The QuickSort/BST recursion

We here consider the recursion

\[ T(n) = n - 1 + 2 \sum_{i=1}^{n-1} T(i) \]  

(1)

with initial conditions \( T(0) = T(1) = 0 \). The recursion comes up in QuickSort and in random BST, but this is a mathpage.

We first multiply out

\[ nT(n) = n(n - 1) + 2 \sum_{i=1}^{n-1} T(i) \]  

(2)

Now we replace \( n \) with \( n + 1 \)

\[ (n + 1)T(n + 1) = (n + 1)n + 2 \sum_{i=1}^{n} T(i) \]  

(3)

Subtracting (2) from (3) gives

\[ (n + 1)T(n + 1) - nT(n) = 2n + 2T(n) \]  

(4)

or

\[ (n + 1)T(n + 1) = (n + 2)T(n) + 2n \]  

(5)

Dividing (5) by \( n + 1)(n + 2) \):

\[ \frac{T(n + 1)}{n + 2} = \frac{T(n)}{n + 1} + \frac{2n}{(n + 1)(n + 2)} \]  

(6)

Now we define the auxiliary function

\[ S(n) = \frac{T(n)}{n + 1} \]  

(7)

so that (6) becomes

\[ S(n + 1) = S(n) + \frac{2n}{(n + 1)(n + 2)} \]  

(8)

with initial conditions \( S(0) = S(1) = 0 \). Thus \( S(n) \) is the sum:

\[ S(n) = \sum_{i=1}^{n-1} \frac{2i}{(i + 1)(i + 2)} \]  

(9)
As the addends are $\sim 2i^{-1}$ the sum is $\sim 2 \ln n$. So

$$S(n) \sim 2 \ln n$$

(10)

and

$$T(n) \sim 2n \ln n$$

(11)

We can be more precise. Using partial fractions we write

$$\frac{2i}{(i + 1)(i + 2)} = \frac{4}{i + 2} - \frac{2}{i + 1}$$

(12)

We express our answer in terms of the harmonic sum

$$H_n := \sum_{j=1}^{n} \frac{1}{j}$$

(13)

so that

$$\sum_{i=1}^{n-1} \frac{1}{i + 2} = H_n + \frac{1}{n + 1} - 1 - \frac{1}{2}$$

(14)

and

$$\sum_{i=1}^{n-1} \frac{1}{i + 1} = H_n - 1$$

(15)

so that (9) becomes

$$S(n) = 2H_n - \frac{4n}{n + 1}$$

(16)

and the final answer is quite clean:

$$T(n) = 2(n + 1)H_n - 4n$$

(17)

Whew! Must recursions don’t end up with such a nice answer!