Market Efficiency and Dynamics

by

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Dedication

To my parents and grandparents.

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Abstract

General equilibrium theory, initiated by Walras over a century ago [Wal74], explains the interaction between supply and demand in an economy. In this dissertation, we look at Fisher Markets, which are a particular case of the general equilibrium theory. We consider two issues in Fisher Markets: strategic behavior and dynamics.

Strategic behavior is usually considered in a game, such as auction, in which case, participants in the game may choose not to report their real preferences in order to improve their payoff. In general equilibrium theory, buyers are usually considered to be non-strategic: given the prices, buyers will maximize their true utility by properly distributing their money on different goods. In this case, the Market equilibrium should be efficient. However, the prices in the market equilibrium are influenced by the demands of the buyers. In principle, buyers can affect prices by changing their demands, which may improve buyers' final utilities. This may result in inefficient outcomes. In this thesis, we investigate this possibility in large Fisher markets. We show that the market will approach full efficiency as the market becomes larger and larger. We also show a similar result for the Walrasian mechanism in large settings.

We also study two dynamics in Fisher Markets in this dissertation:

- *Proportional response* is a buyer-oriented dynamics. Each round, buyers update their spending in proportion to the utilities they received in the last round, where prices are the sum of the spendings. This dissertation establishes new convergence results for two generalizations of proportional response in Fisher markets with buyers having the constant elasticity of substitution (CES) utility functions. The starting points are respectively a new convex and a new convex-concave formulation of such markets. The two generalizations of proportional response correspond to suitable mirror descent algorithms applied to these formulations. Among other results, we analyze a damped generalized proportional response and show a linear rate of convergence in a Fisher

market with buyers whose utility functions cover the full spectrum of CES utilities aside from the extremes of linear and Leontief utilities; when these utilities are included, we obtain an empirical $O(1/T)$ rate of convergence.

- *Tatonnement* is considered the most natural dynamics in Fisher Markets: the price of a good is raised if the demand exceeds the supply of the good, and decreased if it is too small. Implicitly, buyers' demands are assumed to be a best-response to the current prices. This dissertation addresses a lack of robustness in existing convergence results for discrete forms of tatonnement, including the fact that it need not converge when buyers have linear utility functions. This dissertation shows that for Fisher markets with buyers having CES utility functions, including linear utility functions, tatonnement will converge quickly to an approximate equilibrium (i.e., at a linear rate), modulo a suitable large market assumption. The quality of the approximation is a function of the parameters of the large market assumption.

Contents

Introduction

In economics, General equilibrium theory, initiated by Walras over a century ago [Wal74], is intended to model how prices ensure supply and demand are in balance in an economy. Such balancing prices are called equilibrium prices. A complete theory would account for strategic behavior on the part of participants, and allow for dynamics: price adjustments when the economy is out of equilibrium, perhaps due to changing conditions.

In this dissertation, I investigate both these issues in the Fisher market setting, one special case of the general equilibrium theory.

Strategic Issues

General equilibrium theory shows that markets have efficient equilibria, but this depends on agents being non-strategic, i.e. that they declare their true demands when offered goods at particular prices, or in other words, that they are price-takers. An important question is how much the equilibria degrade in the face of strategic behavior, i.e. what is the Price of Anarchy (PoA) of the market viewed as a mechanism?

It is generally understood that as markets become larger, the incentive to report strategically and the effects of strategic behavior become increasingly negligible, when largeness is defined appropriately. For the most part, the existing work provides in the limit results, and hence there is no quantification in terms of the size of the market. In contrast, in this dissertation, we showed that in large Fisher markets, the Nash Equilibrium (NE), which includes pure NE, mixed NE, and Bayes NE can be almost efficient in the following sense:, the Price of Anarchy is exactly bounded by e raised to the maximum proportion of the money held by single person.

In this dissertation, we have also analyzed the Price of Anarchy of a Walrasian auction. In a Walrasian auction, in contrast to a Fisher market, indivisible items are sold to buyers.

We have shown that the PoA tends to 1 at a rate that is a function of the market size. In contrast, in earlier work, Babaioff et al. [BLNPL14] showed that in general (non-large) markets the PoA is at least 2.

Market Dyanmics

A major goal in Algorithmic Game Theory is to justify equilibrium concepts from an algorithmic and complexity perspective. One appealing approach is to identify natural distributed algorithms that converge quickly to an equilibrium. Two natural dynamics have been studied in the context of Fisher markets.

Tatonnement The first, which is perhaps the most intuitive candidate for a natural algorithm, is tatonnement, in which the price of a good is raised if the demand exceeds the supply of the good, and decreased if it is too small. Implicitly, buyers' demands are assumed to be a best-response to the current prices. This highly intuitive algorithm was proposed by Walras well over a century ago [Wal74]. Past work [CF08, CCR12, CCD19] show that for almost all utility functions in the CES domain, tatonnement converges quickly to a market equilibrium. However, it is well known that tatonnement need not converge when buyer utilities are linear. In addition, and unsurprisingly, as one approaches linear settings, the step size employed by the tatonnement algorithm needs to be increasingly small, which leads to a slower rate of convergence, and indicates a lack of robustness in the tatonnement procedure. In this dissertation, we show that in suitable large Fisher markets, this lack of robustness disappears, so long as approximate rather than exact convergence suffices. Furthermore, we obtain fast, i.e. linear, convergence to an approximate equilibrium.

Proportional Response Another natural dynamics in Fisher markets is Proportional Response. In contrast to tatonnement, it is a buyer-oriented update, originally analyzed in an effort to explain the behavior of peer-to-peer networks [LLSB08, WZ07]. Some years ago, Birnbaum et al. [BDX11] showed that the proportional response with linear utility buyers is equivalent to mirror descent on Shmyrev's convex program [Shm09]. In this dissertation, we first generalize Shmyrev's convex program to a new convex and a new convex-concave formulation to allow buyers with any CES utility function. We show that in the substitutes CES domain, Proportional Response is equivalent to mirror descent on the new convex potential function, and we also extend proportional response to the complementary domain. Several of our new results are a consequence of the notion of strong Bregman convexity and a new notion of strong Bregman convex-concave functions, and associated linear rates of convergence.

Chapter 1

Fisher Market Model

In this dissertation, we mainly focus on the Fisher Market, one special case of the general equilibrium theory. In a Fisher market there are buyers who start with money which they have no desire to keep, and sellers who have goods to sell, which they wish to sell in their entirety for money. This is a modest generalization of the notion of Competitive Equilibrium from Equal Incomes (CEEI) [HZ79, Var74]. In prior work on computing equilibria for these markets, there has been a particular focus on Eisenberg-Gale markets, a term coined by Jain and Vazirani, and their generalizations [JV07]; Eisenberg-Gale markets are Fisher markets in which demands are determined by homothetic utility functions. The latter markets have been seen to capture the notion of proportional fairness, as defined in the networking community [Kel97], which is also equivalent to the optimum Nash Social Welfare [NJ50, KN79].

Note that in the rest of this disseration we use bold symbols, e.g., $\mathbf{p}, \mathbf{x}, \mathbf{e}$, to denote vectors.

Fisher Market In a Fisher market, there are n perfectly divisible goods and m buyers. Without loss of generality, the supply of each good j is normalized to be one unit. Each buyer i has a utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$, and a budget of size e_i. At any given price vector $\mathbf{p} \in \mathbb{R}^n_+$, each buyer purchases a maximum utility affordable collection of goods. More precisely, $\mathbf{x}_i \in \mathbb{R}^n_+$ is said to be a demand of buyer i if $\mathbf{x}_i \in \arg\max_{\mathbf{x}':\ \mathbf{x}'\cdot\mathbf{p}\leqslant e_i} u_i(\mathbf{x}')$.

A price vector $\mathbf{p}^* \in \mathbb{R}^n_+$ is called a *market equilibrium* if at $\mathbf{p}^*$, there exists a demand $\mathbf{x}_i$ of each buyer i such that

$$p_j^* > 0 \ \Rightarrow\ \sum_{i=1}^{m} x_{ij} = 1 \quad \text{and} \quad p_j^* = 0 \ \Rightarrow\ \sum_{i=1}^{m} x_{ij} \leqslant 1.$$

The collection of $\mathbf{x}_i$ is said to be an *equilibrium allocation* to the buyers.

CES utilities In this thesis, each buyer i's utility function is of the form

$$u_i(\mathbf{x}_i) = \left(\sum_{j=1}^{n} a_{ij} \cdot (x_{ij})^{\rho_i}\right)^{1/\rho_i},$$

for some $-\infty \leqslant \rho_i \leqslant 1$. $u_i(\mathbf{x}_i)$ is called a Constant Elasticity of Substitution (CES) utility function. They are a class of utility functions often used in economic analysis.

- The limit as $\rho_i \to -\infty$ is called a Leontief utility[1]:

$$u_i(\mathbf{x}_i) = \min_j \frac{x_{ij}}{c_{ij}};$$

- the limit as $\rho_i \to 0$ is called a Cobb-Douglas utility:

$$u_i(\mathbf{x}_i) = \prod_j {x_{ij}}^{a_{ij}}, \text{ with } \textstyle\sum_j a_{ij} = 1;$$

- and the case $\rho_i =$ is called a linear utility:

$$u_i(\mathbf{x}_i) = \sum_j a_{ij} x_{ij}.$$

[1]Here, the utility function $u_i(\mathbf{x}) = \min_j \frac{x_{ij}}{c_{ij}}$ can be seen as the limit of $u_i(\mathbf{x}) = \left(\sum_j \left(\frac{x_{ij}}{c_{ij}}\right)^{\rho_i}\right)^{\frac{1}{\rho_i}}$ as ρ_i tends to $-\infty$.

The utilities with $\rho_i \geqslant 0$ capture goods that are substitutes, and those with $\rho_i \leqslant 0$ goods that are complements. It is sometimes convenient to write

$$c_i = \frac{\rho_i}{\rho_i - 1}. \tag{1.1}$$

Notation Buyer i's spending on good j, denoted by b_{ij}, is given by $b_{ij} = x_{ij} \cdot p_j$. Also, $z_j = \sum_i x_{ij} - 1$ denotes the excess demand for good j. We sometimes index prices, spending, and demands by t to indicate the relevant value at time t. Finally, we use a superscript of * to indicate an equilibrium value.

As mentioned before, Fisher markets are actually a special case of Exchange economies. (To see this, view the money as another good, and the supply of the goods as being initially owned by another agent, who desires only money.)

In general computing equilibria is computationally hard even for Fisher markets [CT09, VY11]. One feasible class is the class of Eisenberg-Gale markets, markets for which the equilibrium computation becomes the solution to a convex program. This class was named in [JV07]; the program was previously identified in [EG59].

Definition 1.0.1. *Eisenberg-Gale markets are those economies for which the equilibria are exactly the solutions to the following convex program, called the Eisenberg-Gale convex program:*

$$\max_{\mathbf{x}} \quad \sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots x_{im})) \tag{1.2}$$
$$s.t. \quad \forall j: \quad \sum_i x_{ij} \leq 1 \quad and \quad \forall i, j: \quad x_{ij} \geq 0.$$

Note that Fisher markets with CES utilities are special cases of Eisenberg-Gale markets.

Chapter 2

Strategic Issues

2.1 Preliminary

When is there no gain to participants in a game from strategizing? One answer applies when players in a game have no prior knowledge; then a game that is strategy proof ensures that truthful actions are a best choice for each player. However, in many settings there is no strategy proof mechanism. Also, even if there is a strategy proof mechanism, with knowledge in hand, other equilibria are possible, for example, the "bullying" Nash Equilibrium as illustrated by the following example: there is one item for sale using a second price auction, the low-value bidder bids an amount at least equal to the value of the high-value bidder, who bids zero; the resulting equilibrium achieves arbitrarily small social welfare compared to the optimal outcome.

To make the notion of gain meaningful one needs to specify what the game or mechanism is seeking to optimize. Social welfare and revenue are common targets. For the above example, the social welfare achieved in the bullying equilibria can be arbitrarily far from the optimum. However, for many classes of games, over the past fifteen years, bounds on the gains from strategizing, a.k.a. the Price of Anarchy (PoA), have been obtained, with much progress coming thanks to the invention of the smoothness methodology [Rou15, Rou12, ST13, FIL$^+$16]; many of the resulting bounds have been shown to be tight. Often these bounds are modest constants, such as $\frac{4}{3}$ [RT02] or 2 [Syr12], etc., but rarely are there provably no losses from strategizing, i.e. a PoA of 1.

This dissertation investigates when bounds close to 1 might be possible. In particular, we study both large Fisher markets and large Walrasian auctions viewed as mechanisms.

Fisher Markets As mentioned before, a Fisher market is a special case of an exchange economy in which the agents are either buyers or sellers. Each buyer is endowed with money but has utility only for non-money goods; each seller is endowed with non-money goods, WLOG with a single distinct good, and has utility only for money. Fisher markets capture settings in which buyers want to spend all their money. In particular, they generalize the competitive equilibrium from equal incomes (CEEI) [HZ79, Var74], in that they allow buyers to have non-equal incomes. While at first sight this might appear rather limiting, we note that much real-world budgeting in large organizations treats budgets as money to be spent in full, with the consequence that unspent money often has no utility to those making the spending decisions. The budgets in GoogleAds and other online platforms can also be viewed as money that is intended to be spent in full.

We consider the outcomes when buyers bid strategically in terms of how they declare their utility functions. We show that the PoA tends to 1 as the setting size increases. The only assumptions are some limitations on the buyers' utility functions: they need to satisfy the gross substitutes property and to be monotone and homogeneous of degree 1.

This result is also obtained via a smoothness-type bound and hence extends to bidders playing no-regret strategies, assuming that the ensuing prices are always bounded away from zero. We ensure this by imposing reserve prices.

Walrasian Auctions Walrasian Auctions are used in settings where there are indivisible goods for sale and agents, called bidders, who want to buy these goods. Each agent has varying preferences for different subsets of the goods, preferences that are represented by valuation functions. The goal of the auction is to identify equilibrium prices; these are prices at which all the goods sell, and each bidder receives a favorite (utility maximizing) collection of goods, where each bidder's utility is quasi-linear: the difference of its valuation for the goods and their cost at the given prices. Such prices, along with an associated allocation of goods, are said to form a Walrasian equilibrium.

Walrasian equilibria for indivisible goods are known to exist when each bidder's demand satisfies the gross substitutes property [GS99], but this is the only substantial class of settings in which they are known to exist.

[BLNPL14] analyzed the PoA of the games induced by Walrasian mechanisms, i.e. the prices were computed by a method, such as an English or Dutch auction, that yields equilibrium prices when these exist. Note that the mechanism can be applied even when Walrasian equilibria do not exist, though the resulting outcome will not be a Walrasian equilibrium. But even when Walrasian equilibria exist, because bidders may strategize, in general the outcome will be a Nash equilibrium rather than a Walrasian one. Among other results, Babaioff et al. showed an upper bound of 4 on the PoA for any Walrasian mechanism when the bids and valuations satisfied the gross substitutes property and overbidding was not allowed.[1] In addition, they obtained lower bounds on the PoA that were greater than 1, even when overbidding was not allowed, which excludes bullying equilibria; e.g. the English auction has a PoA of at least 2.

Babaioff et al. also noted that the prices computed by double auctions, widely used in financial settings, are essentially computing a price that clears the market and maximizes trade; one example they mention is the computation of the opening prices on the New York Stock Exchange, and another is the adjustment of prices of copper and gold in the London market.

By a large auction, we intend an auction in which there are many copies of each good,

[1] They also proved a version of this bound which was parameterized w.r.t. the amount of overbidding.

and in addition the demand set of each bidder is small. The intuition is that then each bidder will have a small influence and hence strategic behavior will have only a small effect on outcomes. In fact, this need not be so. For example, the bullying equilibrium persists: it suffices to increase the numbers of items and bidders for each type to n, and have the buyers of each type follow the same strategy as before.

What allows this bullying behavior to be effective is the precise match between the number of items and the number of low-value bidders. The need for this exact match also arises in the lower-bound examples in [BLNPL14] (as with the bullying equilibrium, it suffices to pump up the examples by a factor of n). To remove these equilibria that demonstrate PoA values larger than 1, it suffices to introduce some uncertainty regarding the numbers of items and/or bidders. Indeed, in a large setting it would seem unlikely that such numbers would be known precisely. We will create this uncertainty by using distributions to determine the number of copies of each good. This technique originates with [Swi01]. In contrast, prior work on non-large markets eliminated the potentially unbounded PoA of the bullying equilibrium by assuming bounds on the possible overbidding [BR11, CKS16, FKL12, ST13].

Our main result on large Walrasian auctions is that the PoA of the Walrasian mechanism tends to 1 as the market size grows. This result assumes that expected valuations are bounded regardless of the size of the market. We specify this more precisely when we state our results in Section 2.5.2. This bound applies to both Nash and Bayes-Nash equilibria; as it is proved by means of a smoothness argument, it extends to mixed Nash and coarse correlated equilibria, and outcomes of no-regret learning.

2.1.1 Related Work

The results on Walrasian auctions generalize earlier work of [Swi01] who showed analogous results for auctions of multiple copies of a single good. In contrast, we consider auctions in which there are multiple goods. Swinkels analyzed discriminatory and non-discriminatory mechanisms. For the latter, he showed that any mechanism that used a combination of the k-th and $(k + 1)$-st prices when there were k copies of the good on sale achieves a PoA that tends to 1 with the auction size[2]. Our result also weakens some of the assumptions in Swinkels work.

The second closely related work on Walrasian auctions is due to [FIL$^+$16]. They also analyze several large settings. Among other results, they analyze auctions in which the PoA tends to 1 as their size grows to ∞. Their results are derived from a new type of smoothness argument. Depending on the result, they require either uncertainty in the number of goods or the number of bidders. In contrast, our main result uses a previously known smoothness technique plus uncertainty in the number of goods. We contrast the techniques in more detail after we present our result in Section 2.5.3. They also show that for traffic routing problems, the PoA of the atomic case tends to that of the non-atomic case as the number of units of traffic grows to ∞.

The idea of uncertainty in the number of agents or items first arose in the Economics literature. [Mye00] used it in the context of voting games, and [Swi01] in the context of auctions. Later, uncertainty in the number of agents was used with the Strategy Proof in the Large concept [AB12].

[HMR$^+$16] considered the effects of non-unique demands on the social welfare, assuming allocations were based on demands. Given a genericity assumption, they showed that in markets with buyers having matroid based valuations the inefficiency was proportional to the number of distinct goods, and so if this was a constant, the efficiency would tend to 1 as the market size grows.

The study of the behavior of large exchange economies was first considered by [RP76], which they modeled as a *replica economy*, the n-fold duplication of a base economy, showing that individual utility gains from strategizing tend to zero as the economy grows. Subsequently, [JM97] showed that with some *regularity* assumptions, the equilibrium allocations

[2]Swinkels did not use the then recently formulated PoA terminology to state his result.

converge to the competitive equilibrium. In contrast, our result proves bounds in terms of a parameter characterizing the size of the economy. More recently, [ANS07] studied the efficiency of exchange economies in the presence of strategic agents; however, their notion of efficiency was weaker than the PoA. They termed an outcome μ-efficient if there was no way of improving everyone's outcome in terms of utility by an additive factor of μ, and showed that with high probability (i.e. $1-\mu$) a μ-efficient outcome would occur when the size of the economy was large enough, so long as each agent was small, agents were truthful with non-zero probability, and some additional more technical conditions. In contrast, the PoA considers the ratio between the social welfare at the competitive equilibrium and the achieved social welfare, namely a ratio of the sum of everyone's outcomes. [AB12] showed that the Strategy Proof in the Large methodology could be applied to exchange economies for agents that are limited to having a finite type space, independent of the size of the economy; in contrast, our results do not require a restriction of this sort. Finally, we note that our bounds apply to classes of Fisher markets, whereas the earlier literature applies to classes of exchange economies, which is a significantly more general setting; nonetheless, there are settings our work handles which are not covered by these prior works.[3]

[BCD$^+$14] analyzed the PoA of strategizing in Fisher markets. The PoA compared the social welfare of the worst resulting Nash Equilibrium to the optimal, i.e. welfare maximizing assignment, under a suitable normalization of utilities. Among other results, they showed lower bounds of $\Omega(\sqrt{n})$ on the PoA when there are n buyers with linear utilities. However, we view the comparison point of an optimal assignment to be too demanding in this setting, as it may not be an assignment that could arise based on a pricing of the goods. In our results we will be comparing the strategic outcomes to those that occur under truthful bidding. Another approach is to bound the gains to individual agents, called the *incentive ratio*; [CDZ11, CDZZ12] showed these values were bounded by small constants in Fisher market settings.

[3]For example, buyers with linear utility functions but an infinite bidding space.

There has been much other work on large settings and their behavior. We mention only a sampling. [Kal04] studied the notion of extensive robustness for large games, and [KS13] investigated large repeated games using the notion of compressed equilibria. [PRU14] studied repeated games and the use of differential privacy as a measure of largeness. In a different direction, [GR08] investigated fault tolerance in large games for λ-continuous and anonymous games.

2.2 Large Fisher Markets and Regret Minimization

2.2.1 Definitions for Large Fisher Markets

In a Fisher market game, each buyer declares a bid function s_i; however, her endowment is public knowledge. The mechanism computes prices and allocations as if the bids were the valuations. The same restrictions will apply to the bid functions and the utilities. The goal of each buyer is to maximize her utility.

Notational remark The demands are induced by the bids, thus we could write $u_i(\mathbf{x}_i(s_i, s_{-i}))$, but for brevity we will write this as $u_i(s_i, s_{-i})$ instead. Also, it will be convenient to write v_i for the truthful bid of u_i, yielding the notation $u_i(v_i, v_{-i})$.

Definition 2.2.1. *The largeness L of a Fisher market is defined to be the ratio $L = \frac{\sum_i e_i}{\max_i e_i}$.*

It is natural to measure the efficiency of outcomes in the Fisher market game using the objective function (1.2), or rather its exponentiated form. More specifically, we compare the geometric means of the buyer's utilities weighted by their budgets at the worst Nash Equilibrium (with bids s) and at the market equilibrium (with bids v), and we call it the **Geometric Price of Anarchy (GPoA)**:

$$\text{GPoA}(M) = \max_{\text{NE with bids } s} \left(\prod_i \left(\frac{u_i(v_i, v_{-i})}{u_i(s_i, s_{-i})} \right)^{e_i} \right)^{\frac{1}{\sum_i e_i}}.$$

Note that in the settings we consider the prices at a market equilibrium are unique. We will also use this product to upper bound a Price of Anarchy notion for a market M, which compares the sum of the utilities at the worst Nash Equilibrium to the sum at the market equilibrium.

$$\text{PoA}(M) = \max_{\text{NE with bids } \mathbf{s}} \frac{\sum_i u_i(v_i, v_{-i})}{\sum_i u_i(s_i, s_{-i})}.$$

For the latter measure to be meaningful, we need to use a common scale for the different buyers' utilities. To this end, we define *consistent scaling.*

Definition 2.2.2. *The bidders' utilities are consistently scaled if there is a parameter $t > 0$ such that for every bidder i, $u_i(v_i, v_{-i}) = te_i$.* [4] *That is, bidder i's utility function is scaled to give it utility te_i at the market equilibrium, where e_i is its budget.*

Finally, we will be considering utility functions that are monotone, homogeneous of degree one (defined below), continuous, concave, and that induce demands that satisfy the gross substitutes condition (see Definition 2.2.4).

Definition 2.2.3. *Utility function $u(\mathbf{x})$ is* homogeneous of degree 1 *if for every $\alpha > 0$, $u(\alpha\mathbf{x}) = \alpha \cdot u(\mathbf{x})$.*

Fact 2.2.1. *The utility functions in Eisenberg Gale programs are homogeneous of degree 1, continuous and concave.*

Definition 2.2.4. *A valuation or bid function satisfies the gross substitutes property if increasing the price for one good only increases the demand for other goods. Formally, for each utility maximizing allocation $\mathbf{x}$ at prices $\mathbf{p} = (p_j, p_{-j})$, at prices (q_j, p_{-j}) such that $q_j > p_j$, there is a utility maximizing allocation $\mathbf{y}$ with $y_{-j} \succeq x_{-j}$ (i.e. $y_k \geqslant x_k$ for $k \neq j$). This definition applies to the Walrasian Equilibrium setting also.*

[4] WLOG, we may assume that $t = 1$.

2.2.2 Regret Minimization

In a regret minimization setting, a single player is playing a repeated game. At each round, the player can choose to play one of K strategies, which are the same from round to round. The outcome of the round is a payoff in the range $[-\chi, \chi]$.

Definition 2.2.5. *An algorithm that chooses the strategy to play is regret minimizing if the outcome of the algorithm, in expectation, is almost as good as the outcome from always playing a single strategy regardless of any one else's actions. Formally, there is a function* $\Phi(|K|, T) = o(T)$ *such that, for any* s^t_{-i}*, for any fixed strategy* $s_i \in K$*,*

$$\sum_{t=1}^{T} u_i(s^t_i, s^t_{-i}) - \sum_{t=1}^{T} u_i(s_i, s^t_{-i}) \geq -\Phi(|K|, T) \cdot \chi,$$

where s^t_i *is the strategy bidder* i *uses at time* t*.*

Theorem 2.2.2. *Regret minimizing algorithms exist. If, at the end of each round, the player learns the payoff for all* K *strategies,* $\Phi(|K|, T) = O(\sqrt{T})$ *can be achieved, and if she learns just the payoff for her strategy,* $\Phi(|K|, T) = O(T^{\frac{2}{3}})$ *can be achieved.*

Note that in large auctions and markets, it is the latter result that seems more applicable.

As shown in [Rou15], if all players play regret minimizing strategies, the resulting outcome observes the PoA bound obtained via a smoothness argument up to the regret minimization error.

2.3 Results

One issue that deserves some consideration when specifying a large setting, and placing some inevitable restrictions on the possible settings, is to determine which parameters should remain bounded as the setting size grows. So as to be able to state asymptotic results, we give results in terms of a parameter L which is allowed to grow arbitrarily large. But in fact all settings are finite, so really when stating that some parameters are bounded, we are making statements about the relative sizes of different parameters.

One common assumption is that the type space is finite. However, it is not clear such an assumption is desirable in the settings we consider, for it would be asserting that the number of possible valuations and bidding strategies is much smaller than the number of bidders. Another standard assumption is that the ratio of the largest to smallest non-zero valuations are bounded. This, for example, would preclude valuations being distributed according to a power law distribution (or any other unbounded distribution), which again seems unduly restrictive if it can be avoided.

Theorem 2.3.1. *Let M be a large Fisher market with largeness L in which the utility and bid functions are homogeneous of degree 1, concave, continuous, monotone and satisfy the gross substitutes property. If its demands as a function of the prices are unique at any $\mathbf{p} > 0$, or if its utility functions are linear, then its Price of Anarchy and its Geometric Price of Anarchy are bounded by*

$$PoA(M) \leqslant e^{m/L}, GPoA(M) \leqslant e^{m/L},$$

where m is the number of distinct goods in the market.

Perhaps surprisingly, there is no need for uncertainty in this setting. Note that these assumptions on the utility functions are satisfied by Cobb-Douglas utilities, and by those CES and Nested CES utilities that meet the weak gross substitutes condition. We note that our results do not extend to Fisher markets with Leontief utilities. For Theorem 4 in [BCD^+14] can be readily adapted to show that for some Fisher markets with Leontief utility functions, when L is large, the PoA is at least m, the number of goods.

Theorem 2.3.2. *Suppose all players use regret minimization algorithms, their utility functions and bid functions are homogeneous of degree 1, concave, continuous, monotone, and* *satisfy the gross substitutes property. Let $\lambda \geqslant 4$ be a parameter. If its demands as a function of the prices are unique at any $\mathbf{p} > 0$, or if its utility functions are linear, then in a large Fisher Market with largeness L and with reserve prices $\mathbf{r}$ such that for any j, $\frac{1}{\lambda} \leq \frac{r_j}{p_j(\mathbf{v})} \leq \frac{1}{4}$,*

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{i} u_i(s_i^t, s_{-i}^t) \geq (e^{-\frac{2m}{L}} - \frac{max_i \Phi(|K_i|, T)}{T}\lambda)\sum_{i} u_i(v_i, v_{-i}),$$

where K_i is the set of strategies used by player i and $v_i \in K_i$.

2.4 PoA for Large Fisher Markets

Theorem 2.3.1, which states that the PoA of an m-good market of largeness L is at most $e^{m/L}$, will follow from the next lemma.

Lemma 2.4.1. *For any bidding profile $\mathbf{s}$ and any value profile $\mathbf{v}$ which are homogeneous of degree 1, concave, continuous, monotone and which satisfy the gross substitutes property,*

$$\sum_{i=1}^{n} e_i \cdot \log(u_i(v_i, s_{-i})) \geq \sum_{i=1}^{n} e_i \cdot \log(u_i(v_i, v_{-i})) - m \cdot \max_i e_i.$$

Proof of Theorem 2.3.1: We prove our results for GPoA and PoA separately. Let $\mathbf{s}$ be a Nash Equilibrium.

- **PoA bound** On exponentiating the expressions on both sides in the statement of Lemma 2.4.1, we obtain

$$\prod_i u_i(v_i, s_{-i})^{e_i} \geq \frac{1}{e^{m\cdot\max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i}.$$

  Therefore, $\prod_i \left(\frac{u_i(v_i,s_{-i})}{u_i(v_i,v_{-i})}\right)^{e_i} \geq \frac{1}{e^{m\cdot\max_i e_i}}$. Using the weighted GM-AM inequality, we obtain

$$\frac{\sum_i e_i \frac{u_i(v_i,s_{-i})}{u_i(v_i,v_{-i})}}{\sum_i e_i} \geq \left(\prod_i \left(\frac{u_i(v_i,s_{-i})}{u_i(v_i,v_{-i})}\right)^{e_i}\right)^{\frac{1}{\sum_i e_i}} \geq \left(\frac{1}{e^{m \max_i e_i}}\right)^{\frac{1}{\sum_i e_i}} = e^{-\frac{m \max_i e_i}{\sum_i e_i}}.$$

Since for all i, $u_i(v_i, v_{-i}) = te_i$, $\sum_i u_i(v_i, s_{-i}) \geq e^{-\frac{m \max_i e_i}{\sum_i e_i}} \sum_i u_i(v_i, v_{-i})$. As Roughgarden encapsulated in the smooth technique, bounds of this sort yield PoA bounds for Nash Equilibria and more generally [Rou15, Rou12, ST13, FIL+16]. We obtain:

$$\mathbb{E}_{\mathbf{s}}\left[\sum_i u_i(s_i, s_{-i})\right] \geq \mathbb{E}_{\mathbf{s}}\left[\sum_i u_i(v_i, s_{-i})\right] \geq e^{-\frac{m \max_i e_i}{\sum_i e_i}} \mathbb{E}\left[\sum_i u_i(v_i, v_{-i})\right];$$

the first inequality follows because (s_i, s_{-i}) is a Nash Equilibrium.

- **GPoA bound** According to Lemma 2.4.1,

$$\prod_i u_i(v_i, s_{-i})^{e_i} \geq \frac{1}{e^{m \cdot \max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i}.$$

Therefore,

$$\begin{aligned}\prod_i \mathbb{E}_{\mathbf{s}}[u_i(v_i, s_{-i})]^{e_i} &\geq \prod_i e^{\mathbb{E}_{\mathbf{s}}[\log u_i(v_i,s_{-i})]e_i} = e^{\sum_i \mathbb{E}_{\mathbf{s}}[\log u_i(v_i,s_{-i})]e_i} \\ &= e^{\mathbb{E}_{\mathbf{s}}[\sum_i \log(u_i(v_i,s_{-i})^{e_i})]} = e^{\mathbb{E}_{\mathbf{s}}[\log(\prod_i u_i(v_i,s_{-i})^{e_i})]} \\ &\geq e^{\mathbb{E}_{\mathbf{s}}[\log \frac{1}{e^{m \cdot \max_i e_i}} \prod_i u_i(v_i,v_{-i})^{e_i}]} = \frac{1}{e^{m \cdot \max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i}.\end{aligned}$$

By applying the Nash equilibrium condition, $\mathbb{E}_{\mathbf{s}}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{\mathbf{s}}[u_i(v_i, s_{-i})]$, the GPoA bound follows.

□

To prove Lemma 2.4.1, we need the following claim; intuitively, it states that a single bidder can cause the prices to increase by only a small amount.

Lemma 2.4.2. $\mathbf{p}(v_i, s_{-i}) \leq \mathbf{p}(s_i, s_{-i}) + \max_i e_i \cdot 1$.

Proof of Lemma 2.4.1: Consider the dual of the Eisenberg-Gale convex program:

$$\min_p \max_x \sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) - \sum_{i,j} p_j x_{ij} + \sum_j p_j$$

$$\text{s.t.} \quad \forall j: \quad p_j \geqslant 0 \quad \text{and} \quad \forall i,j: \quad x_{ij} \geq 0.$$

Let $\mathbf{p}$ denote an arbitrary collection of prices, and $\mathbf{p}^*$ denote the prices with truthful bids. Since $\mathbf{p}^*$ minimizes the dual program,

$$\begin{aligned} \max_{\mathbf{x}\geq 0} \quad & \sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) - \sum_{i,j} p_j x_{ij} + \sum_j p_j \\ & \geq \max_{\mathbf{x}\geq 0} \quad \sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) - \sum_{i,j} p_j^* x_{ij} + \sum_j p_j^*. \end{aligned} \tag{2.1}$$

Let $\tilde{x}_{ij}$ be an allocation over all goods j and bidders i at prices p that maximize (2.1). As u_i is homogeneous of degree 1, u_i is differentiable in the direction $\mathbf{x}_i$. It follows that

$$\lim_{\epsilon \to 0} \frac{[e_i \cdot \log u_i((1+\epsilon)\tilde{\mathbf{x}}_i) - \sum_j p_j (1+\epsilon)\tilde{x}_{ij}] - [e_i \cdot \log u_i(\tilde{\mathbf{x}}_i) - \sum_j p_j \tilde{x}_{ij}]}{\epsilon} = 0. \tag{2.2}$$

The LHS of (2.2) equals $e_i - \sum_j p_j \tilde{x}_{ij}$, implying that $e_i = \sum_j p_j \tilde{x}_{ij}$. Therefore,

$$\begin{aligned} & \max\nolimits_{\mathbf{x}:\forall i \sum x_{ij} p_j = e_i} \quad \textstyle\sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots x_{im})) + \sum_j p_j \\ & \quad \geq \max\nolimits_{\mathbf{x}:\forall i \sum x_{ij} p_j^* = e_i} \quad \textstyle\sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots x_{im})) + \sum_j p_j^*. \end{aligned} \tag{2.3}$$

If all the prices stay the same or increase, a buyer's optimal utility stays the same or decreases. Using the price upper bound from Lemma 2.4.2, it follows that

$$
\begin{aligned}
\sum_{i=1}^{n} e_i \cdot \log(u_i(v_i, s_{-i})) &\geq \sum_{i=1}^{n} \max_{\mathbf{x}:\forall i \sum x_{ij}(p_j(s_i,s_{-i})+\max_{i'} e_{i'})=e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) \\
&= \sum_{i=1}^{n} \max_{\mathbf{x}:\forall i \sum x_{ij}(p_j(s_i,s_{-i})+\max_{i'} e_{i'})=e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) \\
&\quad + \sum_j (p_j(s_i, s_{-i}) + \max_{i'} e_{i'}) - \sum_j (p_j(s_i, s_{-i}) + \max_{i'} e_{i'}) \\
&\geq \sum_{i=1}^{n} \max_{\mathbf{x}:\forall i \sum x_{ij} p_j^* = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) \\
&\quad + \sum_j p_j^* - \sum_j (p_j(s_i, s_{-i}) + \max_{i'} e_{i'}) \qquad \text{by (2.3)} \\
&\geqslant \sum_{i=1}^{n} \max_{\mathbf{x}:\forall i \sum_j x_{ij} p_j^* = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots, x_{im})) - m \max_i e_i \\
&\qquad \text{as } \sum_j p_j^* = \sum_i e_i = \sum_j p_j(s_i, s_{-i}) \\
&= \sum_{i=1}^{n} e_i \log(u_i(v_i, v_{-i})) - m \max_i e_i.
\end{aligned}
$$

□

The proof of Lemma 2.4.2 uses the following notation and follows from Lemmas 2.4.3 and 2.4.4 below. $\mathbf{p}$ denotes the prices when the ith bidder is not participating and the bidding profile is s_{-i}; $\mathbf{x}$ denotes the resulting allocation. Similarly, $\hat{\mathbf{p}}$ denotes the prices when the bidding profile is (s_i, s_{-i}); $\hat{\mathbf{x}}$ denotes the resulting allocation.

As later on we need a generalized version of Lemma 2.4.3 which allows reserve prices $\mathbf{r} \geq 0$, we state and prove the more general version here.

