in L$ have $x_i^* = \Omega(q^{4k})$ and therefore $.48x_i^* < y_i^* < .52x_i^*$ for all $i \in L$. This is the desired move. Besides the congruence condition note that if x_i^* is large in P^* then in both

Theorem. There exists q_1 so that for all $q \geq q_1$ and all initial positions $P = (x_0, \ldots, x_k)$, Paul wins if and only if Paul can survive k moves.

If $w_q(P) > 2^q$ then Paul can neither win nor survive k moves. If $w_q(P) < 2^q - cq^k$ then Paul wins and hence also survives k moves. Therefore we may and shall assume

$$2^q - cq^k < w_q(P) \leq 2^q$$

We let P satisfy the above inequalities and assume Paul can survive k moves. Fix a decision tree of depth k describing Paul's survival and let $\mathcal{P}$ denote the set of position vectors appearing in those first k moves. We let $t = t(P)$ denote the depth (or round number) on which P appears so that the original $P = P^0$ has $t = 0$ and the leaves of the decision tree have $t(P) = k$. Write $w(P)$ for $w_{q-t}(P)$ where $t = t(P)$. For any nonleaf P the two children P^{YES} and P^{NO} satisfy $w(P) = w(P^{YES}) + w(P^{NO})$. The bound on $w(P^0)$ and the upper bound on all $w(P)$ force all P at depth t to have $2^{q-t} - O(q^k) \leq w(P) \leq 2^{q-t}$.

For any $P = (x_0, \ldots, x_k) \in \mathcal{P}$ with $t = t(P)$ let $w_i = x_i \binom{q-t}{\leq k-i}$ and let W be the set of all such w_i. The original P^0 must have some $w_i > (2^q/k)(1 + o(1))$ and $w_k = O(q^k)$. As W has constant (dependent on k) size we find for q sufficiently large $A > q^{5k}$ so that all $w \in W$ have either $w < A3^{-k}$ or $w > A3^k q$ and that the initial P^0 contributes w in both categories.

We now create a new decision tree, with nodes denoted by P^*. The root node is P^0 in both cases. We require $w(P^*)$ to be the same as $w(P)$ for the corresponding P. We require that for the P^* at depth t the values w_i all satisfy either $w_i < A3^{-k+t}$ or $w_i > A3^{k-t}q$, and we shall call such coordinates small or large respectively. We shall further require that small coordinates in P^* be precisely the same value as the same coordinate in the corresponding P. Further, for any coordinate the values x_i, x_i^* must be congruent modulo 2^{k-t}. Finally, we require that if the i-th coordinate of P^* is large then both the i-th and the $(i+1)$-st coordinates of both its children will be large and conversely if the j-th coordinate of a child (in the $*$-tree) is large either the j-th or $(j-1)$-st coordinate of its parent must be large.

If we can accomplish this then with the new decision tree at the end of k rounds in every branch the k-th coordinate will be large so that $x_k = w_k \geq Aq \gg q^k$ so by our main theorem Paul can win that game.

The new decision tree is created top down, formally by induction. Suppose to a position $P = (x_0, \ldots, x_k)$ at depth t we have corresponded a position $P^* = (x_0^*, \ldots, x_k^*)$ and that in the old tree Paul's move is now

We now consider $w_{q-i-1}(P, v^+) = V_i$ as a diophantine equation in the z_j. This equation is of the form

$$\sum_{j=0}^{i} z_j \binom{q-i-1}{k-j} = S$$

where $|S| = O(q^{k+i+1})$ and V_i was chosen so that there is an integer solution. Thus there is a solution with all $|z_j| = O(q^{k+i+1})$. Paul moves this v^+. For q sufficiently large this implies $x_i/3 < y_i < 2x_i/3$ so that this is a legitimate move and the coordinates of the new P will be at least half those of the old P and,

$$w_{q-i-1}(NO(P, v^+)) > \frac{1}{2} w_{q-i}(P) - \Delta_{q-i-1}(P, v^+)$$

so that regardless of Carole's move the new P has $w_{q-i-1}(P)$ in the appropriate interval.

