## The QuickSort/BST recursion

We here consider the recursion

$$
\begin{equation*}
T(n)=n-1+\frac{2}{n} \sum_{i=1}^{n-1} T(i) \tag{1}
\end{equation*}
$$

with initial conditions $T(0)=T(1)=0$. The recursion comes up in QuickSort and in random BST, but this is a mathpage.

We first multiply out

$$
\begin{equation*}
n T(n)=n(n-1)+2 \sum_{i=1}^{n-1} T(i) \tag{2}
\end{equation*}
$$

Now we replace $n$ with $n+1$

$$
\begin{equation*}
(n+1) T(n+1)=(n+1) n+2 \sum_{i=1}^{n} T(i) \tag{3}
\end{equation*}
$$

Subtracting (2) from (3) gives

$$
\begin{equation*}
(n+1) T(n+1)-n T(n)=2 n+2 T(n) \tag{4}
\end{equation*}
$$

or

$$
\begin{equation*}
(n+1) T(n+1)=(n+2) T(n)+2 n \tag{5}
\end{equation*}
$$

Dividing (5) by $n+1)(n+2)$ :

$$
\begin{equation*}
\frac{T(n+1)}{n+2}=\frac{T(n)}{n+1}+\frac{2 n}{(n+1)(n+2)} \tag{6}
\end{equation*}
$$

Now we define the auxilliary function

$$
\begin{equation*}
S(n)=\frac{T(n)}{n+1} \tag{7}
\end{equation*}
$$

so that (6) becomes

$$
\begin{equation*}
S(n+1)=S(n)+\frac{2 n}{(n+1)(n+2)} \tag{8}
\end{equation*}
$$

with initial conditions $S(0)=S(1)=0$. Thus $S(n)$ is the sum:

$$
\begin{equation*}
S(n)=\sum_{i=1}^{n-1} \frac{2 i}{(i+1)(i+2)} \tag{9}
\end{equation*}
$$

As the addends are $\sim 2 i^{-1}$ the sum is $\sim 2 \ln n$. So

$$
\begin{equation*}
S(n) \sim 2 \ln n \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
T(n) \sim 2 n \ln n \tag{11}
\end{equation*}
$$

We can be more precise. Using partial fractions we write

$$
\begin{equation*}
\frac{2 i}{(i+1)(i+2)}=\frac{4}{i+2}-\frac{2}{i+1} \tag{12}
\end{equation*}
$$

We express our answer in terms of the harmonic sum

$$
\begin{equation*}
H_{n}:=\sum_{j=1}^{n} \frac{1}{j} \tag{13}
\end{equation*}
$$

so that

$$
\begin{equation*}
\sum_{i=1}^{n-1} \frac{1}{i+2}=H_{n}+\frac{1}{n+1}-1-\frac{1}{2} \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{n-1} \frac{1}{i+1}=H_{n}-1 \tag{15}
\end{equation*}
$$

so that (9) becomes

$$
\begin{equation*}
S\left(n_{=} 2 H_{n}-\frac{4 n}{n+1}\right. \tag{16}
\end{equation*}
$$

and the final answer is quite clean:

$$
\begin{equation*}
T(n)=2(n+1) H_{n}-4 n \tag{17}
\end{equation*}
$$

Whew! Must recursions don't end up with such a nice answer!