Lemma 2.4.3. *Let* $\mathbf{r} \geq 0$ *be reserve prices. Then* $\mathbf{r} \preceq \mathbf{p} \preceq \hat{\mathbf{p}} = \mathbf{p}(s_i, s_{-i})$.

Lemma 2.4.4. $\hat{\mathbf{p}} \preceq \mathbf{p} + e_i \cdot \mathbf{1}$.

Proof of Lemma 2.4.2: Lemmas 2.4.3 and 2.4.4 also apply to prices $\mathbf{p}(v_i, s_{-i})$ as well as to $\hat{\mathbf{p}}$. So $\mathbf{p}(v_i, s_{-i}) \leqslant \mathbf{p} + e_i \cdot \mathbf{1} \leqslant \mathbf{p}(s_i, s_{-i}) + e_i \cdot \mathbf{1} \leq \mathbf{p}(s_i, s_{-i}) + \max_i e_i \cdot \mathbf{1}$. □

Lemma 2.4.4 follows readily from Lemma 2.4.3.

Proof of Lemma 2.4.4: Since $\mathbf{1} \cdot \mathbf{p} + e_i = \mathbf{1} \cdot \hat{\mathbf{p}}$ and $\mathbf{p} \preceq \hat{\mathbf{p}}$, the lemma follows. $\square$

We finish by proving that Lemma 2.4.3 holds in two scenarios: single-demand WGS utility functions and linear utility functions.

2.4.1 Unique Demand WGS Utility Functions

Proof of Lemma 2.4.3: For a contradiction, we suppose there is an item j such that $p_j > \hat{p}_j$.

Let $\epsilon > 0$ be a very small constant such that $\epsilon < p_k$ for all $p_k \neq 0$ and $\epsilon < \hat{p}_k$ for all $\hat{p}_k \neq 0$.

Let $\mathbf{p}'$ denote the following collection of prices: $p'_k = p_k$ if $p_k \neq 0$, and $p'_k = \epsilon$ otherwise. We consider the resulting demands for a bidder $h \neq i$. Recall that $\mathbf{x}_h$ denotes bidder h's demand at prices $\mathbf{p}$. $\mathbf{x}'_h$ will denote her demand at prices $\mathbf{p}'$. By the WGS property, $x'_{hk} = x_{hk}$ if $p_k \neq 0$, and $x'_{hk} = 0$ if $p_k = 0$, i.e. if $p'_k = \epsilon$.[5]

Analogously, let $\hat{p}'_k = \hat{p}_k$ if $\hat{p}_k \neq 0$, and $\hat{p}'_k = \epsilon$ otherwise. Let $\hat{\mathbf{x}}_h$ denote bidder h's demand at prices $\hat{\mathbf{p}}$, and $\hat{\mathbf{x}}'_h$ her demand at prices $\hat{\mathbf{p}}'$. Again, $\hat{x}'_{hk} = \hat{x}_{hk}$ if $\hat{p}_k \neq 0$, and $\hat{x}'_{hk} = 0$ if $\hat{p}'_k = \epsilon$.

Now, we look at those items l which have the smallest ratio between p'_l and $\hat{p}'_l$.

$$S = \left\{ l \;\middle|\; \frac{\hat{p}'_l}{p'_l} = \min_k \frac{\hat{p}'_k}{p'_k} \right\}.$$

By assumption, $p_j > \hat{p}_j$; therefore $p'_j > \hat{p}'_j$. Thus, for $l \in S$, $\frac{\hat{p}'_l}{p'_l} < 1$. For simplicity, let η

[5]Changing the prices from $\mathbf{p}$ to $\mathbf{p}'$, one by one, by setting p'_k to ϵ, which happens when $p_k = 0$, only increases the demand for other goods, but as no spending is released by this price increase, these demands are in fact unchanged.

denote this ratio. Note that this inequality implies $p'_l > \epsilon$, and thus $p_l = p'_l > 0$. Also,

$$p_l = p'_l > \hat{p}'_l \geq r_l. \tag{2.4}$$

We now consider the following procedure:

First multiply $\mathbf{p}'$ by η. By the homogeneity of the utility function, bidder h's demand at prices $\eta \cdot \mathbf{p}'$ will be $\frac{1}{\eta}\mathbf{x}'_h$. Note that $\eta \cdot p'_l = \hat{p}'_l$ for any $l \in S$ and $\eta \cdot p'_k < \hat{p}'_k$ for any $k \notin S$.

Second, increase the prices of $\eta \cdot \mathbf{p}'$ to $\hat{\mathbf{p}}'$. Since for $l \in S$ the two prices are the same, by the Gross Substitutes property, $\hat{x}'_{hl} \geq \frac{1}{\eta} x'_{hl}$ for any $l \in S$.

Summing over all the bidders except i,

$$\sum_{h \neq i} \hat{x}'_{hl} \geq \frac{1}{\eta} \sum_{h \neq i} x'_{hl} \qquad \text{for } l \in S.$$

By (2.4), $p_l > r_l$ for any $l \in S$; hence $\sum_{h \neq i} x'_{hl} = \sum_{h \neq i} x_{hl} = 1$. So, since $\eta < 1$,

$$\sum_{h \neq i} \hat{x}'_{hl} > \sum_{h \neq i} x'_{hl} = \sum_{h \neq i} x_{hl} = 1 \qquad \text{for } l \in S. \tag{2.5}$$

For all h and l, $\hat{x}_{hl} \geq \hat{x}'_{hl}$. Therefore,

$$\sum_{h} \hat{x}_{hl} \geq \sum_{h \neq i} \hat{x}'_{hl} > \sum_{h \neq i} x_{hl} = 1 \qquad \text{for } l \in S.$$

As $\sum_h \hat{x}_{hl} \leq 1$, this is impossible and yields a contradiction. □

2.4.2 Linear Utility Functions

Proof of Lemma 2.4.3: For a contradiction, we suppose there is an item j such that $p_j > \hat{p}_j$. Now, we look at those items j which have the smallest ratio between p_l and $\hat{p}_l$.

$$S = \left\{ l \;\middle|\; \frac{\hat{p}_l}{p_l} = \min_k \frac{\hat{p}_k}{p_k} \right\}.$$

Note $l < 1$ and $p_l > r_l$. For simplicity, we set $\frac{0}{x} = 0$ for $x > 0$, $\frac{0}{0} = 1$ and $\frac{x}{0} = +\infty$ for $x > 0$.

For linear utility functions, we use the following observation: if at prices $\mathbf{p}$ a bidder's favorite items include some items in S, then at prices $\hat{\mathbf{p}}$ her favorite items will all be in S.

Therefore, as $p_l > r_l$ implying $\sum_{i' \neq i} x_{il}(s_{-i}) = 1$,

$$\sum_{l \in S} p_l = \sum_{l \in S} \sum_{i' \neq i} x_{il}(s_{-i}) p_l \leq \sum_{l \in S} \sum_{i' \neq i} x_{il}(s_i, s_{-i}) \hat{p}_l \leq \sum_{l \in S} \hat{p}_l.$$

This implies that $\min_k \frac{\hat{p}_k}{p_k} = 1$, and the lemma follows. $\square$

2.5 Result for Walrasian Equilibrium

In this section, we will describe our results for the Walrasian Equilibrium.

2.5.1 Definitions for Large Walrasian Auctions

Definition 2.5.1. *An auction A comprises a set of N bidders $B_1, B_2, \ldots, B_N$, and a set of m goods G, with n_j copies of good j, for $1 \leqslant j \leqslant m$. We write $\mathbf{n} = (n_1, n_2, ..., n_m)$, where n_j denotes the number of copies of good j, and we call it the* multiplicity *vector. We also write $\mathbf{n} = (n_j, n_{-j})$, where n_{-j} is the vector denoting the number of copies of goods other than good j. We refer to an instance of a good as an item. For an allocation $\mathbf{x}_i$ to bidder i, which is a* *subset of the available goods, we write* $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})$ *where* x_{ij} *denotes the number of copies of good* j *in allocation* $\mathbf{x}_i$*. There is a set of prices* $\mathbf{p} = (p_1, p_2, \ldots, p_m)$*, one per good; we also write* $\mathbf{p} = (p_j, p_{-j})$*. Each bidder* i *has a valuation function* $v_i : X \to \mathbb{R}_+$*, where* X *is the set of possible assignments, and a quasi-linear utility function* $u_i(\mathbf{x}_i) = v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p}$*.*

A Walrasian equilibrium is a collection of prices $\mathbf{p}$ *and an allocation* $\mathbf{x}_i$ *to each bidder* i *such that (i) the goods are fully allocated but not over-allocated, i.e. for all* j*,* $\sum_i x_{ij} \leq n_j$*, and* $\sum_i x_{ij} = n_j$ *if* $p_j > 0$*, and (ii) each bidder receives a utility maximizing allocation at prices* $\mathbf{p}$*, i.e.* $u_i(\mathbf{x}_i) = v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p} = \max_{\mathbf{x}_i}[v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p}]$*.*

In a Walrasian mechanism for auction A *each bidder declares a bid function* $s_i : X \to \mathbb{R}_+$*. We write* $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ *and* $\mathbf{s} = (s_i, s_{-i})$*. The mechanism computes prices and allocations as if the bids were the valuations.*

Given the bidders and their bids, $\mathbf{p}(\mathbf{n}; \mathbf{s})$ denotes the prices produced by the Walrasian mechanism at hand when there are $\mathbf{n}$ copies of the goods and $\mathbf{s}$ is the bidding profile. Also, $p_j(\mathbf{n}; \mathbf{s})$ denotes the price of good j and $\mathbf{p}(\mathbf{n}; \mathbf{s}) = (p_j(\mathbf{n}; \mathbf{s}), p_{-j}(\mathbf{n}; \mathbf{s}))$. Finally, we let both $\mathbf{x}_i(\mathbf{n}; \mathbf{s})$ and $\mathbf{x}_i(\mathbf{n}; s_i, s_{-i})$ denote the allocation to bidder i provided by the mechanism.

In the auctions we consider, the number $\mathbf{n}$ of copies of each good is determined by a distribution $F(\mathbf{n})$. In order for the auction to be large, we need that the probability that there are exactly r_j copies of the j-th item be small, for every r_j and for every j.

Definition 2.5.2. *A large Walrasian auction is characterized by a distribution* $F(\mathbf{n})$*, a demand bound* k*, and a largeness measure* L*. It satisfies the following two properties.*

i. The demand of every bidder is for at most k *items. Formally, if allocated a set of more than* k *items, the bidder will obtain equal utility with a subset of size* k*.*

ii. The probability that there are exactly c *copies of good* j*, for any* c *and any* j *is bounded by* $1/L$*. Formally, Let* $F(n_j, j|n_{-j})$ *denote the probability that there are exactly* n_j *copies of good* j *when given* n_{-j} *copies of other goods; then* $\max_j \max_{n_j, n_{-j}} F(n_j, j|n_{-j}) \leqslant 1/L$*.*

Note this definition implies that the expected number of copies of each good is at least $\frac{L}{2}$ and it is in this sense that the market is large.

A Bayes-Nash equilibrium is an outcome with no expected gain from an individual deviation:

$$\forall b_i' : \mathbb{E}_{\mathbf{n},v_{-i},s_{-i}}\left[u_i(x_i(\mathbf{n}; s_i, s_{-i}), p((s_i, s_{-i}))\right] \geqslant \mathbb{E}_{\mathbf{n},v_{-i},s_{-i}}\left[u_i(x_i(\mathbf{n}; b_i', s_{-i}), p((b_i', s_{-i}))\right].$$

The social welfare $\mathrm{SW}(\mathbf{x})$ of an allocation $\mathbf{x}$ is the sum of the individual valuations: $\mathrm{SW}(\mathbf{x}) = \sum_i v_i(\mathbf{x}_i)$. We also write SW(OPT) for the (expected) optimal social welfare, the maximum (expected) achievable social welfare, and SW(NE) for the smallest (expected) social welfare achievable at a Bayes-Nash equilibrium.

Finally, the Price of Anarchy is the worst case ratio of SW(OPT) to SW(NE) over all instances in the class of games at hand, namely auctions A_N of N buyers:

$$\mathrm{PoA} = \max_{A_N} \frac{\mathrm{SW(OPT)}}{\mathrm{SW(NE)}}.$$

2.5.2 Result for Walrasian Auctions

Our analysis makes two assumptions; stronger assumptions were made for the large auction results in [Swi01, FIL+16]. [Swi01] also ruled out overbidding by arguing it is a dominated strategy. Our analysis can avoid even this assumption of other players' rationality; however, bounded overbidding is needed for the extension to regret minimizing strategies.

Assumption 2.5.1. [Bounded Expected Valuation] *There is a constant ζ such that for each bidder and each item, her expected value for this single item is at most ζ:*

$$\max_s \mathbb{E}_{v_i}[v_i(s)] \leq \zeta.$$

Note that without this assumption the social welfare would not be bounded, and then it is not clear how to measure the Price of Anarchy. Prior work had assumed $v_i(s) \leqslant \zeta$ for all s and i (i.e. absolutely rather than in expectation).

Observation 2.5.1.

$$\mathbb{E}_{v_i}\left[\max_s\{v_i(s)\}\right] \leq \zeta m.$$

Theorem 2.5.1. *In a large Walrasian auction which satisfies Assumption 2.5.1 and with buyers whose valuation and bid functions are monotone and satisfy the gross substitutes property,*

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)},$$

where $Y = \frac{m}{L}\left[2m\binom{k+1+m}{m}\right]$ *and* $\rho = \frac{\text{SW(OPT)}}{N}$.

Also, if there is only one good, i.e., if $m = 1$*, then*

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)},$$

where $Y = \frac{2(k+2)}{L}$ *and* $\rho = \frac{\text{SW(OPT)}}{N}$.

Remark The gross substitutes assumption is present so as to ensure the auction outcome is a Walrasian equilibrium w.r.t. to the bids, for if it is not then some bidders will be allocated a non-favorite bundle, which seems unattractive as a solution concept.

To achieve $\text{SW(NE)} \geq (1-\epsilon)\text{SW(OPT)}$ where ϵ is small, we need $\frac{L}{\rho \cdot \log L}$ to be large. We can achieve this by considering a sequence of auctions indexed by N, the number of bidders, and requiring ρ to be a constant and L to be sufficiently large. One way to obtain a constant ρ is to make the following two assumptions.

Assumption 2.5.2. [Auction Size] *Let $\mu(n_j)$ be the expected number of copies of good j, for $1 \leqslant j \leqslant m$, and let $\Gamma(n_j)$ be its standard deviation. The assumption is that for each j, $\mu(n_j) = \Theta(N)$ and $\Gamma(n_j) \leqslant (1-\lambda)\mu(n_j)$ for some constant $0 < \lambda \leq 1$. Let $0 < \alpha \leq 1$ be such that $\mu(n_j) \geqslant \alpha N$ for all j and sufficiently large N.*

Assumption 2.5.3. [Value Lower Bound] *There is a parameter $\rho' > 0$ such that for any bidder, its largest expected value for one item is at least ρ':*

$$\max_s \mathbb{E}_{v_i}[v_i(s)] \geq \rho'.$$

Lemma 2.5.1. *Let $\rho = \lambda^2 \alpha \frac{2\lambda+\lambda^2}{(1+\lambda)^2} \rho'$. If Assumptions 2.5.2 and 2.5.3 hold, then* $\mathrm{SW}(\mathrm{OPT}) \geqslant \rho N$.

In previous work, [FIL+16] also made the assumption that ρ is a constant. [Swi01] made assumptions on the value distribution which again imply ρ is a constant although this consequence is not stated in his work.

Corollary 2.5.1. *In a large Walrasian auction which satisfies Assumption 2.5.1 and with buyers whose valuation and bid functions are monotone and satisfy the gross substitutes property, if the number of copies of each good is independently and identically distributed according to the Binomial distribution $B(N, \frac{1}{2})$, and $\rho, k, m = O(1)$, then*

$$\mathrm{SW}(\mathrm{NE}) \geqslant \left(1 - O\left(\frac{\log N}{\sqrt{N}}\right)\right) \mathrm{SW}(\mathrm{OPT}).$$

In order to obtain good bounds when using regret minimization algorithms, we need to be able to bound the possible losses a player makes, which we achieve by bounding the possible overbidding. This is similar to the notion of overbidding previously given in [BLNPL14].

Definition 2.5.3. *Let K be the set of strategies a player uses. She is a (γ, δ)-player if $v \in K$ and, for any $s \in K$ and for any set $\mathbf{x}$,*

$$s(\mathbf{x}) \leq v(\mathbf{x}) \cdot \gamma + \delta.$$

Theorem 2.5.2. *Suppose all players use regret minimization algorithms, they are all (γ, δ)-players and their valuation and bid functions are monotone and satisfy the gross substitutes* *property. Then, in a large Walrasian auction which satisfies Assumption 2.5.1,*

$$\frac{1}{T}\mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\sum_{t=1}^{T} v_i(\mathbf{x}_i(s_i^t, s_{-i}^t))\right] \geq \Bigg(1 - \frac{3\cdot k\cdot (k+1)\cdot \zeta \cdot m}{\rho}\cdot Y \cdot \lceil \log_2 \frac{1}{Y}\rceil - \frac{\max_i \Phi(|K_i|, T)\cdot (km\zeta\gamma + \delta)}{\rho \cdot T}\Bigg)\mathrm{SW(OPT)}.$$

where $Y = \frac{m}{L}\left[2m\binom{k+1+m}{m}\right]$, $\rho = \frac{\mathrm{SW(OPT)}}{N}$, K_i *is the set of strategies used by* i*, and* $v_i \in K_i$.

2.5.3 Proof

Here we prove a slightly weaker version of Theorem 2.5.1 which demonstrates the main ideas (Theorem 2.5.3 below). Our goal is to show that in expectation

$$\sum_i \mathbb{E}\Big[u_i(x_i(v_i, s_{-i}))\Big] \geq \mathrm{SW(OPT)} - \mathrm{R} - O(N\epsilon), \tag{2.6}$$

where R denotes the expected auction revenue under bidding profile (s_i, s_{-i}). For, as encapsulated in the smooth technique for Bayesian settings [ST13], this type of bound yields PoA bounds, and our result in particular:

$$\mathrm{SW}(s_i, s_{-i}) = \mathrm{R} + \sum_i \mathbb{E}\Big[u_i(x_i(s_i, s_{-i}))\Big] \geq R + \sum_i \mathbb{E}\Big[u_i(x_i(v_i, s_{-i}))\Big] \geq \mathrm{SW(OPT)} - O(N\epsilon).$$

Inequality (2.6) follows from two observations. First, with high probability, a buyer has at most a small influence on prices (Lemma 2.5.2), and hence can improve her own utility by at most a small amount via a non-truthful bid (Lemma 2.5.3). Otherwise, by Assumption 2.5.1 and the Gross Substitutes property, her expected utility is bounded by $km\zeta$. The probability bound stems from the distribution F over the number of goods. To obtain the bound, we define (k,ϵ)-good and bad multiplicity vectors $\mathbf{n}$, wr.t. bids $\mathbf{b}$. By counting their number, we will show that the fraction of $(k+1,\epsilon)$-bad vectors is $O(\frac{1}{L\epsilon})$. Also, if the vector is $(k+1,\epsilon)$-good, we will show that a bidder can cause the prices, when they are all bounded by 1, to vary by at most $(k+1)\epsilon$. Essentially, a vector $\mathbf{n}$ is (k,ϵ)-good if changing the supplies by at most k items causes prices $p_j \leqslant 1$ to change in total by at most $k\epsilon$. Then, using the fact that the equilibrium is Walrasian, we can show that for $(k+1,\epsilon)$-good vectors $\mathbf{n}$,

$$u_i(x_i(v_i, s_{-i})) \geq v_i(x_i(v_i, v_{-i})) - \sum_{g \in x_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) - k(k+1)\epsilon.$$

On summing over i and taking expectations, we can then deduce that

$$\sum_i \mathbb{E}\Big[u_i(x_i(v_i, s_{-i}))\Big] \geq \mathrm{SW(OPT)} - \mathrm{R} - N \cdot k \cdot (k+1) \cdot \epsilon - O(\frac{N \cdot k}{L\epsilon}).$$

Recall that the English Walrasian mechanism can be implemented as an ascending auction. The prices it yields can be computed as follows: p_j is the maximum possible increase in the social welfare when the supply of good j is increased by one unit. Similarly, the Dutch Walrasian mechanism can be implemented as a descending auction, and the resulting price p_j is the loss in social welfare when the supply of good j is decreased by one unit.

We will be considering an arbitrary Walrasian mechanism. Necessarily, its prices must lie between those of the Dutch Walrasian and English Walrasian mechanisms. We let $\mathbf{p}^{Eng}(\mathbf{n}; (s_i, s_{-i}))$ denote the price output by the English Walrasian mechanism and $\mathbf{p}^{Dut}(\mathbf{n}; (s_i, s_{-i}))$ the price output by the Dutch Walrasian mechanism.

We define the distance between two price vectors $\mathbf{p}$ and $\mathbf{p'}$ with respect to U as follows:

$$dist^U(\mathbf{p}, \mathbf{p'}) = \sum_{j=1}^{m} \left|\min\{p_j, U\} - \min\{p'_j, U\}\right|,$$

where U is a parameter we set later.

Definition 2.5.4. *Given bidding profile (s_i, s_{-i}), $\mathbf{n} = (n_j, n_{-j})$ is* (ϵ, U)-bad *for good j if, in the English Walrasian mechanism, the distance between the prices is more than ϵ when* *an additional copy of good j is added to the market:*

$$dist^U(\mathbf{p}^{Eng}((n_j, n_{-j}); (s_i, s_{-i})), \mathbf{p}^{Eng}((n_j + 1, n_{-j}); (s_i, s_{-i}))) > \epsilon.$$

Let $\mathbf{k} = (k, k, \ldots, k)$ and $\mathbf{0} = (0, 0, \ldots, 0)$ be m-vectors.

Definition 2.5.5. *Given bidding profile* $\mathbf{s}$*,* $\mathbf{n}$ *is* (k, ϵ, U)-bad *for good j if there is a vector* $\mathbf{n}'$ *which is (ϵ, U)-bad for good j, such that $n'_h \leqslant n_h$ for all h, and $\sum_h n_h \leqslant k + \sum_h n'_h$.* $\mathbf{n}$ *is* (k, ϵ, U)-good *if it is not (k, ϵ, U)-bad.*

Observation 2.5.2. *Given bidding profile* $\mathbf{s}$*, if* $\mathbf{n}$ *is* (k, ϵ, U)-good *then*

$$dist^U(\mathbf{p}^{Eng}(\mathbf{n}; \mathbf{s}), \mathbf{p}^{Eng}(\mathbf{n}'; \mathbf{s})) \leq k\epsilon$$

if $n'_h \leqslant n_h$ for all h, and $\sum_h n_h \leqslant k + \sum_h n'_h$.

For brevity, we sometimes write $u_i(v_i, s_{-i})$ instead of $u_i(\mathbf{x}_i(v_i, s_{-i}))$. For simplicity, let $\Lambda(m, k)$ denote $m \cdot \binom{k+m}{m}$.

Lemma 2.5.2. *In the English Walrasian mechanism with bidding profile* $\mathbf{s}$*, the probability that* $\mathbf{n}$ *is (k, ϵ, U)-bad for some good, or* $\min_j n_j \leqslant k$ *is at most*

$$\frac{m}{L}\left[\frac{U}{\epsilon}\Lambda(m, k) + k + 1\right].$$

Let $|\mathbf{x}_i(\cdot)|$ denotes the total number of items in allocation $\mathbf{x}_i$. Let $\mathbf{d}_i \preceq \mathbf{x}_i(v_i, v_{-i})$ be a minimal set with $v_i(\mathbf{d}_i) = v_i(\mathbf{x}_i(v_i, v_{-i}))$. By Definition 2.5.2(i), $|\mathbf{d}_i| \leq k$.

Lemma 2.5.3. *If* $\mathbf{n}$ *is $(k+1, \epsilon, U)$-good, where $U \geq v_i(s)$ for every single item s, $n_j > k+1$ for all j, and v_i and s_i satisfy the gross substitutes property for all i, then*

$$u_i(v_i, s_{-i}) \geq v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in x_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon,$$

where the sum is over all the items in allocation $\mathbf{x}_i$*.*

Theorem 2.5.3. *In a large Walrasian auction which satisfies Assumption 2.5.1 and with buyers whose valuation and bid functions are monotone and satisfy the gross substitutes property,*

$$\text{SW(NE)} \geq \left(1 - \frac{3k \cdot \zeta \cdot m}{\rho}\sqrt{(k+2)\frac{m}{L}\Lambda(m, k+1)}\right)\text{SW(OPT)},$$

where $\rho = \frac{\text{SW(OPT)}}{N}$*.*

Proof. By Lemma 2.5.3, if $\mathbf{n}$ is $(k+1, \epsilon \cdot \max_g\{v_i(g)\}, \max_g\{v_i(g)\})$-good and $n_j > k+1$ for all j, then

$$\begin{aligned} u_i(v_i, s_{-i}) \;\; &\geq \;\; v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) \\ &\qquad - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_g\{v_i(g)\}. \end{aligned}$$

By Lemma 2.5.2, the probability that $\mathbf{n}$ is $(k+1, \epsilon \cdot \max_g\{v_i(g)\}, \max_g\{v_i(g)\})$-bad or $n_j \leq k+1$ for some j is less than

$$\frac{m}{L}\left[\frac{1}{\epsilon}\Lambda(m, k+1) + k + 2\right],$$

$$\begin{aligned} \text{and} \quad \mathbb{E}_{\mathbf{n}}[u_i(v_i, s_{-i})] \geq \;& \mathbb{E}_{\mathbf{n}}\Bigg[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) \\ &\qquad - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_g\{v_i(g)\}\Bigg] \\ &- \frac{m}{L}\left[\frac{1}{\epsilon}\Lambda(m, k+1) + k + 2\right] \cdot k \cdot \max_g\{v_i(g)\}. \end{aligned}$$

Here, the expectation is taken over the randomness on the multiplicities of the goods; the inequality holds since $u_i(v_i, s_{-i}) \geq 0$ and $v_i(\mathbf{x}_i(v_i, v_{-i})) \leq k \cdot \max_g\{v_i(g)\}$.

Taking the expectation over the valuation of agent i yields

$$\begin{aligned}
\mathbb{E}_{v_i}[\mathbb{E}_{\mathbf{n}}[u_i(v_i, s_{-i})]] \;\; \geq \;\; & \mathbb{E}_{v_i}\Bigg[\mathbb{E}_{\mathbf{n}}\Big[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) \\
& \qquad - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_g\{v_i(g)\}\Big] \\
& \qquad - \frac{m}{L}\left[\frac{1}{\epsilon}\Lambda(m, k+1) + k + 2\right] \cdot k \cdot \max_g\{v_i(g)\}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\geq \;\; & \mathbb{E}_{v_i}\Bigg[\mathbb{E}_{\mathbf{n}}\Big[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i}))\Big]\Bigg] \\
& - \mathbb{E}_{v_i}[\max_g\{v_i(g)\}] \cdot k \cdot (k+1)\epsilon \\
& - \mathbb{E}_{v_i}[\max_g\{v_i(g)\}]\frac{m}{L}\left[\Lambda(m, k+1)\frac{k}{\epsilon} + (k+2)k\right].
\end{aligned}$$

Since $\mathbb{E}_{v_i}[\max_g\{v_i(g)\}] \leq \mathbb{E}_{v_i}[\sum_g v_i(g)] \leq \sum_g \mathbb{E}_{v_i}[v_i(g)] \leq m \cdot \zeta$,

$$\begin{aligned}
\mathbb{E}_{v_i}[\mathbb{E}_{\mathbf{n}}[u_i(v_i, s_{-i})]] \;\; \geq \;\; & \mathbb{E}_{v_i}\Bigg[\mathbb{E}_{\mathbf{n}}\Big[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i}))\Big]\Bigg] \\
& - \zeta \cdot m \cdot k(k+1)\epsilon \\
& - \zeta \cdot m \cdot m \cdot \frac{1}{L}\left[\Lambda(m, k+1)\frac{k}{\epsilon} + (k+2)k\right].
\end{aligned}$$

Let $\mathrm{R}(\mathbf{b})$ denote the expected revenue when the bidding profile is $\mathbf{b}$. Also, recall that $\mathrm{SW}(\mathrm{OPT}) = \rho N$. Now, summing over all the bidders yields

$$
\begin{aligned}
\sum_i \mathbb{E}_{\mathbf{v},\mathbf{s},\mathbf{n}}[u_i(v_i, s_{-i})] \;\; &\geq \;\; \sum_i \mathbb{E}_{\mathbf{v},\mathbf{s},\mathbf{n}}\left[\mathbb{E}_{\mathbf{n}}\left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i}))\right]\right] \\
&\quad - \zeta \cdot m \cdot m \cdot \frac{1}{L}\left[\Lambda(m, k+1)\frac{k}{\epsilon} + (k+2)k\right] \cdot N \\
&\quad - \zeta \cdot m \cdot k(k+1)\epsilon \cdot N \\
&\geq \;\; \Bigg(1 - \frac{\zeta \cdot m \cdot m \cdot \frac{1}{L}\left[\Lambda(m, k+1)\frac{k}{\epsilon} + (k+2)k\right]}{\rho} \\
&\qquad - \frac{\zeta \cdot m \cdot k(k+1)\epsilon}{\rho}\Bigg)\mathrm{SW}(\mathrm{OPT}) - \mathbb{E}_{\mathbf{s},\mathbf{n}}[\mathrm{R}(s_i, s_{-i})].
\end{aligned}
$$

Now, $\mathrm{SW}(\mathrm{NE}) = \mathbb{E}_{\mathbf{s},\mathbf{n}}[\mathrm{R}(s_i, s_{-i})] + \sum_i \mathbb{E}_{\mathbf{v},\mathbf{s},\mathbf{n}}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{\mathbf{s},\mathbf{n}}[\mathrm{R}(s_i, s_{-i})] + \sum_i \mathbb{E}_{\mathbf{v},\mathbf{s},\mathbf{n}}[u_i(v_i, s_{-i})]$; Therefore,

$$
\begin{aligned}
\mathrm{SW}(\mathrm{NE}) \;\; &\geq \;\; \Bigg(1 - \frac{\zeta \cdot m \cdot m \cdot \frac{1}{L}\left[\Lambda(m, k+1)\frac{k}{\epsilon} + (k+2)k\right]}{\rho} \\
&\qquad - \frac{\zeta \cdot m \cdot k(k+1)\epsilon}{\rho}\Bigg)\mathrm{SW}(\mathrm{OPT}).
\end{aligned}
$$

The analysis is using the methodology of the smooth technique for Bayesian settings [ST13]. Now set $\epsilon = \sqrt{\frac{\frac{m}{L}\Lambda(m,k+1)}{k+1}}$. The claimed bound follows. □

Comparison of our methodology with that of [FIL$^+$16] We will be considering the combinatorial auction in [FIL$^+$16] which uses separate auctions for each type of good, more specifically a $(c+1)$-st price auction when there are c copies of the good. To facilitate a comparison, we adjust their notation to match the notation we have been using and reduce its generality[6]. They begin by defining a notion of smooth in the large which in the current

[6] In fact, the comparison applies in full generality.

context amounts to showing

$$\sum_i U(v_i, s_{-i}) \geqslant (1-\epsilon)\mathrm{SW}(\mathrm{OPT}) - R(\mathbf{b}). \tag{2.7}$$

To obtain such bounds, they propose the following methodology: It entails identifying an approximate utility function $U(v_i, s_{-i})$ and then showing the following two bounds:

- The approximate and actual utilities are close: For all $\mathbf{b}$, $|u_i(\mathbf{b}) - U_i(\mathbf{b})| \leqslant \epsilon$.
- The standard smoothness formulation applies to the approximate utility: For all $i, \mathbf{v}, \mathbf{b}$, $\sum_i U(v_i, s_{-i}) \geqslant \mathrm{SW}(\mathrm{OPT}) - R(\mathbf{b})$.

One can then deduce that $\sum_i U(v_i, s_{-i}) \geqslant \mathrm{SW}(\mathrm{OPT}) - R(\mathbf{b}) - N\epsilon$, which, on taking expectations, is exactly the bound we obtain for our auction. With the assumption that $\mathrm{SW}(\mathrm{OPT}) = \rho N$ one obtains (2.7). However, it not clear that we can specify an approximate utility U as specified in the framework of [FIL+16]. In particular, handling the expected bound on valuations in this framework, rather than the fixed bound used by Feldman et al., appears challenging.

2.6 Missing Proofs

2.6.1 Proofs from Section 2.5.2

Proof of Lemma 2.5.1: Let $\#\mathrm{items}_j$ denote the number of copies of good j that are present, and let N_j denote the number of buyers for which good j has the largest expected value (breaking ties arbitrarily). By Chebyshev's Theorem, $\Pr\left[\#\mathrm{items}_j > \mathbb{E}[\#\mathrm{items}_j] - t \cdot \Gamma(\#\mathrm{items}_j)\right] \geq 1 - \frac{1}{t^2}$. We set t equal $1+\lambda$, where λ is the parameter in Assumption 2.5.2. Then by Assumption 2.5.2, $\Pr\left[\#\mathrm{items}_j > \lambda^2 \cdot \mathbb{E}[\#\mathrm{items}_j]\right] \geq \frac{2\lambda+\lambda^2}{(1+\lambda)^2}$, which implies $\Pr[\#\mathrm{items}_j > \lambda^2 \alpha N] \geq \frac{2\lambda+\lambda^2}{(1+\lambda)^2}$. If at least $\lambda^2\alpha N$ copies of good j are available, then by Assumption 2.5.3, there is an assignment with valuation at least $\rho' \cdot \min\{N_j, \lambda^2\alpha N\}$. Therefore, the social welfare is at least $\sum_j \min\{N_j, \lambda^2\alpha N\}\frac{2\lambda+\lambda^2}{(1+\lambda)^2} \cdot \rho' \geqslant \lambda^2\alpha\frac{2\lambda+\lambda^2}{(1+\lambda)^2}N \cdot \rho'$. □

2.6.2 Proofs from Section 2.5.3

In Lemmas 2.6.1 and 2.6.4, we bound the number of (ϵ, U)-bad multiplicity vectors, and then in Lemma 2.5.2 we bound the probability of a (k, ϵ, U)-bad vector. Following this, in Lemma 2.6.5 and 2.6.6, assuming the multiplicity vector is $(k+1, \epsilon, U)$-good, we bound the difference between the English Walrasian mechanism prices and those of the Walrasian mechanism at hand. Next, in Lemma 2.5.3, again for $(k+1, \epsilon, U)$-good multiplicity vectors, we relate $u_i(\mathbf{x}_i(v_i, s_{-i}))$ to $v_i(\mathbf{x}_i(v_i, v_{-i}))$ and the prices paid; we then use this to carry out a PoA analysis.

First, we have following two observations.

Observation 2.6.1. *In the Dutch Walrasian mechanism, if there are zero copies of a good, letting its price be $+\infty$ will not affect the mechanism outcome.*

Observation 2.6.2. *Suppose bidders' demands satisfy the Gross Substitutes property. In both the English and Dutch Walrasian mechanisms, if $n_i \geq n'_i$, then $\mathbf{p}(n_i, n_{-i}) \preceq \mathbf{p}(n'_i, n_{-i})$, where $\mathbf{p} \preceq \mathbf{p}'$ means that, for all j, $p_j \leq p'_j$.*

Lemma 2.6.1. *In the English Walrasian mechanism, given n_{-j} and bidding profile $\mathbf{s}$, the number of values n_j for which (n_j, n_{-j}) is (ϵ, U)-bad for good j is at most $\frac{m}{\epsilon}U$.*

Proof of Lemma 2.6.1: We prove the result by contradiction. Accordingly, let $S = \left\{ n_j \middle| dist^U(\mathbf{p}^{Eng}((n_j, n_{-j}); \mathbf{s}), \mathbf{p}^{Eng}((n_j+1, n_{-j}); \mathbf{s})) > \epsilon \right\}$ and suppose that $|S| > \frac{m}{\epsilon}U$.

