AN OVERLAPPING SCHWARZ ALGORITHM FOR RAVIART-THOMAS VECTOR FIELDS WITH DISCONTINUOUS COEFFICIENTS TR2010-933

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Abstract. Overlapping Schwarz methods form one of two major families of domain decomposition methods. We consider a two-level overlapping Schwarz method for Raviart-Thomas vector fields. The coarse part of the preconditioner is based on the energy-minimizing extensions and the local parts are based on traditional solvers on overlapping subdomains. We show that the condition number grows linearly with the logarithm of the number of degrees of freedom in the individual subdomains and linearly with the relative overlap between the overlapping subdomains. The condition number of the method is also independent of the values and jumps of the coefficients. Numerical results for 2D and 3D problems, which support the theory, are also presented.

Key words. domain decomposition, overlapping Schwarz, Raviart-Thomas finite element, preconditioners

AMS subject classifications. 65F08, 65N30, 65N55

1. Introduction. We consider the following boundary value problem:

$$L\mathbf{u} := -\mathbf{grad}\,(\alpha \operatorname{div} \mathbf{u}) + \beta\,\mathbf{u} = \mathbf{f} \text{ in } \Omega, \qquad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here Ω is a bounded polygon in $\mathbb{R}^2$ or a polyhedron in $\mathbb{R}^3$ and $\mathbf{n}$ is the outward normal vector of its boundary. We assume that $\mathbf{f}$ is in $(L^2(\Omega))^2$ or $(L^2(\Omega))^3$ and that α and β are positive $L^\infty(\Omega)$ functions.

We now consider the weak formulation of the original problem:

$$\mathbf{a}(\mathbf{u},\mathbf{v}) := \int_\Omega \alpha \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\,dx + \beta\,\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx,\ \mathbf{v}\in H_0(\mathrm{div};\Omega). \tag{1.2}$$

$H(\mathrm{div};\Omega)$ is the subspace of $(L^2(\Omega))^2$ or $(L^2(\Omega))^3$ with $\operatorname{div}\mathbf{u} \in L^2(\Omega)$ and $H_0(\mathrm{div};\Omega)$ is the subspace of $H(\mathrm{div};\Omega)$ with vanishing normal components on $\partial\Omega$.

We will consider two-level overlapping Schwarz methods. Such methods were originally considered for scalar elliptic problems; see [25, Chapter 3] and references therein. Later these methods have been widely extended to various problems including vector fields problems; see [2,13,22–24]. Other methods for $H(\mathrm{div})$ problems, such as multigrid methods and iterative substructuring methods, have also been considered; see [3,12,13,28,29]. While many iterative substructuring methods have been studied for variable coefficients cases, there has been no supporting theory for the overlapping Schwarz methods until recently. In order to deal with variable coefficients, we use the modified techniques of [8,9] developed for the almost incompressible elasticity case. A different type of technique is also available for scalar elliptic problems; see [27].

In Section 2, we introduce a finite element approximation of (1.2). In Section 3, we describe some functional tools to help derive the main result. We present the

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algorithm in Section 4 and the main result and proofs in Section 5. Section 6 contains supporting numerical experiments.

2. Raviart-Thomas and Nédélec elements. We introduce two kinds of triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$. $\mathcal{T}_H$ is a shape regular and quasi-uniform coarse triangulation of the domain Ω with the maximum diameter H. Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ and a shape regular and quasi-uniform triangulation of individual coarse mesh elements. Here, h is the minimum diameter of the triangulation. We consider the lowest order Raviart-Thomas and Nédélec elements on this triangulation; see [5, Chap. 3] and [16].

We first consider the Raviart-Thomas elements. The lowest order Raviart-Thomas element space is defined by

$$X_h := \{\mathbf{u} \mid \mathbf{u}_{|t} \in \mathcal{RT}(t), t \in \mathcal{T}_h \text{ and } \mathbf{u} \in H(\text{div};\Omega)\},$$

where $\mathcal{RT}(t)$ is given by

$$\mathcal{RT}(t) := \mathbf{a} + \beta \mathbf{x},$$

for triangular or tetrahedral elements and by

$$\mathcal{RT}(t) := \mathbf{a} + \mathbf{b} \cdot \mathbf{x},$$

for quadrilateral or hexahedral elements.

The degrees of freedom are defined by the average values of the normal components over the edges and the faces of $\mathcal{T}_h$ for two and three dimensions, respectively, i.e., by

$$\lambda(\mathbf{u}) := \frac{1}{|F|} \int_F \mathbf{u} \cdot \mathbf{n} \, ds, \;\; F \subset \partial K.$$

The basis functions of the lowest order Raviart-Thomas element space are supported in two elements of $\mathcal{T}_h$ and their normal component equals 1 on a specified edge (2D) or face (3D) and 0 on the other edges (2D) or faces (3D).

We also define $X_{0;h}$ which is the subspace of X_h with a vanishing normal components on the boundary of the domain Ω, i.e.,

$$X_{0;h}(\Omega) := X_h(\Omega) \cap H_0(\text{div};\Omega).$$

We need to define trace spaces. Let $W_h(\partial\Omega_i)$ be the space of functions which are constant on each edge (2D) or face (3D) of the edges or faces of the elements of $\mathcal{T}_h$ which are contained in $\partial\Omega_i$. We also define $W_{0;h}(\partial\Omega_i)$ as the subspace of $W_h(\partial\Omega_i)$ with mean value zero over $\partial\Omega_i$.

Similarly, we consider the spaces involving **curl**. We can define $H(\mathbf{curl};\Omega)$ which is a subspace of $(L^2(\Omega))^2$ or $(L^2(\Omega))^3$ with $\mathbf{curl}\,\mathbf{u} \in (L^2(\Omega))^2$ or $(L^2(\Omega))^3$and $H_0(\mathbf{curl};\Omega)$, which is a subspace of $H(\mathbf{curl};\Omega)$ with vanishing tangential components on $\partial\Omega$.