Now at the end of k rounds the position P has $w_{q-k}(P) \leq V_k \leq 2^{q-k}$ and the number of pennies is at least $n3^{-k} > cq^k$ so by the main theorem Paul wins. $\square$

4 The First k Moves

Given an initial position $P = (x_0, \ldots, x_k)$ and a number of moves q we would like decide if Paul or Carole wins. Our analysis will be for k fixed (as usual) and q sufficiently large. Many of the cases are easily settled. If $w_q(x_0, \ldots, x_k) > 2^q$ then Carole wins. Now suppose $q \geq q_0$ and

$$w_q(x_0, \ldots, x_k) \leq 2^q - cq^k$$

where c, q_0 are given by the main theorem. Replacing x_k by $x'_k = x_k + cq^k$ we still have $w_q \leq 2^q$ but we now have at least cq^k pennies so by the main theorem Paul wins. Adding chips only makes the game harder for Paul so we conclude that Paul wins the original game. Henceforth we shall assume

$$2^q - cq^k < w_q(x_0, \ldots, x_k) \leq 2^q$$

as these are the only interesting cases remaining.

We shall say that *Paul can survive k moves* if there is a strategy for Paul so that, regardless of Carole's play, the position P at the end of k rounds has $w_{q-k}(P) \leq 2^{q-k}$. Clearly if Paul wins he can survive k moves.

Then Paul wins if and only if

$$V_k \le 2^{q-k}$$

Proof (necessity). Regardless of the play after i rounds ($i < k$) the position P will be of the form $(x_0, \dots, x_i, 0, \dots, 0)$ with $\sum x_j = n$ and

$$w_{q-i}(P) = \sum_{j=0}^{i} x_j \binom{q-i}{\le k-j} = n\binom{q-i}{\le k} - \sum_{j=1}^{i} r_j \binom{q-i}{k-j+1}$$

where $r_j = x_j + \dots + x_i$. To see this note that the position $(n, 0, \dots, 0)$ can be moved to the new P by r_j movements of a chip from position $j-1$ to j and each such move reduces w_{q-1} by $\binom{q-i}{k-j+1}$. As the r_j are integers the definition of A_{q-i} gives that

$$w_{q-i}(P) \equiv n\binom{q-i}{\le k} \bmod A_{q-i}$$

for any play. Carole can play so that, letting P_{q-s} denote the position after s rounds , $w_{q-s}(P_{q-s}) \ge \frac{1}{2} w_{q-(s-1)}(P_{q-(s-1)})$. Because of the congruence condition Carole actually assures with this play that $w_{q-i}(P_{q-i}) \ge V_i$ for $0 \le i \le k$. If the condition fails then after the first k rounds $P = P_{q-k}$ has $w_{q-k}(P) > 2^{q-k}$ and therefore Carole wins.

Proof (sufficiency). We show for $0 \le i \le k$ by induction on i that Paul can assure (regardless of Carole's responses) that

$$2^{q-i} - q^{k+i+1} < w_{q-i}(P_{q-i}) \le V_i$$

and so that the first $i+1$ coordinates of P_{q-i} (i.e., the nonzero ones) are all at least $n3^{-i}$. (The factor 3 simply allows us extra space, it is not best possible. The lower bound also is designed to give extra room.) For $i = 0$ this is true by our assumptions on the initial P. Assume this true for i. Let $P = (x_0, \dots, x_i, 0, \dots, 0)$ be the position after the i-th round. Set $y_j = \lfloor x_j/2 \rfloor$ and $v = (y_0, \dots, y_i, 0, \dots, 0)$ so that $\Delta_{q-i-1}(P, v) < q^k$. Let $z_0, \dots, z_i$ be (for the moment) integer variables and set $v^+ = v + (z_0, \dots, z_i, 0, \dots, 0)$. Then

$$w_{q-i-1}(YES(P, v^+)) = w_{q-i-1}(YES(P, v)) + \sum_{j=0}^{i} z_j \binom{q-i-1}{k-j}$$