The proof uses a new function $pf(\cdot)$: $\quad pf(n_j) = \sum_{q=1}^{m} \min\{p_q^{Eng}((n_j, n_{-j}); \mathbf{s}), U\}.$

$$\text{Then,} \qquad \liminf_{n\to\infty}(pf(0) - pf(n)) = \liminf_{n\to\infty}\sum_{h=0}^{n-1}(pf(h) - pf(h+1))$$
$$\geq \sum_{n_j\in S}(pf(n_j) - pf(n_j+1)) > \frac{m}{\epsilon}U \cdot \epsilon = m \cdot U. \tag{2.8}$$

The first inequality follows as by Observation 2.6.2, $pf(\cdot)$ is a non-increasing function. Further, by construction, $0 \leq pf(h) \leq m \cdot U$ for all h, thus $\liminf_{n\to\infty}(pf(0) - pf(n)) \leq m \cdot U$, contradicting (2.8). □

Lemma 2.6.2.
$$\binom{m+n-1}{n} = \sum_{i=0}^{n}\binom{m+i-2}{i}$$

Lemma 2.6.3.
$$\sum_{n=0}^{k}\binom{m+n-1}{n} = \sum_{n=0}^{k}(k-n+1)\binom{m+n-2}{n}.$$

Proof.

$$\sum_{n=0}^{k}\binom{m+n-1}{n} = \sum_{n=0}^{k}\sum_{i=0}^{n}\binom{m+i-2}{i} = \sum_{i=0}^{k}\sum_{n=i}^{k}\binom{m+i-2}{i}$$
$$= \sum_{i=0}^{k}(k-i+1)\binom{m+i-2}{i}.$$

□

Lemma 2.6.4. *In the English Walrasian mechanism with bidding profile* $\mathbf{s}$*, for a fixed* n_{-j}*, the number of values* n_j *for which* (n_j, n_{-j}) *is* (k,ϵ,U)*-bad for good* j *is at most* $\frac{m}{\epsilon}U \cdot \binom{k+m}{m}$.

Proof of Lemma 2.6.4: Consider the case that $m \geq 2$. For (n_j, n_{-j}) to be (k,ϵ,U)-bad for good j we need an (ϵ, U)-bad vector $\mathbf{n'} \preceq \mathbf{n}$ for good j, with $\sum_{h\neq j} n_h - n'_h = c$ for some $0 \leqslant c \leqslant k$ and $n_j - n'_j \leqslant k - c$. There are $\binom{m-2+c}{c}$ ways of choosing the n'_{-j}. For each n'_{-j}, by Lemma 2.6.1, there are at most $\frac{m}{\epsilon}U$ points that are (ϵ, U)-bad for good j. For each

(ϵ, U)-bad point, there are $k - c + 1$ choices for n_j. This gives a bound of

$$\sum_{c=0}^{k} \frac{m}{\epsilon} U (k - c + 1) \binom{m - 2 + c}{c} = \frac{m}{\epsilon} U \sum_{c=0}^{k} \binom{m - 1 + c}{c} = \frac{m}{\epsilon} U \binom{m + k}{k}$$

(k, ϵ, U)-bad vectors. Note that the first equality follows by Lemma 2.6.3 and the second equality follows by Lemma 2.6.2.

For the case $m = 1$, for this good, each (ϵ, U)-bad point will cause at most $k + 1$ points to be (k, ϵ, U)-bad. This gives a bound of

$$\frac{m}{\epsilon} U (k + 1) = \frac{m}{\epsilon} U \binom{m + k}{k}.$$

(k, ϵ, U)-bad vectors. □

Proof of Lemma 2.5.2: Conditioned on the bidding profile being $\mathbf{b}$,

$$\begin{aligned}
&\sum_{1 \le j \le m} \Pr[(\ \mathbf{n} \textit{ is } (k, \epsilon, U)\textit{-bad for good } j) \cup (n_j \le k)]) \\
&\quad \le \sum_{1 \le j \le m} \Pr[(\mathbf{n} \textit{ is } (k, \epsilon, U)\textit{-bad for good } j)] + \Pr[(n_j \le k)] \\
&\quad \le \sum_{1 \le j \le m} \sum_{\overline{n}_{-j}} \Big(\Pr[(\ \mathbf{n} \textit{ is } (k, \epsilon, U)\textit{-bad for good } j) | n_{-j} = \overline{n}_{-j}] \\
&\qquad\qquad + \Pr[(n_j \le k) | n_{-j} = \overline{n}_{-j}] \Big) \cdot \Pr[n_{-j} = \overline{n}_{-j}] \\
&\quad \le \frac{m}{L} \left[\frac{m}{\epsilon} U \binom{k + m}{m} + k + 1 \right] \qquad \text{(by Lemma 2.6.4).}
\end{aligned}$$

□

Let $n_j^i(s_i, s_{-i})$ denote the number of copies of good j that bidder i receives with bidding profile (s_i, s_{-i}) and $n^i(s_i, s_{-i})$ denote the corresponding vector. Also, let $p^{Eng}(\mathbf{n}; s_{-i})$ denote the market equilibrium prices when bidder i is not present.

Lemma 2.6.5. $p_j^{Eng}(\mathbf{n}; s_{-i}) \le p_j(\mathbf{n}; (s_i, s_{-i}))$.

Proof of Lemma 2.6.5: Consider the situation with $\mathbf{n}' = \mathbf{n} - \mathbf{n}^i(s_i, s_{-i})$ and suppose that agent i is not present. Then $p_j(\mathbf{n}; (s_i, s_{-i}))$ is a market equilibrium.

$$\begin{array}{lll} \text{So} & \forall j & p_j^{Eng}(\mathbf{n}'; s_{-i}) \leq p_j(\mathbf{n}; (s_i, s_{-i})). \\ \text{Since } \mathbf{n} \geq \mathbf{n}', & \forall j & p_j^{Eng}(\mathbf{n}; s_{-i}) \leq p_j^{Eng}(\mathbf{n}'; s_{-i}). \end{array}$$

The lemma follows on combining these two inequalities. □

Lemma 2.6.6. *If* $\mathbf{n}$ *is* $(k+1, \epsilon, U)$*-good for all goods, and* $n_j > k+1$ *for all* j*, then*

$$\forall j \quad \min\{p_j(\mathbf{n}; (v_i, s_{-i})), U\} \leq \min\{p_j(\mathbf{n}; (s_i, s_{-i})), U\} + (k+1)\epsilon.$$

Proof of Lemma 2.6.6: Let $\mathbf{d}^i \preceq \mathbf{n}^i(v_i, s_{-i})$ be a minimal set with $v_i(\mathbf{d}^i) = v_i(\mathbf{n}^i(v_i, s_{-i}))$. By Definition 2.5.2(i), $\sum_j d_j^i \leq k$. First, if $n_j^i(v_i, s_{-i}) > d_j^i$ then $p_j(\mathbf{n}; (v_i, s_{-i})) = 0$, as the pricing is given by a Walrasian Mechanism.

Consider the scenario with $\mathbf{n}'$ copies of goods on offer, where for all j, $n'_j = n_j - d_j^i$ and suppose that bidder i is not present; then $\mathbf{p}(\mathbf{n}; (v_i, s_{-i}))$ is a market equilibrium.

$$\begin{array}{lr} \text{So,} & p_j(\mathbf{n}; (v_i, s_{-i})) \leq p_j^{Dut}(\mathbf{n}'; s_{-i}). \\ \text{For all } j' \neq j, \text{ let } n''_{j'} = n'_{j'} \text{ and let } n''_j = n'_j - 1; \text{ then} & p_j^{Dut}(\mathbf{n}'; s_{-i}) \leq p_j^{Eng}(\mathbf{n}''; s_{-i}), \\ \text{and by Lemma 2.6.5,} & p_j^{Eng}(\mathbf{n}''; s_{-i}) \leq p_j^{Eng}(\mathbf{n}''; s_i, s_{-i}). \end{array}$$

As $\mathbf{n}$ is $(k+1, \epsilon, U)$-good for all goods, and as $\sum_h n_h - n''_h \leq k+1$, using Observation 2.5.2 for the second inequality below, we conclude that

$$\begin{aligned} &\min\{p_j(\mathbf{n}; (v_i, s_{-i})), U\} \leq \min\{p_j^{Eng}(\mathbf{n}''; (s_i, s_{-i})), U\} \\ &\quad \leq \min\{p_j^{Eng}(\mathbf{n}; (s_i, s_{-i})), U\} + (k+1)\epsilon \leq \min\{p_j(\mathbf{n}; (s_i, s_{-i})), U\} + (k+1)\epsilon. \end{aligned} \tag{2.9}$$

□

The second inequality holds by Definition 2.5.4 and 2.5.5.

Proof of Lemma 2.5.3: As we are using a Walrasian mechanism, for any allocation x'_i,

$$v_i(x_i(v_i, s_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, s_{-i})} p_g(\mathbf{n}; (v_i, s_{-i})) \geq v_i(\mathbf{x}'_i) - \sum_{g \in \mathbf{x}'_i} p_g(\mathbf{n}; (v_i, s_{-i})). \tag{2.10}$$

We let G denote the set of goods whose prices $p_g(\mathbf{n}; (v_i, s_{-i}))$ are larger than U. Then,

$$\begin{aligned} u_i(v_i, s_{-i}) &= v_i(\mathbf{x}_i(v_i, s_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, s_{-i})} p_g(\mathbf{n}; (v_i, s_{-i})) \\ &\geq v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G) - \sum_{g \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G} p_g(\mathbf{n}; (v_i, s_{-i})) \\ &\qquad \text{(by (2.10))} \end{aligned} \tag{2.11}$$

Since $\mathbf{n}$ is $(k+1, \epsilon, U)$-good, by Lemma 2.6.6,

$$\min\{p_g(\mathbf{n}; (v_i, s_{-i})), U\} \leq \min\{p_g(\mathbf{n}; (s_i, s_{-i})), U\} + (k+1)\epsilon.$$

Therefore, for any $g \notin G$,

$$\begin{aligned} p_g(\mathbf{n}; (v_i, s_{-i})) &\leq \min\{p_g(\mathbf{n}; (s_i, s_{-i})), U\} + (k+1)\epsilon \\ &\leq p_g(\mathbf{n}; (s_i, s_{-i})) + (k+1)\epsilon. \end{aligned}$$

So,

$$\begin{aligned} & v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G) - \sum_{g \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G} p_g(\mathbf{n}; (v_i, s_{-i})) \\ &\geq v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G) - \sum_{g \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G} p_g(\mathbf{n}; (s_i, s_{-i})) \\ &\quad - |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus G| \cdot (k+1)\epsilon. \end{aligned} \tag{2.12}$$

For any $g \in G$, on applying Lemma 2.6.6, we obtain $U = \min\{p_g(\mathbf{n}; (v_i, s_{-i})), U\} \leq$

$\min\{p_g(\mathbf{n};(s_i,s_{-i})),U\}+(k+1)\epsilon$, which implies $p_g(\mathbf{n};(s_i,s_{-i}))+(k+1)\epsilon \geq U$. Also,

$$v_i(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)-v_i((\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G) \leq v_i((\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\cap G) \leq |(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\cap G|\cdot U,$$

where the first inequality follows by the Gross Substitutes assumption, and the second by Gross Substitutes and because by assumption $v_i(g) \leq U$ for all single items g. Thus,

$$\begin{aligned}
& v_i((\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G) - \sum_{g\in(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G} p_g(\mathbf{n};(s_i,s_{-i})) - |(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G|\cdot(k+1)\epsilon \\
&\quad \geq v_i(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i) - |(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\cap G|\cdot U \\
&\qquad\qquad - \sum_{g\in(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G} p_g(\mathbf{n};(s_i,s_{-i})) - |(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i)\setminus G|\cdot(k+1)\epsilon \\
&\quad \geq v_i(\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i) - \sum_{g\in\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i} p_g(\mathbf{n};(s_i,s_{-i})) - |\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i|\cdot(k+1)\epsilon \\
&\quad \geq v_i(\mathbf{x}_i(v_i,v_{-i})) - \sum_{g\in\mathbf{x}_i(v_i,v_{-i})} p_g(\mathbf{n};(s_i,s_{-i})) - |\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i|\cdot(k+1)\epsilon.
\end{aligned} \tag{2.13}$$

By (2.11), (2.12) and (2.13),

$$u_i(v_i,s_{-i}) \quad\geq\quad v_i(\mathbf{x}_i(v_i,v_{-i})) - \sum_{g\in\mathbf{x}_i(v_i,v_{-i})} p_g(\mathbf{n};(s_i,s_{-i})) - |\mathbf{x}_i(v_i,v_{-i})\cap\mathbf{d}_i|\cdot(k+1)\epsilon.$$

□

Proof of Theorem 2.5.1: By Lemma 2.5.2, the probability that $\mathbf{n}$ is $(k+1,\frac{\max_g\{v_i(g)\}}{2^c},\max_g\{v_i(g)\})$-bad or $n_j \leq k+1$ for some j is less than

$$\begin{aligned}
&\frac{m}{L}\left[\frac{1}{\frac{\max_g\{v_i(g)\}}{2^c}}\max_g\{v_i(g)\}\Lambda(m,k+1)+k+2\right] \\
&\quad = \frac{m}{L}\left[2^c\Lambda(m,k+1)+k+2\right].
\end{aligned}$$

So, for any integer c', by Lemma 2.5.3,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{n}}[u_i(v_i, s_{-i})] \geq \mathbb{E}_{\mathbf{n}}\Bigg[& v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) \\
& - \sum_{c=1}^{c'} \mathbb{1}\Big[\mathbf{n} \text{ is } (k+1, \frac{\max_g\{v_i(g)\}}{2^c}, \max_g\{v_i(g)\})\text{-bad and} \\
& \qquad (k+1, \frac{\max_g\{v_i(g)\}}{2^{c-1}}, \max_g\{v_i(g)\})\text{-good}\Big] \\
& \qquad \cdot \Big|\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i\Big| \cdot (k+1)\frac{\max_g\{v_i(g)\}}{2^{c-1}} \\
& - \mathbb{1}\Big[\mathbf{n} \text{ is } (k+1, \frac{\max_g\{v_i(g)\}}{2^{c'-1}}, \max_g\{v_i(g)\})\text{-good}\Big] \\
& \qquad \cdot \Big|\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i\Big| \cdot (k+1)\frac{\max_g\{v_i(g)\}}{2^{c'}}\Bigg] \\
\geq \mathbb{E}_{\mathbf{n}}\Bigg[& v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i}))\Bigg] \\
& - \sum_{c=1}^{c'} \frac{m}{L}\left[2^c \Lambda(m, k+1) + k + 2\right] \cdot k \cdot (k+1)\frac{\max_g\{v_i(g)\}}{2^{c-1}} \\
& - k \cdot (k+1)\frac{\max_g\{v_i(g)\}}{2^{c'}} \\
\geq \mathbb{E}_{\mathbf{n}}\Bigg[& v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{g \in \mathbf{x}_i(v_i, v_{-i})} p_g(\mathbf{n}; (s_i, s_{-i})) \\
& - c' \cdot \frac{m}{L}\left[2\Lambda(m, k+1) + k + 2\right] \cdot k \cdot (k+1) \cdot \max_g\{v_i(g)\} \\
& - k \cdot (k+1)\frac{\max_g\{v_i(g)\}}{2^{c'}}\Bigg].
\end{aligned}
$$

Summing over all the bidders and averaging over $\mathbf{v}$ and $\mathbf{s}$ gives

$$\sum_i \mathbb{E}_{\mathbf{v},\mathbf{s},\mathbf{n}}[u_i(v_i, s_{-i})] \geq \mathrm{SW}(\mathrm{OPT}) - \mathbb{E}_{\mathbf{s}}[\mathrm{R}(s_i, s_{-i})]$$
$$- N \cdot c' \cdot \frac{m}{L}\left[2\Lambda(m, k+1) + k + 2\right] \cdot k \cdot (k+1) \cdot \zeta \cdot m$$
$$- N \cdot k \cdot (k+1)\frac{1}{2^{c'}} \cdot \zeta \cdot m.$$

Using the smooth technique for Bayesian settings [ST13] yields

$$\mathrm{SW}(\mathrm{NE}) \geq \Bigg(1 - \frac{\zeta \cdot m \cdot k \cdot (k+1)\frac{1}{2^{c'}}}{\rho} - \frac{\zeta \cdot m \cdot c' \cdot \frac{m}{L}\left[2\Lambda(m, k+1) + k + 2\right] \cdot k \cdot (k+1)}{\rho}\Bigg)\mathrm{SW}(\mathrm{OPT}).$$

Let $Y = \frac{m}{L}\left[2\Lambda(m, k+1)\right]$. Set $c' = \lceil \log_2 \frac{1}{Y} - \log_2 \log_2 \frac{1}{Y} \rceil$; then $\frac{1}{2^{c'}} \leq Y \log_2 \frac{1}{Y}$. So,

$$\mathrm{SW}(\mathrm{NE}) \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right)\mathrm{SW}(\mathrm{OPT}).$$

□

2.6.3 Regret Minimization

2.6.3.1 Walrasian Market

We note the following corollary to Theorem 2.5.1.

Corollary 2.6.1. *In a large Walrasian auction which satisfies Assumptions 2.5.1, if v_i and s_i are monotone and satisfy the gross substitutes property for all i, then*

$$\sum_i \mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}[u_i(v_i, s_{-i})] \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right)\mathrm{SW}(\mathrm{OPT}) - \mathbb{E}_{\mathbf{s},\mathbf{n}}[R(s_i, s_{-i})]$$

where $Y = \frac{m}{L}\left[2m\binom{k+1+m}{m}\right]$ *and* $\rho = \frac{\mathrm{SW(OPT)}}{N}$.

Proof of Theorem 2.5.2: Since player i uses a regret minimizing algorithm and she is a (γ,δ)-player,

$$\mathbb{E}_{\mathbf{n}}\left[\sum_{t=1}^{T} v_i(s_i^t, s_{-i}^t)\right] \geq \mathbb{E}_{\mathbf{n}}\left[\sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \Phi(|K_i|, T)\cdot(\max_{\mathbf{x}_i} v_i(\mathbf{x}_i)\cdot\gamma+\delta)\right].$$

Summing over all bidders and integrating w.r.t. $\mathbf{v}$ and $\mathbf{s}$ gives

$$\begin{aligned}
\mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\sum_i\sum_{t=1}^{T} u_i(v_i^t, s_{-i}^t)\right] &\geq \mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\sum_i\sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \Phi(|K_i|, T)\cdot(\max_{\mathbf{x}_i} v_i(\mathbf{x}_i)\cdot\gamma+\delta)\right] \\
&\geq \mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\sum_i\sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \Phi(|K_i|, T)\cdot(km\zeta\gamma+\delta)\right].
\end{aligned}$$

By Corollary 2.6.1,

$$\begin{aligned}
\sum_i \mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}[u_i(v_i, s_{-i})] \quad \geq \quad & \left(1 - \frac{3\cdot k\cdot(k+1)\cdot\zeta\cdot m}{\rho}\cdot Y\cdot\lceil\log_2\frac{1}{Y}\rceil\right)\mathrm{SW(OPT)} \\
& - \mathbb{E}_{\mathbf{s},\mathbf{n}}[\mathrm{R}(s_i, s_{-i})]. \qquad (2.14)
\end{aligned}$$

Therefore, since valuation equals utility plus payment,

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\frac{1}{T}\sum_i\sum_{t=1}^{T} v_i(\mathbf{x}_i(s_i^t, s_{-i}^t))\right] \\
&\quad= \mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\frac{1}{T}\sum_i\sum_{t=1}^{T}(u_i(s_i^t, s_{-i}^t) + \mathrm{R}(s_i^t, s_{-i}^t))\right] \\
&\quad\geq \frac{1}{T}\mathbb{E}_{\mathbf{n},\mathbf{v},\mathbf{s}}\left[\sum_i\left(\sum_{t=1}^{T}(u_i(v_i, s_{-i}^t) + \mathrm{R}(s_i^t, s_{-i}^t)) - \Phi(|K_i|, T)\cdot(km\zeta\gamma+\delta)\right)\right] \\
&\quad\geq \left(1 - \frac{3\cdot k\cdot(k+1)\cdot\zeta\cdot m}{\rho}\cdot Y\cdot\lceil\log_2\frac{1}{Y}\rceil\right)\mathrm{SW(OPT)} \\
&\qquad\qquad - \frac{1}{T}\sum_i \Phi(|K_i|, T)\cdot(km\zeta\gamma+\delta) \qquad\qquad \text{by (2.14)} \\
&\quad\geq \left(1 - \frac{3\cdot k\cdot(k+1)\cdot\zeta\cdot m}{\rho}\cdot Y\cdot\lceil\log_2\frac{1}{Y}\rceil\right)\mathrm{SW(OPT)} \\
&\qquad\qquad - \max_i \Phi(|K_i|, T)\cdot(km\zeta\gamma+\delta)\frac{1}{\rho\cdot T}\mathrm{SW(OPT)} \\
&\quad= \left(1 - \frac{3\cdot k\cdot(k+1)\cdot\zeta\cdot m}{\rho}\cdot Y\cdot\lceil\log_2\frac{1}{Y}\rceil\right. \\
&\qquad\qquad \left. - \frac{\max_i \Phi(|K_i|, T)\cdot(km\zeta\gamma+\delta)}{\rho\cdot T}\right)\mathrm{SW(OPT)}.
\end{aligned}
$$

□

2.6.3.2 Fisher Market with Reserve Prices

Theorem 2.3.2 will follow from the following lemma; its proof is given in Section 2.6.4.

Lemma 2.6.7. *For any bidding profile* $\mathbf{s}$ *and any value profile* $\mathbf{v}$ *which are homogeneous of degree 1, concave, continuous, monotone and gross substitutes, if the reserve prices* $r_j \leq \frac{1}{4}p_j^*$ *for any* j*, then*

$$\sum_i u_i(v_i, s_{-i})) \geq e^{-\frac{2m}{L}}\sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)).$$

Proof of Theorem 2.3.2: Since player i uses a regret minimizing algorithm and the maximum payoff is $\lambda u_i(v_i, v_{-i})$ as $\lambda r_j > p^*_j$ for all j,

$$\sum_{t=1}^{T} u_i(s_i^t, s_{-i}^t) \geq \sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \Phi(|K_i|, T) \cdot \lambda u_i(v_i, v_{-i}).$$

Summing over all the bidders gives

$$\begin{aligned}\sum_i \sum_{t=1}^{T} u_i(s_i^t, s_{-i}^t) &\geq \sum_i \sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \sum_i \Phi(|K_i|, T) \cdot \lambda u_i(v_i, v_{-i}) \\ &\geq \sum_i \sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \sum_i \max_{i'} \Phi(|K_{i'}|, T) \cdot \lambda u_i(v_i, v_{-i}).\end{aligned}$$

By Theorem 2.6.7,

$$\sum_i u_i(v_i, s_{-i})) \geq e^{-\frac{2m}{L}} \sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)).$$

Therefore,

$$\begin{aligned}\sum_i \sum_{t=1}^{T} u_i(s_i^t, s_{-i}^t) &\geq \sum_i \sum_{t=1}^{T} u_i(v_i, s_{-i}^t) - \sum_i \max_{i'} \Phi(|K_{i'}|, T) \cdot \lambda u_i(v_i, v_{-i}) \\ &\geq T \cdot e^{-\frac{2m}{L}} \sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)) - \max_{i'} \Phi(|K_{i'}|, T) \lambda \sum_i u_i(v_i, v_{-i}).\end{aligned}$$

The theorem follows on dividing both sides by T. □

2.6.4 Reserve Prices

Definition 2.6.1. *Eisenberg Gale markets with reserve prices are exactly the solutions to the following convex program:*

$$\max_{\mathbf{x}} \quad \sum_{i=1}^{n} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \cdots x_{im})) + \sum_{j=1}^{m} y_j r_j$$
$$s.t. \quad \forall j : \quad \sum_i x_{ij} + y_j \leq 1$$
$$\forall i, j : \quad x_{ij} \geq 0,$$

where r_j is the reserve price of item j.

The proof of Theorem 2.6.7 uses the following lemma.

Lemma 2.6.8. *For any bidding profile* $\mathbf{s}$ *and any value profile* $\mathbf{v}$ *which are homogeneous of degree 1, concave, continuous, monotone and satisfy the gross substitutes property, if the reserve prices $r_j \leq \frac{1}{4}p_j^*$ for every j, then*

$$\sum_i e_i \log u_i(v_i, s_{-i})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \geqslant -2m \cdot \max_{i'} e_{i'}.$$

Proof of Lemma 2.6.7: On exponentiating the expressions on both sides in the statement of Lemma 2.6.8 we obtain

$$\prod_i u_i(v_i, s_{-i})^{e_i} \geq \frac{1}{e^{2m \cdot \max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i},$$

as $u_i(v_i, v_{-i}) = u_i(x_i(\mathbf{p}^*))$. Therefore,

$$\prod_i \left(\frac{u_i(v_i, s_{-i})}{u_i(v_i, v_{-i})}\right)^{e_i} \geq \frac{1}{e^{2m \cdot \max_i e_i}}.$$

Using the weighted GM-AM inequality, we obtain

$$\frac{\sum_i e_i \frac{u_i(v_i,s_{-i})}{u_i(v_i,v_{-i})}}{\sum_i e_i} \geq \left(\prod_i \left(\frac{u_i(v_i,s_{-i})}{u_i(v_i,v_{-i})}\right)^{e_i}\right)^{\frac{1}{\sum_i e_i}} \geq \left(\frac{1}{e^{2m\max_i e_i}}\right)^{\frac{1}{\sum_i e_i}} = e^{-\frac{2m\max_i e_i}{\sum_i e_i}}.$$

Since $u_i(v_i,v_{-i}) = te_i$, for all i,

$$\sum_i u_i(v_i,s_{-i}) \geq e^{-\frac{2m\max_i e_i}{\sum_i e_i}} \sum_i u_i(v_i,v_{-i}).$$

□

The goal in Lemma 2.6.8 is to bound $\sum_i e_i \log u_i(v_i,v_{-i}) - \sum_i e_i \log u_i(v_i,s_{-i})$. We will be working with the following function, the demand at prices $\mathbf{p}$:

$$x(\mathbf{p}) = (x_1(\mathbf{p}), x_2(\mathbf{p}), \cdots) = \arg\max_{\mathbf{x}} \sum_i e_i \log u_i(\mathbf{x}_i) + \mathbf{1}\cdot\mathbf{p} - \sum_i \mathbf{x}_i\cdot\mathbf{p}. \tag{2.15}$$

Recall that, by Lemma 2.4.2, $p_j(v_i,s_{-i}) \leqslant p_j(\mathbf{s}_i,s_{-i}) + \mathbf{1}\cdot e_i$. Consequently, $u_i(\mathbf{x}_i(\mathbf{p}(v_i,s_{-i}))) \geqslant u_i(\mathbf{x}_i(\mathbf{p}(\mathbf{s}) + \mathbf{1}\cdot\max_{i'} e_{i'}))$, and so it will suffice to bound $\sum_i e_i \log u_i(v_i,v_{-i})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}(\mathbf{s}) + \mathbf{1}\cdot\max_{i'} e_{i'}))$.

We want to apply the bound in (2.3), but then we need prices $\mathbf{q}$ such that $\sum_j q_j = \sum_i e_i$. Accordingly, we will be considering the *scaled* prices $\mathbf{q}(\mathbf{s}) = (\mathbf{p}(\mathbf{s}) + \mathbf{1}\cdot\max_{i'} e_{i'})\cdot\frac{\sum_i e_i}{\sum_j p_j(\mathbf{s}) + \max_{i'} e_{i'}}$ and the *compressed* prices, defined below.

For convenience, in the following definition, we set $\frac{0}{x} = 0$ for $x > 0$, $\frac{0}{0} = 1$ and $\frac{x}{0} = +\infty$ for $x > 0$.

Definition 2.6.2. *Let* $\mathbf{q}$ *be a price vector such that* $\mathbf{1}\cdot\mathbf{q} = \sum_i e_i$. *For* $l < 1$, *the* l*-compressed* *version of* $\mathbf{q}$ *is defined as* $\mathbf{p}'(l, \mathbf{q})$ *where*

$$\begin{aligned} \frac{p_j'(l,\mathbf{q})}{p_j^*} &= l && \textit{if} \quad \frac{q_j}{p_j^*} \le l, \\ \frac{p_j'(l,\mathbf{q})}{p_j^*} &= t && \textit{if} \quad \frac{q_j}{p_j^*} \ge t, \\ \textit{and} \quad \frac{p_j'(l,\mathbf{q})}{p_j^*} &= \frac{q_j}{p_j^*} && \textit{if} \quad l < \frac{q_j}{p_j^*} < t, \end{aligned}$$

where t *is a number bigger than* 1 *such that* $\sum_j p_j'(l, \mathbf{q}) = \sum_i e_i$*, and* $\mathbf{p}^*$ *is the optimal solution* $(\mathbf{1} \cdot \mathbf{p}^* = \sum_i e_i)$*.*

Henceforth, unless noted otherwise, we let $\mathbf{p}$ denote $\mathbf{p}(\mathbf{s})$ and $\mathbf{q}$ denote $\mathbf{q}(\mathbf{s})$.

Lemma 2.6.9. *Let* $\mathbf{q}$ *be the scaled prices* $\mathbf{q} = (\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j(\mathbf{s}) + \max_{i'} e_{i'})}$*. Then*

$$\begin{aligned} \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'})) &= \sum_i e_i \log u_i \left(\mathbf{x}_i(\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \\ &\quad - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \\ &= \sum_i e_i \log u_i(\mathbf{q}) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}. \end{aligned}$$

Proof.

$$\begin{aligned} \sum_i e_i \log u_i(x_i(\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'})) &= \sum_i e_i \log \Big[u_i(\mathbf{x}_i(\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \\ &\qquad \cdot \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})}) \Big] \\ &= \sum_i e_i \log \Big[u_i(\mathbf{x}_i((\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \\ &\qquad \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}})) \cdot \frac{\sum_j e_i}{\sum_j p_j + \max_{i'} e_{i'}} \Big]. \end{aligned}$$

Now

$$\sum_i e_i \log \left[u_i(\mathbf{x}_i((\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}})) \cdot \frac{\sum_j e_i}{\sum_j p_j + \max_{i'} e_{i'}} \right]$$
$$= \sum_i e_i \log u_i(\mathbf{x}_i((\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}})) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}.$$

□

Lemma 2.6.10. *Suppose that* $\sum_j q_j = \sum_j p'_j = \sum_i e_i$. *Then*

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}')) \geqslant \sum_{ij} (p'_j - q_j) x_{ij}(\mathbf{p}').$$

Proof. As $\mathbf{x}(\mathbf{q}) = \arg\max_{\mathbf{x}} \sum_i e_i \log u_i(\mathbf{x}_i) - \sum_i \mathbf{x}_i \cdot \mathbf{q} + \mathbf{1} \cdot \mathbf{q}$,

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'))$$
$$\geq \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}').$$

As in the "PoA" analysis, $\mathbf{x}_i(\mathbf{q}) = \arg\max_{\mathbf{x}_i \cdot \mathbf{q} = e_i} u_i(\mathbf{x}_i)$. So, $\mathbf{x}_i(\mathbf{q}) \cdot \mathbf{q} = e_i$ and $\mathbf{x}_i(\mathbf{p}') \cdot \mathbf{p}' = e_i$. Therefore,

$$\sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}')$$
$$= \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}') + \sum_i \mathbf{p}' \cdot \mathbf{x}_i(\mathbf{p}') - \sum_i \mathbf{p}' \cdot \mathbf{x}_i(\mathbf{p}')$$
$$= \sum_{ij} (p'_j - q_j) x_{ij}(\mathbf{p}').$$

□

Lemma 2.6.11. *Let* $l \leq 1$ *and* $\mathbf{p}' = \mathbf{p}'(l, q)$. *There exists an* $\mathbf{x}(\mathbf{p}')$ *such that: if* $\frac{p'_j(l, \mathbf{q})}{p^*_j} = l$, *then* $\sum_i x_{ij}(\mathbf{p}') \geq \frac{1}{l}$, *and if* $\frac{p'_j(l, \mathbf{q})}{p^*_j} = t$, *then* $\sum_i x_{ij}(\mathbf{p}') \leq \frac{1}{t}$.

Proof. It is straightforward to check this for linear utility functions.

Now we consider the unique-demand WGS utility functions. Let $\widehat{\mathbf{p}}$ denote the prices such that $\widehat{p_j} = p_j$ when $p_j > 0$ and $\widehat{p_j} = \epsilon$ when $p_j = 0$. Note that here ϵ is an arbitrarily small positive value.

By the homogeneity of the utility function, there exists an $\mathbf{x}(l\mathbf{p}^*)$, such that $\sum_i x_{ij}(l\mathbf{p}^*) = \frac{1}{l}$ for all j such that $p_j^* > 0$. Now, we consider $\sum_i x_{ij}(\widehat{l\mathbf{p}^*})$. For those j such that $p_j^* > 0$, by the Gross Substitutes property, $\sum_i x_{ij}(\widehat{l\mathbf{p}^*}) \geq \frac{1}{l}$.

Then, we let the price increase from $\widehat{l\mathbf{p}^*}$ to $\widehat{p_j'(l, \mathbf{q})}$. Also, by Gross Substitutes property, $\sum_i x_{ij}(\widehat{p_j'(l, \mathbf{q})}) \geq \frac{1}{l}$ for those j such that $p_j^* > 0$ and $\frac{p_j'(l,\mathbf{q})}{p_j^*} = l$. By the same reasoning, $\sum_i x_{ij}(\widehat{p_j'(l, \mathbf{q})}) \leq \frac{1}{t}$ for those j such that $p_j^* > 0$ and $\frac{p_j'(l,\mathbf{q})}{p_j^*} = t$.

Furthermore, by the Gross Substitutes property and homogeneity of the utility function, $\sum_i x_{ij}(\widehat{\mathbf{p}'(l, \mathbf{q})}) = 0$ for those j such that $p_j^* = 0$. [7]

So, there exists an $\mathbf{x}(\widehat{\mathbf{p}'})$, where $\mathbf{p}' = \mathbf{p}'(l, \mathbf{q})$, such that for $p_j^* > 0$, if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\widehat{\mathbf{p}'}) \geq \frac{1}{l}$ and if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\widehat{\mathbf{p}'}) \leq \frac{1}{t}$, and for $p_j^* = 0$, $\sum_i x_{ij}(\widehat{\mathbf{p}'}) = 0$.

Since $l < 1$, $\frac{p_j'(l,\mathbf{q})}{p_j^*} \neq l$ when $p_j^* = 0$. Therefore, we have an $\mathbf{x}(\widehat{\mathbf{p}'})$, where $\mathbf{p}' = \mathbf{p}'(l, \mathbf{q})$, such that if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\widehat{\mathbf{p}'}) \geq \frac{1}{l}$ and if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\widehat{\mathbf{p}'}) \leq \frac{1}{t}$.

Next, we will show there exists an $\mathbf{x}(\mathbf{p}')$ which equals $\mathbf{x}(\widehat{\mathbf{p}'})$, where $\mathbf{p}' = \mathbf{p}'(l, \mathbf{q})$.

By the Gross Substitutes property, the demand $\mathbf{x}_i(\widehat{\mathbf{p}'})$ for a given ϵ is also an optimal demand for any $0 < \epsilon' < \epsilon$. This is because for any small positive ϵ, including ϵ', the demand for the goods with price ϵ is 0. And reducing prices on the price ϵ goods only reduces the

[7] Note that $\widehat{p_j'(l, \mathbf{q})} > 0$ for all j. We define prices $\mathbf{p}^{*'}$ as follows: $p_j^{*'} = 0$ if $p_j^* = 0$, and $p_j^{*'} = \widehat{p_j'(l, \mathbf{q})}$ if $p_j^* > 0$. Now we consider a procedure that changes $\mathbf{p}^{*'}$ to $\widehat{\mathbf{p}'(l, \mathbf{q})}$ by increasing those prices $p_j^{*'}$ such that $p_j^{*'} = 0$. Let S_0 be the set of items such that $p_j^{*'} = 0$: $\{j | p_j^{*'} = 0\}$. Next, we argue that all players will spent all their money given prices $\mathbf{p}^{*'}$. For this could be false only if one or more players' demands are only for items in S_0. Because of the zero prices for items in S_0, these items receive infinite demand. However, for those items j in S_0, p_j^*, the equilibrium price, also equals 0, which implies these items would also receive infinite demand at the market equilibrium, contradicting the market equilibrium conditions. Since we know all players will spent all their money given prices $\mathbf{p}^{*'}$, when we increase prices p_j from 0 to positive for $j \in S_0$, for each player, by the Gross Substitutes property, the spending on items $j \notin S$ will not decrease. Consequently, no spending will be released to the items in S_0. Therefore, the spending on the items in S_0 will stay at 0, and hence the allocation will equal 0.

demand for other goods, but as there can be no reduction in spending on the latter goods, in fact the demands are unchanged.

