to the entire argument.) Given that the probes find $E_1, \ldots, E_k$ their birth-times $t_{E_1}, \ldots, t_{E_k}$ are uniform in $[0,t]$, just as with the birth model. As these asymptotics hold for each of the finite number of processings y they hold for T as a whole. $\Box$

The finiteness of the branching process gives that $\sum f(T,c) = 1$, the summation over all broodtrees T. Therefore History almost surely does not abort and the $\lim^*$ distribution of its broodtree approaches the branching process distribution. Thus the $\lim^*$ probability of x surviving is $f(c)$.

References

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common birthdate t_E. Now y is considered processed. History halts, if not aborted, when all $y \in T$ are processed.

Example: $V = \{1, \ldots, n\}$, $x = 1$, $c = 3.7$, $Q = 2$. Suppose $t_{123} = 2.6$, $t_{245} = 1.5$ are the only relevant birthdates. Then History creates a broodtree with root 1 having brood $2,3$ and 2 having brood $4,5$. Then childless $3,4,5$ survive, 2 doesn't survive since it has a brood all surviving and so 1 does survive. In the random greedy algorithm 245 is added to P at time 1.5 so that 123 is not added to P at time 2.6 and so 1 is still surviving at time 3.7.

Claim: If History does not abort then root x survives broodtree T in the sense of §2 if and only if x survives at time c in the sense of §1.

Proof: If $T = \{x\}$ then no $E \in H$ with $x \in E$ has $t_E < c$ so x survives at time c. Otherwise, say $x \in E \in H$ with $t_E < c$. (There may be several such E.) For $y \in E - \{x\}$ procedure History applied to y, t_E would yield the subtree $T(y)$ of T rooted at y. By induction on tree size y survives at time t_E if and only if y survives broodtree $T(y)$.

If x does not survive T then some $E = \{x, y_1, \ldots, y_Q\}$ with $y_1, \ldots, y_Q$ all surviving. By induction all $y_1, \ldots, y_Q$ survive at time t_E. Either E is placed in P at time t_E or $x \in E'$ which is already in P at time t_E. Either way $x \in E^*$ for some $E^* \in P_c$.

Inversely, if x does survive T then for each such E there is a $y = y_1$ that does not survive. By induction $y \in E^*$ for some $E^* \in P_t$, $t = t_E$. Then E is not placed into P at time t_E. As this holds for all such E there is no $E \in P_c$ with $x \in E$. □

Let T be a fixed (abstract) broodtree. Let $f_{x,H}(T, c)$ denote the probability that History does not abort and yields broodtree T. Let $f(T, c)$ denote the probability that the branching process of §2 yields broodtree T.

Theorem: $\lim^* f_{x,H}(T, c) = f(T, c)$.

Proof: During procedure History consider the processing of y with $t_y = t$. All $O(1)$ already processed z have only $o(D)$ common edges with y. Thus $D(1 - o(1))$ edges $E \in H$ are now probed. Of these $o(D)$ contain one of the $O(1)$ currently unprocessed $z' \in T$ so almost surely none of these have $t_E < c$ and so cause History to abort. There are $o(D^2)$ probed pairs E, E' with $|E \cap E'| > 1$ since for each of the $\sim D$ choices of E there are $Q = O(1)$ choices for $z \in E - \{y\}$ and then $o(D)$ choices of E' containing y, z. Thus almost surely no such E, E' have $t_E, t_{E'} < t$ causing History to abort. The number of probed E with $t_E < t$ has Binomial Distribution $B(m, t/D)$ where m is the number of probed sets so $m \sim D$. For fixed k (recall $D \to \infty$) the probability that there are precisely k such E is then $\sim e^{-t}t^k/k!$ as desired. (In some way this approximation of the Binomial by the Poisson is the key one brood and the second (again by monotonicity) a lower bound on the probability of a brood all surviving. As $f(c+\Delta c) = f(c)(1-p)$

$$-(\Delta c)f(c)^{Q+1} \leq f(c+\Delta c) - f(c) \leq -(1-e^{-\Delta c})f(c)f(c+\Delta c)^{Q} \qquad (7)$$

As f is continuous and $1 - e^{-\Delta c} \sim \Delta c$ as $\Delta c \to 0$ this gives that the derivative

$$f'(c) = \lim_{\Delta c \to 0} \frac{f(c+\Delta c) - f(c)}{\Delta c} = -f(c)^{Q+1} \qquad (8)$$

We add an initial condition $f(0) = 1$ since an Eve with birthdate 0 must be childless and hence survives. This differential equation with initial condition has the unique solution

$$f(c) = [1 + Qc]^{-1/Q} \qquad (9)$$

Note that indeed $\lim_{c\to\infty} f(c) = 0$.

4 The Limit of the Discrete

Throughout this section $c > 0$ is fixed. Our object is to show $\lim^* f_{x,H}(c) = f(c)$. We fix x, H. We describe a procedure we call History which either aborts or produces a broodtree. History contains a set T of vertices, initially $T = \{x\}$. The $y \in T$ are either processed or unprocessed (one can imagine a stack here), x is initially unprocessed. To each $y \in T$ is assigned a "birthtime" t_y, we initialize $t_x = c$. Each $y \in T$, $y \neq x$, will also have a parent and Q wombmates (including itself) so that T will have a broodtree structure with x the root.

