Quantum Information Physics II

TR2021-997

Davi Geiger and Zvi M. Kedem

Courant Institute of Mathematical Sciences

New York University, New York, New York 10012

Abstract

We study quantum entropy, a measure of randomness over the degrees of freedom of a quantum state and quantified in phase spaces. We show that it is dimensionless, a relativistic scalar, and it is invariant under coordinate and CPT transformations.

We show that the entropy evolution of a coherent state is increasing with time. We augment time reversal with time translation and show that CPT with time translation can transform particles with decreasing entropy evolution into anti-particles with increasing entropy evolution. We revisit transition probabilities of a two state Hamiltonian and show how it relates to entropy oscillation. We study the entropy of a spin phase space and apply it to the study of the entanglement states.

We also explore the possibility that entropy oscillations trigger the annihilations and the creations of particles.

CONTENTS

INTRODUCTION

A time arrow emerges in physics only when a probabilistic behavior of ensembles of particles is considered in classical physics. In contrast, quantum physics is presented as time reversible even though a probabilistic behavior is intrinsic even to a single-particle. In [10] we proposed a definition of quantum entropy to measure the randomness of a quantum state, while accounting for all its degrees of freedom (DOFs). That entropy is a sum of two components: the coordinate-entropy and the spin-entropy, each defined in its own phase space. We analyzed the possible entropy evolution and conjectured that a law analogous to the classical second law of thermodynamics holds, applicable to all particle physics.

This paper provides more technical depth, to further develop the issues raised in [10]. The results are applicable to both the Quantum Mechanics (QM) and the Quantum Field Theory (QFT) settings, but we will generally present them in the more convenient setting. We first derive the spin-entropy for a single particle and study the case of entanglement for two particles of spin $\frac{1}{2}$. Then we further develop the coordinate-entropy for multiple particles. We show that the coordinate-entropy is invariant under changes in continuous 3D coordinate transformations, continuous Lorentz transformations, and discrete CPT transformations. We then analyze the evolution of coherent states. We study time reflection of particles' evolution and the impact of the transformation into anti-particles. We study entropy oscillations for a two-state Hamiltonian and their relation to Fermi's golden rule. Following the results presented here, we review a conjectured entropy law that the entropy of a quantum system is an increasing function of time, and end with conclusions.

QUANTUM ENTROPY IN PHASE SPACES

Spin-Entropy

The DOFs associated with the spin are captured by the vector or bispinor representation of the states in both frameworks, QM and QFT. The spin matrix associated with a particle can be specified (e.g., [5]) as

$$\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z} \quad \text{and} \quad S^2 = S_x^2 + S_y^2 + S_z^2\,,$$

$$[S_a, S_b] = \mathrm{i}\hbar S_c\,, \quad \text{where } a, b, c \text{ is a cyclic permutation of } x, y, z\,,$$

$$[S^2, S_a] = 0\,, \quad \text{for } a = x, y, z\,.$$

The spin value of a particle is a Casimir invariant but it is not possible to simultaneously know the spin of a particle in all three dimensional directions. Knowing the z-direction spin does not imply that we know the x- or the y-direction spins as captured by the non-commutative property of the spin operators and the Stern-Gerlach experiment [11]. This uncertainty, or randomness, is the intrinsic source for the non-zero spin-entropy. The uncertainty also reflects the close relation between spin matrices, their unitary transformations, and the rotation group SO(3). For spin 0 particles (Higgs bosons), there is no spin uncertainty and so the spin-entropy is 0. For spin $\frac{1}{2}$ particles, any spin state is reachable from any other spin state via a 2×2 unitary transformation, a local isomorphism (and a global homomorphism) to the SO(3) group. For spin 1, the matrices are unitarily similar to SO(3), and one can transform them into generators of SO(3) via unitary transformations. We then characterize the spin phase space by considering simultaneously a spin state along x, y, and z directions, i.e., a spin state $|\xi\rangle$ is represented in spin phase space as $(|\xi\rangle_x\,, |\xi\rangle_y\,, |\xi\rangle_z)$. The spin-entropy is then

$$S = -\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} P(i,j,k)\ln P(i,j,k)\,, \tag{1}$$

where $P(i,j,k) = P_x(i)P_y(j)P_z(k)$, $\{P_a(i) = |\langle i|\xi\rangle_a|^2; i = 1,2,3, a = x,y,z\}$, $\langle i|j\rangle = \delta_{ij}$, $i,j = 1,2,3$, and $|\xi\rangle_z = |\xi\rangle$.

Preparing a spin state to align with a particular direction, say z direction as it is done in many experiments with spin, e.g., [11], implies full knowledge of one of the spin directions. We will show that this knowledge, and thus this preparation, reduce the spin-entropy.

A spin state of a particle with spin value $\frac{1}{2}$ is represented by a set of two orthonormal eigenstates $|+\rangle = (1,0)^\mathsf{T}$ and $|-\rangle = (0,1)^\mathsf{T}$ of the operators (S^2, S_z), with associated eigenvalues $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$.

Theorem 1 (Spin-Entropy $s = \frac{1}{2}$)**.** *A general description of the state of a particle with spin* $\frac{1}{2}$ *is* $|\xi\rangle_z = e^{i\varphi}\left(e^{i\nu}\cos\theta_\alpha\,|+\rangle + \sin\theta_\alpha\,|-\rangle\right) = e^{i\varphi}\left(e^{i\nu}\cos\theta_\alpha,\ \sin\theta_\alpha\right)^\mathsf{T}$*, where* $\theta_\alpha \in [0, \frac{\pi}{2}]$ *and* $\varphi, \nu \in [0, 2\pi)$*. The spin entropy of this state is*

$$S_{\frac{1}{2}} = -\sum_{a=x,y,z}\ \sum_{\text{sign}=-,+} P_a^{\text{sign}}(\theta_\alpha,\nu)\ln P_a^{\text{sign}}(\theta_\alpha,\nu)\,, \tag{2}$$

where $P_x^{\pm}(\theta_\alpha,\nu) = \frac{(1\pm\sin 2\theta_\alpha\cos\nu)}{2}$, $P_y^{\pm}(\theta_\alpha,\nu) = \frac{(1\pm\sin 2\theta_\alpha\sin\nu)}{2}$, $P_z^{+}(\theta_\alpha) = \cos^2\theta_\alpha$*, and* $P_z^{-}(\theta_\alpha) = \sin^2\theta_\alpha$*.*

Proof. A state $|\xi\rangle_z$ assigns a probability distribution $P_z = \left(|\langle 1|\xi\rangle_z|^2,\ |\langle 2|\xi\rangle_z|^2\right)^\mathsf{T} = \left(\cos^2\theta_\alpha,\ \sin^2\theta_\alpha\right)^\mathsf{T}$. To calculate P_x we write the x-basis as eigenvectors of S_x and S^2, i.e., the basis matrix $B_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ made of column eigenvectors. Then, $|\xi\rangle_x = e^{i\varphi}B_x^{-1}|\xi\rangle_z$ with corresponding probabilities $P_x = \frac{1}{2}\left(1+\sin 2\theta_\alpha\,\cos\nu,\ 1-\sin 2\theta_\alpha\,\cos\nu\right)^\mathsf{T}$. Similarly, the eigenvectors of S_y and S^2 form the y-basis $B_y = \frac{1}{\sqrt{2}}\begin{pmatrix}-\mathrm{i} & \mathrm{i}\\ 1 & 1\end{pmatrix}$; and $|\xi\rangle_y = \mathrm{e}^{\mathrm{i}\varphi} B_y^{-1}\,|\xi\rangle_z$, $\mathrm{P}_y = \frac{1}{2}\left(1+\sin 2\theta_\alpha\ \sin\nu,\ 1-\sin 2\theta_\alpha\ \sin\nu\right)^{\mathsf{T}}$. Replacing these expressions for $\mathrm{P}_x, \mathrm{P}_y, \mathrm{P}_z$ into (1) we obtain the spin-entropy (2). For a visualization of the entropy, see Figure 2a. □

For spin $s = 1$ we must evaluate the internal uncertainty of the gauge field $A^\mu(\mathbf{r}, t)$, a vector under Lorentz transformation. For a massless field, the quantized electric field $E(\mathbf{r}, t)$ and magnetic field $B(\mathbf{r}, t)$ form the phase space for each space-time coordinate $(\mathbf{r}, t)$, where their commutation properties leads to an uncertainty relation. For massive particles with spin $s = 1$, the spin matrices ([5])

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-\mathrm{i}&0\\ \mathrm{i}&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix} \qquad S_z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \qquad S^2 = \begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

yield a basis representation for spin with the eigenvectors of S_z and S^2

$$|{\uparrow}\rangle_z = \left(1,\ 0,\ 0\right)^{\mathsf{T}}, \qquad |{\rightarrow}\rangle_z = \left(0,\ 1,\ 0\right)^{\mathsf{T}}, \qquad |{\downarrow}\rangle_z = \left(0,\ 0,\ 1\right)$$