Let $\mathbf{p}' = \mathbf{p}'(l, \mathbf{q})$. By the continuity and homogeneity of the utility function, there exists an optimal allocation $\mathbf{x}(\mathbf{p}')$, which equals $\mathbf{x}(\widehat{\mathbf{p}'})$ and which thereby proves the theorem. For if not, suppose there were a higher utility allocation when $\epsilon = 0$, whose value is u_0, where $u_0 > u_i(\mathbf{x}_i(\widehat{\mathbf{p}'}))$. Then at any $\epsilon > 0$, we could achieve utility $(1 - \varkappa(\epsilon))u_0$, where $\lim_{\epsilon\to 0} \varkappa(\epsilon) = 0$; and as $x_i(\widehat{\mathbf{p}'})$ is an optimal allocation, $u_i(\mathbf{x}_i(\widehat{\mathbf{p}'})) \geq (1 - \varkappa(\epsilon))u_0$. But this holds for every $\epsilon > 0$, so letting $\epsilon \to 0$ yields $u_i(\mathbf{x}_i(\widehat{\mathbf{p}'})) \geq u_0$ and hence $\mathbf{x}(\widehat{\mathbf{p}'})$ is an optimal allocation when $\epsilon = 0$. □

Lemma 2.6.12. *Suppose that* $\sum_j q_j = \sum_i e_i$. *Then*

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) \geqslant \sum_j \left(\frac{1}{l} - 1\right)(lp_j^* - q_j) \cdot \mathbf{1}_{q_j \leqslant lp_j^*}.$$

Proof. By Lemma 2.6.11, there exists an $\mathbf{x}(\mathbf{p}')$, where $\mathbf{p}' = \mathbf{p}'(l, \mathbf{q})$, such that if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\mathbf{p}') \geq \frac{1}{l}$ and if $\frac{p_j'(l,\mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\mathbf{p}') \leq \frac{1}{t}$. Therefore, by lemma 2.6.10,

$$\begin{aligned}
&\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) \\
&\quad \geq \sum_{ij} (p_j'(l, \mathbf{q}) - q_j) x_{ij}(\mathbf{p}'(l, \mathbf{q})) \\
&\quad \geq \sum_j (p_j'(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \leq l} \cdot \frac{1}{l} + \sum_j (p_j'(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \geq t} \cdot \frac{1}{t}.
\end{aligned}$$

Since $\sum_j p_j'(l, \mathbf{q}) = \sum_i e_i = \sum_j q_j$, and as $q_j = p_j'(l, \mathbf{q})$ when $l < \frac{q_j}{p_j^*} < t$,

$$\sum_j (p_j'(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \leq l} = -\sum_j (p_j'(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \geq t}.$$

Thus

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l,\mathbf{q}))) \geqslant \sum_j (\frac{1}{l} - \frac{1}{t})(p'_j(l,\mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p^*_j} \le l}$$
$$\geqslant \sum_j (\frac{1}{l} - 1)(lp^*_j - q_j) \cdot \mathbf{1}_{q_j \le lp^*_j}.$$

□

Lemma 2.6.13. $\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l,\mathbf{q}))) \geqslant \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)$.

Proof. The result follows from (2.3), as $\sum_j p^*_j = \sum_i e_i = \sum_j p'_j$. □

Lemma 2.6.14. *For any* (s_i, s_{-i}) *and for any* j

$$p_j(s_i, s_{-i}) \le p_j(s_{-i}) + e_i.$$

Proof. Let $\mathcal{S}(s_i, s_{-i})$ denote the set of items with $p_j(s_i, s_{-i}) > r_j$. Then for $j \in \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) = 1$ and for $j \notin \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) \le 1$. Similarly, we let $\mathcal{S}(s_{-i})$ denote the set of items with $p_j(s_{-i}) > r_j$. Then, for $j \in \mathcal{S}(s_{-i})$, $\sum_{i' \neq i} x_{i'j}(s_{-i}) = 1$, and for $j \notin \mathcal{S}(s_{-i})$, $\sum_{i' \neq i} x_{i'j}(s_{-i}) \le 1$. By Lemma 2.4.3, if $j \notin \mathcal{S}(s_i, s_{-i})$, $r_j \le p_j(s_{-i}) \le p_j(s_i, s_{-i}) = r_j$. Consequently, if $j \notin \mathcal{S}(s_i, s_{-i})$,

$$p_j(s_{-i}) = p_j(s_i, s_{-i}) = r_j. \tag{2.16}$$

We know

$$\sum_{j,i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) = \sum_{i'} e_{i'} = \sum_{j,i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i}) + e_i. \tag{2.17}$$

We want to show that for each $j \in \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \ge \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$ and $\sum_{j \notin \mathcal{S}(s_i, s_{-i}), i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \ge \sum_{j \notin \mathcal{S}(s_i, s_{-i}), i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$. If this is true, from (2.17), for each $j \in \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \le \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i}) + e_i$. Then for $j \in \mathcal{S}(s_i, s_{-i})$, $p_j(s_i, s_{-i}) = \sum_{i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \leq \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i}) + e_i \leq p_j(s_{-i}) + e_i$; and for $j \notin \mathcal{S}(s_i, s_{-i})$, by (2.16), $r_j = p_j(s_{-i}) = p_j(s_i, s_{-i})$. The result follows.

Next, we will prove for each $j \in \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \geq \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$ and $\sum_{j \notin \mathcal{S}(s_i, s_{-i}), i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \geq \sum_{j \notin \mathcal{S}(s_i, s_{-i}), i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$. It is easy to show that for each $j \in \mathcal{S}(s_i, s_{-i})$, $\sum_{i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \geq \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$ as $\sum_{i'} (x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i})) = p_j(s_i, s_{-i}) \geq p_j(s_{-i}) \geq \sum_{i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$. To prove $\sum_{j \notin \mathcal{S}(s_i, s_{-i}), i'} (x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i})) \geq \sum_{j \notin \mathcal{S}(s_i, s_{-i}), i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i})$, we consider linear utilities and unique-demand utilities separately.

- Linear Utilities: if all buyers have linear utilities, then

$$\sum_{j : p_j(s_i, s_{-i}) = p_j(s_{-i}), i' \neq i} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \geq \sum_{j : p_j(s_i, s_{-i}) = p_j(s_{-i}), i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i}). \quad (2.18)$$

  This is because in going from $\mathbf{p}(s_{-i})$ to $\mathbf{p}(s_i, s_{-i})$, prices can only increase and all buyers will move money to those items whose prices are not increased. Then for those items in $\mathcal{S}(s_i, s_{-i})$ for which $p_j(s_i, s_{-i}) = p_j(s_{-i})$,

$$\begin{aligned} &\sum_{j \in \mathcal{S}(s_i, s_{-i}) : p_j(s_i, s_{-i}) = p_j(s_{-i}), i' \neq i} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \\ &\leq \sum_{j \in \mathcal{S}(s_i, s_{-i}) : p_j(s_i, s_{-i}) = p_j(s_{-i}), i'} x_{i'j}(s_i, s_{-i}) p_j(s_i, s_{-i}) \\ &= \sum_{j \in \mathcal{S}(s_i, s_{-i}) : p_j(s_i, s_{-i}) = p_j(s_{-i})} p_j(s_i, s_{-i}) \\ &= \sum_{j \in \mathcal{S}(s_i, s_{-i}) : p_j(s_i, s_{-i}) = p_j(s_{-i})} p_j(s_{-i}) \\ &= \sum_{j \in \mathcal{S}(s_i, s_{-i}) : p_j(s_i, s_{-i}) = p_j(s_{-i}), i' \neq i} x_{i'j}(s_{-i}) p_j(s_{-i}). \quad (2.19) \end{aligned}$$

Comparing (2.18) and (2.19) yields

$$\sum_{j\notin\mathcal{S}(s_i,s_{-i}),i'\neq i} x_{i'j}(s_i,s_{-i})p_j(s_i,s_{-i}) \geq \sum_{j\notin\mathcal{S}(s_i,s_{-i}),i'\neq i} x_{i'j}(s_{-i})p_j(s_{-i}),$$

which directly implies the result.

- Unique Demand Utilities: In this case, we will show that $\sum_{i'} x_{i'j}(s_i,s_{-i})p_j(s_i,s_{-i}) \geq \sum_{i'\neq i} x_{i'j}(s_{-i})p_j(s_{-i})$ for each $j \notin \mathcal{S}(s_i,s_{-i})$. Recall that by (2.16), for these j, $p_j(s_{-i}) = p_j(s_i,s_{-i}) = r_j$. It is easy to see the claimed result if $p_j(s_i,s_{-i}) = p_j(s_{-i}) = r_j = 0$. For the case that $p_j(s_i,s_{-i}) = p_j(s_{-i}) = r_j > 0$, we first note that given price $\mathbf{p}(s_i,s_{-i})$, the demand $x_{i'j}(s_i,s_{-i})$ is unique. [8] By the gross substitutes property, there exists a demand at price $\mathbf{p}(s_i,s_{-i})$ such that $x_{i'j}(s_i,s_{-i})p_j(s_i,s_{-i}) \geq x_{i'j}(s_{-i})p_j(s_{-i})$. By the uniqueness of $x_{i'j}(s_i,s_{-i})$, this is true for the demand at price $\mathbf{p}(s_i,s_{-i})$.

□

Corollary 2.6.2.

$$\begin{aligned}
&\sum_i e_i \log u_i(v_i,s_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
&\quad \geqslant \sum_j \left((1-l)p_j^* - \left(\frac{1}{l}-1\right)(p_j + \max_{i'} e_{i'}) \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \cdot \mathbf{1}_{(p_j+\max_{i'} e_{i'})\frac{\sum_i e_i}{\sum_j (p_j+\max_{i'} e_{i'})} \leqslant l p_j^*} \\
&\quad\quad - \sum_i e_i \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}.
\end{aligned}$$

Proof. Recall that $\mathbf{q} = (\mathbf{p} + \mathbf{1}\cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j p_j(\mathbf{s}) + \max_{i'} e_{i'}}$, defined in Lemma 2.6.9, denotes

[8] If there exist two distinct demands $x^1_{i'j}(s_i,s_{-i}) \neq x^2_{i'j}(s_i,s_{-i})$, then we consider the procedure increasing the price vector $\mathbf{p}(s_i,s_{-i})$ to $\hat{\mathbf{p}}(s_i,s_{-i})$ in which $\hat{p}_{j'}(s_i,s_{-i}) = p_{j'}(s_i,s_{-i})$ if $p_{j'}(s_i,s_{-i}) > 0$ and $\hat{p}_{j'}(s_i,s_{-i}) = \epsilon$ (for a small enough ϵ) if $p_{j'}(s_i,s_{-i}) = 0$. Then by the gross substitutes property, on increasing the prices from $\mathbf{p}(s_i,s_{-i})$ to $\hat{\mathbf{p}}(s_i,s_{-i})$, the spending on, and the allocation for those items with $p_{j'}(s_i,s_{-i}) > 0$ will not decrease. At the same time, the budget constraint ensures the spending on these items doesn't increase. Therefore, $x^1_{i'j}(s_i,s_{-i})$ and $x^2_{i'j}(s_i,s_{-i})$ will induce two different demands at price $\hat{\mathbf{p}}(s_i,s_{-i})$, which contradicts the unique demand assumption.

the scaled prices, and $\mathbf{p} = \mathbf{p}(s_i, s_{-i})$. Now,

$$
\begin{aligned}
&\sum_i e_i \log u_i(v_i, s_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
&\quad \geqslant \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p} + \mathbf{1} \cdot \max_{i'} e_{i'})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \qquad \text{(by Lemma 2.6.14)} \\
&\quad \geqslant \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \\
&\qquad\qquad - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \qquad \text{(by Lemma 2.6.9)} \\
&\quad \geqslant \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) + \sum_j \left(\frac{1}{l} - 1\right)(lp_j^* - q_j) \cdot \mathbf{1}_{q_j \leqslant lp_j^*} \\
&\qquad - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \qquad \text{(by Lemma 2.6.12)} \\
&\quad \geqslant \sum_j \left(\frac{1}{l} - 1\right)(lp_j^* - q_j) \cdot \mathbf{1}_{q_j \leqslant lp_j^*} - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \qquad \text{(by Lemma 2.6.13).}
\end{aligned}
$$

□

Let $\delta = \sum_j (p_j + \max_{i'} e_{i'}) - \sum_i e_i$. Suppose that the good j reserve price $r_j \leqslant \frac{1}{4} p_j^*$ for all j. Note that $\sum_i e_i + \sum_j r_j \cdot \mathbf{1}_{p_j = r_j} \geqslant \sum_j p_j \geqslant \sum_i e_i$. Clearly, $\sum_j r_j \cdot \mathbf{1}_{p_j = r_j} \geqslant \delta - m \cdot \max_{i'} e_{i'}$.

Lemma 2.6.15.

$$\sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \leqslant \delta.$$

Proof.

$$\sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} = \sum_i e_i \log(1 + \frac{\delta}{\sum_i e_i}) \leq \sum_i e_i \frac{\delta}{\sum_i e_i} = \delta.$$

□

Proof of Lemma 2.6.8: We set $l = \frac{1}{2}$. Recall that $\mathbf{p}^*$ is the optimal pricing without reserve prices and $\mathbf{p}$ is the price of bidding profile (s_i, s_{-i}) with reserve price. Then,

$$
\begin{aligned}
&\sum_i e_i \log u_i(v_i, s_{-i})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
&\quad \geq \sum_j (2-1)(\frac{1}{2}p_j^* - (p_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})}) \\
&\qquad\qquad \cdot \mathbf{1}_{(p_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq \frac{1}{2}p_j^*} - \delta \quad \text{(by Corollary 2.6.2 and Lemma 2.6.15)} \\
&\quad \geq \sum_j (2-1)(\frac{1}{2}p_j^* - (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})}) \\
&\qquad\qquad \cdot \mathbf{1}_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq \frac{1}{2}p_j^*} - \delta \\
&\quad \geq \sum_j (\frac{1}{2}p_j^* - (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})}) \cdot \mathbf{1}_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \leq \frac{1}{2}p_j^*} - \delta \\
&\qquad\qquad \text{(as } \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq 1 \text{ by } \textstyle\sum_i e_i \leq \sum_j p_j) \\
&\quad \geq \sum_j (\frac{1}{2}p_j^* - (r_j + \max_{i'} e_{i'})) \cdot \mathbf{1}_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \leq \frac{1}{2}p_j^*} - \delta \qquad \text{(as } \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq 1) \\
&\quad \geq \sum_j (\frac{1}{2}p_j^* - (r_j + \max_{i'} e_{i'})) \cdot \mathbf{1}_{p_j = r_j \wedge r_j \leq \frac{1}{2}p_j^*} - \delta \\
&\quad \geq \sum_j r_j \cdot \mathbf{1}_{p_j = r_j} - m \cdot \max_{i'} e_{i'} - \delta \qquad \text{(as } \frac{1}{2}p_j^* \geq 2r_j) \\
&\quad \geqslant \delta - m \cdot \max_{i'} e_{i'} - m \cdot \max_{i'} e_{i'} - \delta = -2m \cdot \max_{i'} e_{i'} \quad \text{(as } \sum_j r_j \cdot \mathbf{1}_{p_j = r_j} \geqslant \delta - m \cdot \max_{i'} e_{i'}).
\end{aligned}
$$

□

Chapter 3

Dynamics

3.1 Preliminary

One of the most important results in Algorithmic Game Theory is the PPAD-hardness of finding a Nash Equilibrium in finite games [DGP09, CDT09], which serves as a strong evidence that there is no efficient algorithm to compute Nash Equilibria. Similar hardness results have been established for markets [CSVY06, CDDT09, CT09, VY11, CPY17]. By viewing the players and the environment collectively as implicitly performing a computation, these hardness results indicate that, in general, a market cannot reach an equilibrium quickly. In Kamal Jain's words: "If your laptop cannot find it, neither can the market" [NRTV07, Chapter 2.1].

As a result, a lot of attention has been given to the design of polynomial-time algorithms to find equilibria, either exactly or approximately, for specific families of games and markets. Most of these algorithms can be categorized as either simplex-like (e.g., Lemke-Howson), numerical methods (e.g., the interior-point method or the ellipsoid method), or some carefully-crafted combinatorial algorithm (e.g., flow-based algorithms for computing a market equilibrium for linear utility functions).

However, it seems implausible that these algorithms describe the implicit computations in games or markets. For many markets would appear to have a highly distributed environment, or need to make rapid decisions on an ongoing basis. These features would appear to preclude computations which require centralized coordination, which is essential for the three categories of algorithms above. Consequently, in order to justify equilibrium concepts, we want natural algorithms which could plausibly be running (in an implicit form) in the associated distributed environments.

This dissertation focus on two natural dynamics in Fisher markets: Proportional Response and Tatonnement.

Dynamics, implicitly, are being considered in an ongoing Fisher Market. In an ongoing Fisher Market, each round, buyers will receive a new budget and they will distribute their budget on different goods; sellers will refresh their supply and update their prices. Based on the spending and prices, buyers will receive an allocation, and they will update their spending in the next round.

3.1.1 Related Work

Computer scientists, beginning with the work by Deng et al. [DPS03], showed that computing equilibria was a hard problem in general; see also [DD08, PY10]. This led to much work on polynomial time algorithms for restricted classes of markets, e.g. [DPSV08, Dev04, CMV05, GK06].

The Eisenberg-Gale program for the case of linear utilities was formulated in [EG59] and then generalized to homothetic functions in [Eis61]; further generalizations were given in [JV07]. The maxima of these convex programs correspond to the equilibria of the corresponding markets. In particular, when buyer or agent utilities are homothetic, the optimum of the Eisenberg-Gale program corresponds to the optimum Nash Social Welfare; interestingly, this optimum also appears to provide good outcomes when apportioning indivisible goods [Bud11, CKM+16]. Recently, Cole et al. [CDG+17] identified another variant of the Eisenberg-Gale program that captured the then best currently-known polynomial-time approximate solution for the indivisible setting.

Two natural dynamics have been studied in the context of Fisher markets, tatonnement and proportional response.

The stability of the tatonnement process has been considered to be one of the most fundamental issues in general equilibrium theory. Hahn [Hah82] provides a thorough survey on the topic, and the textbook of Mas-Colell, Whinston and Green [MCWG95] contains a good summary of the classic results.

The longstanding interpretation of tatonnement is that it is a method used by an auctioneer for iteratively updating prices, followed by trading at the equilibrium prices once they are reached. If trading is allowed as the price updating occurs, this is called a *non-tatonnement* process. In recent years, discrete versions of the (non)-tatonnement process have received increased attention. Codenotti et al. [CMV05] considered a tatonnement-like process that required some coordination among different goods and showed polynomial time convergence for a class Fisher markets with weak gross substitutes (WGS) utilities. Cole and Fleischer [CF08] were the first to establish fast convergence for a truly distributed, asynchronous and discrete version of tatonnement, once again for a class of WGS Fisher markets. The continued interest in the plausibility of tatonnement is also reflected in some experiments by Hirota [HHPR05], which showed the predictive accuracy of tatonnement in a non-equilibrium trade setting.

Proportional Response, in contrast, is a buyer-oriented update, originally analyzed in an effort to explain the behavior of peer-to-peer networks [WZ07, Zha11]. Here, buyers update their spending in proportion to the contribution each good makes to its current utility. An $O(1/T)$ rate of convergence was shown in [BDX11] for Fisher markets with buyers having linear utilities, and a faster linear rate of convergence for the substitutes domain excluding linear utilities was shown in [Zha11]. The analysis most similar to ours is the one in [BDX11] which considers convex functions that obey a constraint, which we name L-Bregman convexity w.r.t. a Bregman divergence (see Definition 3.2.1). Our work generalizes this notion substantially.

Other dynamics have been considered. In particular, Dvijotham et al. [DRS17] study sellers best responding in a setting in which they form beliefs about other sellers' strategies. They obtain linear convergence in Fisher markets for most of the CES domain, but not for linear utilities. In the context of network flow control, Low and Lapsley [LL99] adopted an optimization approach to derive a dynamic protocol where both prices (of links) and flow demands of agents are updated, and showed that the protocol converges to a social-welfare maximizing state. The update rules (3.2), (3.3) look quite similar to a game-learning dynamic called *log-linear learning* [Blu93, MS12] (by suitably viewing spendings as probability densities), but due to different contexts (games vs. markets), the actual behaviors and the analyses have significant qualitative differences.

Convex-concave saddle-point problems can be reduced to non-smooth convex minimization problems, for which algorithms yielding $O(1/\sqrt{T})$ convergence rate exist. Its wide applications (e.g., to two-person zero-sum game equilibria) have motivated exploration of properties of the underlying function which support faster converging algorithms [Nem04, Nes05b, Nes05a, Nes07, RS13].

3.2 Proportional Response

Proportional Response is a buyer-oriented update, originally analyzed in an effort to explain the behavior of peer-to-peer networks [WZ07, LLSB08]. Here, buyers update their spending in proportion to the contribution each good makes to its current utility. While its meaning is clear for linear and other separable utilities, for other classes of utilities this needs more interpretation, which we provide in following sections. Here prices are assumed to equal the current spending. An $O(1/T)$ rate of convergence was shown in [BDX11] for Fisher markets with buyers having linear utilities, and for the substitutes domain excluding linear utilities, a faster linear rate (i.e., an $\mathbb{E}(-\Omega(T))$ rate) of convergence was shown in [Zha11].

This dissertation continues the exploration of the connection between distributed dynamic processes and convex optimization, and more specifically the relation of proportional response to mirror descent.

Our first set of results starts by rederiving Zhang's bounds for CES substitutes utilities, by showing that for this setting proportional response amounts to mirror descent on a suitable convex function. To achieve the linear rate of convergence he obtained, we need to go beyond the standard $O(1/T)$ rate of convergence for mirror descent with a Bregman divergence. We proceed by analogy with gradient descent. Gradient descent with a Lipschitz constraint on the gradients guarantees only an $O(1/T)$ rate of convergence, but a faster linear rate of convergence is obtained when the objective function f is strongly convex. For mirror descent with Bregman Divergences we introduce the notion of *strong Bregman convexity* and show that it also leads to a linear convergence rate. It turns out that the convex function associated with the CES substitutes utilities satisfies strong Bregman convexity, thereby obtaining Zhang's bound anew. In addition, for complementary CES utilities, the same now concave function satisfies an analogous strong Bregman concavity property, which also yields a linear rate of convergence for these utilities. In addition, if we include linear utilities in the substitutes utilities, we obtain an $O(1/T)$ rate of convergence; likewise, including Leontief utilities in the complementary utilities also yields an $O(1/T)$ rate.

Next, we seek to handle substitute and complementary CES utilities simultaneously. The challenge we face is that the objective function used for the first set of results becomes a mixed concave-convex function in this setting, and the equilibrium corresponds to a saddle point of this function. We introduce the further notion of strongly-Bregman convex-concave functions, and for these functions we obtain a linear rate of convergence to the saddle point. Again, our objective function for the mixed CES utilities satisfies this property, thereby yielding linear convergence, albeit now for a *damped* proportional response, rather than the undamped proportional response analyzed in the first set of results. Here, including linear utilities and Leontief utilities yields an *empirical* $O(1/T)$ rate of convergence.

We note that our results are not a straightforward application of the existing mirror descent toolbox. The Bregman notions and the related convergence results in this dissertation are new. While the results for strong Bregman convex (resp. concave) functions are natural generalizations of gradient descent (resp. ascent) on standard strong convex (resp. concave) functions, the technique for demonstrating convergence for strong Bregman convex-concave functions appears to be new. It is not evident that suitable damping (i.e., reducing the step-size) permits a clean convergence analysis. Indeed, results showing linear point-wise convergence on convex-concave functions are rare; the only such work we are aware of is [GJLJ17]. We believe the new notions and convergence results for optimization problems may be of wider interest.

3.2.1 Bregman Divergence and Proportional Response

Bregman Divergence and Mirror Descent Let C be a compact and convex set. Given a differentiable convex function $h(\mathbf{x})$ with domain C, the *Bregman divergence* generated by kernel h is denoted by d_h, and is defined as:

$$d_h(\mathbf{x},\mathbf{y}) \;=\; h(\mathbf{x}) \;-\; [\; h(\mathbf{y}) \;+\; \langle\; \nabla h(\mathbf{y}) \;,\; \mathbf{x}-\mathbf{y} \;\rangle \;], \qquad \forall \mathbf{x} \in C \text{ and } \mathbf{y} \in \mathsf{rint}(C),$$

where $\mathsf{rint}(C)$ is the relative interior of C. We note that, in general, d_h is asymmetric, i.e., possibly $d_h(\mathbf{x},\mathbf{y}) \neq d_h(\mathbf{y},\mathbf{x})$. In this dissertation, we use the Kullback-Leibler or KL divergence extensively; it is the Bregman divergence generated by $h(\mathbf{x}) = \sum_j (x_j \cdot \ln x_j - x_j)$. When $\sum_j x_j \;=\; \sum_j y_j$, the explicit formula is:

$$\mathrm{KL}(\mathbf{x}||\mathbf{y}) \;:=\; \sum_j x_j \cdot \ln \frac{x_j}{y_j}.$$

For the problem of minimizing a convex function $f(\mathbf{x})$ subject to $\mathbf{x} \in C$, the mirror descent method w.r.t. Bregman divergence d_h is given by the following update rule:

$$\mathbf{x}^{t+1} \;=\; \arg\min_{\mathbf{x}\in C} \left\{ f(\mathbf{x}^t) \;+\; \left\langle \nabla f(\mathbf{x}^t) \,,\, \mathbf{x} - \mathbf{x}^t \right\rangle \;+\; \frac{1}{\Gamma_t} \cdot d_h(\mathbf{x}, \mathbf{x}^t) \right\}, \qquad (3.1)$$

where $\Gamma_t > 0$, and may be dependent on t.

Proportional Response For linear utility functions, Proportional Response is the dynamic given by the spending update rule:

$$b_{ij}^{t+1} = e_i \cdot \frac{a_{ij} x_{ij}^t}{\sum_k a_{ik} x_{ik}^t} = e_i \cdot \frac{a_{ij} \frac{b_{ij}^t}{p_j^t}}{\sum_k a_{ik} \frac{b_{ik}^t}{p_k^t}} \text{ with } p_k^t = \sum_i b_{ik}^t.$$

For substitutes CES utilities, [Zha11] generalized this rule to:

$$b_{ij}^{t+1} = e_i \cdot \frac{a_{ij} (x_{ij}^t)^{\rho_i}}{\sum_k a_{ik} (x_{ik}^t)^{\rho_i}} = e_i \cdot \frac{a_{ij} \left(\frac{b_{ij}^t}{p_j^t}\right)^{\rho_i}}{\sum_k a_{ik} \left(\frac{b_{ik}^t}{p_k^t}\right)^{\rho_i}} \qquad (3.2)$$

obtaining a linear convergence rate for the resulting dynamic, assuming $0 < \rho_i < 1$. The above rule has a natural distributed interpretation in the Fisher market setting: in each round, each buyer splits her spending on different goods in proportion to the values of $a_{ik}(x_{ik}^t)^{\rho_i}$. The seller of good j then allocates the good to buyers in proportion to the spending received from each buyer.

3.2.2 Results

3.2.2.1 Proportional Response

It is natural to seek to extend the Proportional Response rule (3.2) to the complementary domain, but this rule does not lead to convergent behavior in general. To see this, set $\rho = -1$.

Suppose there are two buyers and two items. Both buyers have the same preference for each item and the same budgets; i.e. $a_{11} = a_{12} = a_{21} = a_{22} = \frac{1}{2}$, and $e_1 = e_2 = 1$. Initially, at time $t = 0$, suppose that $b_{11}^{(0)} = \frac{1}{4}$, $b_{12}^{(0)} = \frac{3}{4}$, $b_{21}^{(0)} = \frac{3}{4}$, and $b_{22}^{(0)} = \frac{1}{4}$. A simple calculation shows that applying update rule (3.2) gives $b_{11}^{(1)} = \frac{3}{4}$, $b_{12}^{(1)} = \frac{1}{4}$, $b_{21}^{(1)} = \frac{1}{4}$, and $b_{22}^{(1)} = \frac{3}{4}$. So this simple example shows that in this setting, the spending will not converge to the market equilibrium; rather, it will cycle among two states.

Instead, we observe that in the substitutes domain, excluding Cobb-Douglas utilities, this rule is the mirror descent updating rule using the KL divergence for the following optimization problem.

$$\min_{\mathbf{b}} \quad \Phi(\mathbf{b}) = -\sum_{ij} \frac{b_{ij}}{\rho_i} \log \frac{a_{ij}(b_{ij})^{\rho_i - 1}}{(\sum_h b_{hj})^{\rho_i}}$$

$$\text{subject to } \textstyle\sum_j b_{ij} = e_i \text{ for all } i, \text{ and } b_{ij} \geq 0 \quad \text{for all } i, j.$$

We exclude Cobb-Douglas utilities, because as $\rho_i \to 0$ the corresponding term in Φ tends to ∞. When restricted to linear utilities, i.e. $\rho_i = 1$ for all i, this is simply Shmyrev's convex program [Shm09] for these markets.

In the complementary domain, we adapt the potential function to

$$\Phi(\mathbf{b}) = -\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{1}{\rho_i} \sum_j b_{ij} \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}} - \sum_{i:\rho_i = -\infty} \sum_j b_{ij} \log \frac{b_{ij}}{c_{ij} p_j(\mathbf{b})}.$$

where c_{ij} is defined in the Leontief utility $u_i(\mathbf{x}_i) = \min_j\{\frac{x_{ij}}{c_{ij}}\}$. The mirror descent updating rule for this function is:

$$b_{ij}^{t+1} = e_i \cdot \frac{\left(\frac{a_{ij}}{(p_j^t)^{\rho_i}}\right)^{\frac{1}{1-\rho_i}}}{\sum_k \left(\frac{a_{ik}}{(p_k^t)^{\rho_i}}\right)^{\frac{1}{1-\rho_i}}} \text{ for } -\infty < \rho_i < 0, \quad \text{and} \quad b_{ij}^{t+1} = e_i \cdot \frac{c_{ij} p_j^t}{\sum_k c_{ik} p_k^t} \text{ for } \rho_i = -\infty,$$

where $p_k^t = \sum_i b_{ik}^t$. (3.3)

Accordingly, we adopt this as the generalization of Proportional Response to the complementary domain. This rule can be easily implemented in the distributed environment of Fisher markets. In each round, each buyer only needs the prices computed in the previous round to compute its update. Thus, to implement the update rule, it suffices to have the sellers broadcast their prices by the end of each round.

Interestingly, this update rule is also the best response action to the current prices for each buyer. We note that this rule can be viewed as a tatonnement update if we define $x_{ij}^{t+1} = b_{ij}^{t+1}/p_j^t$, for then $p_j^{t+1} = \sum_i b_{ij}^{t+1} = \sum_i x_{ij}^{t+1} p_j^t = p_j^t(1 + z_j^{t+1})$. However, this is not the same rule as was used for the tatonnement analyzed in recent works regarding Fisher markets [CF08, CCD19].

For linear utilities, update rule (3.2) was analyzed in [BDX11]. For the substitutes domain excluding linear utilities, a faster linear rate of convergence was shown in [Zha11], but not based on considering the above optimization problem. To obtain a linear rate of convergence for an analysis based on optimizing Φ via a mirror descent with a KL Divergence, we introduce the notion of *strong Bregman convexity.* We also coin the term Bregman convexity for an analogous notion introduced in [BDX11]).

Definition 3.2.1. *The function f is L-Bregman convex w.r.t. Bregman divergence d_h if, for any $\mathbf{y} \in \mathsf{rint}(C)$ and $\mathbf{x} \in C$,*

$$f(\mathbf{y}) + \langle \nabla f(\mathbf{y}) \, , \, \mathbf{x} - \mathbf{y} \rangle \quad \leqslant \quad f(\mathbf{x}) \quad \leqslant \quad f(\mathbf{y}) + \langle \nabla f(\mathbf{y}) \, , \, \mathbf{x} - \mathbf{y} \rangle \; + \; L \cdot d_h(\mathbf{x}, \mathbf{y}).$$

The function f is (σ, L)-strongly Bregman convex w.r.t. Bregman divergence d_h if, $0 < \sigma \leqslant L$, and, for any $\mathbf{y} \in \mathsf{rint}(C)$, $\mathbf{x} \in C$,

$$f(\mathbf{y}) + \langle \nabla f(\mathbf{y}) \, , \, \mathbf{x} - \mathbf{y} \rangle + \sigma \cdot d_h(\mathbf{x}, \mathbf{y}) \;\; \leqslant \;\; f(\mathbf{x}) \;\; \leqslant \;\; f(\mathbf{y}) + \langle \nabla f(\mathbf{y}) \, , \, \mathbf{x} - \mathbf{y} \rangle + L \cdot d_h(\mathbf{x}, \mathbf{y}).$$

If the direction of the inequalities and the signs on the $d_h(\mathbf{x}, \mathbf{y})$ terms are reversed, the *function is said to be Bregman concave (or strongly Bregman concave respectively). ($\mathsf{rint}(C)$ denotes the relative interior of C.)*

Consider the update rule:

$$\mathbf{x}^{t+1} \quad \leftarrow \quad \arg\min_{\mathbf{y}} \left\{ \left\langle \nabla f(\mathbf{x}^t) \, , \, \mathbf{y} - \mathbf{x}^t \right\rangle \; + \; L \cdot d_h(\mathbf{y}, \mathbf{x}^t) \right\}. \tag{3.4}$$

Theorem 3.2.1. *Suppose that f is (σ, L)-strongly Bregman convex w.r.t. d_h. If update rule* (3.4) *is applied, then, for all $t \geqslant 1$,*

$$f(\mathbf{x}^t) - f(\mathbf{x}^*) \quad \leqslant \quad \frac{\sigma}{\left(\frac{L}{L-\sigma}\right)^t - 1} \cdot d_h(\mathbf{x}^*, \mathbf{x}^0).$$

An analogous Theorem for L-Bregman convex functions was given in [BDX11]:

Theorem 3.2.2. *[BDX11] Suppose f is an L-Bregman convex function w.r.t. d, and $\mathbf{x}^T$ is the point reached after T applications of the mirror descent update rule* (3.4). *Then,*

$$f(\mathbf{x}^T) - f(\mathbf{x}^*) \leq \frac{L \cdot d(\mathbf{x}^*, \mathbf{x}^0)}{T}.$$

We show that in the CES substitutes domain, excluding linear utilities, Φ is a strongly Bregman convex function w.r.t. the KL-divergence on spending, thereby providing an alternative derivation of Zhang's result. In addition, in the CES complements domain, excluding Leontief utilities, Φ is a strongly Bregman concave function w.r.t. the KL divergence on spending, which yields a proof of linear convergence in this domain. These analyses are readily modified to give a $1/T$ rate of convergence if we include respectively the linear and Leontief utilities. These results are made precise in the following theorem.

Theorem 3.2.3. *Suppose buyers with substitutes utilities repeatedly update their spending using Proportional Response rule* (3.2)*, and those with complementary utilities use rule* (3.3)*. Then the potential function Φ converges to the market equilibrium as follows.*

- *If all buyers have substitutes CES utilities, then*

$$\Phi(\mathbf{b}^T) - \Phi(\mathbf{b}^*) \leq \frac{1}{T}\sum_i \frac{1}{\rho_i}\mathrm{KL}(b_i^*||b_i^0).$$

- *Suppose that in addition no buyer has a linear utility. Let $\sigma = \min_i\{1-\rho_i\}$. Then,*

$$\Phi(\mathbf{b}^T) - \Phi(\mathbf{b}^*) \leq \frac{\sigma(1-\sigma)^T}{1-(1-\sigma)^T}\sum_i \frac{1}{\rho_i}\mathrm{KL}(b_i^*||b_i^0).$$

- *If all buyers have complementary CES utilities, then*[1]

$$\Phi(\mathbf{b}^*) - \Phi(\mathbf{b}^T) \leq \frac{1}{T}\sum_i \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(b_i^*||b_i^0).$$

- *Suppose that in addition no buyer has a Leontief utility. Let $\sigma = \min_i\left\{\frac{1}{1-\rho_i}\right\}$. Then,*

$$\Phi(\mathbf{b}^*) - \Phi(\mathbf{b}^T) \leq \frac{\sigma(1-\sigma)^T}{1-(1-\sigma)^T}\sum_i \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(b_i^*||b_i^0).$$

The results are shown in Corollaries 3.2.1, 3.2.2, 3.2.3, and 3.2.4, respectively. We also note that as shown in Lemma 13 in [BDX11], if $b_{ij}^0 = e_i/m$ for all i and j, then $\mathrm{KL}(\mathbf{b}^*||\mathbf{b}^0) \leqslant \log mn$, which provides a possibly more intuitive version of the above bounds.

Theorem 3.2.3 does not cover buyers with Cobb-Douglas utilities, because, as already noted, the terms in Φ for such buyers are equal to ∞. Note that these buyers always wish to allocate their spending in fixed proportions regardless of the prices. Thus, arguably, it would be natural for these buyers to always have the equilibrium spending. But even if this were not true initially, after one update this property would hold, and remain true henceforth. Thus the presence of these buyers would seem to have little effect on the convergence. Indeed, the above bounds hold with $\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)$ replaced by $\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^1)$ and T replaced by $T-1$

[1] For $\rho_i = -\infty$, we define $\frac{\rho_i - 1}{\rho_i}$ to equal 1.

on the RHS. But to obtain bounds in terms of $\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)$ appears to require substantially more effort; this analysis is given in Section 3.2.8.1(see Theorems 3.2.7-3.2.10). The rates of convergence are similar to those given in Theorem 3.2.3.