In analyzing History it will be useful to imagine the values t_E, $E \in H$ as hidden from us and that probes are made from an Oracle. When E has not yet been probed the conditional distribution of t_E remains, as initially, uniform in $[0, D]$.

The basic step of History is to take an unprocessed $y \in T$ (so $y = x$ the first time) and process it as follows. Probe the Oracle for the value of all t_E with $y \in E$ but with no $z \in E$ that has already been processed. (This assures all probes are new.) If either

• Some E has $t_E < t_y$ and $y, z \in E$ with $z \in T$

or

• Some E, E' have $t_E, t_{E'} < t_y$ with $y \in E, E'$ and $|E \cap E'| > 1$

then the entire procedure History is aborted. Otherwise for each probed E with $t_E < t_y$ add all $z \in E - \{y\}$ to T as wombmates with parent y and

i Eve has an expected number e^{Qc} descendents, including herself. Pick $K > e^{cQ}\epsilon^{-1}$. $\Box$

In particular, Eve's family tree is finite with probability one.

A rooted tree with the notions of parent, child, root, birthorder and wombmate shall be called a *broodtree*. Given a finite broodtree T we define the notion of a vertex surviving or (its negation) dying. A childless vertex survives. A vertex v dies if and only if it has at least one brood $w_1, \ldots, w_Q$ all of whom survive. Given T one can then work one's way up from the leaves to see if the root survives.

We let $f(c)$ denote the probability that Eve survives in the broodtree T given by the above process. Our objects will be to show

$$\lim_{c\to\infty} f(c) = 0 \tag{4}$$

and that for any fixed c

$$\lim^{*} f_{x,H}(c) = f(c) \tag{5}$$

Together these imply (2) and thus will complete the argument.

3 A Differential Equation

Here we show (4). Let $c \geq 0$, $\Delta c > 0$. We first claim

$$f(c) \geq f(c+\Delta c) \geq f(c)(1-\Delta c) \tag{6}$$

For an Eve with birthdate $c+\Delta c$ to survive she must have no births in $[0, c)$ for which the whole brood survives and no births in $[c, c+\Delta c)$ for which the whole brood survives. The nature of the Poisson distribution makes these independent events. The first has probability precisely $f(c)$. The second has probability at most one (trivially) and at least $1 - \Delta c$ as Eve may be childless in that interval. Thus $f(c)$ is a decreasing and continuous function of c.

Now consider more carefully the probability p that Eve, with birthdate $c + \Delta c$, has a (at least one) birth in $[c, c + \Delta c)$ for which the whole brood survives. We bound $p \leq (\Delta c)f(c)^Q$ since Δc is the expected number of broods and given a brood with birthdate $x \in [c, c + \Delta c)$ the probability that they all survive is $f(x)^Q \leq f(c)^Q$ by monotonicity. But also $p \geq (1 - e^{-\Delta c})f(c + \Delta c)^Q$, the first factor being the probability Eve has at least distinct $B, B' \in V$ their codegree is maximized when $|B \cup B'| = l+1$ so that

$$\text{codeg}(B, B') \leq \binom{n-l-1}{k-l-1} = o(D)$$

Our thanks to Nati Linial for noting that Pippenger's Theorem is the natural setting for the branching process approach. Indeed, our original proof was only for the Erdős-Hanani case. V. Rödl and L. Thoma [3] recently gave a different proof of our main result. They show that the random greedy algorithm gives a packing in some sense close in distribution to that given by the "Rödl nibble" argument of [2].

2 The Continuous Model

We turn to a continuous time branching process that will appropriately model the asymptotic behavior of x surviving. Fix $c > 0$. Begin with Eve, having birthdate c. Time goes backwards. Eve gives birth in the time interval $[0, c)$ with a unit density Poisson distribution. (That is, in an infinitesmal time interval $[a, a + da)$ Eve has probability da of giving birth. Equivalently, Eve has probability $e^{-c}c^k/k!$ of having precisely k births and conditioning on her having k births their times $x_1, \ldots, x_k$ are independently and uniformly distributed on $[0, c)$. A key property of this Poisson process is that if T, T' are disjoint time intervals and X, X' the number of births on these respective intervals then X, X' are independent.) When Eve does give birth at time a she does so to a brood of Q new offspring, all with birthdate a. Offspring in the same brood are called wombmates. (Note this is *not* the same as sibling!) Each offspring with birthdate a gives (or does not give) birth independently in $[0, a)$ with the same Poisson distribution, again always to broods of size Q. The process is iterated, each offspring may in turn beget offspring. Parent, child, descendent all have their natural meaning. For purposes of enumeration we assign each offspring of a brood a birthorder from 1 to Q.
Claim 1. For all $\epsilon > 0$ there exists K so that with probability at least $1 - \epsilon$ Eve has at most K descendents.