Theorem 2 (Massive Spin-Entropy $s = 1$)**.** *A general state of spin $s = 1$ is*

$$|\xi\rangle = \mathrm{e}^{\mathrm{i}\varphi_y}\left(\cos\theta_\alpha\cos\theta_\beta\,\mathrm{e}^{\mathrm{i}\varphi_x}\,|{\uparrow}\rangle_z + \sin\theta_\alpha\,|{\rightarrow}\rangle_z + \cos\theta_\alpha\sin\theta_\beta\,\mathrm{e}^{\mathrm{i}\varphi_z}\,|{\downarrow}\rangle_z\right) \tag{3}$$

where $\theta_\alpha, \theta_\beta \in [0, \frac{\pi}{2}]$, $\varphi_x, \varphi_z, \varphi_y \in [0, 2\pi)$. The spin-entropy of this state is

$$\mathrm{S}_1 = -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mathrm{P}_x(i)\mathrm{P}_y(j)\mathrm{P}_z(k)\ln\left(\mathrm{P}_x(i)\mathrm{P}_y(j)\mathrm{P}_z(k)\right), \tag{4}$$

where

$$P_{\substack{x\\y}} = \frac{1}{4}\begin{pmatrix} 1+\sin^2\theta_\alpha \pm \sin 2\theta_\beta \cos^2\theta_\alpha \cos(\varphi_x-\varphi_z) + \sqrt{2}\sin 2\theta_\alpha(\sin\theta_\beta|^{\cos\varphi_z}_{\sin\varphi_z} + \cos\theta_\beta|^{\cos\varphi_x}_{-\sin\varphi_x}) \\ 2\cos^2\theta_\alpha \mp 2\sin 2\theta_\beta\cos^2\theta_\alpha\cos(\varphi_x-\varphi_z) \\ 1+\sin^2\theta_\alpha \pm \sin 2\theta_\beta \cos^2\theta_\alpha \cos(\varphi_x-\varphi_z) - \sqrt{2}\sin 2\theta_\alpha(\sin\theta_\beta|^{\cos\varphi_z}_{\sin\varphi_z} + \cos\theta_\beta|^{\cos\varphi_x}_{-\sin\varphi_x}) \end{pmatrix}$$

$$\mathrm{P}_z = \begin{pmatrix} \cos^2\theta_\alpha\cos^2\theta_\beta \\ \sin^2\theta_\alpha \\ \cos^2\theta_\alpha\sin^2\theta_\beta \end{pmatrix}$$

Proof. The state $|\xi\rangle$ in the $z-$axis basis is given by (3), and describing it in the $y-$axis basis $B_y^{s=1} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & -1 \\ -\sqrt{2}\,\mathrm{i} & 0 & \sqrt{2}\,\mathrm{i} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$ and in the $x-$axis basis $B_x^{s=1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$, we get $|\xi\rangle_y = (B_y^{s=1})^{-1}|\xi\rangle$, $|\xi\rangle_x = (B_x^{s=1})^{-1}|\xi\rangle$ yielding $\{\mathrm{P}_a(i) = |\langle i|\xi\rangle_a|^2;\ i = 1,2,3,\ a = x,y,z\}$, where $\langle i|j\rangle = \delta_{ij}$, $i,j = 1,2,3$, and $|\xi\rangle_z = |\xi\rangle$. Performing the matrix calculations completes the proof. □

For a visualization of the spin-entropy for some parameters see Figure1.

Theorem 3. *The minimum spin-entropy for* $s = 0, \frac{1}{2}, 1$ *is* $2\theta(s)\ln 2$*, where* $\theta(s) = 1$ *for* $s > 0$ *and zero otherwise.*

Proof. The spin-entropy for $s = 0$ is 0. Relying on the numerical minimization functionality of Wolfram Mathematica®, we obtained (i) for $\mathrm{S}_{\frac{1}{2}}$ the minimum of $2\ln 2$ at $\theta_\alpha = \frac{\pi}{4}$, $\nu = 0$; or $\theta_\alpha = \frac{\pi}{4}$, $\nu = \frac{\pi}{2}$; or $\theta_\alpha = 0$ (describing when x-, y-, or z-axis are oriented along the state); (ii) for S_1 we obtained the minimum of $2\ln 2$ at $\theta_\alpha = \frac{\pi}{2}$ (eigenstate $|\rightarrow\rangle$), $\theta_\alpha = \frac{\pi}{2}$, $\theta_\alpha = 0, \theta_\beta = \varphi_x = \varphi_z = \frac{\pi}{4}$. This concludes the proof. □

We also observed that for $s = 1$ the local minimum $3\ln 2$ occurs at $|\uparrow\rangle$, defined by $\theta_\alpha = 0, \theta_\beta = 0$ and $|\downarrow\rangle$, defined by $\theta_\alpha = 0, \theta_\beta = \frac{\pi}{2}$; and at other states. Thus, preparing a spin state orientation, which allows it to be aligned with a given coordinate system, reduces the entropy either to local or global minimum.

Spin Entanglement

We now examine the spin-entropy of a system with two particles with spin $s = \frac{1}{2}$. It is known that that a product of two independent spin particles $|\xi_{AB}\rangle_z = e^{i\varphi_A}\left(e^{i\nu_A}\cos\alpha_A\,|+\rangle^A + \sin\alpha_A\,|-\rangle^A\right) e^{i\varphi_B}\left(e^{i\nu_B}\cos\alpha_B\,|+\rangle^B + \sin\alpha_B\,|-\rangle^B\right)$ does not cover all two-particle states, described by $\left|\xi_{\{A,B\}}\right\rangle_z = a_{++}\,|+\rangle^A\,|+\rangle^B + a_{+-}\,|+\rangle^A\,|-\rangle^B + a_{-+}\,|-\rangle^A\,|+\rangle^B + a_{--}\,|-\rangle^A\,|-\rangle^B$, where $a_{++}, a_{+-}, a_{-+}, a_{--}$ are complex value coefficients with $|a_{++}|^2 + |a_{+-}|^2 + |a_{-+}|^2 + |a_{--}|^2 = 1$. In particular, entangled states such as $a_{++} = a_{--} = 0$, $a_{+-} = a_{-+} = \frac{1}{\sqrt{2}}$ can not be described by the product of two states. We now explore entangled states further.

Theorem 4 (Two spin $s = \frac{1}{2}$ entanglement)**.** *Consider a system with two identical particles, A and B, each with $s = \frac{1}{2}$ in an entangled state*

$$|\xi\rangle = e^{i\alpha}\left[\cos\theta_{AB}\ |+\rangle^A\,|-\rangle^B - \sin\theta_{AB}\,|-\rangle^A\,|+\rangle^B\right] \tag{5}$$

where $\pm$ varies over the two z-components and $\theta_{AB} \in [0, 2\pi)$. The spin-entropy of $|\xi\rangle$ is

$$\mathrm{S}_{A^\pm,B^\mp}(\theta_{AB}) = -\frac{3}{2}\left[(1-\sin 2\theta_{AB})\ln(1-\sin 2\theta_{AB}) + (1+\sin 2\theta_{AB})\ln(1+\sin 2\theta_{AB})\right]$$

$$+ (4-\sin 2\theta_{AB})\ln 2\,. \tag{6}$$

Proof. The four vectors of the spin basis for two fermions along the z-axis are $|++\rangle = |+\rangle^A\,|+\rangle^B$, $|-+\rangle = |-\rangle^A\,|+\rangle^B$, $|-+\rangle = |-\rangle^A\,|+\rangle^B$, and $|--\rangle = |-\rangle^A\,|-\rangle^B$; and in this basis the state (5) is $|\xi\rangle_z = e^{i\alpha}\left(0,\ \cos\theta_{AB},\ -\sin\theta_{AB},\ 0\right)^{\mathsf{T}}$, with spin matrices

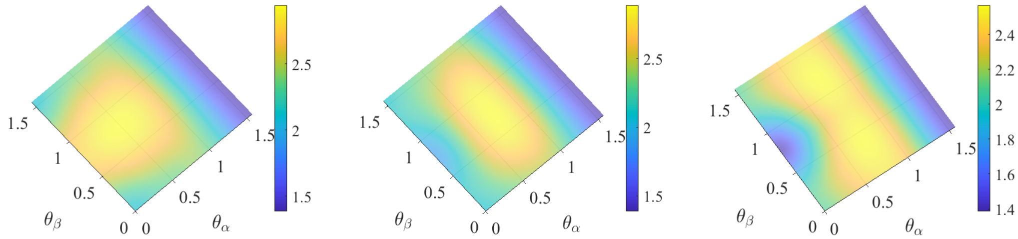
$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

A basis simultaneously constructed from eigenvectors of S_z and S^2 is $\left\{ |{++}\rangle, \frac{1}{\sqrt{2}}(|{+-}\rangle + |{-+}\rangle), \frac{1}{\sqrt{2}}(|{+-}\rangle - |{-+}\rangle), |{--}\rangle \right\}$, and they can be written as a matrix of eigenvector columns $B_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then, the state (5) and its probability distribution is written in this basis as

$$|\xi\rangle_z = B_z^{-1}\,|\xi\rangle = \mathrm{e}^{\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \cos\theta_{AB} - \sin\theta_{AB} \\ \cos\theta_{AB} + \sin\theta_{AB} \\ 0 \end{pmatrix} \quad \rightarrow \quad \mathrm{P}_z = \frac{1}{2}\begin{pmatrix} 0 \\ 1 - \sin 2\theta_{AB} \\ 1 + \sin 2\theta_{AB} \\ 0 \end{pmatrix}.$$

A simultaneous eigenvector basis B_x to S_x, S^2 produces the state $|\xi\rangle$ in such basis as

$$B_x = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 0 & -1 \\ 1 & 0 & \sqrt{2} & 1 \\ 1 & 0 & -\sqrt{2} & 1 \\ 1 & -\sqrt{2} & 0 & -1 \end{pmatrix} \quad \rightarrow \quad |\xi\rangle_x = B_x^{-1}\,|\xi\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{2}\begin{pmatrix} \cos\theta_{AB} - \sin\theta_{AB} \\ 0 \\ \sqrt{2}(\cos\theta_{AB} + \sin\theta_{AB}) \\ \cos\theta_{AB} - \sin\theta_{AB} \end{pmatrix},$$



a. $\varphi_x = 0, \varphi_z = \frac{\pi}{2}$ b. $\varphi_x = \frac{\pi}{3}, \varphi_z = \frac{\pi}{2}$ c. $\varphi_x = \frac{\pi}{4}, \varphi_z = \frac{\pi}{4}$

Figure 1. Spin-Entropy for $s = 1$ (4) vs $(\theta_\alpha, \theta_\beta)$ for a fixed set of parameters φ_x, φ_z. Note that for $\theta_\alpha = \frac{\pi}{2}$ and for $\theta_\alpha = 0, \theta_\beta = \varphi_x = \varphi_z = \frac{\pi}{4}$ the spin-entropy reaches its minimum $2\ln 2 \approx 1.3863$.

