Basic Algorithms, Assignment 11

Solutions

1. Here is a variant implementation `EasyPrim` that doesn't use the parent π. With a given set S the data structure will be a minheap Q consisting of *all* crossing edges $\{x,y\}$ with $key[x,y]=w(x,y)$. The initial step, the minheap of all edges $\{s,x\}$ is the same. Now $V-1$ times we EXTRACT-MAX getting a crossing edge $\{x,y\}$, $x\in S, y\notin S$ of minimal weight. We add edge $\{x,y\}$ to T and y to S. The UPDATE is different. For each $z\in Adj[y]$ we examine $\{y,z\}$. If $z\in S$ then $\{y,z\}$ is no longer a crossing edge so we delete it (caution: this takes time!) from the minheap. If $z\notin S$ then $\{y,z\}$ has become a crossing edge so we add it to the minheap, using its weight as its key.

   Analyze the time for `EasyPrim`. Show that the total time is $O(E\lg E)$.
   `Solution:`The initial step takes $O(V)$ as before. The minheap has at most size E. The EXTRACT-MAX takes $O(\lg E)$ each time, a total of $O(V\lg E)$. Adding or deleting from the minheap takes time $O(\lg E)$. For every edge it can at most be added to the minheap and later deleted from the minheap so the cost for the edge is $O(\lg E)$ which in total gives $O(E\lg E)$. This is the dominant time. [Note that $O(E\lg E)$ and $O(E\lg V)$ are the same as $E\le V^2$ so that in Θ-land `Prim` and `EasyPrim` have the same time – but in reality, with the constants, `Prim` is faster.]

2. Consider Prim's Algorithm for MST on the complete graph with vertex set $\{1,\ldots,n\}$. Assume that edge $\{i,j\}$ has weight $|j-i|^3$. Let the root vertex $r=1$. Show the pattern as Prim's Algorithm is applied.
   `Solution:`The set S, initially $\{1\}$, will grow to $\{1,2\}$,…,$\{1,2,\ldots,i\}$, …, $\{1,\ldots,n\}$. When $S=\{1,\ldots,i\}$ the closest point to S will be $i+1$ with $\pi[i+1]=i$ and $key[i+1]=1$. In particular, Let $n=500$ and consider the situation when the tree created has 211 vertices and π and key have been updated.

   (a) What are these 211 vertices and what are the edges.
   `Solution:`$1,\ldots,211$. The edges are $\{1,2\},\{2,3\},\cdots,\{210,211\}$.

   (b) What are $\pi[309]$ and $key[309]$.
   `Solution:`$\pi[309]=211$ (all other of $1,\ldots,202$ are further) and $key[309]=(309-211)^3$.

3. Find $d=\gcd(144,89)$ and x,y with $144x+89y=1$. [Remark: This is part of a pattern with two consecutive numbers from the Fibonacci

sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots]$
`Solution:`

$$\texttt{EUCLID}(144, 89) =$$

$$\texttt{EUCLID}(89, 55) = \texttt{EUCLID}(55, 34) = \texttt{EUCLID}(34, 21) =$$
$$= \texttt{EUCLID}(21, 13) = \texttt{EUCLID}(13, 8) = \texttt{EUCLID}(8, 5) =$$
$$= \texttt{EUCLID}(5, 3) = \texttt{EUCLID}(3, 2) = \texttt{EUCLID}(2, 1) =$$
$$= \texttt{EUCLID}(1, 0) = 1$$

with all quotients 1 except the last. For $\texttt{EXTENDED} - \texttt{EUCLID}$ we get a chart like Figure 31.1:

a	b	$\lfloor a/b \rfloor$	d	x	y
144	89	1	1	34	-55
89	55	1	1	-21	34
55	34	1	1	13	-21
34	21	1	1	-8	13
21	13	1	1	5	-8
13	8	1	1	-3	5
8	5	1	1	2	-3
5	3	1	1	-1	2
3	2	1	1	1	-1
2	1	2	1	0	1
1	0	-	1	1	0

so $x = 34$ and $y = -55$. (Note that the x's and y's form a Fibonacci like pattern as well!)

4. Find $\frac{311}{507}$ in Z_{1000}.
`Solution:`Here we first find $\texttt{EUCLID}(1000, 507)$:

$$\texttt{EUCLID}(1000, 507) = \texttt{EUCLID}(507, 493) = \texttt{EUCLID}(493, 14) =$$
$$= \texttt{EUCLID}(14, 3) = \texttt{EUCLID}(3, 2) = \texttt{EUCLID}(2, 1) =$$
$$= \texttt{EUCLID}(1, 0) = 1$$

For $\texttt{EXTENDED} - \texttt{EUCLID}$ we get a chart like Figure 31.1:

a	b	$\lfloor a/b \rfloor$	d	x	y
1000	507	1	1	181	-357
507	493	1	1	-176	181
493	14	1	35	5	-176
14	3	1	4	-1	5
3	2	1	1	1	-1
2	1	1	2	0	1
1	0	-	1	1	0

so that $1000(181) - 357(507) = 1$ so in Z_{1000} we have $(-357)(507) = 1$ so $\frac{1}{507} = -357 = 643$. Finally $\frac{311}{507} = 311 \cdot 643 = 199973 = 973$. So the answer is 973. (You might prefer to write it as -27 which is the same.) To check: $973 \cdot 507 = 493311 = 311$.

5. Solve the system
   $x \equiv 34 \bmod 101$
   $x \equiv 59 \bmod 103$.
   `Solution:` We write $x = 103y + 59$ (we could start with either and this one is a bit easier) so that in Z_{101} we want $103y + 59 = 34$ or $2y = -25 = 76$ and $y = 38$. (Usually division is complicated but here it worked out like normal division.) Then $x = 103(38) + 59 = 3973$. The general answer is given as $x \equiv 3973 \bmod 10403$ as $10403 = 103 \cdot 101$.