Fundamental Algorithms, Assignment 7
Solutions

1. Determine an LCS of 10010101 and 010110110.
   `Solution:` We create an eight by eight array giving $C[m, n]$, the length of the LCS between the first m of the first sequence and the first n of the second sequence.

   Here is array. The sequences are placed on top and on the left for convenience. The numbering starts at 0 so that the row zero and column zero are all zeroes.

-	-	0	1	0	1	1	0	1	1	0
-	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1
0	0	1	1	2	2	2	2	2	2	2
0	0	1	1	2	2	2	3	3	3	3
1	0	1	2	2	3	3	3	4	4	4
0	0	1	2	3	3	3	4	4	4	5
1	0	1	2	3	4	4	4	5	5	5
0	0	1	2	3	4	4	5	5	5	6
1	0	1	2	3	4	5	5	6	6	6

   So the length is 6. Start at the bottom right and walk until hitting the edge. At (i, j) go diagonal left if $C[i, j] = C[i-1, j-1] + 1$; if not go left or up, whichever is $C[i, j]$. (We'll go left if they both are.) This gives

-	-	0	1	0	1	1	0	1	1	0
-	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1
0	**0**	1	1	2	2	2	2	2	2	2
0	0	**1**	1	2	2	2	3	3	3	3
1	0	0	**2**	2	3	3	3	4	4	4
0	0	1	2	**3**	**3**	3	4	4	4	5
1	0	1	2	3	4	**4**	4	5	5	5
0	0	1	2	3	4	4	**5**	**5**	5	6
1	0	1	2	3	4	5	5	6	**6**	**6**

   The places where you go diagonally left are the same in both sequences and these give the common sequence **010101**. Note that there is no uniqueness to the sequences themselves.

-	-	**0**	**1**	**0**	1	**1**	**0**	1	**1**	0
-	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1
0	**0**	1	1	2	2	2	2	2	2	2
0	0	**1**	1	2	2	2	3	3	3	3
1	0	0	**2**	2	3	3	3	4	4	4
0	0	1	2	**3**	**3**	3	4	4	4	5
1	0	1	2	3	4	**4**	4	5	5	5
0	0	1	2	3	4	4	**5**	**5**	5	6
1	0	1	2	3	4	5	5	6	**6**	**6**

2. Write all the parenthesizations of $ABCDE$. Associate them in a natural way with (setting $n = 5$) the terms $P(i)P(5-i)$, $i = 1, 2, 3, 4$ given in the recursion for $P(n)$.
`Solution:`Splitting $1-4$ gives $P(1)P(4) = 5$ parenthesizations:

$$A(B(C(DE))), A(B((CD)E)), A((BC)(DE)), A((B(CD))E), A(((BC)D)E)$$

Splitting $4-1$ gives $P(4)P(1) = 5$ parenthesizations:

$$(A(B(CD)))E, (A((BC)D))E, ((AB)(CD))E, (((AB)C)D)E, ((A(BC))D)E$$

Splitting $2-3$ gives $P(2)P(3) = 2$ parenthesizations:

$$(AB)((CD)E), (AB)(C(DE))$$

Splitting $3-2$ gives $P(3)P(2) = 2$ parenthesizations:

$$((AB)C)(DE), (A(BC))(DE)$$

3. Let $x_1, \ldots, x_m$ be a sequence of distinct real numbers. For $1 \leq i \leq m$ let $INC[i]$ denote the length of the longest increasing subsequence ending with x_i. Let $DEC[i]$ denote the length of the longest decreasing subsequence ending with x_i.

   (a) Find an efficient method for finding the values $INC[i]$, $1 \leq i \leq n$. (You should find $INC[i]$ based on the previously found $INC[j]$, $1 \leq j < i$. Your algorithm should take time $O(n^2)$.)
   `Solution:`The longest increasing subsequence ending in x_i is either simply x_i or it is obtained by appending x_i to some subsequence ending in x_j where $j < i$. One can do that if and only if $x_j < x_i$. So we should take $INC[i]$ to be 1 (x_i itself) plus the

maximum of the $INC[j]$, $j < i$, for which $x_j < x_i$. However, if there are no such j (for example, when $i = 1$) the default value should be 1. Each $INC[i]$ then takes a single loop which is time $O(n)$ and so the total time is $O(n^2)$. (Of course, $DEC[i]$ can be found similarly.)

(b) Let LIS denote the length of the longest increasing subsequence of $x_1, \ldots, x_m$. Show how to find LIS from the values $INC[i]$. Similarly, let DIS denote the length of the longest decreasing subsequence of $x_1, \ldots, x_m$. Show how to find DIS from the values $DEC[i]$.

`Solution:` LIS is simply the maximum of all $INC[i]$, $1 \leq i \leq n$, as the subsequence has to end somewhere. Similarly, DIS is simply the maximum of all $DEC[i]$, $1 \leq i \leq n$.

(c) Suppose $i < j$. *Prove* that it is impossible to have $INC[i] = INC[j]$ *and* $DEC[i] = DEC[j]$.

`Solution:` Suppose $x_i < x_j$. Then $INC[j] \geq INC[i] + 1$ since you can take the maximal increasing sequence ending at x_i and append x_j. (That may not be optimal, but $INC[j]$ is at least that length.)

Similarly, suppose $x_i > x_j$. Then $DEC[j] \geq DEC[i] + 1$ since you can take the maximal decreasing sequence ending at x_i and append x_j.

