LECTURE #2

February 4 2014.

Some Warm-up Ideas:

(1) - Aristotle (384 BC)

- Telos ... The end, or the purpose → Teleology
- Flute Story with 3-people { Maker → Creative Happiness; Enabler → Individual Happiness; Player → Collective Happiness.
- Capitalist (VC), Founders (IP), Users ...

  JUSTICE. (... as Honorific). [Question of Valuation]

(2) - C.K. Prahalad (1941).

- N=1, R=G. { Focus on Individual; Access to wide variety of Suppliers.
- N=1. $\underbrace{\text{Data} \rightarrow \text{Desiderata}}_{\text{Sender. } u_S} \rightarrow \underbrace{\text{Acta.}}_{\text{Receiver } u_R}$

  Sender. → Receiver

  u_S u_R.

  ↳ S=1, S=n, S=N.

S=1	S=n	S=N
Monarchy	Aristocracy	Politiae
Tyranny	Oligarchy	Democracy

  Socrates | Polybius.

  Plato | Cicero

  Aristotle |

  Hobbes

  Machiavelli

  ↓

  Rousseau

  ⋮

(3) <u>Social Network:</u>

a) Optimizing Utilities.

b) Deception → Discrepancies. [Social Network].

c) Learning

d) Data → Information Theory + Statistical Inference

e) Models, Model Selection, Model Checking

↳ Verification.

f) Institutions / Markets.

GRAPH THEORY.

- Combinatorial Structure { Correlation of Encounter }
  - Binary Relationship
  - "Friendship"/Similarity
- Strategic Choices
  - { One shot vs. Repeated Encounter.
    - → Game Theory.
  - ✱ Non-cooperative
    - (Revealation Principle)
  - ★ Rationality
    - → Common Knowledge of Rationality (CKR)

a) Topological Properties
   (Spectral)

b) Temporal Properties
   (Random Graphs and their Evolution)

Actors, Players. ⟶ V ⟶ Vertices.

Links, Connections. ⟶ $E \subseteq V \times V$ → Edges.
(Symmetric Binary Relation on V).

We can add other information with $\forall\, v \in V$

E.g. D_v = Data

M_v = Desiderata / Traits.

S_v = Strategy Space

$$u_v : \prod_{v \in V} D_v \times \prod_{v \in V} M_v \times \prod_{v \in V} S_v \to \mathbb{R}_+$$

More simply

$$u_v : \prod_{v \in V} S_v \to \mathbb{R}_+$$

Linked In
Facebook
Patients Like Me

Example of a Binary Relationship:
"FRIENDSHIP"

1) Irreflexive: One is not his own friend.
2) Symmetric: One is a friend to a friend.
3) Nontransitive: A friend's friend is not necessarily a friend.

Friendship can be described by an (undirected) edge in the graph.

Definition. Graph (Network)

A graph, usually denoted $G(V,E)$ or $G=(V,E)$, consists of a set of vertices V together with a set of edges $E \subseteq V \times V$.

→ Mathematically, it describes an irreflexive, symmetric and non-transitive binary relation on a discrete set (non necessarily finite.).

Graphon / Graphlet.

◇ Two vertices u and v are <u>adjacent</u> if

$$\exists_{e \in E} \quad e = (u,v) \qquad\qquad e \in E \subseteq V \times V$$

The vertices u and v are end-points of $e = (u,v)$.

◇ # Vertices $= |V| = n$
Edges $= |E| = m$ $\qquad m \leq \frac{n(n-1)}{2} = \binom{n}{2}$

n = # of ways to choose u
$n-1$ = # of ways to choose $v (\neq u)$
Symmetry → Identify edges $(u,v) = (v,u)$.

TRIADIC CLOSURE

Getting job with your social network.

→ It is good to have lot of weak ties...

acquaintances are more important than close friends.

Story: "Strength of Weak Ties." (1973).

Mark Granovetter (American Sociologist, Prof. Standford U.).

"Getting a Job" Granovetter's PhD Dissertation

Dept of Social Relation, Harvard Univ.

"Weak ties enable reaching populations and audiences with much higher efficiency than what is achievable or accessible via strong ties."

Granovetter's Experiment:

Newton, MA.

282 professional, technical and managerial workers.

N = # individuals out of 282 who found jobs through personal contacts

= 54

- Strong Ties (16.6%)
- Weak Ties
  - → (55.6%) Occasional Contacts
  - → (27.8%) Rare Contacts

Triadic Closure.