Let us define $ans_k(q)$ to be the maximal n for which Paul wins the game so that, from the condition $w_q(P) \le 2^q$, we bound

$$ans_k(q) \le \lfloor \frac{2^q}{\binom{q}{\le k}} \rfloor$$

Suppose

$$n \le \lfloor \frac{2^q}{\binom{q}{\le k}} \rfloor - c_5$$

where c_5 is a large absolute (though, as always, dependent on k) constant. For c_5 sufficiently large we may add cq^k pennies to give a position $P' = (n, 0, \ldots, 0, cq^k)$ which still has $w_q(P) \le 2^q$ and so by the main theorem Paul wins. As adding pennies only can make the game harder for Paul, Paul wins the original game and hence we may bound

$$ans_k(q) = \frac{2^q}{\binom{q}{\le k}} + O(1)$$

The next result, while somewhat technical to state, gives necessary and sufficient conditions for Paul to win. For $1 \le s \le k$ define

$$A_{q-s} = \gcd\left(\binom{q-s}{k}, \binom{q-s}{k-1}, \ldots, \binom{q-s}{k-s+1}\right)$$

Note in particular that

$$A_{q-1} = \binom{q-1}{k}$$

Theorem. Define inductively $V_0, V_1, \ldots, V_k$ by setting

$$V_0 = n\binom{q}{\le k}$$

and letting V_i be the least integer such that

$$V_i \ge \frac{V_{i-1}}{2}$$

and

$$V_i \equiv n\binom{q-i}{\le k} \bmod A_{q-i}$$

Endgame Lemma. Let $(x_0, \ldots, x_k)$ be a position with $x_0 \leq 1$, $x_1 = \ldots = x_{k-1} = 0$ and $w_{j+1}(x_0, \ldots, x_k) = 2^{j+1}$. Then Paul wins the $j+1$-move game.

Proof. By induction on j it suffices to find a move v for Paul with $\Delta_j(P, v) = 0$ since both $YES(P, v)$ and $NO(P, v)$ (ignoring leftmost zeroes) will remain in the above form. If $x_0 = 0$ then $x_k = 2^{j+1}$ and this is simply "Twenty Questions", Paul takes $v = (0, \ldots, 0, 2^j)$. Otherwise

$$2^{j+1} = \binom{j+1}{\leq k} + x_k = \binom{j}{\leq k} + \binom{j}{\leq k-1} + x_k$$

If $j+1 \leq k$ then $x_k = 0$ so Paul has already won. Suppose then that $j+1 > k$. Since both

$$\binom{j}{\leq k}, \binom{j}{\leq k-1} \leq 2^j$$

there exists an integer y with $0 \leq y \leq x$ so that

$$\binom{j}{\leq k} + y = \binom{j}{\leq k-1} + x - y = 2^j$$

Paul plays $v = (1, 0, \ldots, 0, y)$. This completes the proofs of the Endgame Lemma and the Main Theorem.

Example. With $k = 4, j = 7$ the state $(1, 0, 0, 0, 93)$ has $w_8(1, 0, 0, 0, 93) = \binom{8}{\leq 4} + 93 = 256 = 2^8$. Paul solves

$$\binom{7}{\leq 4} + y = 2^7$$

to find $y = 29$ and so he selects $v = (1, 0, 0, 0, 29)$. If Carole says Yes the new position is $(1, 0, 0, 0, 29)$ and if she says No the new position is $(0, 1, 0, 0, 64)$; for both $w_7 = 128$.

3 The Original Game

We return to the original game with k lies and q moves. In our formulation the original position is $P = (n, 0, \ldots, 0)$. For Paul to win we must have $n\binom{q}{\leq k} \leq 2^q$ and we have seen examples where this condition is not sufficient.

at the end of Early End (i.e., $j \sim \frac{1}{2}\sqrt{\ln n}$) there is at most one nonpenny remaining. We first show that at the begining of Early End there are a bounded number of chips in each position $s < k$ - and hence a bounded number of nonpennies. As $e_s(j)$ is bounded it suffices to show $pp_s(j)$ is bounded. We know

$$pp_s(j) = \sum_{i=0}^{s} x_i \Pr[B(q-j,.5) = s-i]$$

$$1 = \sum_{i=0}^{k} x_i \Pr[B(q,.5) \leq k-i]$$

We bound

$$\Pr[B(q-j,.5) = s-i] < \frac{c_4}{q} \Pr[B(q-j,.5) \leq k-i]$$

We bound

$$\Pr[B(q-j,.5) \leq k-i] \leq 2^j \Pr[B(q,.5) \leq k-i]$$

as if $q-j$ coin flips give at most $k-i$ heads with probability 2^{-j} the next j coin flips will be all tails. Together