(d) Deduce the following celebrated results (called the Monotone Subsequence Theorem) of Paul Erdős and George Szekeres: Let $m = ab + 1$. Then any sequence $x_1, \ldots, x_m$ of distinct real numbers either $LIS > a$ or $DIS > b$. (Idea: Assume not and look at the pairs $(INC[i], DEC[i])$.)

`Solution:` If $LIS \leq a$ and $DIS \leq b$ then there are only ab possibilities for the pair $(INC[i], DEC[i])$, but from the previous part we have $ab + 1$ distinct pairs!

4. Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $5, 10, 3, 12, 5, 50, 6$. `Solution:`

The matrix chain product of $\mathrm{A_1 A_2 A_3 \ldots A_n}$ can be broken down to $\mathrm{(A_1 \ldots A_k)(A_{k+1} \ldots A_n)}$. To find an optimal parenthesization for n matrices, we find the subset of k matrices, where $k < n$. And then compose them altogether.

In our algorithm, we have two matrices, one to record the minimum number of operations it takes and the other to recored the parenthesization.

$\text{Matrix}[i][j] = 0\,(i = j)$

$\text{Matrix}[i][j] = \text{min} m[i][k] + m[k+1][j] + p_{i-1}p_kp_j$

$\text{Result}[i][j] = k + 1$ which gives min values to Matrix[i][j]

```
MATRIX-CHAIN-ORDER()
for(t = 1; t < p; t++)
    for(i = 0; i < p - t; i++)
        for(k = i; k < i + t; k++)
            matrix[i][i + t] = matrix[i][k] + matrix[k + 1][i + t] + size[i] * size[k + 1] * size[i + t
            result[i][i + t] = k + 1;
```

Matrix[i][j] as following

0	150	330	405	1655	2010
0	0	360	330	2430	1950
0	0	0	180	930	1770
0	0	0	0	3000	1860
0	0	0	0	0	1500
0	0	0	0	0	0

Result[i][j] as following

0	1	2	2	4	2
0	0	2	2	2	2
0	0	0	3	4	4
0	0	0	0	4	4
0	0	0	0	0	5
0	0	0	0	0	0

Therefore the optimal parenthesization is (AB)((CD)(EF))

For example, Matrix[2][5] gives the optimal matrix chain product of CDEF. The optimal choice comes from the minimum of C(DEF), (CD)(EF), (CDE)F. Take C(DEF) for example. It divides into sub-problem C and DEF. C is given by Matrix[2][2], which is 0 since C is itself. DEF is given by Matrix[3][5], which is 1860. C is a matrix of 3*12. The result of DEF is a matrix of 12*6. Therefore, $p_{i-1}p_kp_j$ equals 3*12*6 = 216. The number of operations taken to get C(DEF) is therefore 1860+216 = 2076. We can also get (CD)(EF) and (CDE)F with the same manner. They are 1770 and 1830. As a result, we take 1770 for Matrix[2][5] and 4 for k+1, which is recorded in Result[2][5].

5. Some exercises in logarithms:

   (a) Write $\lg(4^n/\sqrt{n})$ in simplest form. What is its asymptotic value.
   `Solution:`$n\lg(4) - \frac{1}{2}\lg(n) = 2n - \frac{\lg n}{2}$.

   (b) Which is bigger, 5^{313340} or 7^{271251}? Give reason. (You can use a calculator.)
   `Solution:`The numbers themselves are too big for calculators but compare their lgs, which are around 727000 and 761000 respectively so the second is bigger.

   (c) Simplify $n^2\lg(n^2)$ and $\lg^2(n^3)$.
   `Solution:`$2n^2\lg(n)$ and $(3\lg n)^2 = 9\lg^2 n$.

   (d) Solve (for x) the equation $e^{-x^2/2} = \frac{1}{n}$.
   `Solution:`$-\frac{x^2}{2} = \lg(1/n) = -\lg n$ so $\frac{x^2}{2} = \lg n$ so $x^2 = 2\lg n$ so $x = \sqrt{2}\sqrt{\ln n}$.

   (e) Write $\log_n 2^n$ and $\log_n n^2$ in simple form.
   `Solution:`The first is that x for which $n^x = 2^n$ so $x\lg(n) = n$ so $x = \frac{n}{\ln(n)}$ is the answer. For the second the answer is 2.

   (f) What is the relationship between $\lg n$ and $\log_3 n$?
   `Solution:`$\log_3 n = \frac{\lg n}{\lg 3}$. As $\lg(3) \sim 1.5$ is a constant they are "the same" in Θ-land.

   (g) Assume $i < n$. How many times need i be doubled before it reaches (or exceeds) n?
   `Solution:`If we double x times we reach $i2^x$ so we need $i2^x \geq n$, or $2^x \geq \frac{n}{i}$ or $x \geq \lg(\frac{n}{i})$. As x need be an integer the precise number of times is $\lceil \lg(\frac{n}{i}) \rceil$.

   (h) Write $\lg[n^n e^{-n}\sqrt{2\pi n}]$ precisely as a sum in simplest form. What is it asymptotic to as $n \to \infty$? What is interesting about the bracketed expression?
   `Solution:`This is Stirling's Formula and is asymptotic to $n!$. Precisely

$$\lg[n^n e^{-n}\sqrt{2\pi n}] = n\lg n - n\lg e + \frac{1}{2}\lg(2\pi) + \frac{1}{2}\lg n$$

which is asymptotic to $n \lg n$.