3.2.2.2 Damped Proportional Response

But what if we want to allow a mix of substitutes and complementary utilities? The difficulty we face is that the objective function Φ is no longer either convex or concave. Rather, if we fix the spending of the buyers with complementary utilities, the resulting restricted Φ is convex, while if we fix the spending of the buyers with substitutes utilities, the resulting restricted Φ is concave. As it happens, the equilibrium corresponds to a saddle point of the function Φ. Also, a suitable dynamic will converge to this saddle point. To show this, we introduce a saddle-point convergence analysis. To this end, we define the following notion.

Definition 3.2.2. *Function f is (L_X, L_Y)-convex-concave w.r.t. the pair of Bregman divergences (d_g, d_h), if it satisfies the following constraints.*

1. *For fixed $\mathbf{y}$, $f(\cdot, \mathbf{y})$ is a convex function;*

2. *For fixed $\mathbf{x}$, $f(\mathbf{x}, \cdot)$ is a concave function;*

3. *There exist parameters $L_X, L_Y > 0$ such that for any $\mathbf{x} \in X$, $\mathbf{x}' \in X$, $\mathbf{y} \in Y$ and $\mathbf{y}' \in Y$,*

$$-L_Y \cdot d_h(\mathbf{y}, \mathbf{y}') \overset{(a)}{\leq} f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}', \mathbf{y}') - \langle \nabla f(\mathbf{x}', \mathbf{y}'), (\mathbf{x}, \mathbf{y}) - (\mathbf{x}', \mathbf{y}') \rangle \overset{(b)}{\leq} L_X \cdot d_g(\mathbf{x}, \mathbf{x}'). \tag{3.5}$$

The saddle point is the "optimal" point of the convex-concave function, which is the minimum point in the x-direction and the maximum point in the y-direction, defined formally as follows.

Definition 3.2.3. *(x^*, y^*) is a saddle point of f if and only if*

$$f(x, y^*) \geq f(x^*, y^*) \geq f(x^*, y) \qquad \text{for any } x \in X \text{ and } y \in Y.$$

Now consider the following update rule:

$$\begin{aligned} \mathbf{x}^{t+1} &= \arg\min_{\mathbf{x}} \{\langle \nabla_x f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{x} - \mathbf{x}^t \rangle + 2L_X \cdot d_g(\mathbf{x}, \mathbf{x}^t)\}; \\ \mathbf{y}^{t+1} &= \arg\min_{\mathbf{y}} \{\langle -\nabla_{\mathbf{y}} f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{y} - \mathbf{y}^t \rangle + 2L_Y \cdot d_h(\mathbf{y}, \mathbf{y}^t)\}. \end{aligned} \tag{3.6}$$

We can then show an $O(1/T)$ empirical rate of convergence, as stated in the next theorem.

Theorem 3.2.4. *Suppose that f is (L_X, L_Y)-convex-concave, and there exists a saddle point $(\mathbf{x}^*, \mathbf{y}^*)$. In addition, suppose that $(\mathbf{x}, \mathbf{y})$ is updated according to (3.6). Then:*

(i) $$\sum_{t=1}^{T} \Big(f(\mathbf{x}^t, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^t) \Big) \leq 2L_X \cdot d_g(\mathbf{x}^*, \mathbf{x}^0) + 2L_Y \cdot d_h(\mathbf{y}^*, \mathbf{y}^0).$$

Note that $f(\mathbf{x}^t, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^t) \geq 0$ since $f(\mathbf{x}^t, \mathbf{y}^*) \geq f(\mathbf{x}^*, \mathbf{y}^*) \geq f(\mathbf{x}^*, \mathbf{y}^t)$.

(ii) Also, if $\bar{\mathbf{x}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}^t$ and $\bar{\mathbf{y}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{y}^t$, then:

$$f(\bar{\mathbf{x}}, \mathbf{y}^*) - f(\mathbf{x}^*, \bar{\mathbf{y}}) \leq \frac{1}{T}[2L_X \cdot d_g(\mathbf{x}^*, \mathbf{x}^0) + 2L_Y \cdot d_h(\mathbf{y}^*, \mathbf{y}^0)].$$

Note that the second part of the theorem follows immediately from the first part because $f(\cdot, \mathbf{y}^*)$ is a convex function and $f(\mathbf{x}^*, \cdot)$ is a concave function.

The objective function Φ is $(1,1)$-convex-concave w.r.t. $d_g = \sum_{i:\rho_i > 0} \frac{1}{\rho_i} \mathrm{KL}(b_i || b'_i)$ and $d_h = \sum_{i:\infty < \rho_i < 0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(b_i || b'_i) + \sum_{i:\rho_i = -\infty} \mathrm{KL}(b_i || b'_i)$. Consequently, we obtain an empirical

$O(1/T)$ rate of convergence for the following *Damped* Proportional Response update.

$$
\begin{aligned}
b_{ij}^{t+1} &= e_i \cdot \frac{\left[b_{ij}^t \cdot a_{ij}\left(\frac{b_{ij}^t}{p_j^t}\right)^{\rho_i}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot a_{ik}\left(\frac{b_{ik}^t}{p_k^t}\right)^{\rho_i}\right]^{\frac{1}{2}}}, && \text{for } \rho_i > 0; \\
b_{ij}^{t+1} &= e_i \cdot \frac{\left[b_{ij}^t \cdot \left(\frac{a_{ij}}{(p_j^t)^{\rho_i}}\right)^{\frac{1}{1-\rho_i}}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot \left(\frac{a_{ik}}{(p_k^t)^{\rho_i}}\right)^{\frac{1}{1-\rho_i}}\right]^{\frac{1}{2}}}, && \text{for } -\infty < \rho_i < 0; \\
b_{ij}^{t+1} &= e_i \cdot \frac{\left[b_{ij}^t \cdot \left(\frac{c_{ij}}{p_j^t}\right)^{-1}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot \left(\frac{c_{ik}}{p_k^t}\right)^{-1}\right]^{\frac{1}{2}}}, && \text{for } \rho_i = -\infty; \qquad (3.7)
\end{aligned}
$$

We say it is damped because the update rule uses the geometric mean of the current value and the standard Proportional Response update.

A natural question is whether a linear convergence rate is possible if the linear and Leontief utilities are excluded. The answer is yes, and to obtain this we need a stronger condition on the convex-concave objective function, as given in the following definition.

Definition 3.2.4. *f is a $(\sigma_X, \sigma_Y, L_X, L_Y)$-strongly Bregman convex-concave function, w.r.t. Bregman divergences d_g, d_h, if, for all $\mathbf{x} \in X$, $\mathbf{x}' \in X$, $\mathbf{y} \in Y$, and $\mathbf{y}' \in Y$, function f satisfies:*

$$
\begin{aligned}
&-L_Y \cdot d_h(\mathbf{y}, \mathbf{y}') + \sigma_X \cdot d_g(\mathbf{x}, \mathbf{x}') \le f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}', \mathbf{y}') - \langle \nabla f(\mathbf{x}', \mathbf{y}'), (\mathbf{x}, \mathbf{y}) - (\mathbf{x}', \mathbf{y}') \rangle \\
&\qquad \le L_X \cdot d_g(\mathbf{x}, \mathbf{x}') - \sigma_Y \cdot d_h(\mathbf{y}, \mathbf{y}').
\end{aligned} \tag{3.8}
$$

Theorem 3.2.5. *If f is a $(\sigma_X, \sigma_Y, L_X, L_Y)$-strongly Bregman convex-concave function w.r.t d_g and d_h, and there exists a saddle point, then update rule* (3.6) *converges to the saddle* *point with a linear convergence rate:*

$$\Big(f(\mathbf{x}^T,\mathbf{y}^*) - f(\mathbf{x}^*,\mathbf{y}^T)\Big)$$
$$\leq \left(1 - \frac{\min\left\{\frac{\sigma_X}{L_X}, \frac{\sigma_Y}{L_Y}\right\}}{2}\right)^{T-1} \Big((2L_X - \sigma_X)d_g(\mathbf{x}^*,\mathbf{x}^0) + (2L_Y - \sigma_Y)d_h(\mathbf{y}^*,\mathbf{y}^0)\Big).$$

Φ is $(\min_{i:\rho_i>0}\{1-\rho_i\}, \min_{i:\rho_i<0}\left\{\frac{1}{1-\rho_i}\right\}, 1, 1)$-strongly Bregman convex-concave, and thus we can deduce that the damped Proportional Response achieves a linear convergence rate if linear and Leontief utilities are excluded.

As before, the above results exclude Cobb-Douglas utilities.

Arguably, the buyers with Cobb-Douglas utilities should always have the equilibrium spending, or failing that, should immediately update to this spending. But for mathematical consistency, we suppose they are performing the same type of damped update as the other buyers. In this case, our previous potential function can't be used when we include Cobb-Douglas utility functions. We now need to include a term in the potential function for each buyer with a Cobb-Douglas utility as their spending keeps changing.

We will need the following notation. Let $\mathbf{b}_{>0}$, $\mathbf{b}_{=0}$, and $\mathbf{b}_{<0}$ denote the spending of those buyers with $\rho_i > 0$, $\rho_i = 0$, and $\rho_i < 0$, respectively. Accordingly, we will write $\Phi(\mathbf{b}) = \Phi(\mathbf{b}_{>0}, \mathbf{b}_{=0}, \mathbf{b}_{<0})$. The resulting function is still convex in $\mathbf{b}_{>0}$ and concave in $\mathbf{b}_{<0}$. The construction is given in Section 3.2.8.1. We note that the update rule for the buyers with $\rho_i = 0$ is given by:

$$b_{ij}^{t+1} = e_i \cdot \frac{\left[b_{ij}^t \cdot a_{ij}^t\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot a_{ik}^t\right]^{\frac{1}{2}}}, \quad \text{for } \rho_i = 0.$$

Theorem 3.2.6. *Suppose buyers repeatedly update their spending using the Damped Pro-*

portional Response rule (3.7). *Then*

$$\mathrm{KL}(\mathbf{b}^T_{=0}||\mathbf{b}^*_{=0}) \leq \frac{1}{2^T}\mathrm{KL}(\mathbf{b}^0_{=0}||\mathbf{b}^*_{=0}),$$

and the potential function Φ *converges to the market equilibrium as follows:*

i. $$\sum_{t=1}^{T}\left[\Phi(\mathbf{b}^t_{>0},\mathbf{b}^*_{=0},\mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{>0},\mathbf{b}^*_{=0},\mathbf{b}^t_{<0})\right]$$
$$\leq 4\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:-\infty<\rho_i<0}\frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)$$
$$+\sum_{i:\rho_i>0}\frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i=-\infty}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i).$$

ii. If in addition no buyer has a linear or Leontief utility,

Let $\sigma = \min\left\{ \min_{i:\rho_i>0}\left\{\frac{2}{1+\rho_i}\right\}, \min_{i:\rho_i<0}\left\{\frac{2(\rho_i-1)}{2\rho_i-1}\right\}\right\}$ *(so* $1<\sigma<2$*). Then,*

$$\Phi(\mathbf{b}^T_{>0},\mathbf{b}^*_{=0},\mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{>0},\mathbf{b}^*_{=0},\mathbf{b}^T_{<0})$$
$$\leq \frac{1}{\sigma^{T-1}}\left[\frac{4}{2-\sigma}\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i<0}\frac{2\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i>0}\frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)\right].$$

Weaker bounds are shown in Corollaries 3.2.5 and 3.2.6, respectively. The complete proof is given in Section 3.2.8.1 (see Theorems 3.2.11 and 3.2.12).

3.2.3 Linear Convergence with Strong Bregman Convexity

Our proof will use the following lemmas.

Lemma 3.2.1. *[CT93] If* $\mathbf{x}^+$ *is the optimal point for the optimization problem:*

$$\begin{aligned} &\textit{minimize} \quad g(\mathbf{x}) + d(\mathbf{x},\mathbf{y}) \\ &\textit{subject to} \quad \mathbf{x}\in C, \end{aligned}$$

where C is a compact convex set, then, for any $\mathbf{x} \in C$,

$$g(\mathbf{x}^+) + d(\mathbf{x}^+, \mathbf{y}) + d(\mathbf{x}, \mathbf{x}^+) \leq g(\mathbf{x}) + d(\mathbf{x}, \mathbf{y}).$$

Lemma 3.2.2. *[BDX11] Suppose that f is an L-Bregman convex function w.r.t. $d(\mathbf{x}, \mathbf{x}')$, and $\mathbf{x}^t$ and $\mathbf{x}^{t+1}$ are the points reached after t and $t+1$ applications of the mirror descent update rule* (3.4). *Then*

$$f(\mathbf{x}^{t+1}) \leq f(\mathbf{x}^t).$$

Proof of Theorem 3.2.1. By Lemma 3.2.1 with $\mathbf{y} = \mathbf{x}^t$, $\mathbf{x}^+ = \mathbf{x}^{t+1}$, and $\mathbf{x} = \mathbf{x}^*$,

$$\begin{aligned} &\left\langle \nabla f(\mathbf{x}^t) \ , \ \mathbf{x}^{t+1} - \mathbf{x}^t \right\rangle \ + \ L \cdot d_h(\mathbf{x}^{t+1}, \mathbf{x}^t) \\ &\qquad \leqslant \ \left\langle \nabla f(\mathbf{x}^t) \ , \ \mathbf{x}^* - \mathbf{x}^t \right\rangle \ + \ L \cdot \left[d_h(\mathbf{x}^*, \mathbf{x}^t) - d_h(\mathbf{x}^*, \mathbf{x}^{t+1}) \right]. \end{aligned} \tag{3.9}$$

By strong Bregman-convexity, with $\mathbf{y} = \mathbf{x}^t$, and $\mathbf{x} = \mathbf{x}^{t+1}$,

$$\left\langle \nabla f(\mathbf{x}^t) \ , \ \mathbf{x}^{t+1} - \mathbf{x}^t \right\rangle \ + \ L \cdot d_h(\mathbf{x}^{t+1}, \mathbf{x}^t) \ \ \geqslant \ \ f(\mathbf{x}^{t+1}) - f(\mathbf{x}^t); \tag{3.10}$$

and with $\mathbf{y} = \mathbf{x}^t$ and $\mathbf{x} = \mathbf{x}^*$,

$$\nabla f(\mathbf{x}^t) \cdot (\mathbf{x}^* - \mathbf{x}^t) \ \ \leqslant \ \ f(\mathbf{x}^*) - f(\mathbf{x}^t) - \sigma \cdot d_h(\mathbf{x}^*, \mathbf{x}^t). \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), gives, for $t \geqslant 0$,

$$f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) \ \ \leqslant \ \ (L - \sigma) \cdot d_h(\mathbf{x}^*, \mathbf{x}^t) \ - \ L \cdot d_h(\mathbf{x}^*, \mathbf{x}^{t+1}). \tag{3.12}$$

On multiplying both sides of the above inequality by $\left(\frac{L}{L-\sigma}\right)^t$, and then summing over

$0 \leqslant t < T$, the RHS becomes a telescoping sum, and hence

$$\sum_{t=0}^{T-1} \left(\frac{L}{L-\sigma}\right)^t \cdot \left[f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*)\right] \quad \leqslant \quad (L-\sigma) \cdot d_h(\mathbf{x}^*, \mathbf{x}^0).$$

By Lemma 3.2.2, $f(\mathbf{x}^{t+1}) \leqslant f(\mathbf{x}^t)$; thus:

$$\begin{aligned} & \frac{L-\sigma}{\sigma} \cdot \left[\left(\frac{L}{L-\sigma}\right)^T - 1\right] \cdot \left[f(\mathbf{x}^T) - f(\mathbf{x}^*)\right] \\ = \quad & \left(\sum_{t=0}^{T-1} \left(\frac{L}{L-\sigma}\right)^t\right) \cdot \left[f(\mathbf{x}^T) - f(\mathbf{x}^*)\right] \\ \leqslant \quad & (L-\sigma) \cdot d_h(\mathbf{x}^*, \mathbf{x}^0), \end{aligned}$$

and the result follows. □

3.2.4 Convergence of Proportional Response

We consider the following potential function:

$$\begin{aligned} p_j(\mathbf{b}) &= \textstyle\sum_i b_{ij}, \\ \Phi(\mathbf{b}) &= -\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{1}{\rho_i} \sum_j b_{ij} \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}} - \sum_{i:\rho_i = -\infty} \sum_j b_{ij} \log \frac{b_{ij}}{c_{ij} p_j(\mathbf{b})}. \end{aligned}$$

For those i for which $\rho_i \neq -\infty$,

$$\begin{aligned} \nabla_{b_{ij}} \Phi(\mathbf{b}) &= -\frac{1}{\rho_i} \log a_{ij} - \frac{\rho_i - 1}{\rho_i} (\log b_{ij} + 1) + \log p_j(\mathbf{b}) + \sum_h b_{hj} \frac{1}{p_j(\mathbf{b})} \\ &= \frac{1}{\rho_i} \Big(1 - \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{p_j^{\rho_i}(\mathbf{b})}\Big); \end{aligned}$$

and for those i for which $\rho_i = -\infty$, $\quad \nabla_{b_{ij}} \Phi(\mathbf{b}) = -\log \frac{b_{ij}}{c_{ij} p_j}$.

We deduce:

Lemma 3.2.3.

$$\sum_{i:\rho_i\neq-\infty}\frac{1-\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)-\sum_{i:\rho_i=-\infty}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)$$
$$\leq \Phi(\mathbf{b})-\Phi(\mathbf{b}')-\langle\nabla\Phi(\mathbf{b}'),\mathbf{b}-\mathbf{b}'\rangle$$
$$\leq \sum_{i:\rho_i\neq-\infty}\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i).$$

Proof.

$$\begin{aligned}\Phi(\mathbf{b})&-\Phi(\mathbf{b}')-\langle\nabla\Phi(\mathbf{b}'),\mathbf{b}-\mathbf{b}'\rangle\\ &=-\sum_{i:\rho_i\neq\{0,-\infty\}}\frac{\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)-\sum_{i:\rho_i=-\infty}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)+\mathrm{KL}(\mathbf{p}||\mathbf{p}')\\ &=\sum_{i:\rho_i\neq-\infty}\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)-\Big(\sum_i\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)-\mathrm{KL}(\mathbf{p}||\mathbf{p}')\Big).\end{aligned}$$

Since $\sum_i \mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)\geq \mathrm{KL}(\mathbf{p}||\mathbf{p}')$, the result follows. □

The Substitutes Domain The following lemma states the equivalence between mirror descent and proportional response in the substitutes domain; it follow readily from the definition of $\mathbf{b}^{t+1}$ for Proportional Response (given by (3.2)).

Lemma 3.2.4. *For buyers with CES substitutes utilities, the Proportional Response update is the same as the mirror descent update, given by:*

$$\mathbf{b}_i^{t+1}=\arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i}\left\{\langle\nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t),\mathbf{b}_i-\mathbf{b}_i^t\rangle+\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t)\right\}.$$

The next lemma states several properties of the potential function in the substitutes domain.

Lemma 3.2.5. *i. If $\rho_i>0$ for all i, then Φ is a 1-Bregman convex function w.r.t. $\sum_i\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)$;*

ii. if $0 < \rho_i < 1$ *for all* i*, then* Φ *is a* $(\min_i\{1-\rho_i\}, 1)$*-strong Bregman convex function w.r.t.* $\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)$*;*

iii. $\mathbf{b}$ *is the spending at the market equilibrium if and only if* $\mathbf{b}$ *is the minimum point of* Φ*.*

Proof. The first two claims follow from Lemma 3.2.3 with a little calculation. The proof of the third claim is given in Section 3.2.8.3. □

Let $\mathbf{b}^*$ be the spending at some market equilibrium. Applying Theorem 3.2.2 yields:

Corollary 3.2.1.

$$\Phi(\mathbf{b}^T) - \Phi(\mathbf{b}^*) \leq \frac{1}{T}\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

Furthermore, if there is no buyer with a linear utility function, then, applying Theorem 3.2.1 yields:

Corollary 3.2.2. *Let* $\sigma = \min_i\{1-\rho_i\} > 0$*. Then*

$$\Phi(\mathbf{b}^T) - \Phi(\mathbf{b}^*) \leq \frac{\sigma(1-\sigma)^T}{1-(1-\sigma)^T}\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

We now explain how to recover Zhang's bound [Zha11]. From (3.12),

$$\begin{aligned}\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) &\leqslant \frac{L-\sigma}{L}\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t-1}) \\ &\leqslant \left(\frac{L-\sigma}{L}\right)^t \sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) = \left(\max_i \rho_i\right)^t \sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) \\ &\qquad \text{(as } L = 1 \text{ here).}\end{aligned}$$

In [Zha11], $\phi(t)$ is used to denote $\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t)$. We have obtained the exact same bound on $\sum_i \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t)$ as in [Zha11], and thus can deduce the identical convergence rate.

The Complementary Domain We proceed as in the substitutes domain. First, the following lemma shows the equivalence between the mirror descent and the proportional response (given by (3.3)) in the complementary domain.

Lemma 3.2.6. *For those complementary buyers such that $-\infty < \rho_i < 0$, the Proportional Response update, which is the best response in this domain, is equal to the mirror descent update, given by:*

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ -\langle \nabla_{b_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\};$$

and this also holds for buyers with Leontief utility functions, where now the mirror descent update is given by:

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \{ -\langle \nabla_{b_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \};$$

Next, we show the following properties of the potential function in the complementary domain. The main difference between the complementary case and the substitutes case is that the potential function is a concave function in the complementary domain, while it is a convex function in the substitutes domain.

Lemma 3.2.7. *i. If $\rho_i < 0$ for all i, then Φ is a 1-Bregman concave function w.r.t. $\sum_i \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i')$;*

ii. if $-\infty < \rho_i < 0$ for all i, then Φ is a $(\min_i \{\frac{1}{1-\rho_i}\}, 1)$-strong Bregman concave function w.r.t. $\sum_i \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i')$;

iii. $\mathbf{b}$ is the spending at the market equilibrium if and only if $\mathbf{b}$ is the maximum point of Φ.

Proof. The first two claims follow from Lemma 3.2.3 with a little calculation. The proof of the third claim is given in Section 3.2.8.3. □

Also, let $\mathbf{b}^*$ be the spending at some market equilibrium. Applying Theorem 3.2.2 yields:[2]

Corollary 3.2.3.

$$\Phi(\mathbf{b}^*) - \Phi(\mathbf{b}^T) \leq \frac{1}{T}\sum_i \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

In addition, if there is no buyer with a Leontief utility, applying Theorem 3.2.1 yields:

Corollary 3.2.4. *Let* $\sigma = \min_i\{\frac{1}{1-\rho_i}\}$. *Then,*

$$\Phi(\mathbf{b}^*) - \Phi(\mathbf{b}^T) \leq \frac{\sigma(1-\sigma)^T}{1-(1-\sigma)^T}\sum_i \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

3.2.5 Saddle Point Analysis

Proof of Theorem 3.2.4. Recall that $\mathbf{x}^{t+1} = \arg\min_{\mathbf{x}\in X}\{\langle\nabla_{\mathbf{x}} f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{x}-\mathbf{x}^t\rangle + 2L_X d_g(\mathbf{x}, \mathbf{x}^t)\}$. Applying Lemma 3.2.1 with $\mathbf{x} = \mathbf{x}^*$, $d(\cdot,\cdot) = 2L_X d_g(\cdot,\cdot)$, $\mathbf{y} = \mathbf{x}^t$, $\mathbf{x}^+ = \mathbf{x}^{t+1}$, and $g(\mathbf{x}) = \langle\nabla_x f(\mathbf{x}^t, \mathbf{y}^t)\rangle, \mathbf{x} - \mathbf{x}^t\rangle$ gives

$$\begin{aligned}&\langle\nabla_x f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{x}^{t+1} - \mathbf{x}^t\rangle + 2L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t)\\ &\qquad \leq \langle\nabla_x f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{x}^* - \mathbf{x}^t\rangle + 2L_X d_g(\mathbf{x}^*, \mathbf{x}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}).\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\underbrace{f(\mathbf{x}^t, \mathbf{y}^t) + \langle\nabla f(\mathbf{x}^t, \mathbf{y}^t), (\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - (\mathbf{x}^t, \mathbf{y}^t)\rangle + 2L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t)}_{\texttt{LHS}}\\ &\quad\leq \underbrace{f(\mathbf{x}^t, \mathbf{y}^t) + \langle\nabla f(\mathbf{x}^t, \mathbf{y}^t), (\mathbf{x}^*, \mathbf{y}^{t+1}) - (\mathbf{x}^t, \mathbf{y}^t)\rangle + 2L_X d_g(\mathbf{x}^*, \mathbf{x}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}).}_{\texttt{RHS}}\end{aligned} \tag{3.13}$$

[2] Recall that for $\rho_i = -\infty$, we defined $(\rho_i - 1)/\rho_i = 1$.

Since f is (L_X, L_y)-convex-concave, the third property — see (3.5) — gives:

$$\begin{aligned} f(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) + L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t) &\overset{(1)}{\leq} \texttt{LHS} \leq \texttt{RHS} \\ &\overset{(2)}{\leq} f(\mathbf{x}^*, \mathbf{y}^{t+1}) + L_Y d_h(\mathbf{y}^{t+1}, \mathbf{y}^t) + 2L_X d_g(\mathbf{x}^*, \mathbf{x}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}), \end{aligned} \tag{3.14}$$

where (1) is deduced from (b) in Definition 3.2.2 with $(x', y') = (\mathbf{x}^t, \mathbf{y}^t)$ and (2) is deduced from (a) in Definition 3.2.2 with $(x', y') = (\mathbf{x}^t, \mathbf{y}^t)$ and $(x, y) = (\mathbf{x}^*, \mathbf{y}^{t+1})$.

Now, let's consider $-f(x, y)$ and $\mathbf{y}^{t+1} = \arg\min_{\mathbf{y}\in Y}\{\langle -\nabla_{\mathbf{y}} f(\mathbf{x}^t, \mathbf{y}^t), \mathbf{y}-\mathbf{y}^t\rangle + 2L_Y d_h(\mathbf{y}, \mathbf{y}^t)\}$. Using a similar argument, we obtain:

$$\begin{aligned} &- f(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) + L_Y d_h(\mathbf{y}^{t+1}, \mathbf{y}^t) \\ &\quad \leq -f(\mathbf{x}^{t+1}, \mathbf{y}^*) + L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t) + 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^t) - 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^{t+1}). \end{aligned} \tag{3.15}$$

Adding these two inequalities gives:

$$\begin{aligned} &f(\mathbf{x}^{t+1}, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^{t+1}) \\ &\quad \leq 2L_X d_g(\mathbf{x}^*, \mathbf{x}^t) + 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}) - 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^{t+1}). \end{aligned}$$

Summing over t yields: $\sum_{t=1}^{T} \Big(f(\mathbf{x}^t, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^t)\Big) \leq 2L_X d_g(\mathbf{x}^*, \mathbf{x}^0) + 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^0)$.

□

Proof of Theorem 3.2.5. Using (3.8) instead of (3.5), we deduce the following from (3.13) instead of (3.14):

$$\begin{aligned} &f(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) + L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t) \\ &\quad \leq f(\mathbf{x}^*, \mathbf{y}^{t+1}) + L_Y d_h(\mathbf{y}^{t+1}, \mathbf{y}^t) + (2L_X - \sigma_X) d_g(\mathbf{x}^*, \mathbf{x}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}). \end{aligned}$$

Also, (3.15) is replaced by:

$$
\begin{aligned}
&- f(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) + L_Y d_h(\mathbf{y}^{t+1}, \mathbf{y}^t) \\
&\qquad \le -f(\mathbf{x}^{t+1}, \mathbf{y}^*) + L_X d_g(\mathbf{x}^{t+1}, \mathbf{x}^t) + (2L_Y - \sigma_Y) d_h(\mathbf{y}^*, \mathbf{y}^t) - 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^{t+1}).
\end{aligned}
$$

Summing up these two inequalities gives:

$$
\begin{aligned}
&f(\mathbf{x}^{t+1}, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^{t+1}) \\
&\quad \le (2L_X - \sigma_X) d_g(\mathbf{x}^*, \mathbf{x}^t) + (2L_Y - \sigma_Y) d_h(\mathbf{y}^*, \mathbf{y}^t) - 2L_X d_g(\mathbf{x}^*, \mathbf{x}^{t+1}) - 2L_Y d_h(\mathbf{y}^*, \mathbf{y}^{t+1}).
\end{aligned}
$$

Let $\sigma = \min\left\{\frac{\sigma_X}{L_X}, \frac{\sigma_Y}{L_Y}\right\}$. Then:

$$
\sum_{t=0}^{T-1} \left(\frac{2}{2-\sigma}\right)^t \Big(f(\mathbf{x}^{t+1}, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^{t+1})\Big) \le (2L_X - \sigma_X) d_g(\mathbf{x}^*, \mathbf{x}^0) + (2L_Y - \sigma_Y) d_h(\mathbf{y}^*, \mathbf{y}^0).
$$

Note that $f(\mathbf{x}^t, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^t)$ is positive for each t, so the result follows. □

3.2.6 Analysis of Damped Proportional Response

Excluding Cobb-Douglas Utility Functions First, we consider a simplified situation where there is no buyer with a Cobb-Douglas utility function. We want to use the technique developed in the saddle point analysis to obtain a convergence result. The potential function is the same as before.

We make the following observations.

Lemma 3.2.8. *If $\rho_i > 0$ for buyer i, then the Damped Proportional Response (given by* (3.7)*) is equivalent to mirror descent with a halved step size, defined as follows:*

$$
\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \left\{ \langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\};
$$

if $-\infty < \rho_i < 0$ *for buyer* i*, then the Damped Proportional Response (given by* (3.7)*) is equivalent to mirror descent (really ascent as this is a concave function) with a halved step size defined as follows:*

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ -\langle \nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{2(\rho_i - 1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\};$$

and if $\rho_i = -\infty$ *for buyer* i*, then the Damped Proportional Response (given by* (3.7)*) is equivalent to mirror descent (really ascent as this is a concave function) with a halved step sizem defined as follows:*

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \{ -\langle \nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + 2\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \}.$$

Proof. By calculation. □

By Lemma 3.2.3 and with a simple calculation one can show that, in Definition 3.2.2, if we set $\mathbf{x} = \mathbf{b}_{>0}$, $\mathbf{y} = \mathbf{b}_{<0}$, $d_g(\mathbf{x}) = \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i')$, and $d_h(\mathbf{y}) = \sum_{i:\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i') + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i')$, then Φ is $(1,1)$-convex-concave function.

Furthermore, let $\mathbf{b}_{>0}^*$ and $\mathbf{b}_{<0}^*$ be the market equilibrium of the Fisher market. Then,

$$\mathbf{b}_{>0}^* \text{ minimizes } \Phi(\cdot, \mathbf{b}_{<0}^*), \text{ and } \mathbf{b}_{<0}^* \text{ maximizes } \Phi(\mathbf{b}_{>0}^*, \cdot),$$

which implies $(\mathbf{b}_{>0}^*, \mathbf{b}_{<0}^*)$ is a saddle point of the potential function Φ. Theorem 3.2.4 yields the following corollary.

Corollary 3.2.5. *The Damped Proportional Response (given by* (3.7)*) converges to an equi-*

librium with a convergence rate of:

$$\sum_{t=1}^{T}\left[\Phi(\mathbf{b}^t_{>0},\mathbf{b}^*_{<0})-\Phi(\mathbf{b}^*_{>0},\mathbf{b}^t_{<0})\right]$$
$$\leq \sum_{i:-\infty<\rho_i<0}\frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)+\sum_{i:\rho_i>0}\frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)+\sum_{i:\rho_i=-\infty}2\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i).$$

Moreover, if we assume there is no buyer with either a linear utility or a Leontief utility function, then, by Lemma 3.2.3, $\Psi(\cdot,\cdot)$ is a $(\min_{i:\rho_i>0}\{1-\rho_i\},\min_{i:\rho_i<0}\{\frac{1}{1-\rho_i}\},1,1)$-strong Bregman convex-concave function with $\mathbf{x}=\mathbf{b}_{>0}$, $\mathbf{y}=\mathbf{b}_{<0}$, $d_g(\mathbf{x})=\sum_{i:\rho_i>0}\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)$ and $d_h(\mathbf{y})=\sum_{i:\infty<\rho_i<0}\frac{\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i)$ (see Definition 3.2.4). Theorem 3.2.5 yields the following corollary.

Corollary 3.2.6. *Suppose there is no buyer with either a linear utility or a Leontief utility. Let*

$$\sigma_{>0}=\min_{i:\rho_i>0}\{1-\rho_i\}\ \ \textit{and}\ \ \sigma_{<0}=\min_{i:\rho_i<0}\left\{\frac{1}{1-\rho_i}\right\}.$$

Then the Damped Proportional Response (given by (3.7)*) converges to the equilibrium with a convergence rate of*

$$\Phi(\mathbf{b}^T_{>0},\mathbf{b}^*_{<0})-\Phi(\mathbf{b}^*_{>0},\mathbf{b}^T_{<0})$$
$$\leq\left(1-\frac{\min\{\sigma_{>0},\sigma_{<0}\}}{2}\right)^{T-1}\left[\sum_{i:\rho_i<0}(2-\sigma_{<0})\frac{\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)+\sum_{i:\rho_i>0}(2-\sigma_{>0})\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)\right].$$

The Entire CES Range Now we consider the Damped Proportional Response with a damping factor of 2 over the entire CES range, i.e. including Cobb-Douglas utilities. Recall that we modify our potential function to include terms for the buyers with $\rho_i=0$. However, for this new modified function, the buyers with Cobb-Douglas utility functions don't actually perform mirror descent. Fortunately, we can make two observations.

First, the buyers with Cobb-Douglas utility functions converge quickly to the equilibrium independently of everyone else's spending. Second, the buyers whose utility functions are not Cobb-Douglas will still perform the mirror descent (ascent) procedure.

So, intuitively, in our analysis, we regard the spending of the buyers with Cobb-Douglas utility functions as a parameter, θ, of $f_\theta(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ represents the spending of the strictly substitutes buyers and $\mathbf{y}$ represents the spending of the strictly complementary buyers. Remember, in the case with no Cobb-Douglas utilities, the market equilibrium corresponded to a saddle point. Here, similarly, a market equilibrium corresponds to a saddle point of $f_{\theta^*}(\cdot, \cdot)$, where θ^* is the spending at the market equilibrium of those buyers with Cobb-Douglas utility functions. We prove the following two claims.

1. θ converges to θ^* quickly;

2. when θ tends to θ^*, though $\mathbf{x}$ and $\mathbf{y}$ perform the mirror descent based on the gradient of $f_\theta(\mathbf{x}, \mathbf{y})$ and not of $f_{\theta^*}(\mathbf{x}, \mathbf{y})$, $(\mathbf{x}, \mathbf{y})$ will still converge quickly to $(\mathbf{x}^*, \mathbf{y}^*)$, the saddle point of $f_{\theta^*}(\cdot, \cdot)$.

We thereby show that Damped Proportional Response converges to the market equilibrium even when faced with the entire range of CES utilities.

3.2.7 Other Measures of Convergence

The potential function ϕ appears to be closely related to the Eisenberg-Gale program. In particular, we can show that in the substitutes domain, when applying update rule (3.2), the Proportional Response update, the objective function Ψ for the Eisenberg-Gale program converges at least as fast as Φ, i.e. that $\Psi(\mathbf{x}(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})) - \Psi(\mathbf{x}(\mathbf{b}_{>0}, \mathbf{b}^*_{=0})) \leq \Phi(\mathbf{b}_{>0}, \mathbf{b}^*_{=0}) - \Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})$, and that in the complementary domain, when applying update rule (3.3), the objective function for the dual of the Eisenberg-Gale program converges at least as fast as Φ. These claims are shown in Section 3.2.8.2.

Lemma 3.2.3 allows us to make some observations about the rate of convergence of the spending. For update rule (3.2), in the substitutes domain excluding linear utilities, we can deduce that $\sum_i \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^*) \leqslant \max_i \frac{\rho_i}{1-\rho_i}[\Phi(\mathbf{b}) - \Phi(\mathbf{b}^*)]$, and for update rule (3.3) in the complementary domain excluding Leontief utilities, that $\sum_i \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^*) \leqslant \max_i -\rho_i[\Phi(\mathbf{b}) - \Phi(\mathbf{b}^*)]$. As the equilibrium need not be unique in terms of spending for either linear or Leontief utilities, this lemma is not going to yield a bound on the convergence rate of the spending in these cases, as it can be applied to any equilibrium. Similarly in the combined domain, still excluding linear and Leontief utilities, we can observe that

$$\sum_{i:\rho_i>0} \frac{1-\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^*) + \sum_{i:\rho_i<0} \frac{-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^*) \leqslant [\Phi(\mathbf{b}_{>0}, \mathbf{b}_{<0}^*) - \Phi(\mathbf{b}^*)] + [\Phi(\mathbf{b}^*) - \Phi(\mathbf{b}_{>0}^*, \mathbf{b}_{<0})].$$

In addition, since $\mathrm{KL}(\mathbf{p}||\mathbf{p}^*) \leqslant \mathrm{KL}(\mathbf{b}||\mathbf{b}^*)$ we can immediately obtain analogous bounds on the KL divergence of the prices. Furthermore, for the substitutes domain, including linear utilities, Lemma 3.2.3 also implies that $\mathrm{KL}(\mathbf{p}||\mathbf{p}^*) \leqslant [\Phi(\mathbf{b}) - \Phi(\mathbf{b}^*)]$.

