Lecture # 7 March 25 2014 63

0-1 Laws

Describe a phenomenon, where an event either occurs or does not occur a.s. (almost surely)

TIPPING POINT (PHASE TRANSITION):

With a small change in a critical parameter the event of interest very quickly goes from probability 0 (ALMOST NEVER) to probability 1 (ALMOST SURELY)

GAME OF "FRIENDING":

- Imagine sending a friend request randomly to $(n-1)$ other individuals in a network (with total of n individuals).

- Assume: (a) If the recipient is already a friend, he simply ignores the request.

  (b) Otherwise, he receives your request for the first time and he accepts you as a friend.

  (c) Under no circumstances, does he ignore, decline or unfriend you.

- Tipping Point: After $\Theta(n \ln n)$ requests, one will have a.s. befriended all the other $(n-1)$ individuals.

Coupon Collector's Problem.

"Collect - All - Coupons - And - Win - Contest"

Problem Statement

- There are n distinct coupons.
- Coupons can be collected with replacement.

$$x_1, x_2, \ldots, x_n \quad \text{iid r.v.}$$

$x_i = 1 \rightarrow$ Event that you obtain the i^{th} coupon $\sim$ Bernoulli($1/n$)

- What is the probability that more than t sample trials are needed to collect all coupons?

$\Leftrightarrow$ Given n coupons, how many coupons are expected to be drawn with replacement, before each coupon has been drawn at least once?

Example $\quad n = 52, \quad t = 225$

If you draw a card randomly (with replacement) from a full-deck, then after $t = 225$ draws you would have seen every card at least once almost surely.

$$t = \Theta(n \ln n).$$

t_i = Time to collect i^{th} coupon after collecting $(i-1)^{th}$ coupon.

$\Rightarrow$ t_i's are independent.

$$t = \sum_{i=1}^{n} t_i = \text{Time to collect all coupons.}$$

$p_i = \Pr[\text{Collect a new coupon after } (i-1)^{th}] = \frac{n-i+1}{n}$

$t_i \sim \text{Geometric}(1/p_i) \qquad \Pr[t_i = k] = (1-p_i)^k p_i$

$E(t_i) = 1/p_i = \frac{n}{n-i+1} \qquad Var(t_i) = \frac{1-p_i}{p_i^2} = \frac{(i-1)n}{(n-i+1)^2}$

$E[T] = E(\sum t_i)$ Linearity of Expectation

$= \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n H_n$

$= n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$

$= n \int_1^n \frac{1}{x}\,dx + \gamma n + 1/2 + o(n) = n \ln n + \Theta(n)$

$\gamma = 0.577$ = Euler's Constant.

$Var[T] = Var(\sum t_i)$ t_i's i.i.d.

$\leq \frac{n^2}{n^2} + \frac{n^2}{(n-1)^2} + \cdots + \frac{n^2}{1} = \frac{\pi^2}{6} n^2$

$\sigma(T) = \frac{\pi n}{\sqrt{6}}$

$\Pr(|T - nH_n| \geq c.n) = \Pr\left(|T - nH_n| \geq \left(c\frac{\sqrt{6}}{\pi}\right)\sigma\right)$

$\leq \frac{\pi^2}{6c^2}$

$\Pr(|T - nH_n| \geq 10.n) \leq \frac{\pi^2}{600} \approx \frac{1}{60}$

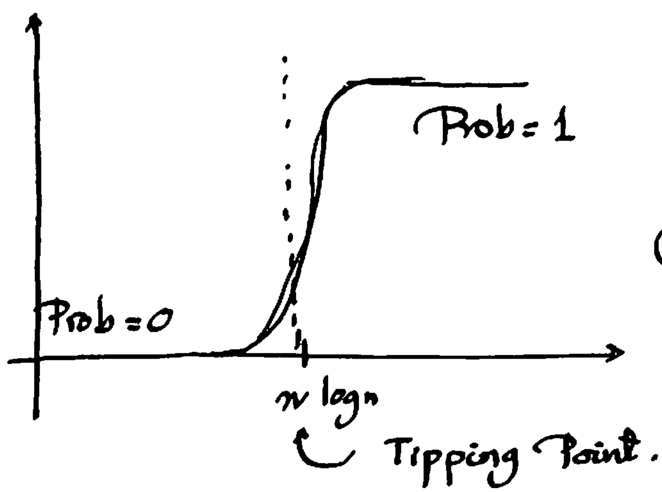
If $T < (1-\varepsilon) n H_n$, you will a.n. get all the coupons.

If $T > (1+\varepsilon) n H_n$, you will a.s. get all the coupons.

GENERALIZATION

T_k = First time k copies of each coupons are collected.

$$T_k \sim n \log n + (k-1)\, n \log\log n + \Theta(n)$$



0-1 Law
(Phase Transition).