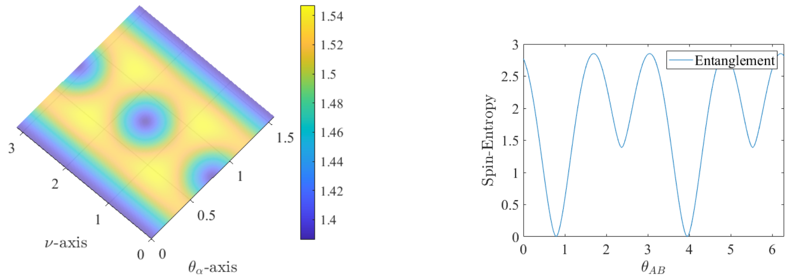
and similarly for the eigenvector basis B_y to S_y, S^2

$$B_y = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & \sqrt{2} & 0 & \mathrm{i} \\ 1 & 0 & \sqrt{2} & 1 \\ 1 & 0 & -\sqrt{2} & 1 \\ \mathrm{i} & \sqrt{2} & 0 & -\mathrm{i} \end{pmatrix} \quad \to \quad |\xi\rangle_y = B_y^{-1}\,|\xi\rangle = \frac{e^{\mathrm{i}\alpha}}{2}\begin{pmatrix} \cos\theta_{AB} - \sin\theta_{AB} \\ 0 \\ \sqrt{2}(\cos\theta_{AB} + \sin\theta_{AB}) \\ \cos\theta_{AB} - \sin\theta_{AB} \end{pmatrix},$$

with probability distributions $\mathrm{P}_x = \mathrm{P}_y = \Big(\frac{1}{4}(1-\sin 2\theta_{AB}),\ 0,\ \frac{1}{2}(1+\sin 2\theta_{AB}),\ \frac{1}{4}(1-\sin 2\theta_{AB})\Big)^{\mathsf{T}}$. Thus, after some manipulations to compute S from (1) we derive (6). For a plot, see Figure 2b. □

Clearly, the eigenvalue 0 of S^2 has only one eigenvector, $\frac{1}{\sqrt{2}}(|{+-}\rangle - |{-+}\rangle)$, and it is common to all three eigenvector basis. Thus, the entangled state for $\theta_{AB} = \frac{\pi}{4}$, $|\xi\rangle_{\theta_{AB}=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(|{+-}\rangle - |{-+}\rangle)$, has the entropy of 0. The minimum entropy for a state with eigenvalue 1 of S^2 occurs for the eigenstates of S^2 that are simultaneously eigenstates of S_x or of S_y or of S_z, such as $\mathrm{S}_{A^\pm,B^\mp}(\theta_{AB} = \frac{3\pi}{4}) = 2\ln 2$.

We will now compare the entropy of a system of two independent particles with that of a system of two entangled particles. Consider two independent states



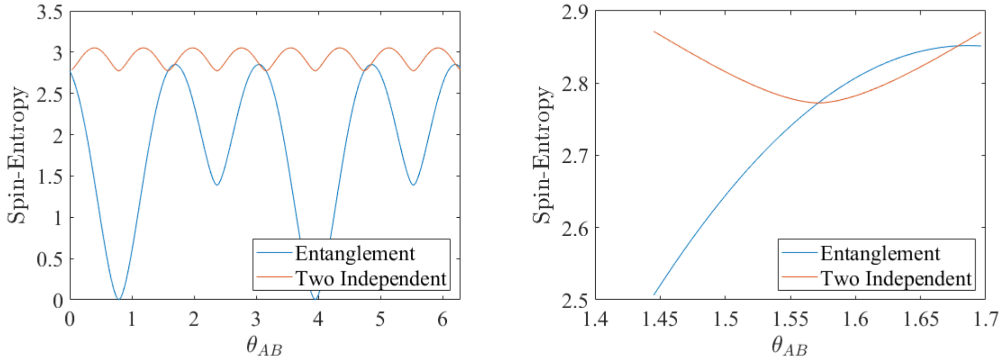
a. Fermion Spin-Entropy (2) vs (θ_α, ν) b. Entangled State Spin-Entropy (6) vs θ_{AB}

Figure 2. a. A surface plot of a fermion with $s = \frac{1}{2}$, with z-axis spin-entropy (2) vs x-,y-axis $(\theta_\alpha \in [0, \frac{\pi}{2}), \nu \in [0, \pi))$. b. A plot of the entanglement spin-entropy (6) vs the angle $\theta_{AB} \in [0, 2\pi]$. Note that for $\theta_{AB} = \frac{\pi}{4} + n\pi; n = 0, 1, \ldots$ the spin-entropy is zero. The entropy also never reaches the largest possible sum of two independent fermions, which is ≈ 3.08.

$\left|\xi^A\right\rangle = \cos\theta_{AB} \left|+\right\rangle - \sin\theta_{AB} \left|-\right\rangle$ and $\left|\xi^B\right\rangle = -\sin\theta_{AB} \left|+\right\rangle + \cos\theta_{AB} \left|-\right\rangle$. The probabilities of finding those particles in $\left|+\right\rangle$ and $\left|-\right\rangle$ are the same as in the entangled state (5). Of course, the product of two independent particles does reach states $\left|+\right\rangle\left|+\right\rangle$ and $\left|-\right\rangle\left|-\right\rangle$ that the entanglement does not, which motivates the examinations of the entropies. The entropies associated with $\left|\xi^A\right\rangle$ and $\left|\xi^B\right\rangle$, $\mathrm{S}^A_{\frac{1}{2}}(\theta_{AB})$ and $\mathrm{S}^B_{\frac{1}{2}}(\theta_{AB})$, are obtained from (2), where for $\left|\xi^A\right\rangle$, $\varphi = \nu = \pi$ and $\theta_\alpha = \theta_{AB}$; and for $\left|\xi^B\right\rangle$, $\varphi = \nu = 0$ and $\theta_\alpha = \theta_{AB} + \frac{\pi}{2}$.

We plot both $\mathrm{S}_{A^\pm,B^\mp}(\theta_{AB})$ and $\mathrm{S}_{A,B}(\theta_{AB}) = \mathrm{S}^A_{\frac{1}{2}}(\theta_{AB}) + \mathrm{S}^B_{\frac{1}{2}}(\theta_{AB})$ in Figure 3. For most values of θ_{AB} the entropy of the entanglement is lower than that of two independent particles. However, as seen in Figure 3b., there is a small interval of values of θ_{AB} where the entanglement spin-entropy is greater than the spin-entropy $\mathrm{S}_{A,B}(\theta_{AB})$. This interval is $[n\frac{\pi}{2}, \theta_{AB}^{\max \mathrm{S}_{A^\pm,B^\mp}}(n)]$ for $n \in \mathbb{Z}^*$, where $\theta_{AB} = n\frac{\pi}{2}$ produces the $\mathrm{S}_{A,B}$ minima and $\theta_{AB}^{\max \mathrm{S}_{A^\pm,B^\mp}}(n) \approx 0.113 + n\frac{\pi}{2} = \arg\max_{\theta_{AB}} \mathrm{S}_{A^\pm,B^\mp}(\theta_{AB})$.

Works exist exploring the information content of entangled physical systems, e.g., [2, 3, 12]. They considered the von Neumann entropy for $\left|\xi\right\rangle$, which is zero, but after tracing out a particle state, they obtain an entropy associated with the two probabilities $\mathrm{P}_{A_+B_-}(\theta_{AB}) = \cos^2\theta_{AB}$ and $\mathrm{P}_{A_-B_+}(\theta_{AB}) = \sin^2\theta_{AB}$. In contrast, our



a. $\mathrm{S}_{A,B}$ (red) and $\mathrm{S}_{A^{\pm},B^{\mp}}$ (blue) vs θ_{AB} b. Close up of a.

Figure 3. A plot of the entanglement spin-entropy (6) in blue overlaid with the two independent $s = \frac{1}{2}$ particles spin-entropy $\mathrm{S}_{A,B}(\theta_{AB}) = \mathrm{S}_A(\theta_{AB}) + \mathrm{S}_B(\theta_{AB})$ in red vs the angle θ_{AB}. In a. $\theta_{AB} \in [0, 2\pi]$ where the entanglement spin-entropy is mostly smaller then $\mathrm{S}_{A,B}(\theta_{AB})$. In b. Close up of the plot in a. for the range $\theta_{AB} \in [1.4, 1.7]$, showing an interval in which entanglement's spin-entropy is greater than $\mathrm{S}_{A,B}(\theta_{AB})$.

proposed entropy captures the information difference between the two entangled states, for $\theta_{AB} = \frac{\pi}{4}$ and for $\theta_{AB} = \frac{3\pi}{4}$. Experiments exploiting the projection of such states along perpendicular directions (in analogy to the [11]) could realize the larger amount of randomness of the $\theta_{AB} = \frac{3\pi}{4}$ entangled state.