We next introduce the Nédélec elements. The lowest order Nédélec element space is defined by

$$N_h := \{\mathbf{u} \mid \mathbf{u}_{|t} \in \mathcal{ND}(t), t \in \mathcal{T}_h \text{ and } \mathbf{u} \in H(\mathbf{curl};\Omega)\},$$

where

$$\mathcal{ND}(t) := \mathbf{a} + \mathbf{x} \times \mathbf{b},$$

for triangular or tetrahedral elements, by

$$\mathcal{ND}(t) := Q_{0,1} \times Q_{1,0},$$

with Q_{k_1,k_2} the space of polynomial of degree k_i in the i-th variable for quadrilateral elements, and by

$$\mathcal{ND}(t) := Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0},$$

with Q_{k_1,k_2,k_3} the space of polynomial of degree k_i in the i-th variable for hexahedral elements.

The degrees of freedom are defined by the average value of the tangential component over the edge e

$$\lambda_e(\mathbf{u}) = \frac{1}{|e|}\int_e \mathbf{u} \cdot \mathbf{t}_e ds, \ e \subset \partial K,$$

for each $K \in \mathcal{T}_h$.

The basis functions of the lowest order Nédélec element space are supported in the union of the elements of $\mathcal{T}_h$ that have the edge in common and their tangential components are 1 on the specified edge and 0 on all other edges.

We also define $N_{0;h}$, which is the subspace of N_h with vanishing tangential components on the boundary of the domain Ω, i.e.,

$$N_{0;h}(\Omega) := N_h(\Omega) \cap H_0(\mathbf{curl}; \Omega).$$

Additionally, let S_h be the continuous P_1 space and $S_{0;h}$ be the subspace of S_h with zero boundary values.

Finally, we define three interpolation operators Π_h^{RT}, Π_h^{ND}, and I^h onto X_h, N_h, and S_h, respectively.

3. Some Functional Tools. We will use some Sobolev spaces and corresponding norms and seminorms for bounded open Lipschitz domains Ω. Let us consider $H^s(\Omega)$ with $s > 0$ and let H be the diameter of Ω. Then, we can define scaled norms:

$$\|u\|_{1;\Omega}^2 = |u|_{1;\Omega}^2 + \frac{1}{H^2}\|u\|_{0;\Omega}^2,$$

$$\|u\|_{\frac{1}{2};\Omega}^2 = |u|_{\frac{1}{2};\Omega}^2 + \frac{1}{H}\|u\|_{0;\Omega}^2.$$

It is known that the normal component of $\mathbf{u} \in H(\mathrm{div}; \Omega)$ is in $H^{-\frac{1}{2}}(\partial\Omega)$; see [5]. The norm for the space $H^{-\frac{1}{2}}(\partial\Omega)$ is given by

$$\|\mathbf{u} \cdot \mathbf{n}\|_{-\frac{1}{2};\Omega} := \sup_{\phi \in H^{\frac{1}{2}}(\partial\Omega), \phi \neq 0} \frac{< \mathbf{u} \cdot \mathbf{n}, \phi >}{\|\phi\|_{\frac{1}{2};\Omega}}.$$

The angle brackets stand for the duality product of $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

LEMMA 3.1 (Helmholtz decompositions). *Let Ω be simply connected and $\partial\Omega$ be connected and Lipschitz and let $H_0(\mathbf{curl}\,; \Omega)$ be the subset of $H(\mathbf{curl}\,; \Omega)$ with vanishing tangential components on $\partial\Omega$ and $H_0(\mathrm{div}\,; \Omega)$ be the subset of $H(\mathrm{div}\,; \Omega)$ with* *vanishing normal components on $\partial\Omega$. Then, $H_0(\mathbf{curl}\,;\Omega)$, $H_0(\mathrm{div}\,;\Omega)$, $H(\mathbf{curl}\,;\Omega)$ and $H(\mathrm{div}\,;\Omega)$ have the following generalized orthogonal Helmholtz decompositions:*

$$\begin{aligned} H_0(\mathbf{curl}\,;\Omega) &= \mathbf{grad}\, H_0^1(\Omega) \oplus H_0^{\perp}(\mathbf{curl}\,;\Omega), \\ H(\mathbf{curl}\,;\Omega) &= \mathbf{grad}\, H^1(\Omega) \oplus H^{\perp}(\mathbf{curl}\,;\Omega) \end{aligned}$$

and

$$\begin{aligned} H_0(\mathrm{div}\,;\Omega) &= \mathbf{curl}\, H_0(\mathbf{curl}\,;\Omega) \oplus H_0^{\perp}(\mathrm{div}\,;\Omega) \\ &= \mathbf{curl}\, H_0^{\perp}(\mathbf{curl}\,;\Omega) \oplus H_0^{\perp}(\mathrm{div}\,;\Omega), \\ H(\mathrm{div}\,;\Omega) &= \mathbf{curl}\, H(\mathbf{curl}\,;\Omega) \oplus H^{\perp}(\mathrm{div}\,;\Omega) \\ &= \mathbf{curl}\, H^{\perp}(\mathbf{curl}\,;\Omega) \oplus H^{\perp}(\mathrm{div}\,;\Omega), \end{aligned}$$

where

$$\begin{aligned} H_0^{\perp}(\mathbf{curl}\,;\Omega) &= H_0(\mathbf{curl}\,;\,\Omega) \cap H(\mathrm{div}_{\,0}\,;\Omega), \\ H^{\perp}(\mathbf{curl}\,;\Omega) &= H(\mathbf{curl}\,;\,\Omega) \cap H_0(\mathrm{div}_{\,0}\,;\Omega) \end{aligned}$$

and

$$\begin{aligned} H_0^{\perp}(\mathrm{div}\,;\Omega) &= H_0(\mathrm{div}\,;\Omega) \cap H(\mathbf{curl}_{\,0}\,;\Omega), \\ H^{\perp}(\mathrm{div}\,;\Omega) &= H(\mathrm{div}\,;\Omega) \cap H_0(\mathbf{curl}_{\,0}\,;\Omega). \end{aligned}$$