$$pp_s(j) = \sum_{i=0}^{s} x_i \Pr[B(q-j,.5) = s-i] < \frac{c_4 2^j}{q} \sum_{i=0}^{s} x_i \Pr[B(q,.5) \leq k-i] < \frac{c_4 2^j}{q}$$

which is less than one in the Early End. Let us define the *nonpenniness* of a state $(y_0, \ldots, y_{k-1}, y_k)$ as $\sum_{i=0}^{k-1}(k-i-1)y_i$ - i.e., the number of moves to the right required to make all nonpennies into pennies. Let M be a bound on the nonpenniness at the start of the Early End - we have shown that M may be taken as an absolute constant. Each round, so long as there is at least two nonpennies remaining, the nonpenniness must decrease by at least one. This is because our alternation of floors and ceilings for Paul assured that if there was more than one nonpenny they could not all be in his set A, nor all not in the set A. (This is the only place where we use the alternation - actually Paul could choose floors and ceilings arbitrarily provided that he makes sure that the nonpennies are neither all in A nor all not in A.) Within M rounds - so certainly by the end of the Early End - Paul reaches a stage where there is at most one nonpenny remaining.

Endgame. For the next Lemma there are no asymptotics - j and even k can be arbitrary.

We select c so that $c2^{-k} \geq c_2$, thus assuring that Paul will survive for the first k rounds.

Now consider the Middle. The probability that $B(q-j,.5) = k-i$ is at least $2^{-(q-j)}$ so that

$$pp_k(j) = \sum_{i=0}^{k} x_i \Pr[B(q-j,.5) = k-i] \geq 2^{-(q-j)} \sum_{i=0}^{k} x_i$$

Here $\sum x_i$ is the number of chips at the beginning of the game. As the maximum weight of a chip is $\binom{q}{\leq k} \leq q^k$ and the total weights of the chips is 2^q the number of chips is at least $2^q/q^k$. Hence

$$pp_k(j) \geq \frac{2^q}{q^k} 2^{-(q-j)} = \frac{2^j}{q^k}$$

so that

$$e_k(j) < c_2 j^k < \frac{2^j}{q^k} \leq pp_k(j)$$

in the Middle and even a bit beyond.

In the Late Middle we must bound a bit more carefully. Our condition of the x's may be written:

$$1 = \sum_{i=0}^{k} x_i \Pr[B(q,.5) \leq k-i]$$

The formula for Perfect Play gives

$$2^{-j} pp_k(j) = \sum_{i=0}^{k} x_i \Pr[B(q-j,.5) = k-i]$$

But for q sufficiently large

$$\Pr[B(q-j,.5) = k-i] > \frac{1}{2} \Pr[B(q-j,.5) \leq k-i] > \frac{1}{4} \Pr[B(q,.5) \leq k-i]$$

Indeed, with $j = o(q)$ these three probabilities are asymptotically equivalent. Thus we may bound

$$pp_k(j) > \frac{1}{4} 2^j > c_2 j^k \geq e_k(j)$$

The above argument applies for the Early End j as well so that Paul may continue applying Fictitious Play. Our object now will be to show that

The $p_i - 2v_i$ are 0 or ± 1 and the nonzero values alternate signs. Hence the sum is at most $\binom{j}{k}$ in absolute value and hence

$$\left| v_k - \frac{p_k}{2} \right| < \frac{1}{2}\binom{j}{k} < j^k$$

Now we bound (not worrying too much about constant factors)

$$\left| fic_k(j) - \frac{1}{2}(fic_k(j+1) + fic_{k-1}(j+1) \right| \leq j^k + 1$$

and so

$$e_k(j) \leq j^k + 1 + \frac{1}{2}e_{k-1}(j+1) + \frac{1}{2}e_k(j+1) \leq c_1 j^k + \frac{1}{2}e_k(j+1)$$

as we can absorb $e_{k-1}(j+1)$, which was previously absolutely bounded, into the constant c_1. Now uniformly over $j \geq 1$ we bound

$$e_k(j) \leq \sum_{x=j}^{q} c_1 x^k 2^{j-x} \leq c_1 j^k \sum_{y=0}^{\infty} \left(\frac{j+y}{j}\right)^k 2^{-y}$$

Here we have set $y = x - j$. Effectively, old errors have been ameliorated by the halving process. The sum is maximized when $j = 1$ but even then $\sum (1+y)^k 2^{-y}$ is convergent so that $e_k(j) \leq c_2 j^k$. $\square$

Paul's Strategy. Paul's strategy is actually quite simple to describe. He play Fictitious Play until there is at most one nonpenny remaining on the board. At that point a specialized (though quite straightforward) strategy that we will call Endgame sees him through to the end. The analysis of this strategy requires proving that Fictitious Play doesn't leave him with a negative number of pennies. We split the analysis into several stages.