Coordinate-Entropy of Multiple Particles

We now consider the coordinate-entropy in phase space, which was defined in [10] as

$$\mathrm{S} = -\int \rho_{\mathrm{r}}(\mathbf{r},t)\rho_k(\mathbf{k},t)\ \ln\left(\rho_{\mathrm{r}}(\mathbf{r},t)\rho_k(\mathbf{k},t)\right)\ \mathrm{d}^3\mathbf{r}\ \mathrm{d}^3\mathbf{k} = \mathrm{S_r} + \mathrm{S}_k\,,$$

where $\mathrm{S_r} = -\int \rho_{\mathrm{r}}(\mathbf{r},t)\ln\rho_{\mathrm{r}}(\mathbf{r},t)\ \mathrm{d}^3\mathbf{r}$, and analogously for $\mathrm{S_k}$, $\rho_{\mathrm{r}}(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and $\rho_k(\mathbf{k},t) = |\tilde{\phi}(\mathbf{k},t)|^2$, with $\psi(\mathbf{r},t)$ and $\tilde{\phi}(\mathbf{k},t)$ representing in QM the wave function and in QFT the coefficients of the Fock states. The momentum is described by the change of variables $\mathbf{p} = \hbar\mathbf{k}$, so that the entropy is dimensionless and invariant under changes of the units of measurements.

A natural extension of this entropy to an N-particle QM system is

$$\begin{aligned} S &= -\int \mathrm{d}^3\mathbf{r}_1\,\mathrm{d}^3\mathbf{k}_1 \ldots \mathrm{d}^3\mathbf{r}_N\,\mathrm{d}^3\mathbf{k}_N\,\rho_\mathrm{r}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\rho_\mathrm{k}(\mathbf{k}_1,\ldots,\mathbf{k}_N,t) \\ &\qquad \times \ln\left(\rho_\mathrm{r}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\rho_\mathrm{k}(\mathbf{k}_1,\ldots,\mathbf{k}_N,t)\right) \\ &= -\int \mathrm{d}^3\mathbf{r}_1 \ldots \int \mathrm{d}^3\mathbf{r}_N\,\rho_\mathrm{r}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\ln\rho_\mathrm{r}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) \\ &\quad -\int \mathrm{d}^3\mathbf{k}_1 \ldots \int \mathrm{d}^3\mathbf{k}_N\,\rho_\mathrm{k}(\mathbf{p}_1,\ldots,\mathbf{k}_N,t)\ln\rho_\mathrm{k}(\mathbf{k}_1,\ldots,\mathbf{k}_N,t)\,, \end{aligned}$$

where $\rho_\mathrm{r}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) = |\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)|^2$ and $\rho_\mathrm{k}(\mathbf{k}_1,\ldots,\mathbf{k}_N,t) = |\phi(\mathbf{k}_1,\ldots,\mathbf{k}_N,t)|^2$ are defined in QM via the projection of the state $|\psi_t\rangle^N$ of N particles (the product of N Hilbert spaces) onto the position $\langle\mathbf{r}_1|\ldots\langle\mathbf{r}_N|$ and the momentum $\langle\mathbf{k}_1|\ldots\langle\mathbf{k}_N|$ coordinate systems.

ENTROPY INVARIANT PROPERTIES

Continuous Transformations of the Phase Space

In the QM setting, we investigate a point transformation of coordinates and a translation in phase space of a quantum reference frame [1].

Consider a point transformation of position coordinates $F : \mathbf{r} \mapsto \mathbf{r}'$. It induces the new conjugate momentum operator [6]

$$\hat{\mathbf{p}}' = -\mathrm{i}\hbar\left[\nabla_{\mathrm{r}'} + \frac{1}{2}J^{-1}(\mathbf{r}')\nabla_{\mathrm{r}'}\cdot J(\mathbf{r}')\right]\,, \tag{7}$$

where $J(\mathbf{r}') = \frac{\partial\mathbf{r}(\mathbf{r}')}{\partial\mathbf{r}'}$ is the Jacobian of F^{-1}.

Theorem 5. *The entropy is invariant under a point transformation of coordinates.*

Proof. Let S be the entropy in phase-space relative to a conjugate Cartesian pair of coordinates $(\mathbf{r},\mathbf{p})$. Let $\mathbf{p}'$ be the momentum conjugate to $\mathbf{r}'$. As the probabilities in infinitesimal volumes are invariant,

$$|\psi'(\mathbf{r}')|^2\,\mathrm{d}^3\mathbf{r}' = |\psi(\mathbf{r}(\mathbf{r}'))|^2\,\mathrm{d}^3\mathbf{r}(\mathbf{r}') \text{ and } |\tilde{\phi}'(\mathbf{p}')|^2\,\mathrm{d}^3\mathbf{p}' = |\tilde{\phi}(\mathbf{p}(\mathbf{p}'))|^2\,\mathrm{d}^3\mathbf{p}(\mathbf{p}')\,. \quad (8)$$

Thus, by Born's rule the probability density functions are $|\psi'(\mathbf{r}')|^2$ and $|\tilde{\phi}'(\mathbf{p}')|^2$. The Jacobian satisfies $\det J(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}' = \mathrm{d}^3\mathbf{r}$, and combining this with (8),

$$\frac{1}{\sqrt{\det J(\mathbf{r}')}}\psi'(\mathbf{r}') = \psi(\mathbf{r}(\mathbf{r}'))\,, \quad (9)$$

so the infinitesimal probability $|\psi'(\mathbf{r}')|^2\,\mathrm{d}^3\mathbf{r}' = \psi(\mathbf{r}(\mathbf{r}'))\det J(\mathbf{r}')\ \mathrm{d}^3\mathbf{r}'$ is invariant. Considering the Fourier basis $\langle\mathbf{p}|\mathbf{r}\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot\mathbf{p}}$ combined with (9) leads to $\tilde{\phi}(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\sqrt{\det J(\mathbf{r}')}\,\psi'(\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}\mathbf{r}'\cdot\mathbf{p}}\ \mathrm{d}^3\mathbf{r}'$.

It was noted in [6] that in the momentum space there is a transformation $G:\mathbf{p}\mapsto\mathbf{p}'$, specified by (7) up to an arbitrary function $g(\mathbf{p}') = \det J(G^{-1})(\mathbf{p}')$ such that $g(\mathbf{p}')\,\mathrm{d}^3\mathbf{p}' = \mathrm{d}^3\mathbf{p}$. Similarly to (9), let $\langle\mathbf{p}'|\psi\rangle = \frac{1}{\sqrt{g(\mathbf{p}')}}\tilde{\phi}'(\mathbf{p}') = \tilde{\phi}(\mathbf{p}(\mathbf{p}'))$, implying that $|\langle\mathbf{p}'|\psi\rangle|^2 g(\mathbf{p}')\ \mathrm{d}^3\mathbf{p}' = |\tilde{\phi}'(\mathbf{p}')|^2\,\mathrm{d}^3\mathbf{p}'$, which is an infinitesimal probability invariant in momentum space. Thus, scaling $\det J(G^{-1})(\mathbf{p}')$ by any function, say $f(\mathbf{p}')$, while also scaling $\langle\mathbf{p}'|\psi\rangle$ according to $\frac{1}{\sqrt{f(\mathbf{p}')}}$ will produce a new transformation G that satisfies the conjugate properties. Thus,

$$\begin{aligned}\mathrm{S_r}+\mathrm{S_p} &= -\int\mathrm{d}^3\mathbf{r}\ \mathrm{d}^3\mathbf{p}\,\rho_\mathrm{r}(\mathbf{r},t)\rho_p(\mathbf{p},t)\ln\big(\rho_\mathrm{r}(\mathbf{r},t)\rho_p(\mathbf{p},t)\big) - 3\ln\hbar\\ &= \mathrm{S_{r'}}+\mathrm{S_{p'}} - \langle\ln\det J^{-1}(\mathbf{r}')\rangle_{\rho'_{r'}} + \langle\ln g(\mathbf{p}')\rangle_{\rho'_{p'}} = \mathrm{S_{r'}}+\mathrm{S_{p'}}\,,\end{aligned}$$

and $g(\mathbf{p}')$ is chosen to satisfy $\langle\ln g(\mathbf{p}')\rangle_{\rho'_{p'}} = \langle\ln\det J^{-1}(\mathbf{r}')\rangle_{\rho'_{r'}}$. □

We next investigate translation transformations. When a quantum reference frame is translated by x_0 along x, the state $|\psi_t\rangle$ in position representation becomes $\psi(x - x_0, t) = \langle x - x_0|\psi_t\rangle = \langle x|\,\hat{T}_P(-x_0)\,|\psi_t\rangle$, where $\hat{T}_P(-x_0) = \mathrm{e}^{\mathrm{i}x_0\hat{P}}$, and $\hat{P}$ is the momentum operator conjugate to $\hat{X}$. When the reference frame is translated by p_0 along p, the state $|\psi_t\rangle$ in momentum representation becomes $\tilde{\phi}(p - p_0, t) = \langle p - p_0|\psi_t\rangle = \langle p|\,\hat{T}_X(-p_0)\,|\psi_t\rangle$, where $\hat{T}_X(-p_0) = \mathrm{e}^{\mathrm{i}p_0\hat{X}}$, and $\hat{X}$ is the position operator conjugate to $\hat{P}$.