The expected number of i-th generation offspring of Eve is given by $Q^i c^i / i!$. The first factor is the choices for birthorder. The remaining is the integral of $dx_1 \cdots dx_i$ over all $0 \leq x_1 < \ldots < x_i < c$. The volume over $0 \leq x_1, \ldots, x_i < c$ is c^i and the denominator may be thought of as the probability that uniform x_j are in a particular order. Summing over

Let's see why this implies (1), indeed why the random greedy algorithm almost surely yields P with $|P| \sim \frac{n}{Q+1}$. Let $\gamma > 0$ be arbitrary. Using (2) pick c, D_0, ϵ so that for any (as defined below) (D_0, ϵ)-good hypergraph H and any $x \in H$ $f_{x,H}(c) < \gamma^2$. The expected number of x surviving at time c is then less than $\gamma^2 n$. With probability at least $1 - \gamma$ the number of x surviving is less than γn. But then the $(Q+1)$-sets of P_c must cover at least $(1-\gamma)n$ vertices so $|P| \geq (1-\gamma)\frac{n}{Q+1}$.

Definitions and Formalisms. A $(Q+1)$-uniform hypergraph H on vertex set V is a family of subsets E of V, called edges, all of size $Q+1$. For $x \in V$ the degree $\deg(x)$ is the number of $E \in H$ with $x \in E$ and for distinct $x, y \in V$ the *codegree* $\text{codeg}(x,y)$ is the number of $E \in H$ with $x, y \in E$. A *packing* P is a family of edges so that $E \cap E' = \emptyset$ for all distinct $E, E' \in P$.

We consider $Q \geq 1$ fixed for all asymptotics. Call a hypergraph H (D_0, ϵ)-good if there exists $D \geq D_0$ so that

$$D(1-\epsilon) \leq \deg(x) \leq D(1+\epsilon)$$

for all $x \in V$ and

$$\text{codeg}(x,y) < \epsilon D$$

for all distinct $x, y \in V$. The notation $z = o(1)$ has the interpretation that for all $\epsilon' > 0$ there exist D_0, ϵ so that $|z| \leq \epsilon'$ for any (D_0, ϵ)-good H. The notation $\lim^*_{H,x} z = a$ means for all $\epsilon' > 0$ there exist D_0, ϵ so that $|z - a| < \epsilon'$ for any (D_0, ϵ)-good H and any $x \in V(H)$.

Erdős-Hanani-Rödl. The original impetus for this work came from a 1963 conjecture of Paul Erdős and Haim Hanani [1]. Let $m(n,k,l)$ denote the maximal size of a family F of k-element subsets of an n-set so that no l-set is contained in more than one of the k-sets. In graphtheoretical terms $m(n,k,l)$ is the maximal number of k-cliques one can pack into the complete l-graph on n points. Elementary counting gives $m(n,k,l) \leq \binom{n}{l}/\binom{k}{l}$. They conjectured and Vojtech Rödl[2] proved that for every fixed $2 \leq l < k$

$$\lim_{n\to\infty} m(n,k,l)\frac{\binom{k}{l}}{\binom{n}{l}} = 1 \tag{3}$$

Pippenger's Theorem generalized this result. Let $[U]^l$ denote $\{W \subseteq U : |W| = l\}$. Fix an n-set Ω, set $V = [\Omega]^l$ and for each $A \in [\Omega]^k$ let $E_A = [A]^l \subset V$. Let H be the hypergraph of all such E_A. Then V has size $n' = \binom{n}{l}$. H is $\binom{k}{l}$-uniform. Every $B \in V$ has $\deg(B) = \binom{n-l}{k-l} = D$. For

Asymptotic Packing via A Branching Process

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Abstract

It is shown that under certain side conditions the natural random greedy algorithm almost always provides an asymptotically optimal packing of disjoint hyperedges from a hypergraph H.

1 The Problem

Let H be a $(Q+1)$-uniform hypergraph on vertex set V of size n. (Definitions and formalisms are given below.) Assume $\deg(x) = D(1+o(1))$ with $D \to \infty$ for all $x \in V$ and that every pair $x, y \in V$ lie in only $o(D)$ common edges. N. Pippenger[4] showed that there exists a packing P with

$$|P| \geq \frac{n}{Q+1}(1 - o(1)) \tag{1}$$

Here we give a new proof of this result. In fact, we show that a random greedy algorithm almost always gives P of the desired size.

We describe the algorithm with a handy parametrization. For every edge $E \in H$ assign independently and uniformly a real "birthtime" $t_E \in [0, D]$. (Technically we further require the t_E to be distinct. This occurs with probability one.) This orders H. Take them one by one, accepting E if it does not overlap an $E' \in H$ previously accepted. The family P of accepted E will be a packing and we need show $|P| \sim \frac{n}{Q+1}$ almost surely. Rather than consider the full process we stop the process at "time" c, only considering those E with $t_E < c$. This gives a family P_c. Let $x \in V$. We say *x survives at time c* if no $E \in P_c$ contains x. Let $f_{x,H}(c)$ denote the probability of x so surviving. Our object will be to show (formalisms below)

$$\lim_{c \to \infty} \overset{*}{\lim_{x,H}} f_{x,H}(c) = 0 \tag{2}$$