Here, $H(\mathbf{curl}_{\,0}\,;\Omega)$, $H(\mathrm{div}_{\,0}\,;\Omega)$, $H_0(\mathbf{curl}_{\,0}\,;\Omega)$ and $H_0(\mathrm{div}_{\,0}\,;\Omega)$ are defined as follows:

$$\begin{aligned} H(\mathbf{curl}_{\,0}\,;\Omega) &= \{\mathbf{u} \in H(\mathbf{curl}\,;\Omega), \mathbf{curl}\,\mathbf{u} = 0\}, \\ H_0(\mathbf{curl}_{\,0}\,;\Omega) &= \{\mathbf{u} \in H_0(\mathbf{curl}\,;\Omega), \mathbf{curl}\,\mathbf{u} = 0\} \end{aligned}$$

and

$$\begin{aligned} H(\mathrm{div}_{\,0}\,;\Omega) &= \{\mathbf{u} \in H(\mathrm{div}\,;\Omega), \mathrm{div}\,\mathbf{u} = 0\}, \\ H_0(\mathrm{div}_{\,0}\,;\Omega) &= \{\mathbf{u} \in H_0(\mathrm{div}\,;\Omega), \mathrm{div}\,\mathbf{u} = 0\}. \end{aligned}$$

Proof. See [7, Proposition 1, p.215]. □

REMARK 3.1. *There is another kind of stable decomposition. A drawback of this decomposition is lack of L^2-stability; see [14, Lemma 3.10] for details.*

LEMMA 3.2 (Discrete Helmholtz decompositions). *If Ω is simply connected with a connected Lipschitz boundary, then we have similar decompositions for the finite element spaces as in the continuous cases. Thus,*

$$\begin{aligned} N_{0;h}(\Omega) &= \mathbf{grad}\, S_{0;h}(\Omega) \oplus N_{0;h}^{\perp}(\Omega), \\ N_h(\Omega) &= \mathbf{grad}\, S_h(\Omega) \oplus N_h^{\perp}(\Omega) \end{aligned}$$

and

$$\begin{aligned} X_{0;h}(\Omega) &= \mathbf{curl}\, N_{0;h}(\Omega) \oplus X_{0;h}^{\perp}(\Omega) \\ &= \mathbf{curl}\, N_{0;h}^{\perp}(\Omega) \oplus X_{0;h}^{\perp}(\Omega), \\ X_h(\Omega) &= \mathbf{curl}\, N_h(\Omega) \oplus X_h^{\perp}(\Omega) \\ &= \mathbf{curl}\, N_h^{\perp}(\Omega) \oplus X_h^{\perp}(\Omega), \end{aligned}$$

where $X_{0;h}^{\perp}(\Omega)$, $N_{0;h}^{\perp}(\Omega)$, $X_h^{\perp}(\Omega)$ and $N_h^{\perp}(\Omega)$ are orthogonal complements.

Proof. See [12, Theorem 2.36]. □

REMARK 3.2. *There is a discrete version of Remark 3.1 as well; see [14, Lemma 5.1]. This discrete decomposition has one additional term compared to the discrete Helmholtz decomposition. The term is a kind of error part which comes from Scott-Zhang interpolations in [20].*

LEMMA 3.3. *Let Ω be simply connected with a connected Lipschitz boundary. Then, for $\mathbf{u} \in H^{\perp}(\mathbf{curl}\,;\Omega) \cup N_h^{\perp}(\Omega)$ and $\mathbf{u} \in H^{\perp}(\mathrm{div}\,;\Omega) \cup X_h^{\perp}(\Omega)$, we have the following estimates:*

$$\|\mathbf{u}\|_{0;\Omega} \leq CH_{\Omega}\|\mathbf{curl}\,\mathbf{u}\|_{0;\Omega}, \forall \mathbf{u} \in H^{\perp}(\mathbf{curl}\,;\Omega) \cup N_h^{\perp}(\Omega)$$

and

$$\|\mathbf{u}\|_{0;\Omega} \leq CH_{\Omega}\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega}, \forall \mathbf{u} \in H^{\perp}(\mathrm{div}\,;\Omega) \cup X_h^{\perp}(\Omega),$$

where H_{Ω} is the diameter of Ω.

Proof. See [1, Prop. 4.6]. □

LEMMA 3.4. *If Ω is convex, the spaces $H^{\perp}(\mathbf{curl}\,;\Omega)$ and $H^{\perp}(\mathrm{div}\,;\Omega)$ are continuously embedded in $(H^1(\Omega)^d)$, where d is the dimension of Ω.*

Proof. See [1, Theorem 2.17] and [13, Lemma 4.1]. □

4. Overlapping Schwarz Algorithm.

We recall that the domain Ω is a bounded polygon or polyhedron in $\mathbb{R}^2$ or $\mathbb{R}^3$. We consider a decomposition of the domain Ω into N nonoverlapping subdomains Ω_i of diameter H_i. For each subdomain, h_i is the minimum diameter of the triangulation. We also consider extended subregions Ω_i' obtained from Ω_i by adding layers of elements. Thus, $\partial\Omega_i'$ does not cut through any elements. We will use [25, Assumption 3.1 and 3.2].