Theorem 6 (Frames of reference)**.** *The entropy of a state is invariant under a change of a quantum reference frame by translations along x and along p.*

Proof. Let $|\psi_t\rangle$ be a state and S its entropy. We start by showing that $\mathrm{S}_x = -\int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x,t)|^2 \ln |\psi(x,t)|^2$ is invariant under two types of translations:

(i) translations along x by any x_0

$$\mathrm{S}_{x+x_0} = -\int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x + x_0, t)|^2 \ln |\psi(x + x_0, t)|^2 = \mathrm{S}_x\,,$$

verified by changing variables under the infinite integration interval.

(ii) translations along p by any p_0

$$\begin{aligned}\psi_{p_0}(x,t) &= \langle x|\,\hat{T}_X(p_0)\,|\psi_t\rangle = \int_{-\infty}^{\infty} \langle x|\,\hat{T}_X(p_0)\,|p\rangle\,\langle p|\psi_t\rangle\,\mathrm{d}p \\ &= \int_{-\infty}^{\infty} \langle x|p + p_0\rangle\,\tilde{\phi}(p,t)\,\mathrm{d}p = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\,x\,(p+p_0)}\tilde{\phi}(p,t)\,\mathrm{d}p \\ &= \psi(x,t)\,\mathrm{e}^{\mathrm{i}\,x\,p_0}\,,\end{aligned}$$

implying $|\psi_{p_0}(x,t)|^2 = |\psi(x,t)|^2$.

Similarly, by applying both translations to $\mathrm{S_p} = -\int_{-\infty}^{\infty} \mathrm{d}p\, |\tilde{\phi}(p,t)|^2 \ln |\tilde{\phi}(p,t)|^2$ we conclude that $\mathrm{S_p}$ is invariant under them too. Therefore $\mathrm{S} = \mathrm{S}_x + \mathrm{S_p} - 3\ln\hbar$ is invariant under translations in both x and p. □

CPT Transformations

We will be focusing on fermions, and thus on the Dirac spinors equation, though most of the ideas apply to bosons as well. The QFT Dirac Hamiltonian is

$$\mathcal{H}^{\mathcal{D}} = \int \mathrm{d}^3\mathbf{r}\, \Psi^\dagger(\mathbf{r},t)\left(-\mathrm{i}\hbar\gamma^0\vec{\gamma}\cdot\nabla + mc\gamma^0\right)\Psi(\mathbf{r},t)\,.$$

A QFT solution $\Psi(\mathbf{r},t)$ satisfies $[\mathcal{H}^{\mathcal{D}},\Psi(\mathbf{r},t)] = -\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$ and the C, P, and T symmetries provide new solutions from $\Psi(\mathbf{r},t)$. As usual, $\Psi^{\mathrm{C}}(\mathbf{r},t) = C\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t) = P\Psi(-\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t) = T\Psi^*(\mathbf{r},-t)$, and $\psi^{\mathrm{CPT}}(-\mathbf{r},-t) = CPT\overline{\psi}^{\mathsf{T}}(-\mathbf{r},-t)$, For completeness, we briefly review the three operations, Charge Conjugation, Parity Change, and Time Reversal.

Charge Conjugation transforms particles $\Psi(\mathbf{r},t)$ into antiparticles $\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = (\Psi^\dagger\gamma^0)^{\mathsf{T}}(\mathbf{r},t)$. As $C\gamma^\mu C^{-1} = -\gamma^{\mu\mathsf{T}}$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$ is also a solution for the same Hamiltonian. In the standard representation $C = \mathrm{i}\gamma^2\gamma^0$, up to a phase. Parity Change $P = \gamma^0$, up to a sign, effects the transformation $\mathbf{r} \mapsto -\mathbf{r}$. Time Reversal effects $t \mapsto -t$ and is carried by the operator $\mathcal{T} = T\hat{K}$, where $\hat{K}$ applies conjugation. In the standard representation $T = \mathrm{i}\gamma^1\gamma^3$, up to a phase.

Theorem 7 (Invariance of the entropy under CPT-transformations)**.** *Given a quantum field $\Psi(\mathbf{r},t)$, its Fourier transform $\Phi(\mathbf{k},t)$, and its entropy S_t, the entropies of $\Psi^*(\mathbf{r},t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t)$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t)$, and $\Psi^{\mathrm{CPT}}(-\mathbf{r},-t)$ and their corresponding Fourier transforms are all equal to S_t.*

Proof. The probability densities of $\Psi^*(\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t)$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$, and

$\Psi^{\mathrm{CPT}}(-\mathbf{r},-t)$ are

$$
\begin{aligned}
\rho_{\mathrm{r}}^{*}(\mathbf{r},t) &= \Psi^{\mathsf{T}}(\mathbf{r},t)\Psi^{*}(\mathbf{r},t) = \Psi^{\dagger}(\mathbf{r},t)\Psi(\mathbf{r},t) = \rho(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{C}}(\mathbf{r},t) &= \left(\overline{\Psi}^{\mathsf{T}}\right)^{\dagger}(\mathbf{r},t)C^{\dagger}C\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \overline{\Psi}^{*}(\mathbf{r},t)\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{P}}(-\mathbf{r},t) &= \Psi^{\dagger}(\mathbf{r},t)(\gamma^{0})^{\dagger}\gamma^{0}\Psi(\mathbf{r},t) = \Psi^{\dagger}(\mathbf{r},t)\Psi(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{T}}(\mathbf{r},-t) &= \Psi^{\mathsf{T}}(\mathbf{r},t)T^{\dagger}T\Psi^{*}(\mathbf{r},t) = \Psi^{\mathsf{T}}(\mathbf{r},t)\Psi^{*}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{CPT}}(-\mathbf{r},-t) &= \left(\overline{\Psi}^{\mathsf{T}}\right)^{\dagger}(\mathbf{r},t)(CPT)^{\dagger}(CPT)\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,.
\end{aligned}
\tag{10}
$$

As the densities are equal, so are the associated entropies.

Equations (10) also hold for $\Phi(\mathbf{k},t)$ and its density. Thus, both entropies terms in $\mathrm{S}_t = \mathrm{S}_r + \mathrm{S}_k$ are invariant under all CPT transformations. □

Lorentz Transformations

Theorem 8. *The entropy is a relativistic scalar.*

Proof. The probability elements $\mathrm{dP}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,\mathrm{d}^3\mathbf{r}$ and $\mathrm{dP}(\mathbf{k},t) = \rho_{\mathrm{k}}(\mathbf{k},t)\,\mathrm{d}^3\mathbf{k}$ are invariant under Lorentz transformations since event probabilities do not depend on the frame of reference. Consider a slice of the phase space with frequency $\omega_{\mathrm{k}} = \sqrt{\mathbf{k}^2c^2 + \left(\frac{mc^2}{\hbar}\right)^2}$. The volume elements $\frac{1}{\omega_{\mathrm{k}}}\,\mathrm{d}^3\mathbf{k}$ and $\omega_{\mathrm{k}}\,\mathrm{d}^3\mathbf{r}$, are invariant under the Lorentz group [13], that is, $\frac{1}{\omega_{\mathrm{k}}}\,\mathrm{d}^3\mathbf{k} = \frac{1}{\omega_{\mathrm{k}'}}\,\mathrm{d}^3\mathbf{k}'$ and $\omega_{\mathrm{k}}\,\mathrm{d}^3\mathbf{r} = \omega_{\mathrm{k}'}\,\mathrm{d}^3\mathbf{r}'$, implying $\mathrm{d}V = \mathrm{d}^3\mathbf{k}\,\mathrm{d}^3\mathbf{r} = \mathrm{d}^3\mathbf{k}'\,\mathrm{d}^3\mathbf{r}' = \mathrm{d}V'$, where $\mathbf{r}'$, $\mathbf{k}'$, and $\omega_{\mathrm{k}'}$ result from applying a Lorentz transformation to $\mathbf{r}$, $\mathbf{k}$, and ω_{k}; $\frac{1}{\omega_{\mathrm{k}}}\rho_{\mathrm{r}}(\mathbf{r},t)$ and $\omega_{\mathrm{k}}\rho_{\mathrm{k}}(\mathbf{k},t)$ are also invariant under the group. Thus, the phase space density $\rho_{\mathrm{r}}(\mathbf{r},t)\rho_{\mathrm{k}}(\mathbf{k},t)$ is an invariant to Lorentz transformations. Therefore the entropy is a relativistic scalar. □

Note that in QFT, one scales the operator $\Phi(\mathbf{k},t)$ by $\sqrt{2\omega_{\mathrm{k}}}$, that is, one scales the creation and the annihilation operators $\alpha^{\dagger}(\mathbf{k}) = \sqrt{\omega_{\mathrm{k}}}\,\mathbf{a}^{\dagger}(\mathbf{k})$ and $\alpha(\mathbf{k}) = \sqrt{\omega}\,\mathbf{a}(\mathbf{k})$.