- First Steps. $0 \leq q - j < k$, the first k rounds.
- Middle. $k \leq q - j$ and $j > (\ln q)^2$
- Late Middle. $(\ln q)^2 \geq j > \sqrt{\ln q}$
- Early End. $\sqrt{\ln q} \geq j \geq \frac{1}{2}\sqrt{\ln q}$
- Endgame. $\frac{1}{2}\sqrt{\ln q} \geq j \geq 0$

To show that Fictitious Play can be actually played by Paul we must show for each j that $fic_k(j) \geq 0$. We shall do this by showing the inequality

$$e_k(j) \leq pp_k(j)$$

We first consider the First Steps. We have shown $e_k(j) \leq c_2 j^k \leq c_2 q^k$. But in the First Steps

$$pp_k(j) \geq pp_k(q) 2^{-(q-j)} \geq x_k 2^{-k} \geq c 2^{-k} q^k$$

We shall show that Fictitious Play is fairly close to Perfect Play. For $0 \leq i \leq k$ and $q \geq j \geq 0$ we define the error functions

$$e_i(j) = |pp_i(j) - fic_i(j)|$$

Lemma. There is a constant c_2 so that for all $j \geq 1$

$$e_k(j) \leq c_2 j^k$$

We first note that for all j

$$e_0(j) \leq 1$$

With perfect play the zeroth coordinate is $x_0 2^{-(q-j)}$ with j questions remaining (i.e., it halves each round) while with fictitious play it is either $\lfloor x_0 2^{-(q-j)} \rfloor$ or $\lceil x_0 2^{-(q-j)} \rceil$ since each round it halves with roundoff. We also note trivially that all $e_i(q) = 0$ as the game has not yet begun.

Now let $1 \leq i \leq k$. Then the inductive definition of perfect play gives

$$pp_i(j) - \frac{1}{2}(pp_i(j+1) + pp_{i-1}(j+1)) = 0$$

In contrast, now with $1 \leq i < k$

$$\left| fic_i(j) - \frac{1}{2}(fic_i(j+1) + fic_{i-1}(j+1)) \right| \leq 1$$

since both v_i and v_{i-1} may be at most $\frac{1}{2}$ away from $p_i/2$ and $p_{i-1}/2$ respectively. Subtracting we bound for $1 \leq i < k$

$$e_i(j) \leq 1 + \frac{1}{2} e_i(j+1) + \frac{1}{2} e_{i-1}(j+1)$$

Set $M_i = 2^{i+1} - 1$ so that $M_0 = 1$ and $M_i \leq 1 + \frac{1}{2} M_i + \frac{1}{2} M_{i-1}$. (It is only important for the argument that the M_i be constants.) Then a double induction (first on i then on j) gives that for $0 \leq i < k$ and $q \geq j \geq 0$

$$e_i(j) \leq M_i$$

Pennies are special. In fictitious play having chosen $v_0, \ldots, v_{k-1}$ we determine v_k by the equation

$$0 = \Delta_j(P, v) = (2v_k - p_k) + \sum_{i=0}^{k-1} (p_i - 2v_i) \binom{j}{k-i}$$

In general, to find v_k we get an equation to solve of the form $\Delta_j = 2v_k - A = 0$. We claim A will always be even. For any integral vector v since $w_j(YES(P,v)) + w_j(NO(P,v)) = 2^{j+1}$ is even, $\Delta_j(P,v) = w_j(YES(P,v)) - w_j(NO(P,v))$ is also even and hence A must be even. The problem is: A, and hence v_k, might be *negative*. An an example, again with $k = 2, j = 10$, consider the position $P = (29,8,9)$, again with $w_{11}(P) = 29(67)+8(12)+9 = 2^{11}$. Now if Paul selects $v_0 = 15, v_1 = 4$ then

