Solutions of Homework 3
Social Networks

Exercise 3.3 In $G(n,p)$ the probability of a vertex having degree k is $\binom{n}{k} p^k (1 - p)^{n-k}$. Show by direct calculation that the expected degree is np. Where is the mode of the binomial distribution? The mode is the point at which the probability is maximum. Compute directly the variance of the distribution.

Solution:

$$E(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=1}^{n-1} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$
$$= np \sum_{k=1}^{n-1} \frac{n!}{(k-1)!\,[(n-1)-(k-1)]!} p^{k-1} (1-p)^{(n-1)-(k-1)}$$
$$= np(p+1-p)^{n-1}$$
$$= np$$

$$Var(k) = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

Exercise 3.5 In $G\left(n, \frac{1}{n}\right)$, what is the probability that there is a vertex of degree log n? Give an exact formula; also derive simple approximations.

Solution:

$$P(vertex\ of\ degree \log n) = \binom{n}{\log n} \left(\frac{1}{n}\right)^{\log n} \left(1 - \frac{1}{n}\right)^{n - \log n}$$
$$= \binom{n}{\log n} \frac{e^{-1}}{(n-1)^{\log n}}$$

For further simplify, please refer to Gamma Function and General Definition of Factorial

Exercise 3.12 Carry out an argument, similar to the one we used for triangles, to show that $p = \frac{1}{n^{2/3}}$ is a threshold for the existence of a 4-clique. A 4-clique consist of four vertices with all $\binom{4}{2}$ edges present.

Solution:

$$E[x^2] = E\left[\left(\sum_{ijkl} \Delta_{ijkl}\right)^2\right]$$

1. One vertex: $E[x^2] = E^2(x)$

2. Two vertices one edge: $E[x^2] = o(1)$
3. Three vertices two edge: $E[x^2] = o(1)$
4. Four vertices: $E[x^2] = 1/24$

$$E[x^2] = E^2(x) + o(1) + o(1) + \frac{1}{24} = E^2(x) + o(1) + \frac{1}{24}$$

$$Var(x) = E[x^2] - E^2(x) = o(1) + \frac{1}{24}$$

Therefore, $Prob(x = 0) \leq Prob(|x - E(x)| \geq E(x))$

$$Prob(x = 0) \leq \frac{Var(x)}{E^2(x)} \leq \frac{o(1) + \frac{1}{24}}{(\frac{1}{24})^2} \leq o(1) + \frac{1}{24}$$

Thus, for $p \geq \frac{1}{n^{2/3}}$ G(n,p) has a 4-clique with nonzero probability.