In this way, the density operator $\Phi^{\dagger}(\mathbf{k},t)\Phi(\mathbf{k},t)$ scales with ω_{k} and becomes a relativistic scalar. Also, with such a scaling, the infinitesimal probability of finding a particle with momentum $\mathbf{p} = \hbar\mathbf{k}$ in the original reference frame is invariant under the Lorentz transformation, though it would be found with momentum $\mathbf{p}' = \hbar\mathbf{k}'$.

QCURVES AND ENTROPY-PARTITION

We introduced in [10] the concept of a *QCurve* to specify a curve (or path) in a Hilbert space parametrized by time. In QM, a QCurve is represented by a triple $\big(|\psi_0\rangle, U(t), \delta t\big)$ where $|\psi_0\rangle$ is the initial state, $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$ is the evolution operator, and $[0, \delta t]$ is the time interval of the evolution. Alternatively, we can represent the initial state by $(\langle\mathbf{r}|\psi_0\rangle, \langle\mathbf{k}|\psi_0\rangle)$ and in QFT as $(\Psi(\mathbf{r},0)\,|\text{state}\rangle, \Phi(\mathbf{k},0)\,|\text{state}\rangle)$.

Definition 1 (Partition of $\mathcal{E}$ from [10])**.** Let $\mathcal{E}$ to be the set of all QCurves. We define a partition of $\mathcal{E}$ based on the entropy evolution into four blocks:

- $\mathcal{C}$**:** Set of the QCurves for which the entropy is a constant.
- $\mathcal{I}$**:** Set of the QCurves for which the entropy is increasing, but it is not a constant.
- $\mathcal{D}$**:** Set of the QCurves for which the entropy is decreasing, but it is not a constant.
- $\mathcal{O}$**:** Set of oscillating QCurves, with the entropy strictly increasing in some subinterval of $[0, \delta t]$ and strictly decreasing in another subinterval of $[0, \delta t]$.

Consider stationary states $|\psi_t\rangle = |\psi_E\rangle\,\mathrm{e}^{-\mathrm{i}\omega t}$ with $\omega = E/\hbar$, where E is an energy eigenvalue of the Hamiltonian, and $|\psi_E\rangle$ is the time-independent eigenstate of the Hamiltonian associated with E.

Theorem 9. *All stationary states are in* $\mathcal{C}$.

Proof. Follows from the time invariance of the probabilities. □

Dirac's free-particle Hamiltonian in QM [8] is

$$H = -\mathrm{i}\hbar\gamma^0\vec{\gamma}\cdot\nabla + mc\gamma^0\,. \tag{11}$$

It can be diagonalized in the spatial Fourier domain $|\mathbf{k}\rangle$ basis to obtain

$$\omega(\mathbf{k}) = \pm c\sqrt{k^2 + \frac{m^2}{\hbar^2}c^2}\,, \tag{12}$$

where $\omega(\mathbf{k})$ is the frequency component of the Hamiltonian. We focus on the positive energy solutions and so the group velocity becomes

$$\mathbf{v_g}(\mathbf{k}) = \nabla_{\mathrm{k}}\omega(\mathbf{k}) = \frac{\hbar}{m}\frac{\mathbf{k}}{\sqrt{1+(\frac{\hbar k}{mc})^2}}\,. \tag{13}$$

In (16) we will use the Taylor expansion of (12) up to the second order, thus requiring the Hessian $\mathbf{H}(\mathbf{k})$, with the entries

$$\mathbf{H}_{ij}(\mathbf{k}) = \frac{\partial^2\omega(\mathbf{k})}{\partial k_i\partial k_j} = \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{3}{2}}\left[\delta_{i,j}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)-\left(\frac{\hbar k_i}{mc}\right)\left(\frac{\hbar k_j}{mc}\right)\right] \tag{14}$$

for the positive energy solution. The three (positive) eigenvalues of $\mathbf{H}(\mathbf{k})$ are

$$\lambda_1 = \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{3}{2}} = \hbar\frac{m^2}{\left(m^2+\mu^2(k)\right)^{\frac{3}{2}}}\,,$$

$$\lambda_{2,3} = \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{1}{2}} = \hbar\frac{1}{(m^2+\mu^2(k))^{\frac{1}{2}}}\,,$$

where $\mu(k) = \hbar k/c$ is the kinetic energy in mass units. The Hessian is positive definite for positive energy, and so gives a measure of dispersion of the wave.

We now consider initial solutions that are localized in space, $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0) = \psi_0(\mathbf{r}-\mathbf{r}_0)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}$, where $\mathbf{r}_0$ is the mean value of $\mathbf{r}$. Assume that the variance, $\int \mathrm{d}^3\mathbf{r}\,(\mathbf{r}-\mathbf{r}_0)^2\rho_{\mathrm{r}}(\mathbf{r})$, is finite, where $\rho_{\mathrm{r}}(\mathbf{r}) = |\psi_0(\mathbf{r})|^2$. In a Cartesian representation, we can write the initial state in the spatial frequency domain as $\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0) = \phi_0(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}_0}$, where $\phi_0(\mathbf{k})$ is the Fourier transform of $\psi_0(\mathbf{r})$, and so the variance of $\rho_{\mathrm{k}}(\mathbf{k}) = |\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)|^2$ is also finite, with the mean in the spatial frequency center $\mathbf{k}_0$.

The time evolution of $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0)$ according a Hamiltonian with a dispersion relation $\omega(\mathbf{k})$ and written via the inverse Fourier transform is

$$\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0,t) = \frac{1}{(\sqrt{2\pi})^3}\int \Phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}\omega(\mathbf{k})t}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{k}\,. \tag{15}$$

As $\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)$ fades away exponentially from $\mathbf{k}=\mathbf{k}_0$, we expand (12) in a Taylor series and approximate it as

$$\omega(\mathbf{k}) \approx \mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0 + \mathbf{v_g}(\mathbf{k}_0)\cdot(\mathbf{k}-\mathbf{k}_0) + \frac{1}{2}(\mathbf{k}-\mathbf{k}_0)^{\mathsf{T}}\mathbf{H}(\mathbf{k}_0)\,(\mathbf{k}-\mathbf{k}_0)\,, \tag{16}$$

where $\mathbf{v_p}(\mathbf{k}_0)$, $\mathbf{v_g}(\mathbf{k}_0)$, and $\mathbf{H}(\mathbf{k}_0)$ are the phase velocity $\frac{\omega(\mathbf{k}_0)}{|\mathbf{k}_0|}\hat{\mathbf{k}}_0$, the group velocity (13), and the Hessian (14) of the dispersion relation $\omega(\mathbf{k})$, respectively. Then after inserting (16) into (15), we obtain the quantum dispersion transform

$$\begin{aligned}
\phi_{\mathbf{r}^t_{\mathrm{k}_0}}(\mathbf{k}-\mathbf{k}_0,t) &\approx \frac{1}{Z_k}\,\mathrm{e}^{-\mathrm{i}t\mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0}\,\Phi_{\mathbf{r}^t_{\mathrm{k}_0}}(\mathbf{k}-\mathbf{k}_0)\,\mathcal{N}\left(\mathbf{k}\mid\mathbf{k}_0,-\mathrm{i}t^{-1}\mathbf{H}^{-1}(\mathbf{k}_0)\right),\\
\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0},t) &\approx \frac{1}{Z_{\mathrm{r}}}\mathrm{e}^{-\mathrm{i}t\mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0}\,\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}) * \mathcal{N}\left(\mathbf{r}\mid\mathbf{r}^t_{\mathbf{k}_0},\mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right),
\end{aligned} \tag{17}$$

where $\mathbf{r}^t_{\mathbf{k}_0} = \mathbf{r}_0 + \mathbf{v_g}(\mathbf{k}_0)t$, $\Phi_{\mathbf{r}^t_{\mathrm{k}_0}}(\mathbf{k}-\mathbf{k}_0) = \phi_0(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}^t_{\mathrm{k}_0}}$, with Fourier transform $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0})$; $*$ denotes a convolution, Z_r and Z_k normalize the amplitudes, and $\mathcal{N}$ is a normal distribution. Consequently, $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}, t)$ is the spatial Fourier transform of $\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0, t)$.

The probability densities associated with the probability amplitudes in (17) are

$$\begin{aligned}
\rho_{\mathrm{r}}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}, t) &= \frac{1}{Z_{\mathrm{r}}^2}|\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}) * \mathcal{N}\left(\mathbf{r} \mid \mathbf{r}^t_{\mathbf{k}_0}, \mathrm{i}\, t\, \mathbf{H}(\mathbf{k}_0)\right)|^2\,, \\
\rho_{\mathrm{k}}(\mathbf{k}-\mathbf{k}_0, t) &= \frac{1}{Z_k^2}|\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0)|^2\,.
\end{aligned} \tag{18}$$

Lemma 1 (Dispersion Transform and Reference Frames)**.** *The entropy associated with* (18) *is equal to the entropy associated with the simplified probability densities*

$$\begin{aligned}
\rho^{\mathrm{S}}_{\mathrm{r}}(\mathbf{r}, t) &= \frac{1}{Z^2}|\psi_0(\mathbf{r}) * \mathcal{N}\left(\mathbf{r} \mid 0, \mathrm{i}\, t\, \mathbf{H}(\mathbf{k}_0)\right)|^2\,, \\
\rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t) &= \frac{1}{Z_k^2}|\Phi_0(\mathbf{k})|^2 = \rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t=0)\,.
\end{aligned} \tag{19}$$

Proof. Consider (18). If the frame of reference is translating the position by $\mathbf{r}^t_{\mathbf{k}_0} = \mathbf{r}_0 + \mathbf{v}_{\mathbf{g}}(\mathbf{k}_0)t$ and the momentum by $\hbar\mathbf{k}_0$, we get the simplified density functions (19).