Definition: Consider an "augmented" undirected graph

$$G = (V, E, E')$$

in which $E' \subseteq E \subseteq V \times V$.

$$\begin{cases} E : \text{The edges/ties} \\ E' : \text{The strong ties} \\ E \setminus E' : \text{The weak ties.} \end{cases}$$

$(u,v) \in E \Rightarrow$ u and v are friends
(either acquaintances or close friends)

$(u,v) \in E' \Rightarrow$ u and v are close friends.

The Strong Triadic Closure Property:

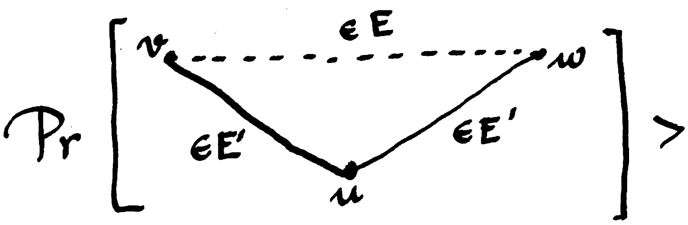
States that

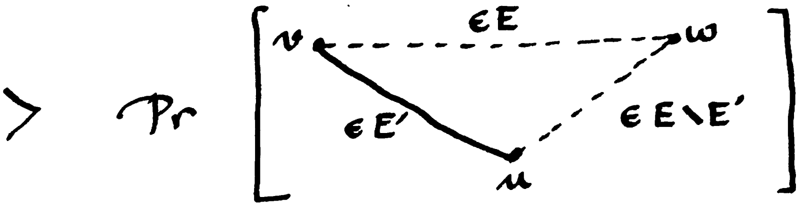
$$(u,v) \in E' \wedge (u,w) \in E' \Rightarrow \cancel{(v,w) \in E'}$$
$$(v,w) \in E \quad \text{a.s.}$$

Equivalently,

$$\Pr\left[(v,w) \in E \mid (u,v) \in E' \wedge (u,w) \in E'\right]$$
$$> \Pr\left[(v,w) \in E\right]$$

- The knowledge that v and w have a common close friend, namely u, raises the (conditional) probability that v and w are at least acquaintances.
- Even though neither v nor w can be deceptive to u does not imply they will not be deceptive to each other.





$$\Pr\left[(v,w)\in E \ \wedge\ (u,v)\in E' \;\middle|\; (u,w)\in E'\right]$$
$$> \Pr\left[(v,w)\in E \wedge (u,v)\in E' \;\middle|\; (u,w)\in E\setminus E'\right]$$

w = Applicant
v = Employer (potentially)
u : Recommender

Consider a relation R

$(w,u)\in R$ = Event w obtained a job through a referral by u.

1) w applies for a job from v. $(w,v)\in V\times V$
2) w asks u for a recommendation. u agrees to write a letter. $[(w,u)\in E]$
3) v gets a letter from u.
   - $(v,u)\notin E$ ignores letter as deceptive
   - $(v,u)\in E'$ uses letter as non deceptive
   - $(v,w)\notin E$ only information is what he gets from u
   - $(v,w)\in E$ has an additional source of information.

$$\Pr[(w,u)\in R \mid (w,u)\in E'] \to 16.6\%$$

(18)

$$< \Pr[(w,u)\in R \mid (w,u)\in E\setminus E'] \to 83.4\%$$

<u>STRENGTH OF WEAK TIES.</u>

$$\Pr[(w,u)\in R \mid (w,u)\in E']$$

$$\approx \Pr[(w,v)\notin E \wedge (u,v)\in E' \mid (w,u)\in E']$$

$$< \Pr[(w,v)\notin E \wedge (u,v)\in E' \mid (w,u)\in E\setminus E']$$

$$\approx \Pr[(w,u)\in R \mid (w,u)\in E\setminus E']$$

Strong Ties: $(w,u)\in E' \wedge (u,v)\in E' \Rightarrow (w,v)\in E$

v: Likely to be an acquaintance
Can use information in addition to what u provides in the referral
(3-person Sender-Receiver-Verifier Game)

Weak Ties: $(w,u)\in E\setminus E' \wedge (u,v)\in E' \not\Rightarrow (w,v)\notin E$

v: Unlikely to be an acquaintance
Goes by u's referral only.
(2-person Sender-Receiver Game)

- In a social network, let v and w be two close friends of yours (u) → Connected by strong ties to you. Then it is likely that v and w are acquaintances → Connected by weak ties to each others.