ASSUMPTION 4.1. *For $i = 1, \cdots, N$, there exists $\delta_i > 0$, such that, if x belongs to Ω_i', then*

$$\mathrm{dist}(x, \partial\Omega_j' \backslash \partial\Omega) \geq \delta_i,$$

for a suitable $j = j(x)$, possibly equal to i, with $x \in \Omega_j'$. The maximum of the ratios H_i/δ_i is denoted by

$$\frac{H}{\delta} = \max_{1 \leq i \leq N} \left\{ \frac{H_i}{\delta_i} \right\}.$$

Moreover, we can consider the factor $\dfrac{H}{h}$ as follows:

$$\frac{H}{h} := \max_{1 \leq i \leq N} \left\{ \frac{H_i}{h_i} \right\}.$$

ASSUMPTION 4.2 (Finite Covering). *The partition $\{\Omega_i'\}$ can be colored using a finite number of N^c colors, in such a way that subregions with the same color are disjoint.*

These assumptions will involve overlap parameters δ_i and diameter parameters H_i.

ASSUMPTION 4.3. *We will assume that each subdomain Ω_i is a convex polygon or polyhedron. In each subdomain Ω_i, we assume that the coefficients α and β are* *constant and that they thus only have jumps across the interface* Γ *which is defined by*

$$\Gamma = \left(\bigcup_{i=0}^{N} \partial\Omega_i \right) \backslash \partial\Omega.$$

Consider the variational problem (1.2). Restricting to the finite element space of the lowest order Raviart-Thomas elements, we obtain the stiffness matrix A.

4.1. The Coarse Component. Instead of the conventional coarse basis, we will use energy-minimal, discrete harmonic extensions to define the new coarse basis functions. The coarse part of the preconditioner is of the form $R_0^T A_0^{-1} R_0$. We need to define R_0 and A_0.

For each face (or edge) F_{ij}, a subset of the interface Γ, we can write a submatrix of the stiffness matrix A. It is corresponding to the two subdomains which have F_{ij} in common:

$$\begin{bmatrix} A_{II}^{(i)} & 0 & A_{IF_{ij}}^{(i)} \\ 0 & A_{II}^{(j)} & A_{IF_{ij}}^{(j)} \\ A_{F_{ij}I}^{(i)} & A_{F_{ij}I}^{(j)} & A_{F_{ij}F_{ij}} \end{bmatrix}.$$

Let $\widetilde{u}_{ij} := [u_I^{(i)T} \; u_I^{(j)T} \; u_{F_{ij}}^T]^T$. $\widetilde{u}_{ij}$ is the discrete harmonic extension if $A_{II}^{(i)} u_I^{(i)} + A_{IF_{ij}}^{(i)} u_{F_{ij}} = 0$ and $A_{II}^{(j)} u_I^{(j)} + A_{IF_{ij}}^{(j)} u_{F_{ij}} = 0$; cf. [25, Chap 4.4]. We can write $\widetilde{u}_{ij}$ as $\widetilde{u}_{ij} = [(E_i u_{F_{ij}})^T \; (E_j u_{F_{ij}})^T \; u_{F_{ij}}^T]^T$ where $E_i := -{A_{II}^{(i)}}^{-1} A_{IF_{ij}}^{(i)}$ and $E_j := -{A_{II}^{(j)}}^{-1} A_{IF_{ij}}^{(j)}$. Also, let u_{ij} be the extension of $\widetilde{u}_{ij}$ to a global vector obtained by an extension by zero. We note that if we know $u_{F_{ij}}$, then u_{ij} is completely determined. We choose $u_{F_{ij}}^T = [1, 1, \cdots, 1]$ to define a coarse basis vector u_{ij} for the face (or edge) F_{ij}. We can now define A_0 and R_0, after introducing a suitable global indexing, by

$$(A_0)_{mn} := u_{ij}^T A u_{kl},$$

where F_{ij} and F_{kl} are the m-th and n-th face of Γ, respectively.

Furthermore, let

$$R_0 := \begin{bmatrix} & \vdots & \\ - & u_{ij}^T & - \\ & \vdots & \end{bmatrix}.$$

Hence, we can now construct the coarse part $R_0^T A_0^{-1} R_0$ by using the above definitions.

4.2. Local Components. For the local components, each R_i is a rectangular matrix with elements equal to 0 or 1. Each R_i just provides the indices relevant to an individual extended subdomain Ω_i'. It means that each R_i extracts the degrees of freedom of Ω_i', the extended subregion obtained from Ω_i by adding layers of elements. We can then define a submatrix of the original stiffness matrix A by the following formula:

$$A_i = R_i A R_i^T.$$

Thus, A_i is just the principal minor of the original stiffness matrix A defined by R_i. By using these matrices, we can build the local part $\sum_{i=1}^{N} R_i^T A_i^{-1} R_i$ of the preconditioner.

4.3. The Additive Schwarz Operator. We now construct our preconditioner. Let $P_i = R_i^T A_i^{-1} R_i A$. The preconditioned linear operator has the following form:

$$P_{ad} = \sum_{i=0}^{N} P_i = \sum_{i=0}^{N} R_i^T A_i^{-1} R_i A.$$

We use one global coarse solver for A_0^{-1} and N local solvers for the A_i^{-1} when applying the operator P_{ad} to a vector. By using a suitable indexing, we can perform most work of the preconditioned conjugate gradient method locally and in parallel except for the work of the coarse part and the communication between neighboring pairs of subdomains; see [21], [25, Chap. 3].