Theorem 6 shows that the entropy in position and momentum is invariant under translations of the position $\mathbf{r}$ and the spatial frequency $\mathbf{k}$, and that completes the proof. □

The time invariance of the density $\rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t)$, and therefore of S_{k}, reflects the conservation law of momentum for free particles.

We now focus on the case of coherent states, represented by $|\alpha\rangle$, the eigenstates of the annihilator operator. The coherent state $|0\rangle$, associated with eigenvalue $\alpha = 0$, yields in position and momentum space representations

$$\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0) = \langle \mathbf{r}|0\rangle = \frac{1}{2^3\pi^{\frac{3}{2}}(\det\mathbf{\Sigma})^{\frac{1}{2}}}\,\mathcal{N}\left(\mathbf{r}\mid\mathbf{r}_0,\mathbf{\Sigma}\right)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}\,,$$
$$\Phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0) = \langle \mathbf{k}|0\rangle = \frac{1}{2^3\pi^{\frac{3}{2}}(\det\mathbf{\Sigma}^{-1})^{\frac{1}{2}}}\,\mathcal{N}\left(\mathbf{k}\mid\mathbf{k}_0,\mathbf{\Sigma}^{-1}\right)\,\mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}_0}\,, \tag{20}$$

where $\mathbf{\Sigma}$ is the spatial covariance matrix.

Theorem 10. *A QCurve with an initial coherent state* (20) *and evolving according to* (11) *is in* $\mathcal{I}$.

Proof. To describe the evolution of the initial states (20), we apply (17). Then, after applying Lemma 1,

$$\rho^{\mathrm{S}}_{\mathrm{r}}(\mathbf{r},t) = \frac{1}{Z_2^2}\mathcal{N}\left(\mathbf{r}\mid 0,\mathbf{\Sigma}+\mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right)\mathcal{N}\left(\mathbf{r}\mid 0,\mathbf{\Sigma}-\mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right) = \mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\mathbf{\Sigma}(t)\right)\,,$$
$$\rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k},t) = \mathcal{N}\left(\mathbf{k}\mid 0,\mathbf{\Sigma}^{-1}\right)\,,$$

where $\mathbf{\Sigma}(t) = \mathbf{\Sigma} + t^2\mathbf{H}(\mathbf{k}_0)\mathbf{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0)$. Then

$$\begin{aligned}
\mathrm{S} &= \mathrm{S}_\mathrm{r} + \mathrm{S}_\mathrm{k} \\
&= -\int \mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\mathbf{\Sigma}(t)\right)\ln\mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\mathbf{\Sigma}(t)\right)\mathrm{d}^3\mathbf{r} \\
&\quad -\int \mathcal{N}\left(\mathbf{k}\mid 0,\mathbf{\Sigma}^{-1}\right)\ln\mathcal{N}\left(\mathbf{k}\mid 0,2\mathbf{\Sigma}^{-1}\right)\mathrm{d}^3\mathbf{k} \\
&= 3(1+\ln\pi) + \frac{1}{2}\ln\det\left(\mathbf{I}+t^2(\mathbf{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0))^2\right)\,.
\end{aligned}$$

As $\det\left(\mathbf{I}+t^2(\mathbf{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0))^2\right) > 0$, the entropy increases over time. □

The theorem suggests that quantum physics has an inherent mechanism to increase entropy for free particles, due to the spatial dispersion property of the Hamiltonian. Note that at $t = 0$ a coherent state (20) reaches the minimum possible coordinate-entropy value, while the spin-entropy remains constant.

Time Reflection

We consider a time-independent Hamiltonian, investigate the discrete symmetries C and P, and propose that Time Reversal be augmented with Time Translation, say by δt. We refer to the mapping $t \mapsto -t + \delta t$ as Time Reflection, because as t varies from 0 to δt, $t'(t) = -t + \delta t$ varies as a reflection from δt to 0. We define the Time Reflection quantum field

$$\Psi^{\mathrm{T}_\delta}(\mathbf{r}, -t + \delta t) = \mathcal{T}\Psi(\mathbf{r}, t) = T\Psi^*(\mathbf{r}, t)\,.$$

Note that in contrast to the case of Time Reversal, $\Psi^{\mathrm{T}_\delta}(\mathbf{r}, t') = \mathcal{T}\Psi(\mathbf{r}, -t' + \delta t)$, and the entropies associated with $\Psi(\mathbf{r}, t)$ and $\Psi^{\mathrm{T}_\delta}(\mathbf{r}, t)$ are generally not equal. Thus, an instantaneous Time Reflection transformation will cause entropy changes.

We next consider a composition of the three transformation, Charge Conjugation, Parity Change, and Time Reflection.

Definition 2 ($\Psi^{\mathrm{CPT}_\delta}$)**.** Let the CPT_δ quantum field be

$$\Psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, -t + \delta t) = \eta_\delta\, CPT\, \overline{\Psi}^{\mathsf{T}}(\mathbf{r}, t) = \eta\gamma^5\, (\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, t)\,, \tag{21}$$

where η is the product of the phases of each operation, η_δ is the phase of time translation, and $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

Definition 3 (Q_{CPT_δ})**.** Let Q_{CPT_δ} be $\big(\psi(\mathbf{r}, 0), U(t), [0, \delta t]\big) \mapsto \big(\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0), U(t), [0, \delta t]\big)$.

Using (21) we see that,

$$\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0) = \eta\gamma^5\, (\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, -0 + \delta t) = \eta\, \gamma^5\, (\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, \delta t)\,. \tag{22}$$

Theorem 11 (Time Reflection)**.** *Consider a* CPT *invariant quantum field theory (QFT) with energy conservation, such as Standard Model or Wightman axiomatic QFT [14]. Let* $e_0 = (\psi(\mathbf{r}, 0), U(t), [0, \delta t])$ *be a QCurve solution to such QFT. Then,* $e_1 = Q_{\mathrm{CPT}_\delta}(e_0)$ *is (i) a solution to such QFT, (ii) if* e_0 *is in* $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$ *then* e_1 *is respectively in* $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$, *making* $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$ *reflections of* $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$, *respectively.*

Proof. Let $t' = -t + \delta t$. The QCurve e_1 describes the evolution $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ during the period $[0, \delta t]$.

Since e_0 is a solution to a QFT that is CPT-invariant and time-translation invariant, e_1 is also a solution to the QFT, proving (i).

The time evolution of $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0)$ from 0 to δt is described by $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ and by (22) $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t') = \eta\, \gamma^5\, (\Psi^\dagger)^\mathsf{T}(\mathbf{r}, -t' + \delta t) = \eta\, \gamma^5\, \Psi^*(\mathbf{r}, \delta t - t')$. Thus, the evolution of $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ as t' evolves from 0 to δt, by Theorem 7, has the same entropies as $\psi(\mathbf{r}, \delta t - t')$. Since $\psi(\mathbf{r}, \delta t - t')$ traverses the same path as $\psi(\mathbf{r}, t')$ but in the opposite time direction, we conclude that e_1 produces the time evolution states $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ in the time interval $[0, \delta t]$ traversing the same path and with the same entropies as $\psi(\mathbf{r}, t')$, but in the opposite time directions.

Applying the above to a QCurve respectively in $\mathcal{I}$, $\mathcal{D}$, $\mathcal{C}$, $\mathcal{O}$, results in a QCurve respectively in $\mathcal{D}$, $\mathcal{I}$, $\mathcal{C}$, $\mathcal{O}$. Thus, we conclude the proof of (ii). □

For a visualization see Figure 1 of [10].

Entropy Oscillations

Theorem 12 (Coefficients for two states)**.** *Consider a particle in an eigenstate* $|\psi_{E_1}\rangle$ *of a Hamiltonian* H *that has only two eigenstates* $|\psi_{E_1}\rangle$ *and* $|\psi_{E_2}\rangle$ *with eigenvalues* $E_1 = \hbar\omega_1$ *and* $E_2 = \hbar\omega_2$, *respectively. Let this particle interact with an external field (such as the impact of a Gauge Field), requiring an additional Hamiltonian term* H^{I} *to describe the evolution of this system.*

Let $\omega^{\mathrm{I}}_{i,j} = \frac{1}{\hbar}\left\langle\psi_{E_i}\right| H^{\mathrm{I}}\left|\psi_{E_j}\right\rangle$, $\omega^{\mathrm{total}}_1 = \omega_1 + \omega^{\mathrm{I}}_{11}$, $\omega^{\mathrm{total}}_2 = \omega_2 + \omega^{\mathrm{I}}_{22}$, $\eta = \sqrt{\left(\omega^{\mathrm{total}}_1 - \omega^{\mathrm{total}}_2\right)^2 + 4(\omega^{\mathrm{I}}_{12})^2}$, *and* $\lambda_\pm = \frac{\omega^{\mathrm{total}}_1+\omega^{\mathrm{total}}_2\pm\eta}{2}$. *The probability of the particle to be in state* $\left|\psi_{E_2}\right\rangle$ *at time t is*

$$\frac{4(\omega^{\mathrm{I}}_{12})^2}{\eta^2}\sin^2\frac{(\lambda_+ - \lambda_-)t}{2}.$$