  → Signaling between v and w can be verified by u for possible deception.

- If v and w have a large subgroup of common friends, then it is probable that they are acquaintances
  - The probability increasing with the size of the set of mutual friends.
  - Many cliques of size 3 … K_3's
  - Possibility of 3-player games … Verification against deception.

Density of a Graph:

a) Random Graphs.
b) Edge Probabilities
c) Emergence of various Graph Properties.
  - Phase Transitions …

- The number of vertices adjacent to a given vertex v is called its <u>degree</u>, and is denoted:

$$d(v) = |\{u \mid (u,v) \in E\}|$$

$$\sum_{v \in V} d(v) = 2|E| = 2m$$

- <u>Average degree</u> of the graph

$$\bar{d} = \frac{\sum d(v)}{|V|} = \frac{2m}{n}$$

- A graph is <u>complete</u> if each of its vertices is adjacent to all others.

$$|E(K_n)| = \frac{n(n-1)}{2} = \binom{n}{2}$$

- A graph's <u>density</u> (or <u>sparsity</u>) indicates the extent to which a graph is complete.

Density = Number of edges divided by the number of possible total

$$= \frac{|E|}{\binom{n}{2}} = \frac{2m}{n(n-1)} = \frac{\bar{d}}{n-1}$$

$$\bar{d} = \text{Density}\,(n-1)$$

Social Value

One's position within a network assigns a social value to one ... Rank

$$f(u)$$

A scalar function

$$f : V \to \mathbb{R}.$$

Relative Value:

$$\Delta f(u) = \frac{1}{d_u} \sum_{u \sim v} \left(f(u) - f(v) \right)$$

$$= f(u) - \frac{\sum_{u \sim v} f(v)}{d_u} = f(u) - \sum_{v \in V} P_{u,v} f(v)$$

$$P_{u,v} = \begin{cases} 1/d_u & \text{if } (u,v) \in E \\ 0 & \text{o.w.} \end{cases}$$

How dissimilar am I from my friends given that I have invested d_u friends/acquaintances?

Adjacency Matrices:

Every graph $G = (V, E)$ with $|V| = n$ vertices has associated with it a symmetric adjacency matrix, which is a

Binary $n \times n$ matrix $A \in \{0,1\}^{n \times n}$

in which

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ & \text{i.e. } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

$(v_i, v_j) = (v_j, v_i)$ Symmetry $\Rightarrow a_{ij} = a_{ji}$.

$$A^T = A.$$

$$d_{ii} = d(v_i) = |\{v_j \mid (v_i, v_j) \in E\}| = \sum_j a_{ij}$$

$$D = \begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix} = \text{Diagonal Matrix.}$$

$$\operatorname{Tr} D = \sum_{v_i \in V} d(v_i) = 2m.$$

$$P(u,v) = \begin{cases} 1/d_u & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$P = D^{-1}A$$

$$I - P = D^{-1}D - D^{-1}A = D^{-1}(D-A) = \Delta \qquad \left\{ \begin{aligned} D - A &= D\Delta \\ &= L \end{aligned} \right.$$

$$\Delta f(u) = \frac{1}{d_u} \sum_{u \sim v} (f(u) - f(v))$$

Δ = (Discrete) Laplace Operator

Laplacian, $\mathcal{L} = D^{1/2} \Delta D^{-1/2}$
$= D^{-1/2}(D-A)D^{-1/2}$

$$\begin{aligned} \mathcal{L} g(u) &= D^{1/2} \Delta D^{-1/2} g(u) \\ &= \frac{1}{\sqrt{d_u}} \sum_{\substack{v \\ u \sim v}} \left(\frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}} \right) \end{aligned}$$

$g : V \to \mathbb{R}$, $f = D^{-1/2} g : V \to \mathbb{R}$

$$\frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} = \frac{\langle g, D^{-1/2} L D^{-1/2} g \rangle}{\langle g, g \rangle} = \frac{\langle f, Lf \rangle}{\langle D^{1/2} f, D^{1/2} f \rangle}$$

$$= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}$$

Dirichlet Sum of G

= Rayleigh Quotient.