5. Technical tools and the main result. We will consider the 3D case only; the arguments are quite similar for 2D. We recall that the coefficients α and β are constants in each subdomain Ω_i. We can then write the weak problem in the following way:

$$\mathbf{a}(\mathbf{u},\mathbf{v}) := \sum_{i=1}^{N} \alpha_i \int_{\Omega_i} \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, dx + \beta_i \int_{\Omega_i} \mathbf{u}\cdot\mathbf{v}\, dx,\ \mathbf{u},\mathbf{v} \in H_0(\operatorname{div};\Omega).$$

We can also define local energy bilinear forms:

$$\mathbf{a}_i(\mathbf{u},\mathbf{u}) := \alpha_i \int_{\Omega_i} \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{u}\, dx + \beta_i \int_{\Omega_i} \mathbf{u}\cdot\mathbf{u}\, dx$$

and

$$\begin{aligned}\widetilde{\mathbf{a}}_i(\mathbf{u},\mathbf{u}) :=& \int_{\Omega_i'} \alpha \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{u}\, dx + \int_{\Omega_i'} \beta\, \mathbf{u}\cdot\mathbf{u}\, dx \\ =& \ \alpha_i \int_{\Omega_i} \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{u}\, dx + \beta_i \int_{\Omega_i} \mathbf{u}\cdot\mathbf{u}\, dx \\ &+ \sum_{\Omega_i'\cap\Omega_j\neq\phi} \alpha_j \int_{\Omega_i'\cap\Omega_j} \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{u}\, dx + \beta_j \int_{\Omega_i'\cap\Omega_j} \mathbf{u}\cdot\mathbf{u}\, dx.\end{aligned}$$

5.1. Technical Tools. We will develop some useful technical tools.

LEMMA 5.1 (Divergence free extension). *We recall that each Ω_i is a convex polyhedron. Then, there exists an extension operator $\widetilde{\mathcal{H}}_i : W_{0;h}(\partial\Omega_i) \rightarrow X_h$ which satisfies*

$$(\widetilde{\mathcal{H}}_i\mu)\cdot\mathbf{n}_{|F} = \mu,\ \operatorname{div}\widetilde{\mathcal{H}}_i\mu = 0,$$

and $\forall F \subset \partial\Omega_i$, $\forall\mu \in W_{0;h}(\partial\Omega_i)$, and with $\mu \equiv 0$ on $\partial\Omega_i\backslash F$ and which satisfies the following estimate:

$$\|\widetilde{\mathcal{H}}_i\mu\|_{0;\Omega_i} \leq C\|\mu\|_{-\frac{1}{2};\partial\Omega_i}.$$

Proof. Consider the following Neumann boundary value problem:

$$-\Delta\phi = 0 \text{ in } \Omega_i,$$

$$\frac{\partial \phi}{\partial n} = \mu \text{ on } \partial\Omega_i.$$

Because μ has zero mean value on $\partial\Omega_i$, this problem is well posed under the additional condition, $\int_{\Omega_i} \phi = 0$. Let $\mathbf{u} := \nabla\phi$ and define $\widetilde{\mathcal{H}}_i$ as $\widetilde{\mathcal{H}}_i\mu := \Pi_h^{RT}\mathbf{u}$, where Π_h^{RT} is the interpolation operator into the Raviart-Thomas finite element space. Then, by an elliptic regularity result,

$$\|\mathbf{u}\|_{0;\Omega_i} = |\phi|_{1;\Omega_i} \leq C\|\mu\|_{-\frac{1}{2};\partial\Omega_i}. \tag{5.1}$$

For all $s < \frac{1}{2}$, $\mu \in H^s(\partial\Omega_i)$ because μ is piecewise constant on $\partial\Omega_i$. By using a regularity result in [6, Corollary 23.5], we also have

$$\|\phi\|_{\frac{3}{2}+s;\Omega_i} \leq C\|\mu\|_{s;\partial\Omega_i}.$$

Hence, $\mathbf{u} \in H^{\frac{1}{2}+s}$ for a positive s. We can conclude that $\Pi_h^{RT}\mathbf{u}$ is then well-defined.

Moreover, by a property of Raviart-Thomas interpolation [4, p.150 Property 5.3], we have

$$\operatorname{div}\widetilde{\mathcal{H}}_i\mu = \operatorname{div}\Pi_h^{RT}\mathbf{u} = \Pi_h \operatorname{div}\mathbf{u} = 0,$$

where Π_h is the L^2-projection onto the space of piecewise constant functions.

We obtain the estimate by using an error estimate for Raviart-Thomas elements and an inverse estimate:

$$\|\mathbf{u} - \Pi_h^{RT}\mathbf{u}\|_{0;\Omega_i} \leq Ch^{\frac{1}{2}+s}|\mathbf{u}|_{\frac{1}{2}+s;\Omega_i} \leq h^{\frac{1}{2}+s}\|\phi\|_{\frac{3}{2}+s;\Omega_i} \leq C\|\mu\|_{-\frac{1}{2};\partial\Omega_i}, \tag{5.2}$$

for some $0 < s \leq \frac{1}{2}$; see [19, Sect. 3.4.2] and [5, Sect. III.3.4].

