Solution of Homework 1
Social Networks

Exercise 5.23 Given an undirected graph with a component consisting of a single edge find two eigenvalues of the Laplacian L = D − A where D is a diagonal matrix with vertex degrees on the diagonal and A is the adjacency matrix of the graph.

Solution: Assume this graph has two vertices connected by a single edge. Then the Laplacian will be L = D – A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, L is symmetric

To find the eigenvalues, $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)^2 - 1 = 0$

$\lambda = 0,2$

Exercise 5.24 A researcher was interested in determining the importance of various edges in an undirected graph. He computed the stationary probability for a random walk on the graph and let pi be the probability of being at vertex i. If vertex i was of degree d_i, the frequency that edge (i, j) was traversed from i to j would to $\frac{1}{d_i}p_i$ and the frequency that the edge was traversed in the opposite direction would be $\frac{1}{d_j}p_j$. Thus he assigned an importance of $|\frac{1}{d_i}p_i - \frac{1}{d_j}p_j|$ to the edge. What is wrong with his idea?

Solution: In a configuration such as v_i to v_j, which only has one edge. Then $|\frac{1}{d_i}p_i - \frac{1}{d_j}p_j|$ = 0 doesn't make any sense. In order to correct this, we can do $|\frac{1}{d_i}p_i + \frac{1}{d_j}p_j|$, since we don't care which way the edge was traversed.

Exercise 5.38 Consider modifying personal page rank as follows. Start with the uniform restart distribution and calculate the steady state probabilities. Then run the personalized page rank algorithm using the stationary distribution calculated instead of the uniform distribution. Keep repeating until the process converges. That is, we get a stationary probability distribution such that if we use the stationary probability distribution for the restart distribution we will get the stationary probability distribution back. Does this process converge? What is the resulting distribution? What distribution do we get for the graph consisting of two vertices u and v with a single edge from u to v?

Solution:

1) The original page rank is computed as:

$$\boldsymbol{p} = \alpha\frac{1}{n}\mathbf{1} + (1-\alpha)G\boldsymbol{p} \quad \Longrightarrow \quad \boldsymbol{p}(\mathbf{1} - (1-\alpha)G) = \alpha\frac{1}{n}\mathbf{1}$$

So, $\boldsymbol{p} = \frac{1}{n}(1,1,\dots,1)\alpha(\mathbf{1} - (1-\alpha)G)^{-1}$

The adjusted page rank can be computed recursively:

$$p_0 = \frac{1}{n}(1,1,\dots,1)$$

$$p_{k+1} = \alpha(\mathbf{1} - (1-\alpha)G)^{-1} \cdot p_k$$

So, $p = p_\infty = \lim_{k\to\infty} p_0 \cdot (\alpha(\mathbf{1} - (1-\alpha)G)^{-1})^k$

This process converges to normalized degree vector:

$$p = \frac{1}{2||E||}\{degree_0, degree_1, \dots, degree_n\}$$

2) Distribution for the graph consisting of two vertices u and v with a single edge from u to v will be $p = \left\{\frac{1}{2}, \frac{1}{2}\right\}$.