$$\Delta_{10}((29,8,9),(15,4,v_2)) = \binom{10}{2} + (2v_2 - 9) = 0$$

has the solution

$$v_2 = \frac{9-45}{2} = -18$$

In Fictitious Play we imagine Paul and Carole continuing to play formally (i.e., with state P Paul selects v and then Carole changes the state to either $YES(P,v)$ or $NO(P,v)$), even though the number of pennies may turn negative. Note that the other coordinates will remain positive. We let

$$fic(j) = (fic_0(j), fic_1(j), \ldots, f_k(j))$$

denote the state P when there are j rounds remaining in the game. Thus $fic(q)$ is simply the initial state of the game. Actually, there are many possible values of $fic(j)$ dependent on both Paul's choices of floor or ceiling and Carole's choices of Yes or No. When we give (as we shall) inequalities involving $fic_i(j)$ we mean that these inequalities hold regardless of Paul and Carole's choices. We shall show that, under our conditions, Fictitious Play will *not* leave us with negative numbers $f_k(j)$ of pennies.

Perfect Play. When the state is P Paul selects $v = \frac{P}{2}$. Again, we imagine Paul and Carole playing formally. (Another useful image is that the chips may be split into halves, quarters, etc.) In Perfect Play $YES(P,v) = NO(P,v)$ so that we may define uniquely the state

$$pp(j) = (pp_0(j), pp_1(j), \ldots, pp_k(j))$$

when j rounds remain. These are defined inductively by

$$pp(j) = YES(pp(j+1), \frac{pp(j+1)}{2})$$

In perfect play the number of chips that move to the right is precisely the expected number had one flipped a fair coin. Hence

$$pp_k(j) = \sum_{i=0}^{k} x_i \Pr[B(q-j,.5) = k-i]$$

Here is the core of Paul's strategy. Initially $w_q(P) = 2^q$. If at any stage of the game there are j moves left and the state is P with $w_j(P) > 2^j$ then Carole has won. Suppose there are $j+1$ moves to go and $w_{j+1}(P) = 2^{j+1}$. Paul selects v and now Carole has the choice of whether the new position is $YES(P,v)$ or $NO(P,v)$. If $\Delta_j(P,v) \neq 0$ then one of those positions will have w_j value bigger than 2^j, Carole can select it and she wins. Paul's only hope (which turns out often to succeed) is if for each j when there are $j+1$ moves remaining he selects v with $\Delta_j(P,v) = 0$. If he can do that then by induction (going down from q to 0) $w_j(P) = 2^j$ where P is the state with j questions remaining.

A calculation gives

$$\Delta_j(P,v) = \sum_{i=0}^{k} (v_i - (p_i - v_i)) \binom{j}{k-i}$$

We may think of Paul deciding for each chip c whether to place c in A. Suppose c is in position i. If he does place c in A then he adds $\binom{j}{k-i}$ to Δ_j. If he leaves c out of A he subtracts the same amount from Δ_j. His objective is to make these decisions so that their effects balance out precisely. The chips at position k have a special function, we shall call them *pennies*. Placing a penny in or out of A will either add or subtract one from Δ_j. Now we introduce Fictitious Play and Perfect Play. As usual we assume there are $j+1$ moves remaining in the game.

Fictitious Play. Paul selects for $0 \leq i < k$

$$v_i = \lfloor \frac{p_i}{2} \rfloor \text{ or } \lceil \frac{p_i}{2} \rceil$$

He alternates the choice of floor or ceiling among those i for which p_i is odd. (This only comes in near the end of the argument.) He now picks v_k so that $\Delta_j = 0$.

As an example, let $k = 2, j = 10$ and consider the position $P = (3, 7, 1763)$ which has $w_{11}(P) = 3(67) + 7(12) + 1763 = 2^{11}$. Paul selects, say, $v_0 = 2, v_1 = 3$. Then

$$\Delta_{10}((3,7,1763),(2,3,v_2)) = \binom{10}{2} - \binom{10}{1} + (2v_2 - 1763) = 0$$

has the solution

$$v_2 = \frac{1763 - 35}{2} = 864$$

2 The Main Result

Our object will be to give a partial converse to the above statement. Let us first give an example that shows that the complete converse is *not* valid. Let $k = 1, n = 5$ (i.e., $x_0 = 5, x_1 = 0$ and let $q = 5$ so that $w_5(5,0) = 5(6) = 30 \leq 2^5$. Carole is thinking of a number from one to five, she may lie once, and Paul has five questions. The first question that best splits the possibilities is "Is $x \leq 2$?". If Carole says "No" the new position is $(3,2)$ and $w_4(3,2) = 3(5) + 2 > 2^4$ so that Carole wins. In a certain sense, this example shows that there is a problem with integrality - we can't split five possibilities into two equal groups!