3.2.8 Missing Proofs

3.2.8.1 Proportional Response Including Cobb-Douglas Utilities

The Potential Function and its Properties The new potential function, $\Phi(\mathbf{b})$, which includes buyers with Cobb-Douglas utilities, is defined as follows:

$$p_j(\mathbf{b}) = \sum_i b_{ij},$$

$$\Phi(\mathbf{b}) = -\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{1}{\rho_i} \sum_j b_{ij} \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}} - \sum_{i:\rho_i=-\infty} \sum_j b_{ij} \log \frac{b_{ij}}{c_{ij} p_j(\mathbf{b})} + \sum_{i:\rho_i=0} \sum_j b_{ij} \log p_j(\mathbf{b}).$$

Note that for those i for which $\rho_i \neq \{0, -\infty\}$,

$$\begin{aligned}\nabla_{b_{ij}}\Phi(\mathbf{b}) &= -\frac{1}{\rho_i}\log a_{ij} - \frac{\rho_i - 1}{\rho_i}(\log b_{ij} + 1) + \log p_j(\mathbf{b}) + \sum_h b_{hj}\frac{1}{p_j(\mathbf{b})} \\ &= \frac{1}{\rho_i}\Big(1 - \log\frac{a_{ij}b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}}\Big);\end{aligned}$$

for those i for which $\rho_i = -\infty$,

$$\nabla_{b_{ij}}\Phi(\mathbf{b}) = -\log\frac{b_{ij}}{c_{ij}p_j(\mathbf{b})};$$

and for those i for which $\rho_i = 0$,

$$\nabla_{b_{ij}}\Phi(\mathbf{b}) = 1 + \log p_j(\mathbf{b}).$$

Then, we can deduce that

$$\begin{aligned}&\Phi(\mathbf{b}) - \Phi(\mathbf{b}') - \langle\nabla\Phi(\mathbf{b}'), \mathbf{b} - \mathbf{b}'\rangle \\ &\quad= -\sum_{i:\rho_i\neq\{0,-\infty\}}\frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) - \sum_{i:\rho_i=-\infty}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) + \mathrm{KL}(\mathbf{p}||\mathbf{p}') \\ &\quad= \sum_{i:\rho_i\neq\{0,-\infty\}}\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) + \sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) \\ &\qquad\qquad - \Big(\sum_i \mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) - \mathrm{KL}(\mathbf{p}||\mathbf{p}')\Big).\end{aligned}$$

Since $\sum_i \mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) \geq \mathrm{KL}(\mathbf{p}||\mathbf{p}')$,

$$\begin{aligned}&\sum_{i:\rho_i\neq\{0,-\infty\}}\frac{1-\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) - \sum_{i:\rho_i=-\infty}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) \\ &\qquad\leq \Phi(\mathbf{b}) - \Phi(\mathbf{b}') - \langle\nabla\Phi(\mathbf{b}'), \mathbf{b} - \mathbf{b}'\rangle \\ &\qquad\qquad\leq \sum_{i:\rho_i\neq\{0,-\infty\}}\frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i) + \sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}'_i).\end{aligned} \tag{3.16}$$

Proportional Response in the Substitutes Domain with Cobb-Douglas Utility Functions As in the analysis of Proportional Response in the substitutes domain without Cobb-Douglas utility functions, the following lemma states the equivalence between mirror descent and proportional response in the substitutes domain; it follows readily from the definition of $\mathbf{b}^{t+1}$ for Proportional Response (given by (3.2)).

Lemma 3.2.9. *For buyers with strict CES substitutes utilities ($\rho_i > 0$), the Proportional Response update is the same as the mirror descent update, given by:*

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \{\langle \nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t\rangle + \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t)\}.$$

Buyers with Cobb-Douglas utility functions use the following update rule:

$$b_{ij}^{t+1} = e_i \frac{a_{ij}}{\sum_{j'} a_{ij'}}.$$

Note that this update rule is the limit of (3.2) and (3.3) when ρ tends to 0. With a simple calculation, one can show the following properties:

1. $\mathbf{b}_{=0}^t$ will be equal to $\mathbf{b}_{=0}^*$ for $t > 0$, which, for $\rho_i = 0$ and $t > 0$, implies that:

$$\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) = 0; \tag{3.17}$$

2. $\mathbf{b}_{>0}$ is the spending at the market equilibrium if and only if $\mathbf{b}_{>0}$ is the minimum point of $\Phi(\cdot, \mathbf{b}_{=0}^*)$ (see Section 3.2.8.3 for more information).

We will show the following result.

Theorem 3.2.7.

$$\Phi(\mathbf{b}_{>0}^T, \mathbf{b}_{=0}^*) - \Phi(\mathbf{b}_{>0}^*, \mathbf{b}_{=0}^*) \le \frac{1}{T}\left(\sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\right).$$

We first show the following lemma.

Lemma 3.2.10. *For* $t > 0$, $\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}) \le \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0})$.

Proof. By Lemma 3.2.9, we know that for those i for which $\rho_i > 0$,

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ \langle \nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t\rangle + \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\}.$$

Therefore, by Lemma 3.2.1 with $x^+ = \mathbf{b}^{t+1}_{>0}$, $x = \mathbf{b}^t_{>0}$, and $y = \mathbf{b}^t_{>0}$,

$$\begin{aligned} &\langle \nabla_{\mathbf{b}_{>0}}\Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), \mathbf{b}^{t+1}_{>0} - \mathbf{b}^t_{>0}\rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^t) \\ &\quad \le \langle \nabla_{\mathbf{b}_{>0}}\Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), \mathbf{b}^t_{>0} - \mathbf{b}^t_{>0}\rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^t||\mathbf{b}_i^t) - \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^t||\mathbf{b}_i^{t+1}) \le 0. \end{aligned}$$

Also, we know that for $t > 0$, $\mathbf{b}^t_{=0} = \mathbf{b}^*_{=0}$. Then, for $t > 0$:

$$\langle \nabla\Phi(\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0}), (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}) - (\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0})\rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^t) \le 0.$$

Applying (3.16) with $\mathbf{b} = (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0})$ and $\mathbf{b}' = (\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0})$ yields:

$$\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}) - \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0}) \le 0,$$

which gives the result. □

Proof of Theorem 3.2.7. By Lemma 3.2.9,

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ \langle \nabla_{\mathbf{b}_i}\Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t\rangle + \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\}.$$

Then, by Lemma 3.2.1 with $x^+ = \mathbf{b}^{t+1}_{>0}$, $x = \mathbf{b}^*_{>0}$, and $y = \mathbf{b}^t_{>0}$,

$$
\begin{aligned}
&\langle \nabla_{\mathbf{b}_{>0}} \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), \mathbf{b}^{t+1}_{>0} - \mathbf{b}^t_{>0} \rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) \\
&\quad \leq \langle \nabla_{\mathbf{b}_{>0}} \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), \mathbf{b}^*_{>0} - \mathbf{b}^t_{>0} \rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i).
\end{aligned}
$$

This is equivalent to:

$$
\begin{aligned}
&\underbrace{\langle \nabla \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}) - (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}) \rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)}_{\texttt{LHS}} \\
&\quad \leq \underbrace{\langle \nabla \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}), (\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}) - (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}) \rangle + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i)}_{\texttt{RHS}}.
\end{aligned}
\tag{3.18}
$$

From the second inequality in (3.16) with $\mathbf{b} = (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0})$ and $\mathbf{b}' = (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0})$, the LHS term is lower bounded by:

$$
\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}) - \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}) - \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)
\tag{3.19}
$$

and from the first inequality with $\mathbf{b} = (\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})$ and $\mathbf{b}' = (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0})$, the RHS term is upper bounded by:

$$
\Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}) - \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}) - \underbrace{\sum_{i:\rho_i>0} \frac{1-\rho_i}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i)}_{B} + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i).
\tag{3.20}
$$

As $\texttt{LHS} \leq \texttt{RHS}$, and as B is positive, we have:

$$\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}) - \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) \leq \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) - \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t+1}). \tag{3.21}$$

Summing over t gives:

$$\begin{aligned}\sum_{t=0}^{T-1}\Big(\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*})\Big) &\leq \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{t=0}^{T-1}\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) \\ &= \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).\end{aligned}$$

The second equality holds because of (3.17).

By Lemma 3.2.10,

$$\Phi(\mathbf{b}_{>0}^{T}, \mathbf{b}_{=0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}) \leq \frac{1}{T}\left(\sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\right).$$

□

Theorem 3.2.8. *Suppose there is no buyer with a linear utility function. Let* $\sigma = \left(\min_{i:\rho_i>0}\left\{\frac{1}{\rho_i}\right\}\right)$. *Then,*

$$\Phi(\mathbf{b}_{>0}^{T}, \mathbf{b}_{=0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}) \leq \frac{\sigma - 1}{\sigma^T - 1}\left(\sum_{i:\rho_i>0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\right).$$

Proof. If there is no buyer with a linear utility function, then we do not drop B in (3.20). So, instead of (3.21), we have:

$$\begin{aligned}&\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}) - \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) \\ &\qquad \leq \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}) + \sum_{i:\rho_i>0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) - \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t+1}).\end{aligned} \tag{3.22}$$

Multiplying both sides by $\left(\min_{i:\rho_i>0}\left\{\frac{1}{\rho_i}\right\}\right)^t$ and summing over all t yields:

$$\sum_{t=0}^{T-1}\left(\min_{i:\rho_i>0}\left\{\frac{1}{\rho_i}\right\}\right)^t\left(\Phi(\mathbf{b}_{>0}^{t+1},\mathbf{b}_{=0}^*)-\Phi(\mathbf{b}_{>0}^*,\mathbf{b}_{=0}^*)\right)\leq\sum_{i:\rho_i>0}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)+\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

Recall that $\sigma=\left(\min_{i:\rho_i>0}\left\{\frac{1}{\rho_i}\right\}\right)$. By Lemma 3.2.10,

$$\Phi(\mathbf{b}_{>0}^T,\mathbf{b}_{=0}^*)-\Phi(\mathbf{b}_{>0}^*,\mathbf{b}_{=0}^*)\leq\frac{\sigma-1}{\sigma^T-1}\left(\sum_{i:\rho_i>0}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)+\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\right).$$

□

Proportional Response in the Complementary Domain with Cobb-Douglas Utility Functions The argument in this case is quite similar to the one in the previous subsection.

First, the following lemma shows the equivalence between mirror descent and proportional response (given by (3.3)) in the complementary domain.

Lemma 3.2.11. *For those complementary buyers such that $-\infty<\rho_i<0$, Proportional Response, which is the best response in this domain, is equivalent to the mirror descent update, given by:*

$$\mathbf{b}_i^{t+1}=\arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i}\left\{-\langle\nabla_{b_i}\Phi(\mathbf{b}^t),\mathbf{b}_i-\mathbf{b}_i^t\rangle+\frac{\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t)\right\};$$

and for those buyers with Leontief utility functions, Proportional Response, which is also the best response in this domain, is equivalent to the mirror descent update, given by:

$$\mathbf{b}_i^{t+1}=\arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i}\left\{-\langle\nabla_{b_i}\Phi(\mathbf{b}^t),\mathbf{b}_i-\mathbf{b}_i^t\rangle+\mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t)\right\}.$$

Similar to the substitutes domain, buyers with Cobb-Douglas utility functions use the following update rule:

$$b_{ij}^{t+1} = e_i \frac{a_{ij}}{\sum_{j'} a_{ij'}}.$$

We also have the following properties:

1. $\mathbf{b}_{=0}^{t}$ will be equal to $\mathbf{b}_{=0}^{*}$ for $t > 0$, which, for $\rho_i = 0$ and $t > 0$, implies that:

$$\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) = 0; \tag{3.23}$$

2. $\mathbf{b}_{<0}$ is the spending at the market equilibrium if and only if $\mathbf{b}_{<0}$ is the maximum point of $\Phi(\cdot, \mathbf{b}_{=0}^*)$ (see Section 3.2.8.3 for more information).

We show the following result.

Theorem 3.2.9.

$$\begin{aligned}
&\Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^*) - \Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^T) \\
&\qquad \le \frac{1}{T}\Big(\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\Big).
\end{aligned}$$

We first show the following lemma.

Lemma 3.2.12. *For $t > 0$, $\Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^t) \le \Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1})$.*

Proof. By Lemma 3.2.11, we know that for those i for which $-\infty < \rho_i < 0$,

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t) \right\},$$

and for those i for which $\rho_i = -\infty$,

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t) \}.$$

Therefore, by Lemma 3.2.1, with $x^+ = \mathbf{b}_{<0}^{t+1}$, $x = \mathbf{b}_{<0}^{t}$, and $y = \mathbf{b}_{<0}^{t}$,

$$
\begin{aligned}
&-\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{<0}^{t+1} - \mathbf{b}_{<0}^t \rangle + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) \\
&\quad \leq -\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{<0}^{t} - \mathbf{b}_{<0}^t \rangle + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t} || \mathbf{b}_i^t) \\
&\qquad - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t} || \mathbf{b}_i^{t+1}) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t} || \mathbf{b}_i^t) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t} || \mathbf{b}_i^{t+1}) \\
&\quad \leq 0.
\end{aligned}
$$

We know that for $t > 0$, $\mathbf{b}_{=0}^t = \mathbf{b}_{=0}^*$. Therefore, for $t > 0$,

$$
\begin{aligned}
-\langle \nabla \Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^t), (\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) - (\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^t) \rangle + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) \\
\leq 0.
\end{aligned}
$$

Using (3.16) yields:

$$
\Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^t) - \Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) \leq 0.
$$

□

Proof of Theorem 3.2.9. First, by Lemma 3.2.11, we know that for those i for which $-\infty < \rho_i < 0$,

$$
\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \left\{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\},
$$

and for those i for which $\rho_i = -\infty$,

$$
\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \},
$$

Therefore, by Lemma 3.2.1, with $x^+ = \mathbf{b}_{<0}^{t+1}$, $x = \mathbf{b}_{<0}^*$, and $y = \mathbf{b}_{<0}^t$,

$$
\begin{aligned}
&-\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{<0}^{t+1} - \mathbf{b}_{<0}^t \rangle + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) \\
&\quad \leq -\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{<0}^* - \mathbf{b}_{<0}^t \rangle + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) \\
&\qquad\qquad - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}).
\end{aligned}
$$

This is equivalent to:

$$
\begin{aligned}
&\underbrace{\begin{aligned}&-\langle \nabla \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), (\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) - (\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) \rangle \\ &+ \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t)\end{aligned}}_{\texttt{LHS}} \\
&\qquad \underbrace{\begin{aligned}&\leq -\langle \nabla \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), (\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^*) - (\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) \rangle \\ &\qquad + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}) \\ &\qquad + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}).\end{aligned}}_{\texttt{RHS}}
\end{aligned} \tag{3.24}
$$

From the first inequality in (3.16), the LHS term is lower bounded by:

$$
\begin{aligned}
&-\Phi(\mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) + \Phi(\mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) \\
&\qquad + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t)
\end{aligned} \tag{3.25}
$$

and from the second inequality, the RHS term is upper bounded by:

$$
\begin{aligned}
&- \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) + \Phi(\mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}) + \underbrace{\sum_{i:-\infty<\rho_i<0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i)}_{D} + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) \\
&\quad + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^{t+1}_i) \\
&\quad + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^{t+1}_i).
\end{aligned} \tag{3.26}
$$

As $\mathtt{LHS} \leq \mathtt{RHS}$, and as D is negative, we have:

$$
\begin{aligned}
-\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) \quad &\leq -\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) \\
&\quad + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) - \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^{t+1}_i) \\
&\quad + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^{t+1}_i).
\end{aligned} \tag{3.27}
$$

Summing over all t yields:

$$
\begin{aligned}
\sum_{t=0}^{T-1} \Big(\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0})\Big) &\leq \sum_{t=0}^{T-1} \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) \\
&\qquad + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) \\
&\leq \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) \\
&\qquad + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i).
\end{aligned}
$$

The second inequality holds because of (3.23).

By Lemma 3.2.12,

$$\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^T_{<0}) \leq \frac{1}{T}\Big(\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)\Big).$$

□

Theorem 3.2.10. *Suppose there is no buyer with a Leontief utility function. Let* $\sigma = \min_{i:\rho_i<0}\left\{\frac{\rho_i-1}{\rho_i}\right\}$. *Then:*

$$\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^T_{<0}) \leq \frac{\sigma - 1}{\sigma^T - 1}\Big(\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i<0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)\Big).$$

Proof. If there is no buyer with a Leontief utility function, then we do not drop D in (3.26). Therefore, instead of (3.27), we have:

$$\begin{aligned} -\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) \quad & \leq -\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) \\ & \quad + \sum_{i:\rho_i<0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^t_i) - \sum_{i:\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^{t+1}_i). \end{aligned} \tag{3.28}$$

Recall $\sigma = \min_{i:\rho_i<0}\left\{\frac{\rho_i-1}{\rho_i}\right\}$. Then, multiplying both sides by σ^t and summing over t yields:

$$\sum_{t=0}^{T-1} \sigma^t \Big(\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0})\Big) \leq \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i<0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i).$$

By Lemma 3.2.12,

$$\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^T_{<0}) \leq \frac{\sigma - 1}{\sigma^T - 1}\Big(\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i) + \sum_{i:\rho_i<0} \mathrm{KL}(\mathbf{b}^*_i||\mathbf{b}^0_i)\Big).$$

□

Proportional Response with the Entire CES Range Formally, in this case, Damped Proportional Response is defined as follows:

$$
\begin{aligned}
b_{ij}^{t+1} &= e_i \frac{\left[b_{ij}^t \cdot a_{ij}\left(\frac{b_{ij}^t}{p_j^t}\right)^{\rho_i}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot a_{ik}\left(\frac{b_{ik}^t}{p_k^t}\right)^{\rho_i}\right]^{\frac{1}{2}}}, && \text{for } \rho_i > 0; \\
b_{ij}^{t+1} &= e_i \frac{\left[b_{ij}^t \cdot a_{ij}^t\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot a_{ik}^t\right]^{\frac{1}{2}}}, && \text{for } \rho_i = 0; \\
b_{ij}^{t+1} &= e_i \frac{\left[b_{ij}^t \cdot \left(\frac{a_{ij}}{p_j^{t\,\rho_i}}\right)^{\frac{1}{1-\rho_i}}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot \left(\frac{a_{ik}}{p_k^{t\,\rho_i}}\right)^{\frac{1}{1-\rho_i}}\right]^{\frac{1}{2}}}, && \text{for } -\infty < \rho_i < 0; \\
b_{ij}^{t+1} &= e_i \frac{\left[b_{ij}^t \cdot \left(\frac{c_{ij}}{p_j^t}\right)^{-1}\right]^{\frac{1}{2}}}{\sum_k \left[b_{ik}^t \cdot \left(\frac{c_{ik}}{p_k^t}\right)^{-1}\right]^{\frac{1}{2}}}, && \text{for } \rho_i = -\infty; \\
p_j^{t+1} &= \sum_j b_{ij}^{t+1}
\end{aligned}
$$

Similarly to the Cobb-Douglas free domain, we have the following observations.

Lemma 3.2.13. *If $\rho_i > 0$ for buyer i, then Damped Proportional Response is equivalent to mirror descent with a halved step size, defined as follows:*

$$
\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \left\{ \langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\};
$$

if $-\infty < \rho_i < 0$ for buyer i, then Damped Proportional Response is equivalent to mirror descent with a halved step size, defined as follows:

$$
\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \left\{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \right\}.
$$

and if $\rho_i = -\infty$ for buyer i, then Damped Proportional Response is equivalent to mirror *descent with a halved step size, defined as follows:*

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + 2\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \}.$$

Proof. By calculation. □

However, we find that for the buyers with Cobb-Douglas utility functions, their updating rule cannot be written in mirror descent form. Instead, we make a separate argument for these buyers.

Let $\mathbf{b}_i^*$ be the equilibrium spending of buyer i. If $\rho_i = 0$ for buyer i, then her updating rule only depends on her previous spending and her preferences, and it is independent of the other buyers. Consequently, as we show in the following lemma, the convergence rate of Damped Proportional Response for the buyers with Cobb-Douglas utilities will be fast.

Lemma 3.2.14. *If $\rho_i = 0$ for buyer i, then:*

$$\mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^0) \geq \sum_{t=1}^{T} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) \quad \textit{and} \quad \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) \geq 2\mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}).$$

Proof. First, we want to show that for any buyer i with $\rho_i = 0$, Damped Proportional Response is equivalent to mirror descent on $\Psi(\mathbf{b}_i) = -\sum_j b_{ij} \log \frac{a_{ij}}{b_{ij}}$[3] with halved step size:

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \{ \langle \nabla \Psi(\mathbf{b}_i^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + 2\mathrm{KL}(\mathbf{b}_i || \mathbf{b}_i^t) \},$$

Note that

$$\nabla_{b_{ij}} \Psi(\mathbf{b}_i) = -\log \frac{a_{ij}}{b_{ij}} + 1.$$

Then, by calculation, $b_{ij}^{t+1} = e_i \frac{(b_{ij}^t \cdot a_{ij})^{\frac{1}{2}}}{\sum_k (b_{ik}^t \cdot a_{ik})^{\frac{1}{2}}}$, which is exactly the Damped Proportional Re-

[3]By simple calculation, the optimal point of this function is $b_{ij}^* = e_i \frac{a_{ij}}{\sum_{j'} a_{ij'}}$, which is the optimal spending in the market equilibrium.

sponse update rule.

Furthermore, it is easy to see that Ψ is a convex function and it satisfies the following equality:

$$\Psi(\mathbf{b}_i) - \Psi(\mathbf{b}'_i) - \langle \nabla\Psi(b'_i), \mathbf{b}_i - \mathbf{b}'_i \rangle = \mathrm{KL}(\mathbf{b}_i, \mathbf{b}'_i). \tag{3.29}$$

Therefore, setting $\mathbf{b}_i = \mathbf{b}_i^{t+1}$ and $\mathbf{b}'_i = \mathbf{b}_i^t$ gives

$$\Psi(\mathbf{b}_i^{t+1}) - \Psi(\mathbf{b}_i^t) = \langle \nabla\Psi(\mathbf{b}_i^t), \mathbf{b}_i^{t+1} - \mathbf{b}_i^t \rangle + 2\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^t) - \mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^t)$$

and then by Lemma 3.2.1 with $g(\cdot) = \langle \nabla\Psi(\mathbf{b}_i^t), \cdot - \mathbf{b}_i^t \rangle$, $x^+ = \mathbf{b}_i^{t+1}$, $x = \mathbf{b}_i^*$, $y = \mathbf{b}_i^t$, and $d(\cdot, \cdot) = 2\mathrm{KL}(\cdot||\cdot)$:

$$\Psi(\mathbf{b}_i^{t+1}) - \Psi(\mathbf{b}_i^t) \leq \langle \nabla\Psi(\mathbf{b}_i^t), \mathbf{b}_i^* - \mathbf{b}_i^t \rangle + 2\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) - 2\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t+1}) - \mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^t). \tag{3.30}$$

Setting $\mathbf{b}_i = \mathbf{b}_i^*$, $\mathbf{b}'_i = \mathbf{b}_i^t$ in (3.29) gives:

$$\Psi(\mathbf{b}_i^t) - \Psi(\mathbf{b}_i^*) = -\langle \nabla\Psi(\mathbf{b}_i^t), \mathbf{b}_i^* - \mathbf{b}_i^t \rangle - \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t).$$

And combining this with (3.30) gives:

$$\Psi(\mathbf{b}_i^{t+1}) - \Psi(\mathbf{b}_i^*) \leq \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) - 2\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t+1}). \tag{3.31}$$

Since $\Psi(\cdot)$ is a convex function and $\mathbf{b}_i^*$ is the minimum point for Ψ, (3.31) implies:

$$\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) \geq 2\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^{t+1}). \tag{3.32}$$

Note this inequality holds for any $t \geq 0$. So, for any T,

$$\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) \geq \sum_{t=1}^{T} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t).$$

□

For the next part, recall that $\mathbf{b}_{>0}$, $\mathbf{b}_{=0}$ and $\mathbf{b}_{<0}$ denote the spending of those buyers with $\rho_i > 0$, $\rho_i = 0$, and $\rho_i < 0$, respectively, and that we rewrote $\Phi(\mathbf{b})$ as $\Phi(\mathbf{b}_{>0}, \mathbf{b}_{=0}, \mathbf{b}_{<0})$.

With a simple calculation one can show:

- For fixed $\mathbf{b}_{=0}$ and $\mathbf{b}_{<0}$, $\Phi(\cdot, \mathbf{b}_{=0}, \mathbf{b}_{<0})$ is a convex function.
- For fixed $\mathbf{b}_{>0}$ and $\mathbf{b}_{=0}$, $\Phi(\mathbf{b}_{>0}, \mathbf{b}_{=0}, \cdot)$ is a concave function.
- Let $\mathbf{b}^*_{>0}$, $\mathbf{b}^*_{=0}$ and $\mathbf{b}^*_{<0}$ be the market equilibrium of the Fisher market; then
  - $\mathbf{b}^*_{>0}$ minimizes $\Phi(\cdot, \mathbf{b}^*_{=0}, \mathbf{b}^*_{<0})$;
  - $\mathbf{b}^*_{<0}$ maximizes $\Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}, \cdot)$.

Theorem 3.2.11. *Damped Proportional Response converges to the equilibrium with a convergence rate of:*

$$\sum_{t=1}^{T} \left[\Phi(\mathbf{b}^t_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^t_{<0}) \right]$$
$$\leq 4 \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)$$
$$+ \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0).$$

Proof. First, let's look at $\mathbf{b}^t_{>0}$. By Lemma 3.2.13, we know that for those i for which $\rho_i > 0$,

$$\mathbf{b}_i^{t+1} = \arg\min_{\mathbf{b}_i : \sum_j b_{ij} = e_i} \left\{ \langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}_i^t \rangle + \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i||\mathbf{b}_i^t) \right\}.$$

Therefore, by Lemma 3.2.1 with $x^+ = \mathbf{b}_{>0}^{t+1}$, $x = \mathbf{b}_{>0}^*$, $y = \mathbf{b}_{>0}^t$,

$$\begin{aligned}
&\langle \nabla_{\mathbf{b}_{>0}} \Phi(\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{>0}^{t+1} - \mathbf{b}_{>0}^t \rangle + \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t) \\
&\quad \leq \langle \nabla_{\mathbf{b}_{>0}} \Phi(\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), \mathbf{b}_{>0}^* - \mathbf{b}_{>0}^t \rangle + \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) - \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1}).
\end{aligned}$$

This is equivalent to:

$$\begin{aligned}
&\underbrace{\langle \nabla \Phi(\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), (\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) - (\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) \rangle + \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t)}_{\texttt{LHS}} \\
&\quad \underbrace{\begin{aligned}\leq \langle \nabla \Phi(\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t), (\mathbf{b}_{>0}^*, \mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) - (\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) \rangle \\
+ \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) - \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^{t+1})\end{aligned}}_{\texttt{RHS}}.
\end{aligned} \tag{3.33}$$

From the second inequality in (3.16), the LHS term is lower bounded by:

$$\begin{aligned}
&\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^*, \mathbf{b}_{<0}^{t+1}) - \Phi(\mathbf{b}_{>0}^t, \mathbf{b}_{=0}^t, \mathbf{b}_{<0}^t) - \underbrace{\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t)}_{A} \\
&\qquad - \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^* || \mathbf{b}_i^t) + \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}_i^{t+1} || \mathbf{b}_i^t)
\end{aligned} \tag{3.34}$$

and from the first inequality, the RHS term is upper bounded by:

$$\Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) - \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}) - \underbrace{\sum_{i:\rho_i>0} \frac{1-\rho_i}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i)}_{B}$$
$$- \sum_{i:-\infty<\rho_i<0} \frac{1-\rho_i}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)$$
$$+ \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i). \tag{3.35}$$

As $\texttt{LHS} \leq \texttt{RHS}$, as the portion of A for $\rho_i < 0$ is negative, and as B is positive, we have

$$\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i)$$
$$\leq \Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)$$
$$+ \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{2}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i). \tag{3.36}$$

Next, let's look at $\mathbf{b}^t_{<0}$. The argument used here is similar to that for $\mathbf{b}^t_{>0}$. First, by Lemma 3.2.13, we know that for those i for which $-\infty < \rho_i < 0$,

$$\mathbf{b}^{t+1}_i = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \left\{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}^t_i \rangle + \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}_i || \mathbf{b}^t_i) \right\},$$

and for those i for which $\rho_i = -\infty$,

$$\mathbf{b}^{t+1}_i = \arg\min_{\mathbf{b}_i:\sum_j b_{ij}=e_i} \{ -\langle \nabla_{\mathbf{b}_i} \Phi(\mathbf{b}^t), \mathbf{b}_i - \mathbf{b}^t_i \rangle + 2\mathrm{KL}(\mathbf{b}_i || \mathbf{b}^t_i) \},$$

Therefore, by Lemma 3.2.1,

$$
\begin{aligned}
&-\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}), \mathbf{b}^{t+1}_{<0} - \mathbf{b}^t_{<0} \rangle + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) \\
&\quad \leq -\langle \nabla_{\mathbf{b}_{<0}} \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}), \mathbf{b}^*_{<0} - \mathbf{b}^t_{<0} \rangle + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) \\
&\qquad - \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i) + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i).
\end{aligned}
$$

This is equivalent to:

$$
\begin{aligned}
&\underbrace{\begin{aligned}&-\langle \nabla \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}), (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) - (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}) \rangle \\ &+ \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)\end{aligned}}_{\texttt{LHS}} \\
&\quad \underbrace{\begin{aligned}&\leq -\langle \nabla \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}), (\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - (\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}) \rangle \\ &\qquad + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i) \\ &\qquad + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^t_i) - \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^*_i || \mathbf{b}^{t+1}_i).\end{aligned}}_{\texttt{RHS}}
\end{aligned} \tag{3.37}
$$

From the first inequality in (3.16), the LHS term is lower bounded by:

$$
\begin{aligned}
&-\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^*_{=0}, \mathbf{b}^{t+1}_{<0}) + \Phi(\mathbf{b}^t_{>0}, \mathbf{b}^t_{=0}, \mathbf{b}^t_{<0}) - \underbrace{\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{\rho_i - 1}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)}_{C} - \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) \\
&\quad + \sum_{i:\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i} \mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i)
\end{aligned} \tag{3.38}
$$

and from the second inequality, the RHS term is upper bounded by:

$$
\begin{aligned}
&-\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{*}) + \Phi(\mathbf{b}_{>0}^{t}, \mathbf{b}_{=0}^{t}, \mathbf{b}_{<0}^{t}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^{t}) + \underbrace{\sum_{i:-\infty<\rho_i<0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t})}_{D} \\
&\quad + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}) \\
&\quad + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}). \qquad (3.39)
\end{aligned}
$$

As $\texttt{LHS} \leq \texttt{RHS}$, as the portion of C for $\rho_i > 0$ is negative, and as D is negative, we have:

$$
\begin{aligned}
&-\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{t+1}) + \sum_{i:-\infty<\rho_i<0} \frac{\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^{t}) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^{t}) \\
&\quad \leq -\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{*}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{t+1}||\mathbf{b}_i^{t}) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) \\
&\quad\quad + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}) \\
&\quad\quad + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}). \qquad (3.40)
\end{aligned}
$$

Summing (3.36) and (3.40) gives:

$$
\begin{aligned}
&\Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{t+1}) \\
&\quad \leq 2\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}) \\
&\quad\quad + \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}) \\
&\quad\quad + \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t}) - \sum_{i:\rho_i=-\infty} 2\mathrm{KL}(\mathbf{b}_i^{*}||\mathbf{b}_i^{t+1}).
\end{aligned}
$$

Summing over all t yields:

$$\sum_{t=0}^{T-1} \Phi(\mathbf{b}_{>0}^{t+1}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{t+1})$$

$$\leq 2\sum_{t=0}^{T-1} \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^t) + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)$$

$$\leq 4\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:-\infty<\rho_i<0} \frac{2(\rho_i-1)}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i=-\infty} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0),$$

where Lemma 3.2.14 is used in bounding the first sum on the right hand side. □

Theorem 3.2.12. *Suppose there is no buyer with either a linear utility or a Leontief utility. Let*

$$\sigma = \min\left\{\min_{i:\rho_i>0}\left\{\frac{2}{1+\rho_i}\right\},\ \min_{i:\rho_i<0}\left\{\frac{2(\rho_i-1)}{2\rho_i-1}\right\}\right\} \quad \textit{(and so } 1<\sigma<2\textit{);}$$

then Damped Proportional Response converges to the equilibrium with a convergence rate of:

$$\Phi(\mathbf{b}_{>0}^{T}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{*}) - \Phi(\mathbf{b}_{>0}^{*}, \mathbf{b}_{=0}^{*}, \mathbf{b}_{<0}^{T}) \leq \frac{1}{\sigma^{T-1}}\left[\frac{4}{2-\sigma}\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i<0} \frac{2\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0) + \sum_{i:\rho_i>0} \frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}_i^*||\mathbf{b}_i^0)\right].$$

Proof. In this case, on combining (3.34) and (3.35), (3.36) is changed to:

$$
\begin{aligned}
&\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{t+1}_{<0}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i=0} \mathrm{KL}(b^*_i || b^t_i) \\
&\quad \leq \Phi(\mathbf{b}^{*}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{t+1}_{<0}) + \sum_{i:\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i>0} \frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) \\
&\qquad - \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^{t+1}_i). \qquad (3.41)
\end{aligned}
$$

Also, on combining (3.38) and (3.39), (3.40) is changed to:

$$
\begin{aligned}
&-\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{t+1}_{<0}) + \sum_{i:\rho_i<0} \frac{\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) \\
&\quad \leq -\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{*}_{<0}) + \sum_{i:\rho_i>0} \frac{1}{\rho_i}\mathrm{KL}(\mathbf{b}^{t+1}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) \\
&\qquad + \sum_{i:\rho_i<0} \frac{2\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^{t+1}_i). \qquad (3.42)
\end{aligned}
$$

Combining (3.41) and (3.42) yields:

$$
\begin{aligned}
&\Phi(\mathbf{b}^{t+1}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{*}_{<0}) - \Phi(\mathbf{b}^{*}_{>0}, \mathbf{b}^{*}_{=0}, \mathbf{b}^{t+1}_{<0}) \\
&\quad \leq 2\sum_{i:\rho_i=0} \mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) + \sum_{i:\rho_i<0} \frac{2\rho_i - 1}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i<0} \frac{2(\rho_i - 1)}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^{t+1}_i) \\
&\qquad + \sum_{i:\rho_i>0} \frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^t_i) - \sum_{i:\rho_i>0} \frac{2}{\rho_i}\mathrm{KL}(\mathbf{b}^{*}_i || \mathbf{b}^{t+1}_i).
\end{aligned}
$$

Recall that $\sigma = \min\left\{\min_{i:\rho_i>0}\left\{\frac{2}{1+\rho_i}\right\}\ ,\ \min_{i:\rho_i<0}\left\{\frac{2(\rho_i-1)}{2\rho_i-1}\right\}\right\}$; then,

$$\begin{aligned}\sum_{t=0}^{T-1}\sigma^t\Big(\Phi(\mathbf{b}^{t+1}_{>0},\mathbf{b}^*_{=0},\mathbf{b}^*_{<0})-\Phi(\mathbf{b}^*_{>0},\mathbf{b}^*_{=0},\mathbf{b}^{t+1}_{<0})\Big)&\\ \leq 2\sum_{t=0}^{T-1}\sum_{i:\rho_i=0}\sigma^t\cdot\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^t_i)&\\ +\sum_{i:\rho_i<0}\frac{2\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)+\sum_{i:\rho_i>0}\frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i).&\end{aligned}$$

By (3.32) and $1<\sigma<2$,

$$\begin{aligned}\sum_{t=0}^{T-1}\sigma^t\Big(\Phi(\mathbf{b}^{t+1}_{>0},\mathbf{b}^*_{=0},\mathbf{b}^*_{<0})-\Phi(\mathbf{b}^*_{>0},\mathbf{b}^*_{=0},\mathbf{b}^{t+1}_{<0})\Big)&\\ \leq\frac{4}{2-\sigma}\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)+\sum_{i:\rho_i<0}\frac{2\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)+\sum_{i:\rho_i>0}\frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i).&\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi(\mathbf{b}^T_{>0},\mathbf{b}^*_{=0},\mathbf{b}^*_{<0})-\Phi(\mathbf{b}^*_{>0},\mathbf{b}^*_{=0},\mathbf{b}^T_{<0})&\\ \leq\frac{1}{\sigma^{T-1}}\left[\frac{4}{2-\sigma}\sum_{i:\rho_i=0}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)+\sum_{i:\rho_i<0}\frac{2\rho_i-1}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)+\sum_{i:\rho_i>0}\frac{1+\rho_i}{\rho_i}\mathrm{KL}(\mathbf{b}^*_i\|\mathbf{b}^0_i)\right].&\end{aligned}$$

□

3.2.8.2 Relationship between the Eisenberg-Gale Program and our Potential Function

In this section, we show a relationship between our potential function and the objective functions in the Eisenberg-Gale convex program and its dual program.

