1. Consider the recursion \( T(n) = 9T(n/3) + n^2 \) with initial value \( T(1) = 1 \). Calculate the precise values of \( T(3), T(9), T(27), T(81), T(243) \). Make a good (and correct) guess as to the general formula for \( T(3^i) \) and write this as \( T(n) \). (Don’t worry about when \( n \) is not a power of three.) Now use the Master Theorem to give, in Thetaland, the asymptotics of \( T(n) \). Check that the two answers are consistent.

Solution: \( T(3) = 9(1) + 3^2 = 18 = 2(9) \), \( T(9) = 9(18) + 9^2 = 243 = 3(81) \), \( T(27) = 9(243) + 729 = 2916 = 4(729) \), \( T(81) = 32805 = 5(6561) \), \( T(243) = 354294 = 6(59049) \). In general, \( T(3^i) = (i+1)3^{2i} \).

With \( n = 3^i \) we have \( 3^{2i} = n^2 \) and \( i = \log_3 n \) so the formula is \( T(n) = n^2(1 + \log_3 n) \). In Thetaland, \( T(n) = \Theta(n^2 \log n) \). With the Master Theorem, as \( \log_3 9 = 2 \) we are in the special case which gives indeed \( T(n) = \Theta(n^2 \log n) \).

Another approach is via the auxiliary function \( S(n) \) discussed in class. Here \( S(n) = T(n)/n^2 \). Dividing the original recursion by \( n^2 \) gives

\[
\frac{T(n)}{n^2} = \frac{T(n/3)}{(n/3)^2} + 1
\]

so that

\( S(n) = S(n/3) + 1 \) with initial value \( S(1) = T(1)/1^2 = 1 \)

so that

\( S(n)1 + \log_3 n \) and so \( T(n) = n^2(1 + \log_3 n) \)

2. Use the Master Theorem to give, in Thetaland, the asymptotics of these recursions:

(a) \( T(n) = 6T(n/2) + n\sqrt{n} \)

Solution: As \( \log_2 6 = \lg 6 \) \( \frac{\lg 6}{\lg 2} = 2.58 \cdots > 3/2 \) we have Low Overhead and \( T(n) = \Theta(n^{\log_2 6}) \).

(b) \( T(n) = 4T(n/2) + n^5 \)

Solution: \( \log_2 4 = 2 < 5 \) so we have High Overhead and \( T(n) = \Theta(n^5) \).

(c) \( T(n) = 4T(n/2) + 7n^2 + 2n + 1 \)

Solution: \( \log_2 4 = 2 \) and the Overhead is \( \Theta(n^2) \) so \( T(n) = \Theta(n^2 \log n) \).
3. Write the following sums in the form $\Theta(g(n))$ with $g(n)$ one of the standard functions. In each case give reasonable (they needn’t be optimal) positive $c_1, c_2$ so that the sum is between $c_1g(n)$ and $c_2g(n)$ for $n$ large.

4. Set $K = \lfloor \sqrt{N} \rfloor$. Let $A[1 \cdots N]$ be an (unsorted) array of numbers. Consider the following algorithm to output the $K+1$-th largest value:

\begin{verbatim}
BUILD-MAX-HEAP[A]
FOR I=1 TO K
    EXTRACT-MAX[A]
END FOR
RETURN A[1]
\end{verbatim}

(a) What is the time (by which we mean the number of flips of data) for the EXTRACT-MAX as a function of $N$ and $I$. (Caution: The heap is getting smaller!)

Solution: At a given $I$ the heap has size $N - I + 1$ so that the “time” is $\lg(N - I + 1)$

(b) Express the total time for the FOR loop as a summation over $I$. Find the asymptotics of the sum.

Solution: We get

$$\sum_{i=1}^{k} \lg(n - i + 1)$$

as the sum. But since $1 \leq i \leq k = \sqrt{n}$ all $n - i + 1 \sim n$ so that all $\lg(n - i + 1) \sim \lg(n)$ so that the sum is asymptotically $k \lg(n) = \sqrt{n} \lg(n)$.

(c) Analyze the total time this algorithm takes. Your answer should be $\Theta(g(n))$ for some “nice” $g(n)$.

Solution: The BUILD-MAX-HEAP[A] takes $O(n)$ and the FOR loop takes $O(\sqrt{n} \lg n)$ so the BUILD-MAX-HEAP[A] dominates and the total time is $O(n)$. (It is interesting to note that this is faster than totally sorting $A$ in time $O(n \lg n)$ and then taking $A[K+1].$)

(a) $n^2 + (n + 1)^2 + \ldots + (2n)^2$

Solution: $\Theta(n^3)$. There are $\sim n$ terms all between $n^2$ and $4n^2$ so the sum is between $n^3(1 + o(1))$ and $4n^3(1 + o(1))$.

(b) $\lg^2(1) + \lg^2(2) + \ldots + \lg^2(n)$

Solution: $\Theta(n \lg^2 n)$. There are $n$ terms all at most $\lg^2(n)$ so
an upper bound is $n \lg^2(n)$. Lopping off the bottom half of the terms we still have $n/2$ terms and each is at least $\lg^2(n/2) = (\lg(n) - 1)^2 \sim \lg^2 n$ so the lower bound is $(1 + o(1))(\frac{n}{2}) \lg^2 n$.

(c) $1^3 + \ldots + n^3$.

Solution: $T(n) = \Theta(n^4)$. Upper bound $n^4$ as $n$ terms, each at most $n$. Lopping off bottom half yields $n/2$ terms, each at least $(n/2)^3$ giving a lower bound $(n/2)(n/2)^3 = n^4/16$. 