We finally obtain our estimate by using (5.1) and (5.2):

$$\|\widetilde{\mathcal{H}}_i\mu\|_{0;\Omega_i} = \|\Pi_h^{RT}\mathbf{u}\|_{0;\Omega_i} \leq \|\mathbf{u}\|_{0;\Omega_i} + \|\mathbf{u} - \Pi_h^{RT}\mathbf{u}\|_{0;\Omega_i} \leq C\|\mu\|_{-\frac{1}{2};\partial\Omega_i};$$

see also [29, Lemma 4.3] and [28, Lemma 2.6]. ∎

LEMMA 5.2 (Discrete harmonic extension). *There exists a discrete harmonic extension operator $\mathcal{H}_i : W_{0;h}(\partial\Omega_i) \to X_h(\Omega_i)$, which satisfies*

$$(\mathcal{H}_i\mu)\cdot\mathbf{n}_{|F} = \mu.$$

$\forall\mu \in W_{0;h}(\partial\Omega_i)$ *and $F \subset \partial\Omega_i$, and for which*

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|^2_{0;\Omega_i} + \beta_i\|\mathcal{H}_i\mu\|^2_{0;\Omega_i} \leq C\beta_i\|\mu\|^2_{-\frac{1}{2};\partial\Omega_i}.$$

Proof. $\mathcal{H}_i$ is the minimal-energy extension for the given subdomain. Therefore, we obtain the following estimate:

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|^2_{0;\Omega_i} + \beta_i\|\mathcal{H}_i\mu\|^2_{0;\Omega_i} \leq \alpha_i\|\operatorname{div}\widetilde{\mathcal{H}}_i\mu\|^2_{0;\Omega_i} + \beta_i\|\widetilde{\mathcal{H}}_i\mu\|^2_{0;\Omega_i}.$$

But $\operatorname{div}\widetilde{\mathcal{H}}_i\mu = 0$. Therefore, by Lemma 5.1,

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|^2_{0;\Omega_i} + \beta_i\|\mathcal{H}_i\mu\|^2_{0;\Omega_i} \leq C\beta_i\|\mu\|^2_{-\frac{1}{2};\partial\Omega_i}.$$

∎

The condition $\mu \in W_{0;h}(\partial\Omega_i)$, which means that $\int_{\partial\Omega_i} \mu ds = 0$, is very important; cf. [9,15,26]. This means that it is important to find a suitable $\mathbf{v}$ which makes the flux of $\mathbf{u} - \mathbf{v}$ zero across $\partial\Omega_i$. To make this possible let us consider the coarse interpolation operator Π_H^{RT} onto the Raviart-Thomas space of the coarse mesh. For a given F, a coarse face contained in the interface Γ, we define

$$\lambda_F(\Pi_H^{RT}\mathbf{u}) := \frac{1}{|F|}\int_F \mathbf{u}\cdot\mathbf{n}ds.$$

Trivially,

$$\int_F (\mathbf{u} - \Pi_H^{RT}\mathbf{u})\cdot\mathbf{n}\,ds = 0.$$

We will need some estimates for Π_H^{RT}.

LEMMA 5.3 (Stability estimate for the coarse interpolation). *For all $\mathbf{u} \in X_h$, we have the following estimates:*

$$\|\mathrm{div}\,(\Pi_H^{RT}\mathbf{u})\|_{0;\Omega_i}^2 \le \|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 \tag{5.3}$$

and

$$\|\Pi_H^{RT}\mathbf{u}\|_{0;\Omega_i}^2 \le C((1+\log\frac{H_i}{h_i})\|\mathbf{u}\|_{0;\Omega_i}^2 + H_i^2\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2). \tag{5.4}$$

The constant C depends only on the aspect ratio of the elements of $\mathcal{T}_H$ and the elements of $\mathcal{T}_h$.

Proof. The first estimate (5.3) follows by using the following property of Raviart-Thomas interpolation:

$$\mathrm{div}\,(\Pi_H^{RT}\mathbf{u}) = \Pi_H(\mathrm{div}\,\mathbf{u}),$$

where Π_H is the L^2 projection onto the space of piecewise constant on the coarse mesh; see [4, p.150 5.3].

For the second estimate (5.4), we use Green's identity and the face basis function; see [11], [25, Lemma 4.25]. We also use the fact that the L^2 norms of functions in the Raviart-Thomas finite element space can be bounded from above and below by the l_2-norm of their degrees of freedom; cf. [19, Proposition 6.3.1]. For details, see [28, Lemma 2.4] and [29, Lemma 4.1]. ∎

LEMMA 5.4. *Let $\mathbf{u} \in X_h$ and θ_i be a continuous, piecewise linear scalar function supported in Ω_i. Then,*

$$\|\Pi_h^{ND}(\theta_i\mathbf{u})\|_{0;\Omega_i}^2 \le C\|\theta_i\mathbf{u}\|_{0;\Omega_i}^2,$$

$$\|\mathbf{curl}\,(\Pi_h^{ND}(\theta_i\mathbf{u}))\|_{0;\Omega_i}^2 \le C\|\mathbf{curl}\,(\theta_i\mathbf{u})\|_{0;\Omega_i}^2,$$

$$\|\Pi_h^{RT}(\theta_i\mathbf{u})\|_{0;\Omega_i}^2 \le C\|\theta_i\mathbf{u}\|_{0;\Omega_i}^2,$$

and

$$\|\mathrm{div}\,(\Pi_h^{RT}(\theta_i\mathbf{u}))\|_{0;\Omega_i}^2 \le C\|\mathrm{div}\,(\theta_i\mathbf{u})\|_{0;\Omega_i}^2,$$

where Π_h^{ND} and Π_h^{RT} are the interpolation operators onto the lowest order Nédélec finite element space and the lowest order Raviart-Thomas finite element space, respectively.