Let $u_i(\mathbf{x}_i)$ be the utility of buyer i when the allocation is $\mathbf{x}_i$. Note that $u_i(\mathbf{x}_i)=(\sum_j a_{ij}\cdot x_{ij}^{\rho_i})^{\frac{1}{\rho_i}}$ for $1\geq\rho_i>0$ and $0>\rho_i>-\infty$. For $\rho_i=0$, $u_i(\mathbf{x}_i)=\prod_j x_{ij}^{a_{ij}}$ with $\sum_j a_{ij}=1$. For

$\rho_i = -\infty$, $u_i(\mathbf{x}_i) = \min_j \left\{ \frac{x_{ij}}{c_{ij}} \right\}$. Our potential function is:

$p_j(\mathbf{b}) = \sum_i b_{ij}$,

$$\Phi(\mathbf{b}) = -\sum_{i:\rho_i \neq \{0,-\infty\}} \frac{1}{\rho_i} \sum_j b_{ij} \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}} - \sum_{i:\rho_i = -\infty} \sum_j b_{ij} \log \frac{b_{ij}}{c_{ij} p_j(\mathbf{b})} + \sum_{i:\rho_i = 0} \sum_j b_{ij} \log p_j(\mathbf{b}).$$

Recall that the goal is to minimize $\Phi(\mathbf{b})$ in the substitutes domain and maximize $\Phi(\mathbf{b})$ in the complementary domain.

The objective function for the Eisenberg-Gale program is:

$$\Psi(\mathbf{x}) = \sum_i e_i \log u_i(\mathbf{x}_i).$$

Recall that the goal is to maximize $\Psi(\mathbf{x})$.

The objective function for the dual of the Eisenberg-Gale convex program is:

$$\Upsilon(\mathbf{p}) = \max_{\mathbf{x}} \left(\sum_i e_i \log u_i(\mathbf{x}_i) + \sum_j p_j \left(1 - \sum_i x_{ij} \right) \right).$$

Recall that the goal is to minimize $\Upsilon(\mathbf{p})$.

Substitutes Domain In the substitutes domain ($\rho_i \geq 0$), let $\mathbf{b}$ be the spending of the buyers; recall that it satisfies $\sum_j b_{ij} = e_i$ for all i. We consider the corresponding allocation $\mathbf{x}(\mathbf{b})$ and the Eisenberg-Gale program, in which $x_{ij} = \frac{b_{ij}}{p_j(\mathbf{b})}$ and $p_j(\mathbf{b}) = \sum_h b_{hj}$. We have the following result.

Theorem 3.2.13.

$$\Psi(\mathbf{x}(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})) - \Psi(\mathbf{x}(\mathbf{b}_{>0}, \mathbf{b}^*_{=0})) \leq \Phi(\mathbf{b}_{>0}, \mathbf{b}^*_{=0}) - \Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}).$$

Proof. First, as $\sum_j b_{ij} = e_i$, using the concavity of the log function yields:

$$\begin{aligned}
\Psi(\mathbf{x}(\mathbf{b})) &= \sum_{i:\rho_i>0} \frac{e_i}{\rho_i} \log\left(\sum_j a_{ij}\left(\frac{b_{ij}}{\mathbf{p}(\mathbf{b})}\right)^{\rho_i}\right) + \sum_{i:\rho_i=0}\sum_j e_i a_{ij} \log \frac{b_{ij}}{\mathbf{p}(\mathbf{b})} \\
&\geq \sum_{i:\rho_i>0} \frac{e_i}{\rho_i} \sum_j \frac{b_{ij}}{e_i} \log \frac{a_{ij} b_{ij}^{\rho_i - 1} e_i}{(\mathbf{p}(\mathbf{b}))^{\rho_i}} + \sum_{i:\rho_i=0}\sum_j e_i a_{ij} \log \frac{b_{ij}}{\mathbf{p}(\mathbf{b})} \\
&= \sum_{ij:\rho_i>0} \frac{b_{ij}}{\rho_i} \log \frac{a_{ij} b_{ij}^{\rho_i-1}}{(\mathbf{p}(\mathbf{b}))^{\rho_i}} + \sum_{i:\rho_i>0} \frac{e_i}{\rho_i} \log e_i + \sum_{i:\rho_i=0}\sum_j e_i a_{ij} \log \frac{b_{ij}}{\mathbf{p}(\mathbf{b})}.
\end{aligned}$$

Note that if for each i such that $\rho_i > 0$, $a_{ij}\frac{b_{ij}^{\rho_i-1}}{\sum_h b_{hj}^{\rho_i}}$ are the same for all j with $b_{ij} > 0$, then the inequality above will become an equality. Also, at the market equilibrium $\mathbf{b}^*$, this condition holds. Therefore,

$$\begin{aligned}
&\Psi(\mathbf{x}(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})) - \Psi(\mathbf{x}(\mathbf{b}_{>0}, \mathbf{b}^*_{=0})) \\
&\leq \sum_{ij:\rho_i>0} \frac{b^*_{ij}}{\rho_i} \log \frac{a_{ij} {b^*_{ij}}^{\rho_i-1}}{(\mathbf{p}(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}))^{\rho_i}} + \sum_{i:\rho_i>0} \frac{e_i}{\rho_i} \log e_i + \sum_{i:\rho_i=0}\sum_j e_i a_{ij} \log \frac{b^*_{ij}}{\mathbf{p}(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0})} \\
&\quad - \sum_{ij:\rho_i>0} \frac{b_{ij}}{\rho_i} \log \frac{a_{ij} b_{ij}^{\rho_i-1}}{(\mathbf{p}(\mathbf{b}_{>0}, \mathbf{b}^*_{=0}))^{\rho_i}} - \sum_{i:\rho_i>0} \frac{e_i}{\rho_i} \log e_i - \sum_{i:\rho_i=0}\sum_j e_i a_{ij} \log \frac{b^*_{ij}}{\mathbf{p}(\mathbf{b}_{>0}, \mathbf{b}^*_{=0})} \\
&= -\Phi(\mathbf{b}^*_{>0}, \mathbf{b}^*_{=0}) + \Phi(\mathbf{b}_{>0}, \mathbf{b}^*_{=0}).
\end{aligned}$$

The last equality follows because $b^*_{ij} = e_i a_{ij}$ for $\rho_i = 0$. □

Complementary Domain In the complementary domain ($\rho_i \leq 0$), again $\mathbf{b}$ satisfies $\sum_j b_{ij} = e_i$ for all i. Here we consider the corresponding price $\mathbf{p}(\mathbf{b})$ and the dual of the Eisenberg-Gale program, in which $p_j(\mathbf{b}) = \sum_h b_{hj}$. We have the following result.

Theorem 3.2.14.

$$\Upsilon(\mathbf{p}(\mathbf{b}^*_{=0}, \mathbf{b}_{<0})) - \Upsilon(\mathbf{p}(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0})) \leq \Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \Phi(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}).$$

Proof. In the proof of Lemma 5.1 in [CT16], it was shown that the maximum point $\mathbf{x}$ in

$\Upsilon(\mathbf{p})$ satisfies $\sum_j x_{ij}p_j = e_i$ for all i. Therefore,

$$\Upsilon(\mathbf{p}) = \max_{\mathbf{x}:\forall i(\sum_j x_{ij}p_j=e_i)} \sum_{i:0>\rho_i>-\infty} e_i \log\Big(\sum_j a_{ij}x_{ij}^{\rho_i}\Big)^{\frac{1}{\rho_i}} + \sum_{i:\rho_i=-\infty} e_i \log \min_j \Big\{\frac{x_{ij}}{c_{ij}}\Big\} + \sum_{i:\rho_i=0}\sum_j e_i a_{ij}\log x_{ij} + \sum_j p_j - \sum_i e_i.$$

Let $b_{ij} = x_{ij}p_j$. Then,

$$\Upsilon(\mathbf{p}) = \max_{\mathbf{b}:\forall i(\sum_j b_{ij}=e_i)} \sum_{i:0>\rho_i>-\infty} e_i \log\Big(\sum_j a_{ij}\Big(\frac{b_{ij}}{p_j}\Big)^{\rho_i}\Big)^{\frac{1}{\rho_i}} + \sum_{i:\rho_i=-\infty} e_i \log \min \Big\{\frac{b_{ij}}{p_j c_{ij}}\Big\} + \sum_{i:\rho_i=0}\sum_j e_i a_{ij}\log \frac{b_{ij}}{p_j} + \sum_j p_j - \sum_i e_i.$$

Let $\mathbf{b}(\mathbf{p})$ be the spending that maximizes

$$\sum_{i:0>\rho_i>-\infty} e_i \log\Big(\sum_j a_{ij}\Big(\frac{b_{ij}}{p_j}\Big)^{\rho_i}\Big)^{\frac{1}{\rho_i}} + \sum_{i:\rho_i=-\infty} e_i \log \min \Big\{\frac{b_{ij}}{p_j c_{ij}}\Big\} + \sum_{i:\rho_i=0}\sum_j e_i a_{ij}\log \frac{b_{ij}}{p_j}$$

under the constraint $\forall i(\sum_j b_{ij} = e_i)$, so

$$\begin{aligned}
\Upsilon(\mathbf{p}) &= \sum_{i:0>\rho_i>-\infty} e_i \log\Big(\sum_j a_{ij}\Big(\frac{b_{ij}(\mathbf{p})}{p_j}\Big)^{\rho_i}\Big)^{\frac{1}{\rho_i}} + \sum_{i:\rho_i=-\infty} e_i \log \min \Big\{\frac{b_{ij}(\mathbf{p})}{p_j c_{ij}}\Big\} \\
&\quad + \sum_{i:\rho_i=0}\sum_j e_i a_{ij}\log \frac{b_{ij}(\mathbf{p})}{p_j} + \sum_j p_j - \sum_i e_i \\
&= \sum_{i:0>\rho_i>-\infty} e_i \log\Big(\sum_j a_{ij}\Big(\frac{b_{ij}(\mathbf{p})}{p_j}\Big)^{\rho_i}\Big)^{\frac{1}{\rho_i}} + \sum_{i:\rho_i=-\infty} e_i \log \min \Big\{\frac{b_{ij}(\mathbf{p})}{p_j c_{ij}}\Big\} \\
&\quad + \sum_{i:\rho_i=0}\sum_j b^*_{ij}\log \frac{b^*_{ij}}{p_j} + \sum_j p_j - \sum_i e_i. \qquad (3.43)
\end{aligned}$$

The second equality holds because, for those buyers with Cobb-Douglas utility functions, their optimal spending is always equal to $b^*_{ij} = e_i a_{ij}$ which is independent of the prices. With a simple calculation, one can show the following:

1. For those i such that $0 > \rho_i > -\infty$, $a_{ij}\frac{b_{ij}(\mathbf{p})^{\rho_i-1}}{p_j^{\rho_i}}$ are the same for different j by the

definition of $\mathbf{b}(\mathbf{p})$. Therefore,

$$\begin{aligned} e_i \log \Big(\sum_j a_{ij} \Big(\frac{b_{ij}(\mathbf{p})}{p_j} \Big)^{\rho_i} \Big)^{\frac{1}{\rho_i}} &= \frac{e_i}{\rho_i} \sum_j \frac{b_{ij}(\mathbf{p})}{e_i} \log e_i a_{ij} \frac{b_{ij}(\mathbf{p})^{\rho_i - 1}}{p_j^{\rho_i}} \\ &= \frac{1}{\rho_i} \sum_j b_{ij}(\mathbf{p}) \log a_{ij} \frac{b_{ij}(\mathbf{p})^{\rho_i - 1}}{p_j^{\rho_i}} + \frac{e_i}{\rho_i} \log e_i. \end{aligned} \tag{3.44}$$

For those i such that $\rho_i = -\infty$, $\frac{b_{ij}(p)}{p_j c_{ij}}$ are the same for different j again by the definition of $\mathbf{b}(\mathbf{p})$. Therefore,

$$e_i \log \min \Big\{ \frac{b_{ij}(\mathbf{p})}{p_j c_{ij}} \Big\} = \sum_j b_{ij}(\mathbf{p}) \log \frac{b_{ij}(\mathbf{p})}{p_j c_{ij}}. \tag{3.45}$$

2. For those i such that $0 > \rho_i > -\infty$, we focus on the function $\frac{1}{\rho_i} b_{ij} \log a_{ij} \frac{b_{ij}^{\rho_i - 1}}{p_j^{\rho_i}}$. By calculation, given $\mathbf{p}$, this function is a convex function. In addition, the minimal point $\mathbf{b}_i$ of the function under the constraint $\sum_j b_{ij} = e_i$ is $\mathbf{b}_i(\mathbf{p})$. Therefore, combining with (3.44) yields

$$e_i \log \Big(\sum_j a_{ij} \Big(\frac{b_{ij}(\mathbf{p})}{p_j} \Big)^{\rho_i} \Big)^{\frac{1}{\rho_i}} \leq \frac{1}{\rho_i} \sum_j b_{ij} \log a_{ij} \frac{b_{ij}^{\rho_i - 1}}{p_j^{\rho_i}} + \frac{e_i}{\rho_i} \log e_i. \tag{3.46}$$

The inequality becomes an equality if $\mathbf{b}_i = \mathbf{b}_i(\mathbf{p})$.

For those i such that $\rho_i = -\infty$, we focus on the function $\sum_j b_{ij} \log \frac{b_{ij}}{p_j c_{ij}}$. Again, given $\mathbf{p}$, this function is a convex function, and the minimal point $\mathbf{b}_i$ of the function under the constraint $\sum_j b_{ij} = e_i$ is $\mathbf{b}_i(\mathbf{p})$. Therefore,

$$e_i \log \min \Big\{ \frac{b_{ij}(\mathbf{p})}{p_j c_{ij}} \Big\} \leq \sum_j b_{ij} \log \frac{b_{ij}}{p_j c_{ij}}. \tag{3.47}$$

Also, the inequality becomes an equality if $\mathbf{b}_i = \mathbf{b}_i(\mathbf{p})$.

Combining (3.43), (3.46) and (3.47)

$$
\begin{aligned}
\Upsilon(\mathbf{p}(\mathbf{b}^*_{=0}, \mathbf{b}_{<0})) \leq & \sum_{i:0>\rho_i>-\infty} \left(\frac{1}{\rho_i} \sum_j b_{ij} \log a_{ij} \frac{b_{ij}^{\rho_i - 1}}{(p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}))^{\rho_i}} + \frac{e_i}{\rho_i} \log e_i \right) \\
& + \sum_{i:\rho_i=-\infty} \sum_j b_{ij} \log \frac{b_{ij}}{p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}) c_{ij}} \\
& + \sum_{i:\rho_i=0} \sum_j b^*_{ij} \log \frac{b^*_{ij}}{p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0})} + \sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}) - \sum_i e_i \\
= & -\Phi(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}) + \sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}) + \sum_{i:\rho_i>-\infty} \frac{e_i}{\rho_i} \log e_i \\
& + \sum_{i:\rho_i=0} \sum_j b^*_{ij} \log b^*_{ij} - \sum_i e_i. \qquad (3.48)
\end{aligned}
$$

Since we know $\mathbf{b}^* = \mathbf{b}(\mathbf{p}(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}))$, this leads to equality in (3.46) and (3.47) in this case. Therefore,

$$
\begin{aligned}
\Upsilon(\mathbf{p}(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0})) = & \sum_{i:0>\rho_i>-\infty} \left(\frac{1}{\rho_i} \sum_j b_{ij} \log a_{ij} \frac{b_{ij}^{\rho_i - 1}}{(p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}))^{\rho_i}} + \frac{e_i}{\rho_i} \log e_i \right) \\
& + \sum_{i:\rho_i=-\infty} \sum_j b_{ij} \log \frac{b_{ij}}{p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) c_{ij}} \\
& + \sum_{i:\rho_i=0} \sum_j b^*_{ij} \log \frac{b^*_{ij}}{p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0})} + \sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) - \sum_i e_i \\
= & -\Phi(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) + \sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) + \sum_{i:\rho_i>-\infty} \frac{e_i}{\rho_i} \log e_i \\
& + \sum_{i:\rho_i=0} \sum_j b^*_{ij} \log b^*_{ij} - \sum_i e_i. \qquad (3.49)
\end{aligned}
$$

Note that $\sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}_{<0}) = \sum_j p_j(\mathbf{b}^*_{=0}, \mathbf{b}^*_{<0}) = \sum_i e_i$. On taking the difference of (3.48) and (3.49), the theorem follows. □

3.2.8.3 Correspondence between Market Equilibrium and Minimal Point, Maximal Point and Saddle Point

Theorem 3.2.15. *The minimal point in the substitutes domain, the maximal point in the complementary domain, and the saddle point in the mixed case all corresponds to the respective market equilibria.*

Proof. Suppose we have a market equilibrium $(\mathbf{b}, \mathbf{p})$. A market equilibrium satisfies two conditions:

- At a market equilibrium, each buyer maximizes her utility function. With some calculation, we see this is equivalent to the following conditions.
  - for $\rho_i = 1$: $b_{ij} > 0$ only if j maximizes $\left\{\frac{a_{ij}}{p_j}\right\}$;
  - for $\rho_i = 0$: $b_{ij} = \lambda_i a_{ij}$;
  - for $\rho_i = -\infty$: $b_{ij} = \lambda_i c_{ij} p_j$;
  - for other ρ_i: $b_{ij} = \lambda_i a_{ij} \left(\frac{b_{ij}}{p_j}\right)^{\rho_i}$.

  Also, note that for the potential function $\Phi(\cdot)$, for those i for which $\rho_i \neq \{0, -\infty\}$,

$$\begin{aligned}\nabla_{b_{ij}} \Phi(\mathbf{b}) &= -\frac{1}{\rho_i} \log a_{ij} - \frac{\rho_i - 1}{\rho_i} (\log b_{ij} + 1) + \log p_j(\mathbf{b}) + \sum_h b_{hj} \frac{1}{p_j(\mathbf{b})} \\ &= \frac{1}{\rho_i} \Big(1 - \log \frac{a_{ij} b_{ij}^{\rho_i - 1}}{[p_j(\mathbf{b})]^{\rho_i}}\Big);\end{aligned}$$

  and for those i for which $\rho_i = -\infty$,

$$\nabla_{b_{ij}} \Phi(\mathbf{b}) = -\log \frac{b_{ij}}{c_{ij} p_j(\mathbf{b})}.$$

  Therefore, the optimal point $(\mathbf{b}^*, \mathbf{p}^*)$ of potential function $\Phi(\cdot)$ satisfies

  - for $\rho_i = 1$: $b_{ij}^* > 0$ only if j maximizes $\left\{\frac{a_{ij}}{p_j^*}\right\}$;

– for $\rho_i = -\infty$: $b^*_{ij} = \lambda'_i c_{ij} p^*_j$;

– for other ρ_i: $b^*_{ij} = \lambda'_i a_{ij} \left(\frac{b^*_{ij}}{p^*_j}\right)^{\rho_i}$.

We can conclude that for $\rho_i \neq 0$, buyers' spending, $\mathbf{b}^*$, maximizes these utilities under price $\mathbf{p}^*$. For $\rho_i = 0$, $b^*_i = e_i \frac{a_{ij}}{\sum_{j'} a_{ij}}$, in which case, it also maximizes this buyer's utility.

- At a market equilibrium, the allocation is equal to 1 if the price is strictly positive and the allocation can be less than 1 if the price is 0. Therefore, if, for each item, the total spending on this item equals its price, then the condition holds. This is ensured by our constraint on $\Phi(\cdot)$: $\forall j,\ p_j = \sum_i b_{ij}$.

Consequently, the optimal point $(\mathbf{b}^*, \mathbf{p}^*)$ of potential function $\Phi(\cdot)$ is a market equilibrium.

□

3.3 Tatonnement

3.3.1 Preliminary

Tatonnement is perhaps the most intuitive candidate for a natural algorithm in Fisher Markets. It raises the price of a good if its demand exceeds its supply, and decreases it if the demand is too small. Implicitly, buyers' demands are assumed to be a best-response to the current prices. This highly intuitive algorithm was proposed by Walras well over a century ago [Wal74].

This dissertation focuses on discrete versions of tatonnement. Unfortunately, as is well known, discrete versions of tatonnement need not converge when buyer utilities are linear, as shown in the following simple example.

Example 1. *There are two items, both with unit supply, and one buyer with 2 units of money whose utility equals the sum of the amount of the two items it receives. Suppose we use the update rule* $p'_j = p_j \cdot e^{\lambda \min\{x_j - 1, 1\}}$, *where* x_j *is the demand for good* j, *and* $\lambda > 0$ *is* *a parameter; this is essentially the version of the tatonnement rule we will consider in this dissertation, and a version that has been analyzed previously. Suppose the prices for the two items are initially* $p_1 = e^{\lambda/2}$ *and* $p_2 = e^{-\lambda/2}$*, respectively. Then the demand for good 1 is 0 and the demand for good 2 is* $2e^{\lambda/2}$*, so following one round of updates, the prices become* $p_1 = e^{-\lambda/2}$ *and* $p_2 = e^{\lambda/2}$*. On subsequent updates the prices keep interchanging, so there is no convergence.*

In addition, and unsurprisingly, as one approaches linear settings, the step size employed by the tatonnement algorithm needs to be increasingly small, which leads to a slower rate of convergence, and indicates a lack of robustness in the tatonnement procedure.

In this dissertation, we show that in suitable large Fisher markets, this lack of robustness disappears, so long as approximate rather than exact convergence suffices. In addition, we obtain fast, i.e. linear, convergence to an approximate equilibrium. To see why approximate convergence is a reasonable and even the right goal, consider dynamical settings; in these settings, the equilibrium state can be expected to change over time, and then the natural convergence question becomes how closely can one track the moving equilibrium? The answer is that it is a function of the rate of change and the market parameters, as analyzed by Cheung et al. [CHN19]. Clearly, in this type of setting, similar results will arise with an approximate convergence result.

Our large market assumption requires that for goods with high elasticity, price changes cause a relatively small change in spending. In the case of buyers with linear utility functions, where the elasticity parameters are unbounded, this occurs if the buyers are heterogeneous, meaning the collection of their utility functions is quite varied; also, we need each individual buyer to constitute a small portion of the market and for the number of buyers to be large compared to the number of goods.

To explain the intuition behind our results, we recall that Cheung, Cole and Devanur [CCD19] showed that for many types of economies, including those we consider here, a suitable tatonnement update is equivalent to a form of mirror descent on a suitable convex function (actually, mirror ascent on a concave function). To achieve convergence with mirror descent, one needs the function F to have a bounded Lipschitz parameter L, namely that

$$||\nabla F(\mathbf{p}) - \nabla F(\mathbf{q})|| \leqslant L||\mathbf{p} - \mathbf{q}||,$$

for any two price vectors $\mathbf{p}$ and $\mathbf{q}$. The rate of convergence will depend inversely on L. Our large market assumption ensures this property so long as $||\mathbf{p} - \mathbf{q}||$ is not too small.

In addition, the boundedness of the Lipschitz parameter holds only if the prices are bounded away from 0. Prior analyses implicitly bounded this parameter by showing the prices are bounded away from zero, though this bound depended on the initial prices and the particulars of the market. In this dissertation, we assume there are minimum or reserve prices which provides an alternate way to implicitly bound this parameter.

Furthermore, to obtain a linear rate of convergence one needs the function $F(p)$ to be strongly convex w.r.t. the equilibrium point $\mathbf{p}^*$, namely:

$$F(\mathbf{p}) - f(\mathbf{p}^*) \geqslant \langle \nabla F(\mathbf{p}), \mathbf{p} - \mathbf{p}^* \rangle + \alpha||\mathbf{p} - \mathbf{p}^*||^2.$$

Again, our large market assumption ensures this property so long as $||\mathbf{p} - \mathbf{p}^*||$ is not too small.

3.3.2 Large Market Assumption

Our large market assumption states that for the buyers with linear or close to linear utilities, the spending on a single good does not vary too much as prices change. We define "close to linear" in terms of a bound $\sigma > 0$ on the ρ_i parameters.

Assumption 3.3.1. *[Large Market Assumption] There is a (small) constant $\epsilon > 0$ such that* *for those buyers with parameter $\rho_i \geqslant \sigma$,*

$$\sum_{i:\rho_i \geq \sigma} |b_{ij}^t - b_{ij}^{t+1}| \leq \epsilon \sum_i b_{ij}^t + \epsilon r_j.$$

In addition, the total available money $E \geqslant \max_j r_j$.

Remark We validate our assumption in the following two settings. In the first setting the market has only a few buyers with ρ_i bigger than σ. In this case, it's easy to see that the assumption holds if we set $\epsilon = \max \left\{ \frac{\sum_{i:\rho_i \geq \sigma} e_i}{r_j} \right\}$.

Our second setting is a large linear market. The property we want is that for each good j, when there are price changes by factors of at most $e^{\pm\lambda}$, a relatively small weight of buyers will be added to and removed from those currently purchasing good j (where the weight is measured in terms of the buyers' budgets.)

More specifically, b_{ij}^t differs from b_{ij}^{t+1} only if one of the following occur:

- there exists a k such that $\frac{a_{ij}}{p_j^t} \leq \frac{a_{ik}}{p_k^t}$ and $\frac{a_{ij}}{p_j^{t+1}} \geq \frac{a_{ik}}{p_k^{t+1}}$
- there exists a k such that $\frac{a_{ij}}{p_j^t} \geq \frac{a_{ik}}{p_k^t}$ and $\frac{a_{ij}}{p_j^{t+1}} \leq \frac{a_{ik}}{p_k^{t+1}}$.

Note that our price update rule ensures that $\frac{p_j^{t+1}}{p_k^{t+1}} \in [e^{-2\lambda}, e^{2\lambda}] \frac{p_j^t}{p_k^t}$. Therefore, b_{ij}^t differs from b_{ij}^{t+1} only if there exists a k such that $\frac{a_{ij}}{a_{ik}} \in [e^{-2\lambda}, e^{2\lambda}] \frac{p_j^t}{p_k^t}$. Also, since one of b_{ij}^t and b_{ij}^{t+1} is non-zero, for all s, $\frac{a_{ij}}{a_{is}} \geq \frac{p_j^t}{p_s^t} e^{-2\lambda}$. Let $q_s \triangleq \frac{p_j^t}{p_s^t}$. We conclude that

$$\sum_i |b_{ij}^t - b_{ij}^{t+1}| \leq \sum_{i:\begin{cases} \exists k: \frac{a_{ij}}{a_{ik}} \in \left[e^{-2\lambda} q_k, e^{2\lambda} q_k\right] \\ \text{and } \forall s \ \left(\frac{a_{ij}}{a_{is}} \geq q_s e^{-2\lambda}\right) \end{cases}} e_i, \tag{3.50}$$

and

$$\sum_i b_{ij}^t \geq \sum_{i:\forall s \neq j \ \left(\frac{a_{ij}}{a_{is}} > q_s\right)} e_i. \tag{3.51}$$

If the buyers are diverse, meaning that for any pair of goods, j and k say, the ratios $\frac{a_{ij}}{a_{ik}}$ vary substantially across the buyers, then so long as there are many buyers satisfying the condition $\forall s \neq j \ \left(\frac{a_{ij}}{a_{is}} > q_s\right)$ in (3.51), it seems reasonable that their purchasing power be much larger than that of the buyers satisfying the condition $\exists k : \frac{a_{ij}}{a_{ik}} \in \left[e^{-2\lambda}q_k, e^{2\lambda}q_k\right]$ and $\forall s \ \left(\frac{a_{ij}}{a_{is}} \geq q_s e^{-2\lambda}\right)$ in (3.50). While if there are few buyers satisfying the first condition, then it is reasonable to assume that only a small number of buyers will switch their purchase to or from good j, and that this changed spending will be much smaller than r_j.

This motivates setting ϵ to be greater than or equal to

$$\max_{j,\mathbf{q}:q_k\in[\frac{r_k}{E},\frac{E}{r_k}]} \left\{ \sum_{i:\begin{cases}\exists k:\ \frac{a_{ij}}{a_{ik}}\in\left[e^{-2\lambda}q_k, e^{2\lambda}q_k\right] \\ \text{and } \forall s\ \left(\frac{a_{ij}}{a_{is}}\geq q_s e^{-2\lambda}\right)\end{cases}} e_i \Bigg/ \sum_{i:\ \forall s\left(\frac{a_{ij}}{a_{is}}>q_s\right)} e_i + r_j \right\},$$

causing our assumption to hold.

Our analysis is carried out with respect to the following potential function, which is the dual of the Eisenberg-Gale convex program:

$$\mathbf{F}(\mathbf{p}) = \sum_j p_j + \sum_i e_i \log \max_{\mathbf{x}_i\cdot\mathbf{p}=e_i} u_i(x_i)$$

using the tatonnement update rule:

$$p_j^{t+1} = p_j^t \cdot e^{\Delta_j^t},$$

where $\Delta_j^t = \lambda \min\{z_j^t, 1\}$ and $\lambda \leq 1$, unless this update would reduce p_j^{t+1} below the reserve price, in which case Δ_j^t is chosen so that $p_j^{t+1} = r_j$.

Our main result shows an initial linear rate of convergence toward the equilibrium, and also shows that so long as the current prices are not too close to the equilibrium then there is good progress toward the equilibrium. The latter statement can also be viewed as a result regarding the tracking of a slowly moving equilibrium.

Before we state the main result, we define a parameter $C(\kappa)$ introduced in [CCD19]. Here κ is an upper bound on the ratio $\max_j \frac{p_j^*}{r_j}$. $C(\kappa) = \min\left\{\frac{h_c(\kappa)}{c}, \frac{\kappa-1-\log\kappa}{(\kappa-1)^2}\right\}$, where $h_c(\kappa) = \frac{1-\kappa^c+c(\kappa-1)}{(\kappa-1)^2}$ for any $\kappa \geq 0$ except $\kappa = 1$, $h_c(1) = \frac{c(1-c)}{2}$, and $c = \max_i c_i$ [4]. Note that $c < 1$.

Theorem 3.3.1. *For any* $0 < \theta < 1$, *if* $\frac{\lambda\sigma}{1-\sigma} \leqslant 1$ *and* $\kappa \geqslant \max_j \frac{p_j^*}{r_j}$, *then*

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) \leq (1-\alpha)^t \left(\mathbf{F}(\mathbf{p}^0) - \mathbf{F}(\mathbf{p}^*)\right) + 2\frac{\lambda\epsilon^2\mathcal{M}}{\alpha\theta},$$

where $\alpha = \frac{\left(1-\lambda-2\lambda\cdot\max\left\{\frac{\sigma}{1-\sigma},1\right\}-\epsilon-2\theta\right)}{\max_j\left\{\max\left\{2,\frac{1}{2C(\kappa)}\right\}\frac{E}{\lambda r_j}\right\}}$ *and* $\mathcal{M} = \max\left\{\sum_j p_j^0, \left(\left(e^\lambda - 2\lambda\right)\frac{1+2\lambda-e^\lambda}{\lambda} + \lambda\right)\left(E + \sum_j r_j\right)\right\}$. *Furthermore, if* $\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) \geqslant \frac{4\lambda\epsilon^2\mathcal{M}}{\alpha\theta}$ *then*

$$\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^*) \leq \left(1 - \frac{\alpha}{2}\right)\left(\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*)\right).$$

Moreover, if $\epsilon \leq \frac{1}{4}$ *and we set* $\sigma = \frac{1}{2}$, $\theta = \lambda = \frac{1}{20}$, *then* $\alpha = \frac{\frac{3}{4}-\epsilon}{\max_j\left\{\max\left\{2,\frac{1}{2C(\kappa)}\right\}\frac{E}{\lambda r_j}\right\}}$ *and* $\mathcal{M} \leq \max\left\{\sum_j p_j^0, E + \sum_j r_j\right\}$.

3.3.3 A High Level Overview of the Analysis

The analysis is largely based on deriving two bounds. The first is a progress lemma, a lower bound on the reduction in value of $F(p)$ due to the time t update, stated in Lemma 3.3.1 below. The second is an upper bound on the distance to the equilibrium, stated in Lemma 3.3.2 below. We also need to relate the sum of the prices at time $t+1$ to the corresponding sum for time 0, as stated in Lemma 3.3.3. Our main result then follows fairly readily.

With a slight abuse of notation, we let $u_i(\mathbf{b}_i, \mathbf{p})$ denote buyer i's utility when spending $\mathbf{b}_i$ at prices $\mathbf{p}$.

[4] $c_i = \frac{\rho_i}{\rho_i - 1}$ is defined in (1.1).

We analyze the change to the potential function due to the time t updates. Note that since buyers best respond, $\max_{x_i \cdot p = e_i} u_i(x_i) = u_i(\mathbf{b}_i^t, \mathbf{p}^t)$. So,

$$\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^t) = \sum_j (p_j^{t+1} - p_j^t) + \sum_i e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)}. \tag{3.52}$$

Lemma 3.3.1. *For any* $0 < \sigma < 1$ *such that* $\frac{\lambda\sigma}{1-\sigma} \leq 1$, *if* $|\Delta_j^t| \leq \lambda |\min\{z_j^t, 1\}|$ *and* $\operatorname{sign}(\Delta_j^t) = \operatorname{sign}(\min\{z_j^t, 1\})$, *then*

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^{t+1}) \geq \left(1 - \lambda - 2\lambda \cdot \max\left\{\frac{\sigma}{1-\sigma}, 1\right\}\right) \sum_j p_j^t z_j^t \Delta_j^t - \sum_{i:\rho_i \geq \sigma} \rho_i \sum_j \left(b_{ij}^t - b_{ij}^{t+1}\right) \Delta_j^t.$$

To obtain an upper bound on $\mathbf{F}(\mathbf{p}) - \mathbf{F}(\mathbf{p}^*)$ we follow the approach taken in [CCD19]. They pointed out that if $\frac{p_j^*}{p_j} \leq \kappa$ for all j, then

$$\mathbf{F}(\mathbf{p}^*) - \mathbf{F}(\mathbf{p}) - \langle \nabla \mathbf{F}(\mathbf{p}), \mathbf{p}^* - \mathbf{p} \rangle \geq \sum_j C(\kappa) x_j \frac{(p_j^* - p_j)^2}{p_j}, \tag{3.53}$$

where $C(\kappa)$ was specified above. We note that this is a strong convexity bound of the type we need for a linear convergence rate.

We will show:

Lemma 3.3.2. *If* $\kappa \geqslant \max_j \frac{p_j^*}{r_j}$, *then*

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) \leq \sum_j \max\left\{2, \frac{1}{2C(\kappa)}\right\} \frac{E}{\lambda r_j} p_j^t z_j^t \Delta_j^t.$$

Finally, the bound on the sum of the prices is stated in the next lemma.

Lemma 3.3.3. *Using the definition of* $\mathcal{M}$ *from Theorem 3.3.1 gives*

$$\sum_j p_j^{t+1} \leqslant \max\left\{\sum_j p_j^0, \left(\left(e^\lambda - 2\lambda\right) \frac{1 + 2\lambda - e^\lambda}{\lambda} + \lambda\right) \left(E + \sum_j r_j\right)\right\} = \mathcal{M}.$$

We are now ready to prove our main result.