Main Theorem. There are constants c, q_0 (dependent on k) so that the following holds for all $q \geq q_0$: If $w_q(x_0, \ldots, x_k) \leq 2^q$ and

$$x_k > cq^k$$

then Paul wins.

If Paul wins for some $(x_0, ..., x_k)$ then he surely wins if x_k is decreased to any $x'_k < x_k$. Thus it suffices to prove the Main Theorem under the stronger assumption

$$w_q(x_0, \ldots, x_k) = 2^q$$

Henceforth we shall make this assumption.

Let $P = (x_0, \ldots, x_k)$, $v = (v_0, \ldots, v_k)$ be vectors. We define

$$YES(P, v) = (v_0, v_1 + x_0 - v_0, v_2 + x_1 - v_1, \ldots, v_k + x_{k-1} - v_{k-1})$$

$$NO(P, v) = YES(P, P - v) = (x_0 - v_0, x_1 - v_1 + v_0, \ldots, x_k - v_k + v_{k-1})$$

When the current state is P and Paul selects a set of chips consisting of v_i chips on position i then $YES(P, v)$ is the new position if Carole answers yes while $NO(P, v)$ is the new position if Carole answers no. The definitions above apply to any real values vectors. For $j > 0$ and any P, v we calculate

$$w_j(YES(P, v)) + w_j(NO(P, v)) = w_j(x_0, x_1 + x_0, x_2 + x_1, \ldots, x_k + x_{k-1})$$

$$= \sum_{i=0}^{k} x_i \left(\binom{j}{\leq k - i} + \binom{j}{\leq k - i - 1} \right) = \sum_{i=0}^{k} x_i \binom{j+1}{\leq k - i} = w_{j+1}(P)$$

We further define

$$\Delta_j(P, v) = w_j(YES(P, v)) - w_j(NO(P, v))$$

The coin flips are done separately each round. Now a strategy for Paul has a probability of winning. For each chip c let X_c be the indicator random variable for c to remain on the board at the end of the game. Regardless of Paul's strategy each chip will move forward with probability .5 each turn - if the coin flip "matches" whether $c \in A$ - and the movements on the different turns are mutually independent. If c starts at position j its position at the end of the game is given by $j + B(q, .5)$, or "off the board" if this is larger than k. Thus $E[X_c]$, the probability of remaining on the board, is precisely the weight of the chip c. Let $X = \sum X_c$, the sum over all chips c. Linearity of Expectation gives $E[X] = \sum E[X_c]$, which is the weight of the state which we assume to be greater than one. In particular, that implies we cannot have $X \leq 1$ always so that with positive probability Carole will win.

However, this is a perfect information game and so with perfect play either Paul or Carole will always win. Since no strategy allows Paul to always win there is a strategy (not randomized) so that Carole always wins! $\square$

We introduce a useful notation:

$$\binom{j}{\leq s} = \sum_{t=0}^{s} \binom{j}{t}$$

Note that $\binom{j}{\leq 0} = 1$ and that if $s \geq j$ then $\binom{j}{\leq s} = 2^j$. The critical property is:

$$\Pr[B(j, .5) \leq s] = \binom{j}{\leq s} 2^{-j}$$

Let $j \geq 0$. We define weight function

$$w_j(x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k} x_i \binom{j}{\leq k - i}$$

Note that in a game with j rounds this is 2^j times the previously defined weight. The integrality of this weight function will prove useful. We will continue to use q to represent the total number of rounds in the game and we will use j to represent the number of rounds remaining at some intermediate point. It will be useful in analysis to consider the function w_j defined when the x_i are arbitrary real numbers. Now we may rephrase our theorem:

If $w_q(x_0, \ldots, x_k) > 2^q$ then Carole wins.

positions" of the n, q, k game. In this sense x_i gives a count on those x for which if x is the answer then Carole has already lied i times.