Proof. The Hamiltonians in the basis $\left|\psi_{E_1}\right\rangle, \left|\psi_{E_2}\right\rangle$ are

$$H = \hbar\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix} \qquad \text{and} \qquad H^{\mathrm{I}} = \hbar\begin{pmatrix}\omega^{\mathrm{I}}_{11} & \omega^{\mathrm{I}}_{12}\\ \omega^{\mathrm{I}}_{12} & \omega^{\mathrm{I}}_{22}\end{pmatrix},$$

where the real values satisfy $\omega^{\mathrm{I}}_{21} = \omega^{\mathrm{I}}_{12}$ as H^{I} is Hermitian. The eigenvalues of the symmetric matrix $H' = H + H^{\mathrm{I}}$ are $\hbar\lambda_\pm$, and so we can decompose it as

$$H' = \hbar\begin{pmatrix}\omega^{\mathrm{total}}_1 & \omega^{\mathrm{I}}_{12}\\ \omega^{\mathrm{I}}_{12} & \omega^{\mathrm{total}}_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hbar\lambda_+ & 0\\ 0 & \hbar\lambda_-\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, \tag{23}$$

where

$$\theta = \frac{1}{2}\arcsin\frac{2\omega^{\mathrm{I}}_{12}}{\eta}. \tag{24}$$

The time evolution of $\left|\psi_{E_1}\right\rangle$ is $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}\frac{(H+H^{\mathrm{I}})}{\hbar}t}\left|\psi_{E_1}\right\rangle = \sum_{k=1}^{2}\alpha_k(t)\left|\psi_{E_k}\right\rangle$, and projecting on $\left\langle\psi_{E_j}\right|$, we get $\alpha_j(t) = \left\langle\psi_{E_j}\right|\mathrm{e}^{-\mathrm{i}\frac{(H+H^{\mathrm{I}})}{\hbar}t}\left|\psi_{E_1}\right\rangle$. From (23),

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\frac{H'}{\hbar}t} &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda_+t} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\lambda_-t}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda_+t}\cos^2\theta + \mathrm{e}^{-\mathrm{i}\lambda_-t}\sin^2\theta & \frac{\mathrm{e}^{-\mathrm{i}\lambda_+t}-\mathrm{e}^{-\mathrm{i}\lambda_-t}}{2}\sin 2\theta\\ \frac{\mathrm{e}^{-\mathrm{i}\lambda_+t}-\mathrm{e}^{-\mathrm{i}\lambda_-t}}{2}\sin 2\theta & \mathrm{e}^{-\mathrm{i}\lambda_+t}\sin^2\theta + \mathrm{e}^{-\mathrm{i}\lambda_-t}\cos^2\theta\end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = e^{-i\frac{H'}{\hbar}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta\, e^{-i\lambda_+ t} + \sin^2\theta\, e^{-i\lambda_- t} \\ \sin 2\theta \left(\frac{e^{-i\lambda_+ t} - e^{-i\lambda_- t}}{2} \right) \end{pmatrix},$$

and so

$$\begin{pmatrix} |\alpha_1(t)|^2 \\ |\alpha_2(t)|^2 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\sin^2 2\theta\,(1 - \cos(\lambda_- - \lambda_+)t) \\ \frac{1}{2}\sin^2 2\theta\,(1 - \cos(\lambda_- - \lambda_+)t) \end{pmatrix}.$$

As $1 - \cos(\lambda_- - \lambda_+)t = 2\sin^2\frac{(\lambda_+ - \lambda_-)t}{2}$, the probability of being in state $|\psi_{E_2}\rangle$ at time t is $|\alpha_2(t)|^2 = \sin^2 2\theta\, \sin^2\frac{(\lambda_+ - \lambda_-)t}{2}$. Using (24), completes the proof. □

If $\omega_1 \gg \omega^{\mathrm{I}}_{11}$, $\omega_2 \gg \omega^{\mathrm{I}}_{22}$, and $|\omega_1 - \omega_2| \gg \omega^{\mathrm{I}}_{12}$, then $\lambda_{+,-} \approx \omega_{1,2}$, and the coefficient of transition becomes $|\alpha_2(t)|^2 \approx \frac{4(\omega^{\mathrm{I}}_{12})^2}{(\omega_1 - \omega_2)^2}\sin^2\frac{(\omega_2 - \omega_1)t}{2}$, which is Fermi's golden rule [7, 9].

In [10] we showed that when the probability of two states oscillates as above, the entropy oscillates too.

The derivation of $\alpha_2(t)$ can be expanded to multiple states. However, for multiple states, the sum over all the frequencies $\lambda_k - \lambda_i$ may cancel the oscillations unless some frequencies dominate the sum, such as when the transition to the ground state dominates other transitions. Thus, to obtain the entropy oscillation in the presence of multiple transitions may require approximations similar to the ones that are usually used in derivations of Fermi's golden rule.

The spin-entropy of a state with two or more particles, such as (6), and its unitary time evolution (forming their QCurves) can be studied with the same technique as above. How such evolution leads to one of the four block partition needs to be further studied.

An Entropy Law and a Time Arrow

In classical statistical mechanics, the entropy provides a time arrow through the second law of thermodynamics [4]. We have shown that due to the dispersion property of the fermions Hamiltonian some states in quantum mechanics, such as coherent states, already obey such a law. However, current quantum physics is described as time reversible. In [10] we conjectured the following

Law (The Entropy Law)**.** *The entropy of a quantum system is an increasing function of time.*

The law may help explain why particles are created and/or annihilated in scenarios such as high-speed collision $e^+ + e^- \to 2\gamma$, kaons decay into mesons, and photon creation and emission when the electron in the hydrogen atom transitions from an excited state to the ground state. In those scenarios, while such final states are reachable in a unitary evolution of the initial state, it seems that only those evolutions in which entropy increase are realized. According to the S-matrix formulation [13], similar to Fermi's golden rule in QM, these final states are among the possible transition states. We note that similarly to Fermi's golden rule, these are also entropy oscillation scenarios in which the evolution is blocked from entering a time interval of decreasing entropy. The creation and/or annihilation of a particles seem to occur when the entropy of the evolution from the initial to the final state is oscillating, and but for such events the entropy would decrease, which the conjectured law forbids.

Furthermore, the spin-entropy evolution of system of particles or fields is also subject to this law which may have implications in all physical scenarios including quantum information and quantum computing.

CONCLUSIONS

The concepts of entropy in phase spaces in [10] were developed here. First for the spin-entropy and studied the case of two entangled spin $\frac{1}{2}$ particles. The spin-entropy obtained differs from Von Newman's entropy, which is the one currently adopted in quantum information and quantum computing and together with the proposed law may impact these fields. We then extended the coordinate-entropy in QM to multiple particles. We proved that the coordinate-entropy is invariant under coordinate transformations, Lorentz transformations, and CPT transformations. We analyzed the entropy evolution of coherent states, showing that the Dirac's Hamiltonian has a mechanism to disperse the information and to increase entropy. We proved that Time Reflection transforms QCurves in $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$ into QCurves in $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$, respectively. We proved that for a two-state Hamiltonian, the addition of a Hamiltonian term not only causes a state oscillation (as suggested by Fermi's golden rule when the appropriate approximations hold) but also causes entropy oscillation. In light of the technical advancements here, we reviewed the conjectured entropy law [10]. According to that law, not only a time arrow would emerge, but should the formation of new particles be triggered by the entropy law, the history of the universe would have to be revised through such a lens. Perhaps, the collapse of a wave function occurs not due to measurements, but instead due to the restrictions posed by the entropy law.

---

[1] Y. Aharonov and T. Kaufherr. Quantum frames of reference. *Phys. Rev. D*, 30(2):368, 1984.

[2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher. Concentrating partial entanglement by local operations. *Phys. Rev. A*, 53:2046–2052, Apr 1996.

[3] P. Calabrese and J. Cardy. Entanglement entropy and quantum field theory. *J. Stat. Mech.*, 2004(06):P06002, 2004.

[4] R. Clausius. *The mechanical theory of heat: With its applications to the steam-engine and to the physical properties of bodies*. J. van Voorst, 1867.

[5] C. Cohen-Tannoudji, B. Diu, and F. Laloe. *Quantum Mechanics: Volume 1, Basic concepts, tool, and applications*. Wiley-VCH, 2 edition, 2019.

[6] B. S. DeWitt. Point transformations in quantum mechanics. *Phys. Rev.*, 85(4):653, 1952.

[7] P. A. M. Dirac. The quantum theory of the emission and absorption of radiation. *Proc. R. Soc. Lond. A*, 114:243–265, 1927.

[8] P. A. M. Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 126(801):360–365, 1930.

[9] E. Fermi. *Nuclear physics: A course given by Enrico Fermi at the University of Chicago*. University of Chicago Press, 1950.

[10] D. Geiger and Z. M. Kedem. Quantum information physics I. New York University, Dept. of Computer Science,TR 2021-996, 2021.

[11] W. Gerlach and O. Stern. Der experimentelle Nachweis des magnetischen Moments des Silberatoms. *Z. Phys.*, 8(1):110–111, 1922.

[12] M. Srednicki. Entropy and area. *Phys. Rev. Lett.*, 71:666–669, Aug 1993.

[13] S. Weinberg. *The quantum theory of fields: Volume 1, (Foundations)*. Cambridge University Press, 1995.

[14] A. S. Wightman. Hilbert's sixth problem: Mathematical treatment of the axioms of physics. In *Proceedings of Symposia in Pure Mathematics*, volume 28, pages 147–240. AMS, 1976.