Proof. We use error estimates of the operators Π_h^{ND} and Π_h^{RT} and an inverse inequality; see [4, Lemma 5.5]. For more details, see [24, Lemma 4.3] and [25, Lemma 10.8 and Lemma 10.13]. □

DEFINITION 5.1 (Projection Operators). *Let $\Theta_{\mathbf{curl}}^{\perp}$ and $\Theta_{\mathrm{div}}^{\perp}$ be the orthogonal projections from $H(\mathbf{curl}\,;\Omega)$ onto $H^{\perp}(\mathbf{curl}\,;\Omega)$ and from $H(\mathrm{div}\,;\Omega)$ onto $H^{\perp}(\mathrm{div}\,;\Omega)$, respectively. We next define a projection P_h^{ND} from $H(\mathbf{curl}\,;\Omega)$ onto V_{ND}^{+} and P_h^{RT} from $H(\mathrm{div}\,;\Omega)$ onto V_{RT}^{+}, with $V_{ND}^{+}=\Theta_{\mathbf{curl}}^{\perp}(N_h^{\perp})$ and $V_{RT}^{+}=\Theta_{\mathrm{div}}^{\perp}(X_h^{\perp})$.*

REMARK 5.1. *We can easily check that $\mathbf{curl}\,(P_h^{ND}\mathbf{u}^{\perp})=\mathbf{curl}\,(\Theta_{\mathbf{curl}}^{\perp}\mathbf{u}^{\perp})=\mathbf{curl}\,\mathbf{u}^{\perp}$ and $\mathrm{div}\,(P_h^{RT}\mathbf{v}^{\perp})=\mathrm{div}\,(\Theta_{\mathrm{div}}^{\perp}\mathbf{v}^{\perp})=\mathrm{div}\,\mathbf{v}^{\perp}$ whenever $\mathbf{u}^{\perp}\in N_h^{\perp}$ and $\mathbf{v}^{\perp}\in X_h^{\perp}$.*

LEMMA 5.5. *Let Ω_i be convex. Then, we have the following error estimates:*

$$\|\mathbf{u}_h^{\perp}-P_h^{ND}\mathbf{u}_h^{\perp}\|_{0;\Omega_i}\leq Ch_i\|\mathbf{curl}\,\mathbf{u}_h^{\perp}\|_{0;\Omega_i},\ \forall\mathbf{u}_h^{\perp}\in N_h^{\perp}(\Omega_i)$$

and

$$\|\mathbf{v}_h^{\perp}-P_h^{RT}\mathbf{v}_h^{\perp}\|_{0;\Omega_i}\leq Ch_i\|\mathrm{div}\,\mathbf{v}_h^{\perp}\|_{0;\Omega_i},\ \forall\mathbf{v}_h^{\perp}\in X_h^{\perp}(\Omega_i),$$

with C independent of h and $\mathbf{u}_h^{\perp}$.

Proof. We can use the almost same idea in [24, Lemma 3.3] and [13, Lemma 4.2, 4.3 and 4.4]. □

LEMMA 5.6. *Let $\Omega_{i,\delta_i}\subset\Omega_i$ be the set of all points which are within a distance δ_i of the boundary of Ω_i. Assume that all subdomains are convex. Then, there exists a constant C such that $\forall\mathbf{u}^{\perp}\in N_h^{\perp}$ and $\forall\mathbf{v}^{\perp}\in X_h^{\perp}$,*

$$\frac{1}{\delta_i^2}\|\mathbf{u}^{\perp}\|^2_{0;\Omega_i\cap\Omega_{i,\delta_i}}\leq C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^{\perp}\|^2_{0;\Omega_i}$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^{\perp}\|^2_{0;\Omega_i\cap\Omega_{i,\delta_i}}\leq C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^{\perp}\|^2_{0;\Omega_i}.$$

Similarly, for the subdomain Ω_j which has a face in common with Ω_i, we have

$$\frac{1}{\delta_i^2}\|\mathbf{u}^{\perp}\|^2_{0;\Omega_i'\cap\Omega_j}\leq C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^{\perp}\|^2_{0;\Omega_j}$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^{\perp}\|^2_{0;\Omega_i'\cap\Omega_j}\leq C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^{\perp}\|^2_{0;\Omega_j}.$$

Moreover, for $\forall m\in I_{jl}$, we have

$$\frac{1}{\delta_i^2}\|\mathbf{u}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}\leq C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^{\perp}\|^2_{0;\Omega_m},$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}\leq C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^{\perp}\|^2_{0;\Omega_m},$$

where $\Psi_{jl} =: \bigcap_{m \in I_{jl}} \Omega'_m$ *with* I_{jl} *the set of indices of the subdomains which have an edge* E_{jl} *common with* Ω_i*.*

Proof. By the triangle inequality,

$$\|\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}} \le 2(\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}} + \|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}}).$$

Consider the first term. By Lemma 5.5,

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}} \le \frac{1}{\delta_i^2}\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i} \le \frac{h_i^2}{\delta_i^2}\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i}.$$

$\frac{h_i}{\delta_i}$ is bounded by 1. Hence, the first term is bounded by $C\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i}$.

For the second term, we will use an argument similar to that of [25, Lemma 3.10]. By a Friedrichs inequality, Lemma 3.4, and Remark 5.1, we have

$$\begin{aligned}
\frac{1}{\delta_i^2}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}} &\le C(|P_h^{ND}\mathbf{u}^\perp|^2_{1;\Omega_i \cap \Omega_{i,\delta_i}} + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\partial\Omega_i}) \\
&\le C(|P_h^{ND}\mathbf{u}^\perp|^2_{1;\Omega_i} + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\partial\Omega_i}) \\
&\le C(\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i} + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\partial\Omega_i}) \\
&= C(\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i} + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\partial\Omega_i}).
\end{aligned}$$