Proof of Theorem 3.3.1. We will be applying Lemma 3.3.1, and we begin by bounding the second term on the RHS of the expression there.

$$\sum_{i:\rho_i\geq\sigma}\rho_i\sum_j\left(b_{ij}^t-b_{ij}^{t+1}\right)\Delta_j^t \leq \sum_{i:\rho_i\geq\sigma}\rho_i\sum_j\left|b_{ij}^t-b_{ij}^{t+1}\right|\left|\Delta_j^t\right|$$
$$\leq \sum_{i:\rho_i\geq\sigma}\sum_j\left|b_{ij}^t-b_{ij}^{t+1}\right|\left|\Delta_j^t\right|.$$

By Assumption 3.3.1 for the first inequality, and because $p_j^t \geq r_j$ for the second inequality,

$$\sum_{i:\rho_i\geq\sigma}\rho_i\sum_j\left(b_{ij}^t-b_{ij}^{t+1}\right)\Delta_j^t \leq \sum_j(\epsilon\sum_i b_{ij}^t+\epsilon r_j)\left|\Delta_j^t\right| = \sum_j(\epsilon p_j^t(1+z_j^t)+\epsilon r_j)\left|\Delta_j^t\right|$$
$$\leq \sum_j 2\epsilon p_j^t\left|\Delta_j^t\right|+\epsilon p_j^t z_j^t\left|\Delta_j^t\right| \leq \sum_j 2\epsilon p_j^t\left|\Delta_j^t\right|+\epsilon p_j^t z_j^t\Delta_j^t. \quad (3.54)$$

We use the following result: for any $\theta > 0$,

$$\sum_j 2\epsilon p_j^t\left|\Delta_j^t\right| \leq \sum_j 2\theta p_j^t z_j^t\Delta_j^t + 2\frac{\lambda\epsilon^2}{\theta}\sum_j p_j^t. \quad (3.55)$$

This holds because if $\epsilon p_j^t\left|\Delta_j^t\right| \geq \theta p_j^t z_j^t\Delta_j^t$, then $\epsilon \geq \theta|z_j^t|$. Therefore

$$2\epsilon p_j^t\left|\Delta_j^t\right| \leq 2\frac{\lambda\epsilon^2}{\theta}p_j^t.$$

Substituting (3.54) and (3.55) in Lemma 3.3.1 yields

$$\mathbf{F}(\mathbf{p}^t)-\mathbf{F}(\mathbf{p}^{t+1}) \geq \left(1-\lambda-2\lambda\cdot\max\left\{\frac{\sigma}{1-\sigma},1\right\}-\epsilon-2\theta\right)\sum_j p_j^t z_j^t\Delta_j^t - 2\frac{\lambda\epsilon^2}{\theta}\sum_j p_j^t$$

Applying Lemma 3.3.3 gives

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^{t+1}) \geq \left(1 - \lambda - 2\lambda \cdot \max\left\{\frac{\sigma}{1-\sigma}, 1\right\} - \epsilon - 2\theta\right) \sum_j p_j^t z_j^t \Delta_j^t - 2\frac{\lambda\epsilon^2}{\theta}\mathcal{M}. \quad (3.56)$$

Applying Lemma 3.3.2 and recalling that $\alpha = \frac{\left(1-\lambda-2\lambda\cdot\max\left\{\frac{\sigma}{1-\sigma},1\right\}-\epsilon-2\theta\right)}{\max_j\left\{\max\left\{2,\frac{1}{2C(\kappa_j)}\right\}\frac{E}{\lambda r_j}\right\}}$ yields:

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^{t+1}) \geq \alpha[F(\mathbf{p}^t) - F(\mathbf{p}^*)] - 2\frac{\lambda\epsilon^2}{\theta}\mathcal{M}.$$

Our first claim follows readily:

$$\begin{aligned}
\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^*) &\leq (1-\alpha)\left(\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*)\right) + 2\frac{\lambda\epsilon^2}{\theta}\mathcal{M} \quad (3.57)\\
&\leq (1-\alpha)^t\left(\mathbf{F}(\mathbf{p}^0) - \mathbf{F}(\mathbf{p}^*)\right) + 2\frac{\lambda\epsilon^2\mathcal{M}}{\theta}\left(1 + (1-\alpha) + (1-\alpha)^2 + \cdots\right)\\
&\leq (1-\alpha)^t\left(\mathbf{F}(\mathbf{p}^0) - \mathbf{F}(\mathbf{p}^*)\right) + 2\frac{\lambda\epsilon^2\mathcal{M}}{\alpha\theta}.
\end{aligned}$$

To prove the second claim, recall that we are assuming $\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) \leq 4\frac{\lambda\epsilon^2}{\alpha\theta}\mathcal{M}$. Then, by (3.57),

$$\begin{aligned}
\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^*) &\leq (1-\alpha)\left(\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*)\right) + \frac{\alpha}{2}\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*)\\
&\leqslant (1-\frac{\alpha}{2})\left(\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*)\right).
\end{aligned}$$

□

3.3.4 The Proof of Lemma 3.3.1, the Progress Lemma

The starting point for our analysis is (3.52). The first step is to bound $\log\frac{u_i(\mathbf{b}_i^{t+1},\mathbf{p}_i^{t+1})}{u_i(\mathbf{b}_i^t,\mathbf{p}_i^t)}$. The next four lemmas provide a variety of bounds depending on the value of ρ_i and other parameters.

Lemma 3.3.4. *If buyer i has a linear utility function, then*

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}_i^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}_i^t)} \leq -\sum_j b_{ij}^t \Delta_j^t + \sum_j (b_{ij}^t - b_{ij}^{t+1}) \Delta_j^t.$$

Lemma 3.3.5. *For any $0 < \rho_i < 1$, if $|\Delta_j^t| \leq 1$ for all j and t, then*

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} \leq -\sum_j b_{ij}^t \Delta_j^t + \sum_j b_{ij}^t \rho_i \left(\Delta_j^t\right)^2 - \rho_i \sum_j b_{ij}^{t+1} \Delta_j^t + \rho_i \sum_j b_{ij}^t \Delta_j^t.$$

Lemma 3.3.6. *For any $\rho_i > 0$, if $|\lambda c_i| \leq 1$ and $|\Delta_j^t| \leq 1$ for all j and t, then*

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} \leq -\sum_j b_{ij}^t \Delta_j^t - \sum_j b_{ij}^t c_i \left(\Delta_j^t\right)^2.$$

Lemma 3.3.7. *If buyer i has a complementary utility function, then*

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}_i^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}_i^t)} \leq -\sum_j b_{ij}^t \Delta_j^t.$$

We are now ready to prove Lemma 3.3.1.

Proof of Lemma 3.3.1: Recall that $c_i = \frac{\rho_i}{\rho_i - 1}$, and σ is a threshold designating the buyers to which Assumption 3.3.1 applies, namely those with $\rho_i \geqslant \sigma$. We apply Lemma (3.3.6) to the buyers with $0 < \rho_i \leq \sigma$. In order to apply Lemma (3.3.6), it suffices to have

$$\lambda \cdot \frac{\sigma}{1 - \sigma} \leq 1.$$

Therefore, by Lemmas 3.3.4–3.3.7 and equation (3.52), for any $0 < \sigma < 1$ such that $\lambda\frac{\sigma}{1-\sigma} \leq 1$,

$$\begin{aligned}
\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^t) &= \sum_j p_j(e^{\Delta_j^t} - 1) - \sum_{ij} b_{ij}^t \Delta_j^t \\
&\quad - \sum_{ij:0<\rho_i<\sigma} b_{ij}^t c_i (\Delta_j^t)^2 + \sum_{ij:\sigma\leq\rho_i<1} b_{ij}^t \rho_i (\Delta_j^t)^2 \\
&\quad + \sum_{i:\rho_i\geq\sigma} \rho_i \sum_j \left(b_{ij}^t - b_{ij}^{t+1}\right) \Delta_j^t \\
&= \sum_j p_j(e^{\Delta_j^t} - \Delta_j^t - 1) - \sum_j (\sum_i b_{ij}^t - p_j^t) \Delta_j^t \\
&\quad - \sum_{ij:0<\rho_i<\sigma} b_{ij}^t c_i (\Delta_j^t)^2 + \sum_{ij:\sigma\leq\rho_i<1} b_{ij}^t \rho_i (\Delta_j^t)^2 \\
&\quad + \sum_{i:\rho_i\geq\sigma} \rho_i \sum_j \left(b_{ij}^t - b_{ij}^{t+1}\right) \Delta_j^t.
\end{aligned}$$

Note that $e^{\Delta_j^t} - \Delta_j^t - 1 \leq \left(\Delta_j^t\right)^2$ as $|\Delta_j^t| \leq 1$, and $\sum_i b_{ij}^t - p_j^t = p_j^t z_j^t$. Therefore,

$$\begin{aligned}
\mathbf{F}(\mathbf{p}^{t+1}) - \mathbf{F}(\mathbf{p}^t) \leq & \underbrace{\sum_j p_j \left(\Delta_j^t\right)^2}_{A} - \underbrace{\sum_j p_j^t z_j^t \Delta_j^t}_{B} \\
& \underbrace{- \sum_{ij:\rho_i<\sigma} b_{ij}^t c_i (\Delta_j^t)^2}_{C} + \underbrace{\sum_{ij:\sigma\leq\rho_i<1} b_{ij}^t \rho_i (\Delta_j^t)^2}_{D} \\
& + \sum_{i:\rho_i\geq\sigma} \rho_i \sum_j \left(b_{ij}^t - b_{ij}^{t+1}\right) \Delta_j^t.
\end{aligned}$$

It is easy to see that $A \leq \lambda B$ if $|\Delta_j^t| \leq \lambda|\min\{z_j^t, 1\}|$ and $\text{sign}(\Delta_j^t) = \text{sign}(\min\{z_j^t, 1\})$. Next, we will bound C and D in terms of B. To this end, we note that we can omit the portion of C with $\rho_i \leqslant 0$ as for these terms $c_i \geqslant 0$ and consequently removing them only increases the RHS expression. We then note that for $\rho_i > 0$,

$$-c_i = -\frac{\rho_i}{\rho_i - 1} \leqslant -\frac{\sigma}{\sigma - 1} = \frac{\sigma}{1-\sigma}.$$

For term D we use the simple bound $\rho_i \leqslant 1$. Thus terms C and D are bounded by

$$\sum_{i,j:\rho_i>0} \max\left\{\frac{\sigma}{1-\sigma}, 1\right\} b_{ij}^t \left(\Delta_j^t\right)^2.$$

We now give a bound on this expression in terms of B.

Claim 3.3.8. *If* $|\Delta_j^t| \le \lambda|\min\{z_j^t, 1\}|$ *and* $\operatorname{sign}(\Delta_j^t) = \operatorname{sign}(\min\{z_j^t, 1\})$, *then*

$$\sum_j p_j^t z_j^t \Delta_j^t \ge \frac{1}{2\lambda}\left(\sum_i b_{ij}^t\right)(\Delta_j^t)^2.$$

Thus

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^{t+1}) \ge \sum_j p_j^t z_j^t \Delta_j^t \left(1 - \lambda - 2\lambda \max\left\{\frac{\sigma}{1-\sigma}, 1\right\}\right).$$

□

3.3.5 Bounding the Distance to the Optimum

In this section, we provide an upper bound on $\mathbf{F}(\mathbf{p}) - \mathbf{F}(\mathbf{p}^*)$. Note that $\nabla\mathbf{F}(\mathbf{p}) = -\mathbf{z}$. Equation (3.53) yields

$$\begin{aligned}\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) &\le \sum_j z_j^t(p_j^* - p_j^t) - \sum_j C(\kappa_j)x_j^t \frac{(p_j^* - p_j^t)^2}{p_j^t} \\ &\le \max_{\mathbf{p}'\ge\mathbf{r}} \sum_j \left(z_j^t(p_j' - p_j^t) - C(\kappa_j)x_j^t \frac{(p_j' - p_j^t)^2}{p_j^t}\right). \qquad (3.58)\end{aligned}$$

In [CCD19], they prove that

$$\max_{p'} \left(z_j^t(p_j' - p_j^t) - C(\kappa_j)x_j^t \frac{(p_j' - p_j^t)^2}{p_j^t}\right) \le \max\left\{2, \frac{1}{2C(\kappa_j)}\right\} \left(z_j^t\right)^2 p_j^t. \qquad (3.59)$$

Here, we want to prove one more upper bound.

Lemma 3.3.9. *If $p_j^{t+1} = r_j$ and $z_j^t \le 0$, then*

$$\max_{p'_j \ge r_j} \left(z_j^t (p'_j - p_j^t) - C(\kappa_j) x_j^t \frac{(p'_j - p_j^t)^2}{p_j^t} \right) \le z_j^t p_j^t \log \frac{p_j^{t+1}}{p_j^t}.$$

It's easy to see that our update rule has $\Delta_j^t = \log \frac{p_j^{t+1}}{p_j^t}$. Now, combining (3.58), (3.59), and Lemma 3.3.9, gives Lemma 3.3.2, as we show below.

Proof of Lemma 3.3.9: Let

$$\mathcal{I}(p'_j) = z_j^t (p'_j - p_j^t) - C(\kappa_j) x_j^t \frac{(p'_j - p_j^t)^2}{p_j^t}$$

and in the setting without reserve prices, let

$$p_j^{opt} = \arg\max_{p'_j} \left\{ \mathcal{I}(p'_j) \right\}.$$

In this case, the maximum value of $\mathcal{I}(p'_j)$ is

$$\frac{\left(z_j^t\right)^2 p_j^t}{4C(\kappa_j) x_j^t},$$

and the optimum value is

$$p_j^{opt} = p_j^t + \frac{z_j^t p_j^t}{2C(\kappa_j) x_j^t}.$$

Case 1: $p_j^{opt} \ge r_j$.

This implies

$$p_j^t + \frac{z_j^t p_j^t}{2C(\kappa_j) x_j^t} \ge r_j.$$

Therefore, since $z_j^t \leq 0$, $-\frac{z_j p_j}{2C(\kappa_j)x_j} \leq p_j - r_j$, and

$$\frac{\left(z_j^t\right)^2 p_j^t}{4C(\kappa_j)x_j} \leq \frac{1}{2} z_j^t p_j^t \frac{r_j - p_j^t}{p_j^t} \leq \frac{1}{2} z_j^t p_j^t \log \frac{r_j}{p_j^t}.$$

Case 2: $p_j^{opt} < r_j$.

Note that $\mathcal{I}(p_j')$ is a quadratic function. On the domain $p_j' \geqslant r_j$, it achieves its maximum value when $p_j' = r_j$. Therefore, the maximum value is

$$\begin{aligned}
\frac{\left(z_j^t\right)^2 p_j^t}{4C(\kappa_j)x_j} - \frac{C(\kappa_j)x_j^t}{p_j^t}(r_j - p_j^{opt})^2 &= \frac{\left(z_j^t\right)^2 p_j^t}{4C(\kappa_j)x_j} - \frac{C(\kappa_j)x_j^t}{p_j^t}\left(r_j - p_j^t - \frac{z_j^t p_j^t}{2C(\kappa_j)x_j^t}\right)^2 \\
&= -C(\kappa_j)x_j^t \frac{(r_j - p_j^t)^2}{p_j^t} + z_j^t(r_j - p_j^t)
\end{aligned}$$

Since $C(\kappa_j)x_j^t \frac{(r_j - p_j^t)^2}{p_j^t} \geq 0$, this is less than

$$z_j^t(r_j - p_j^t) \leq z_j^t p_j^t \frac{r_j - p_j^t}{p_j^t} \leq z_j^t p_j^t \log \frac{r_j}{p_j^t}.$$

□

Proof of Lemma 3.3.2: **Case 1**: $p_j^{t+1} > r_j^t$.

Then $\Delta_j^t = \lambda \min\{z_j^t, 1\}$. Note that $z_j^t = \frac{\sum_i b_{ij}^t - p_j^t}{p_j^t} \leq \frac{E}{r_j}$. Therefore, $|\Delta_j^t| \geq \frac{\lambda r_j}{E}|z_j^t|$. Since Δ_j^t and z_j^t are both positive or both negative, combining with (3.58) and (3.59) yields

$$\mathbf{F}(\mathbf{p}^t) - \mathbf{F}(\mathbf{p}^*) \leq \max\left\{2, \frac{1}{2C(\kappa_j)}\right\}(z_j^t)^2 p_j^t \leq \max\left\{2, \frac{1}{2C(\kappa_j)}\right\}\frac{E}{\lambda r_j} p_j^t z_j^t \Delta_j^t.$$

Case 2: $p_j^{t+1} = r_j^t$.

By (3.58) and using Lemma 3.3.9, the result follows as $\Delta_j^t = \log \frac{p_j^{t+1}}{p_j^t}$, and $E \geqslant r_j$ by assumption. □

3.3.6 Dynamical Markets

In this section, we will study dynamical markets. For each round, there can be a small change to the supplies, budgets and buyers' preferences. We will seek to show that tatonnement can cause the prices to pursue the market equilibrium. Note that, in general, we need to modify the potential function to account for the possibly changing supplies w_j^t for item j at time t; the new potential function is

$$\sum_j w_j^t p_j^t + \sum_i e_i \log \max_{\mathbf{x}_i \cdot \mathbf{p}^t = e_i^t} u_i(x_i),$$

and our update rule will be

$$p_j^{t+1} = p_j^t e^{\Delta_j^t}$$

where $\Delta_j^t = \lambda \max \left\{ \frac{z_j^t}{w_j}, 1 \right\}$.

Cheung, Hoefer, and Nakhe [CHN19] analyzed the following settings:

- *Supply Change* If at time t, the supplies change by at most ϵ, then the potential function changes by at most $(P + E)\epsilon$, where P is the maximum price at time $t + 1$;
- *Budget Change* If at time t, the sum of the absolute values of the changes to the buyers' budgets is at most ϵ, then the potential function changes by at most $C\epsilon$, where C is the maximum possible ratio between a buyer's utility at time $t + 1$ and her utility at the market equilibrium at time $t + 1$;
- *Utility Change* If at time t, given any prices, the ratio of the utility difference when best responding is bounded by χ, then the potential function changes by at most $2E\chi$.

In order to analyze the effect of these changes over time, in this dissertation, we let D denote the maximum change to the potential function at each round. We let $\mathbf{p}^{t,*}$ denote the equilibrium prices at time t. We have the following theorem.

Theorem 3.3.2. *For any $0 < \theta < 1$, if $\frac{\lambda\sigma}{1-\sigma} \leqslant 1$ and $\kappa \geqslant \max_j \frac{p_j^*}{r_j}$, then*

$$\mathbf{F}^{t+1}(\mathbf{p}^{t+1}) - \mathbf{F}^{t+1}(\mathbf{p}^{t+1,*}) \leq (1-\alpha)^t \left(\mathbf{F}(\mathbf{p}^0) - \mathbf{F}(\mathbf{p}^{0,*})\right) + \frac{1}{\alpha}\left(2\frac{\lambda\epsilon^2\mathcal{M}}{\theta} + D\right),$$

where $\alpha = \frac{\left(1-\lambda-2\lambda\cdot\max\left\{\frac{\sigma}{1-\sigma},1\right\}-\epsilon-2\theta\right)}{\max_j\left\{\max\left\{2,\frac{1}{2C(\kappa)}\right\}\frac{E}{\lambda r_j}\right\}}$ *and* $\mathcal{M} = \max\left\{\sum_j \hat{w}_j p_j^0, \left(\left(e^\lambda - 2\lambda\right)\frac{1+2\lambda-e^\lambda}{\lambda} + \lambda\right)\left(E + \sum_j r_j\right)\right\}$. *Also, here E will be the maximum possible total money over time, and $\hat{w}_j$ will be the maximum supply of item j over time. Furthermore, if $\mathbf{F}^t(\mathbf{p}^t) - \mathbf{F}^t(\mathbf{p}^{t,*}) \geqslant \frac{2}{\alpha}\left(\frac{2\lambda\epsilon^2\mathcal{M}}{\theta} + D\right)$ then*

$$\mathbf{F}^{t+1}(\mathbf{p}^{t+1}) - \mathbf{F}^{t+1}(\mathbf{p}^{t+1,*}) \leq \left(1 - \frac{\alpha}{2}\right)\left(\mathbf{F}^t(\mathbf{p}^t) - \mathbf{F}^t(\mathbf{p}^{t,*})\right).$$

Proof. This theorem follows directly from the proof of Theorem 3.3.1 if we replace $2\frac{\lambda\epsilon^2\mathcal{M}}{\theta}$ by $2\frac{\lambda\epsilon^2\mathcal{M}}{\theta} + D$ in (3.56). □

3.3.7 Missing Proofs

Proof of Lemma 3.3.3:

$$\sum_j p_j^{t+1} = \sum_j p_j^t e^{\Delta_j^t} = \sum_j p_j^t\left(e^{\Delta_j^t} - 1 - \Delta_j^t\right) + \sum_j p_j^t\left(1 + \Delta_j^t\right).$$

If $p_j^{t+1} = r_j$, then $p_j^t\left(1+\Delta_j^t\right) \leq p_j^t e^{\Delta_j^t} = r_j \leq (1-\lambda)p_j^t + \lambda r_j$. Otherwise, $p_j^t\left(1+\Delta_j^t\right) \leq p_j^t\left(1+\lambda z_j^t\right) = (1-\lambda)p_j^t + \lambda\sum_i b_{ij}^t$. This implies

$$\sum_j p_j^{t+1} \leq \sum_j p_j^t\left(e^{\Delta_j^t} - 1 - \Delta_j^t\right) + \lambda\sum_j r_j + (1-\lambda)\sum_j p_j^t + \lambda\sum_{ij} b_{ij}^t.$$

Since $|\Delta_j^t| \leq \lambda$, $e^{\Delta_j^t} - 1 - \Delta_j^t \leq \max\left\{e^\lambda - 1 - \lambda, e^{-\lambda} - 1 + \lambda\right\} \leq e^\lambda - 1 - \lambda$.

$$\sum_j p_j^{t+1} \leq \left(e^\lambda - 1 - \lambda\right)\sum_j p_j^t + \lambda\sum_j r_j + (1-\lambda)\sum_j p_j^t + \lambda\sum_{ij} b_{ij}^t. \qquad (3.60)$$

If $\sum_j p_j^t \geq \left(\frac{1+2\lambda-e^\lambda}{\lambda}\right)\left(\sum_{ij} b_{ij}^t + \sum_j r_j\right)$, then rearranging (3.60) gives $\sum_j p_j^{t+1} \leq \sum_j p_j^t$. Otherwise, replacing $\sum_j p_j^t$ by $\left(\frac{1+2\lambda-e^\lambda}{\lambda}\right)\left(\sum_{ij} b_{ij}^t + \sum_j r_j\right)$ gives $\sum_j p_j^{t+1} \leq \left(\left(e^\lambda - 2\lambda\right)\frac{1+2\lambda-e^\lambda}{\lambda} + \lambda\right)\left(E + \sum_j r_j\right)$. Thus $\sum_j p_j^{t+1} \leqslant \max\left\{\sum_j p_j^t, \left(\left(e^\lambda - 2\lambda\right)\frac{1+2\lambda-e^\lambda}{\lambda} + \lambda\right)\left(E + \sum_j r_j\right)\right\}$ and the result follows by induction on t. □

Proof of Lemma 3.3.4: For simplicity, we can assume that at any given time each buyer will spend all her money on just one item. To handle the general case, we partition each buyer into several buyers, each of whom buys one good. Then the same result follows.

So, here we use $j(i, \mathbf{p})$ to denote the item with max utility-per-dollar for buyer i at price $\mathbf{p}$ and buyer i spends the whole budget on this item. Note that

$$\begin{aligned} e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}_i^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}_i^t)} &= e_i \log \frac{a_{ij(i,\mathbf{p}^{t+1})}}{p_{j(i,\mathbf{p}^{t+1})}^{t+1}} - e_i \log \frac{a_{ij(i,\mathbf{p}^t)}}{p_{j(i,\mathbf{p}^t)}^t} \\ &= e_i \log \frac{a_{ij(i,\mathbf{p}^{t+1})} p_{j(i,\mathbf{p}^t)}^{t+1}}{a_{ij(i,\mathbf{p}^t)} p_{j(i,\mathbf{p}^{t+1})}^{t+1}} - b_{ij(i,\mathbf{p}^t)}^t \log \frac{p_{j(i,\mathbf{p}^t)}^{t+1}}{p_{j(i,\mathbf{p}^t)}^t} \\ &= e_i \log \frac{a_{ij(i,\mathbf{p}^{t+1})} p_{j(i,\mathbf{p}^t)}^{t+1}}{a_{ij(i,\mathbf{p}^t)} p_{j(i,\mathbf{p}^{t+1})}^{t+1}} - b_{ij(i,\mathbf{p}^t)}^t \Delta_{j(i,\mathbf{p}^t)}^t. \end{aligned} \tag{3.61}$$

We also know that

$$\frac{a_{ij(i,\mathbf{p}^t)}}{p_{j(i,\mathbf{p}^t)}^{t+1}} = \frac{a_{ij(i,\mathbf{p}^t)}}{p_{j(i,\mathbf{p}^t)}^t e^{\Delta_{j(i,\mathbf{p}^t)}^t}} \geq \frac{a_{ij(i,\mathbf{p}^{t+1})}}{p_{j(i,\mathbf{p}^{t+1})}^t e^{\Delta_{j(i,\mathbf{p}^t)}^t}} = \frac{a_{ij(i,\mathbf{p}^{t+1})} e^{\Delta_{j(i,\mathbf{p}^{t+1})}^t}}{p_{j(i,\mathbf{p}^{t+1})}^{t+1} e^{\Delta_{j(i,\mathbf{p}^t)}^t}}.$$

Therefore,

$$e_i \log \frac{a_{ij(i,\mathbf{p}^{t+1})} p_{j(i,\mathbf{p}^t)}^{t+1}}{a_{ij(i,\mathbf{p}^t)} p_{j(i,\mathbf{p}^{t+1})}^{t+1}} \leq e_i(\Delta_{j(i,\mathbf{p}^t)}^t - \Delta_{j(i,\mathbf{p}^{t+1})}^t) = \sum_j (b_{ij}^t - b_{ij}^{t+1})\Delta_j^t. \tag{3.62}$$

To see the final equality, note that $b_{ij(i,\mathbf{p}^t)} = e_i$ and $b_{ij} = 0$ for all other j; so $\sum_j b_{ij}^t \Delta_j^t = e_i \Delta_{j(i,\mathbf{p}^t)}^t$; likewise, $\sum_j b_{ij}^{t+1} \Delta_j^t = e_i \Delta_{j(i,\mathbf{p}^{t+1})}^t$.

Combining (3.61) and (3.62) yields the result. □

The remaining lemmas use the following observations from [CCD19].

$$\max_{\mathbf{x}_i \cdot \mathbf{p} = e_i} u_i(x_i) = \begin{cases} e_i \left(\sum_j a_{ij}^{1-c_i} p_j^{c_i} \right)^{-\frac{1}{c_i}} & \rho_i < 1 \\ e_i \max_j \{ \frac{a_{ij}}{p_j} \} & \rho_i = 1 \end{cases}, \tag{3.63}$$

And the best response to price $\mathbf{p}$ for $\rho_i < 1$ is

$$b_{ij} = e_i \frac{a_{ij}^{1-c_i} p_j^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} p_{j'}^{c_i}}. \tag{3.64}$$

Proof of Lemma 3.3.5: First, we decompose the LHS into two parts:

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})} + e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)}.$$

We start by bounding the first term. Note that

$$u_i(\mathbf{b}_i, \mathbf{p}) = \left(\sum_j a_{ij} \left(\frac{b_{ij}}{p_j} \right)^{\rho_i} \right)^{\frac{1}{\rho_i}}.$$

Using (3.64) yields:

$$\begin{aligned} e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})} &= \frac{e_i}{\rho_i} \log \frac{\sum_j a_{ij} \left(\frac{e_i \frac{a_{ij}^{1-c_i} (p_j^{t+1})^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^{t+1})^{c_i}}}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij} \left(\frac{e_i \frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^t)^{c_i}}}{p_j^{t+1}} \right)^{\rho_i}} \\ &= \frac{e_i}{\rho_i} \log \frac{\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^{t+1})^{c_i}}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{p_j^{t+1}} \right)^{\rho_i}} + e_i \log \frac{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^t)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^{t+1})^{c_i}}. \end{aligned}$$

Recall that $c_i = \frac{\rho_i}{\rho_i - 1}$. By calculation, $1 + (1 - c_i)\rho_i = 1 - c_i$ and $(c_i - 1)\rho_i = c_i$. So,

$$\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^{t+1})^{c_i}}{p_j^{t+1}} \right)^{\rho_i} = \sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i} \qquad \text{and}$$

$$\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^{t})^{c_i}}{p_j^{t+1}} \right)^{\rho_i} = \sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}$$

Therefore,

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^{t}, \mathbf{p}^{t+1})} = \underbrace{e_i \frac{1-\rho_i}{\rho_i} \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}}_{\mathcal{A}} + \underbrace{e_i \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}}_{\mathcal{B}}.$$

Note that, by (3.64), $b_{ij}^{t+1} = e_i \frac{a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} \left(p_{j'}^{t+1}\right)^{c_i}}$ and $b_{ij}^{t} = e_i \frac{a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} \left(p_{j'}^{t}\right)^{c_i}}$. Therefore,

$$\begin{aligned}
\mathcal{A} &= -e_i \frac{1-\rho_i}{\rho_i} \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}} \\
&= -e_i \frac{1-\rho_i}{\rho_i} \log \frac{\sum_j a_{ij}^{1-c_i} (p_j^{t+1})^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}} \\
&= -e_i \frac{1-\rho_i}{\rho_i} \log \sum_j \frac{b_{ij}^{t+1}}{e_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{c_i},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= -e_i \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^{t}\right)^{c_i}} \\
&= -e_i \log \sum_j \frac{b_{ij}^{t}}{e_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})} \\
&\qquad = -e_i \frac{1-\rho_i}{\rho_i} \log \sum_j \frac{b_{ij}^{t+1}}{e_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{\rho_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{c_i} - e_i \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{\rho_i}.
\end{aligned}
$$

As the log function is concave, $\log \sum_i a_i x_i \geq \sum_i a_i \log x_i$ when $\sum_i a_i = 1$; this yields:

$$
\begin{aligned}
e_i \frac{1-\rho_i}{\rho_i} \log \sum_j \frac{b_{ij}^{t+1}}{e_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{\rho_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{c_i} &\geq \sum_j b_{ij}^{t+1} (\rho_i + c_i) \frac{1-\rho_i}{\rho_i} \log \frac{p_j^t}{p_j^{t+1}} \\
&= -\rho_i \sum_j b_{ij}^{t+1} \log \frac{p_j^t}{p_j^{t+1}} \\
\text{and} \qquad e_i \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^t}{p_j^{t+1}}\right)^{\rho_i} &\geq \rho_i \sum_j b_{ij}^t \log \frac{p_j^t}{p_j^{t+1}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})} &\leq \rho_i \sum_j b_{ij}^{t+1} \log \frac{p_j^t}{p_j^{t+1}} - \rho_i \sum_j b_{ij}^t \log \frac{p_j^t}{p_j^{t+1}} \\
&= -\rho_i \sum_j b_{ij}^{t+1} \Delta_j^t + \rho_i \sum_j b_{ij}^t \Delta_j^t. \qquad (3.65)
\end{aligned}
$$

Now let's look at the second part, $e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)}$.

$$e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = \frac{e_i}{\rho_i} \log \frac{\sum_j a_{ij} \left(\frac{e_i \frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^t)^{c_i}}}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij} \left(\frac{e_i \frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} (p_{j'}^t)^{c_i}}}{p_j^{t}} \right)^{\rho_i}}$$

$$= \frac{e_i}{\rho_i} \log \frac{\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij} \left(\frac{a_{ij}^{1-c_i} (p_j^t)^{c_i}}{p_j^{t}} \right)^{\rho_i}}.$$

Recall that $1 + (1 - c_i)\rho_i = 1 - c_i$ and $(c_i - 1)\rho_i = c_i$. So,

$$e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = \frac{e_i}{\rho_i} \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^t\right)^{c_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^t\right)^{c_i}}.$$

Remember that $b_{ij}^t = e_i \frac{a_{ij}^{1-c_i} \left(p_j^t\right)^{c_i}}{\sum_{j'} a_{ij'}^{1-c_i} \left(p_{j'}^t\right)^{c_i}}$. Therefore,

$$e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = \frac{e_i}{\rho_i} \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i} \leq \frac{e_i}{\rho_i} \sum_j \frac{b_{ij}^t}{e_i} \left(\left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i} - 1 \right),$$

using the fact that $\log x \leq x - 1$ for the last inequality, and noting that $\sum_j b_{ij}^t = e_i$.

As $\frac{p_j^{t+1}}{p_j^t} = e^{\Delta_j^t}$ and $|\Delta_j^t| \leq 1$,

$$\frac{\left(\frac{p_j^t}{p_j^{t+1}} \right)^{\rho_i} - 1}{\rho_i} \leq -\Delta_j^t + \rho_i \left(\Delta_j^t\right)^2.$$

Therefore,

$$e_i \log \frac{u_i(\mathbf{b}_i^t, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} \le -\sum_j b_{ij}^t \Delta_j^t + \sum_j b_{ij}^t \rho_i \left(\Delta_j^t\right)^2. \tag{3.66}$$

Combining (3.65) and (3.66) gives the result. □

The following claim is used in the final two results.

Claim 3.3.10. *If* $\rho < 1$,

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = -\frac{e_i}{c_i} \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^{t+1}}{p_j^t}\right)^{c_i}.$$

Proof. As $\rho_i < 1$, by (3.63), $\max_{x_i \cdot p = e_i} u_i(x_i) = e_i \left(\sum_j a_{ij}^{1-c_i} p_j^{c_i}\right)^{-\frac{1}{c_i}}$. Thus,

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = -\frac{e_i}{c_i} \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^t\right)^{c_i}}.$$

Substituting from (3.64) gives

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} = -\frac{e_i}{c_i} \log \frac{\sum_j a_{ij}^{1-c_i} \left(p_j^{t+1}\right)^{c_i}}{\sum_j a_{ij}^{1-c_i} \left(p_j^t\right)^{c_i}} = -\frac{e_i}{c_i} \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^{t+1}}{p_j^t}\right)^{c_i}.$$

□

Proof of Lemma 3.3.6: By applying Claim 3.3.10, and noting that $p_j^{t+1} = p_j^t e^{\Delta_j^t}$, $|\Delta_j^t| \le$

λ, and $c_i < 0$, yields

$$\begin{aligned}
e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}^t)} &= -\frac{e_i}{c_i} \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^{t+1}}{p_j^t} \right)^{c_i} \\
&\leq -\frac{e_i}{c_i} \sum_j \frac{b_{ij}^t}{e_i} \left[\left(\frac{p_j^{t+1}}{p_j^t} \right)^{c_i} - 1 \right] \\
&\qquad \text{(using } \log x \leq x - 1\text{)} \\
&= -\frac{e_i}{c_i} \sum_j \frac{b_{ij}^t}{e_i} \left(e^{c_i \Delta_j^t} - 1 \right) \\
&\leq -\sum_j b_{ij}^t \frac{c_i \Delta_j^t + \left(c_i \Delta_j^t \right)^2}{c_i} \\
&\qquad \text{(using } e^x \leq 1 + x + x^2 \text{ if } -1 \leq x = c_i \Delta_j^t \leq 1\text{)} \\
&= -\sum_j b_{ij}^t \Delta_j^t - \sum_j b_{ij}^t c_i \left(\Delta_j^t \right)^2 .
\end{aligned}$$

□

Proof of Lemma 3.3.7: As $\rho \leqslant 0$, by Claim 3.3.10,

$$e_i \log \frac{u_i(\mathbf{b}_i^{t+1}, \mathbf{p}_i^{t+1})}{u_i(\mathbf{b}_i^t, \mathbf{p}_i^t)} = -\frac{e_i}{c_i} \log \sum_j \frac{b_{ij}^t}{e_i} \left(\frac{p_j^{t+1}}{p_j^t} \right)^{c_i} \leq -\frac{e_i}{c_i} \sum_j \frac{b_{ij}^t}{e_i} \log \left(\frac{p_j^{t+1}}{p_j^t} \right)^{c_i} \leq -\sum_j b_{ij}^t \Delta_j^t.$$

The first inequality holds as log is a concave function, $c_i > 0$ for complementary buyers, and $\sum_j \frac{b_{ij}^t}{e_i} = 1$; the final inequality uses $\Delta_j^t = \log \frac{p_j^{t+1}}{p_j^t}$. □

Proof of Claim 3.3.8: If $z_j = -1$ then $\sum_j b_{ij}^t = 0$ and the claim holds. Otherwise, $z_j > -1$ and

$$\begin{aligned}
p_j^t z_j^t \Delta_j^t = \left(\sum_i b_{ij}^t - p_j \right) \Delta_j^t &= \left(\sum_j b_{ij}^t \right) \left(1 - \frac{1}{1 + z_j^t} \right) \Delta_j^t \\
&\geq \left(\sum_i b_{ij}^t \right) \left(1 - \frac{1}{1 + \frac{\Delta_j^t}{\lambda}} \right) \Delta_j^t.
\end{aligned}$$

If $z_j^t \geq 0$, then $0 \leq \frac{\Delta_j^t}{\lambda} \leq 1$. This implies

$$1 - \frac{1}{1 + \frac{\Delta_j^t}{\lambda}} \geq \frac{\Delta_j^t}{2\lambda};$$

and if $z_j^t < 0$, then $-\lambda < \Delta_j^t \leq 0$. This implies

$$1 - \frac{1}{1 + \frac{\Delta_j^t}{\lambda}} \leq \frac{\Delta_j^t}{\lambda} < \frac{\Delta_j^t}{2\lambda}.$$

The result now follows. □

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