We like to think of this game in terms of chips. Imagine a board with positions marked (from left to right) $0, 1, \ldots, k$. There is one chip for each possible answer x. A chip is placed on position i when if x is the answer Carole can lie at most $k - i$ more times. Thus the $x_0, \ldots, x_k$ game starts with x_i chips on position i for each i. In this context how is the game played? Each round (q is now the number of rounds) Paul selects a set A of chips, corresponding to asking the question "Is $x \in A$?". A "No" answer by Carole would mean that, for each $x \in A$, if x is the answer then it has been lied about one more time. This corresponds to moving all chips in A one position to right. Chips that were in position k are removed from the board. A "Yes" answer by Carole corresponds to moving all chips not in A one position to the right. That is: Paul selects a set A of chips and Carole selects whether to move all chips in A or all chips not in A one position to the right. Carole is not permitted to move all the chips off the board (though this would not occur in actual play). Paul wins if at the end of the game there is precisely one chip remaining on the board. We define the *state* to be the vector $P = (x_0, \ldots, x_k)$, or, in the chip board formulation, the picture with x_i chips on position i, $0 \leq i \leq k$. The state will change during the game as the chips are moved.

Work on liar games has been inspired in the last generation by comments in the autobiography of Stan Ulam [6]. This author was involved in one of the early papers, Kleitman *et. al.* [3]. Pelc [4] has completely solved the case where Carole can lie at most $k = 1$ time. There has been a spurt of recent work, most notably [1,2] The specific names Paul and Carole were not randomly chosen. The initials P and C refer to Pusher-Chooser games investigated by this author in, e.g., [5]. Paul may be considered the Great Questioner - Paul Erdős. And Carole may be thought of as her acronym - Oracle!

A Fundamental Inequality. We define the weight of a chip on position i as $\Pr[B(q, .5) \leq k - i]$. Here $B(q, .5)$ is the standard Binomial Distribution, the number of heads in q flips of a fair coin. The weight of a state is defined as the sum of the weights of the chips.

Theorem. If a state has weight more than one then Carole wins.

Proof. We first imagine Carole announcing a random strategy - whatever set A Paul selects Carole will then flip a fair coin to decide whether to move the chips of A or the chips not in A one position to the right. (If by this strategy all chips are removed we will agree that Carole has lost.)

Ulam's Searching Game with a Fixed Number of Lies
Joel Spencer

1 Basics

Our investigations concern a game with two players, named Paul and Carole and three parameters n, q, k, known to both players. Carole thinks of an integer x from one to n. Paul has q questions with which to determine x. The questions must be of the form "Is $x \in A$?", where $A \subseteq \{1, \ldots, n\}$. He (Paul) may use previous answers before deciding his next question. Carole is permitted to lie but she (Carole) may lie at most k times through the entire course of the game. Paul wins if at the end of the q questions there is a unique possible value for x. We allow Carole to play an adversary strategy - i.e., Carole does not actually pick an x but answers all questions so that there is at least one x that she could have had picked. Now the game is one of perfect information and so we can say for given n, q, k that either Paul or Carole will win the game. The question is - who wins? Note that when $k = 0$ the game reverts to the classical "Twenty Questions" and Paul wins if and only if $n \leq 2^q$. *Throughout this paper we shall consider k a fixed positive integer.*

In §3 we give, for fixed k and q sufficiently large dependent on k, necessary and sufficient conditions on n for Paul to win. Mathematically, however, we think of the "Main Theorem" of §2 as the central result and the results of §3 as basically corollaries.

We shall actually analyse a generalization of this game with the single parameter n replaced by a sequence of nonnegative integers $x_0, x_1, \ldots, x_k$. Let A_i, $0 \leq i \leq k$, be disjoint sets, with $|A_i| = x_i$, these sets known to both players. Now Carole selects $x \in A_0 \cup \ldots \cup A_k$. If $x \in A_i$ then Carole is permitted to lie at most $k - i$ times. Again, Carole can play an adversary strategy so that either Paul or Carole will win the game. The n, q, k games corresponds to $x_0 = n$, $x_1 = \ldots = x_k = 0$. The more general use of $x_0, \ldots, x_k$, besides its intrinsic interest, is useful for analysing "middle