By a trace estimate and by combining [25, Lemma A.6], the embedding $L^2(\partial\Omega_i) \subset H^{\frac{1}{2}}(\partial\Omega_i)$ with scaling, Lemma 3.4, Lemma 3.3, and Remark 5.1, we find

$$\begin{aligned}
\frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\partial\Omega_i} &\le C(\frac{H_i}{\delta_i}|P_h^{ND}\mathbf{u}^\perp|^2_{1;\Omega_i} + \frac{1}{\delta_i H_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i}) \\
&\le C(\frac{H_i}{\delta_i}\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i} + \frac{1}{\delta_i H_i}\|P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i}) \\
&\le C(\frac{H_i}{\delta_i}\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i} + \frac{1}{\delta_i H_i}H_i^2\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i}) \\
&\le C\frac{H_i}{\delta_i}\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|^2_{0;\Omega_i} = C\frac{H_i}{\delta_i}\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i}.
\end{aligned}$$

Therefore,

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp\|^2_{0;\Omega_i \cap \Omega_{i,\delta_i}} \le C(1 + \frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i}.$$

We can use the exactly same idea for all the other estimates. ∎

5.2. Stability Estimates. We consider the coarse part first.

Lemma 5.7 (Coarse Estimate). *Let* $\mathbf{u}_0$ *be the discrete harmonic extension of the given interface values of* $\Pi_H^{RT}\mathbf{u}$*. Then,*

$$\mathbf{a}(\mathbf{u}_0, \mathbf{u}_0) \le C(1 + \log\frac{H}{h})\mathbf{a}(\mathbf{u}, \mathbf{u}), \tag{5.5}$$

where C *is independent of* H*,* h *and the jumps in the coefficients.*

Proof. First, let us assume that $H_i^2\beta_i \le \alpha_i$. Let $\mathbf{u}_H := \Pi_H^{RT}\mathbf{u}$. We note that $\mathbf{u}_0$ is the discrete harmonic extension with the same interface value as $\mathbf{u}_H$ on $\partial\Omega_i$.

By the minimal-energy property of the discrete harmonic extension and Lemma 5.3, we find

$$\begin{aligned}\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0) &\le \mathbf{a}_i(\mathbf{u}_H,\mathbf{u}_H)\\ &\le C(1+\log\frac{H_i}{h_i})(\beta_i\|\mathbf{u}\|_{0;\Omega_i}^2 + (\alpha_i + H_i^2\beta_i)\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2)\\ &\le C(1+\log\frac{H_i}{h_i})(\alpha_i\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + \beta_i\|\mathbf{u}\|_{0;\Omega_i}^2).\end{aligned}$$

Hence, we obtain

$$\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0) \le C(1+\log\frac{H_i}{h_i})\mathbf{a}_i(\mathbf{u},\mathbf{u}). \tag{5.6}$$

We now assume that $H_i^2\beta_i \ge \alpha_i$. We use a method similar to that of [18, Lemma 4.1].

We introduce piecewise linear scalar cut-off functions χ_1 and χ_2. The two functions satisfy the following conditions: χ_1 is equal to 1 on all interior small faces of F_{ij} and $\chi_1|_{\partial\Omega_i\setminus F_{ij}} = 0$. The extension of χ_1 takes values between 0 and 1; c.f. [25, Sect. 4.6.3]. χ_2 has the value 1 on $\partial\Omega_i$. Also, $\chi_k|_{\Omega_i\setminus\Omega_{i,d_k}} = 0$ for some $h_i \le d_k \le H_i$. Moreover, $\|\nabla\chi_k\|_\infty \le \dfrac{C}{d_k}$ for $k = 1$ and 2.

Then, the following estimates hold; cf. [9, Sect. 4 and 5], [10, Sect. 4], and [25, Lemma 4.25]:

$$\|\chi_1\|_{0;\Omega_i}^2 \le CH_i^2 d_1,$$

$$|\chi_1|_{1;\Omega_i}^2 \le C(1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}.$$

Let ϕ_{ij} be the coarse basis function corresponding to the face F_{ij}. This means that the normal component of ϕ_{ij} is 1 on F_{ij} and 0 on other faces of $\Omega_i \cup \Omega_j$ and the interior values of ϕ_{ij} are obtained by the discrete harmonic extension. We know that $\|\phi_{ij}\|_{0;\Omega_i}^2 \le CH_i^3$ and $\|\mathrm{div}\,\phi_{ij}\|_{0;\Omega_i}^2 \le CH_i$. The function $\mathbf{u}_0$ can be expressed as follows:

$$\mathbf{u}_0 = \sum_{F_{ij}\subset\Gamma} \lambda_{F_{ij}}\phi_{ij}.$$

Hence, it is enough to consider of these terms one by one. We provide bounds of the coefficient and the energy of the basis functions separately.

We first consider the coefficients. We modify the proof of [28, Lemma 2.4]. Let f_k be the small faces which contain edges of F_{ij}. We note that on f_k, χ_1 has values between 0 and 1. Also we know that the number of such faces, n_F, is bounded by $C(H/h)$; for details, see [28, Lemma 2.4]. We find

$$\begin{aligned}|F_{ij}|\lambda_{F_{ij}}(\mathbf{u}) &= \int_{F_{ij}} \mathbf{u}\cdot\mathbf{n}\,ds = \int_{F_{ij}} \chi_1\mathbf{u}\cdot\mathbf{n}\,ds + \sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k})\\ &= \int_{\Omega_i} \chi_1\mathrm{div}\,\mathbf{u} + \nabla\chi_1\cdot\mathbf{u}\,dx + \sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k}),\end{aligned}$$

where $|c_k| < 1$. We note that $(\sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k}))^2$ is bounded by $CH_i\|\mathbf{u}\|_{0;\Omega_i}^2$; see [28, (2.16)].

Hence,

$$
\begin{aligned}
|\lambda_{F_{ij}}(\mathbf{u})|^2 &\le C\frac{1}{H_i^4}((\int_{\Omega_i}\chi_1 \mathrm{div}\,\mathbf{u}\,dx)^2 + (\int_{\Omega_i}\nabla\chi_1\cdot\mathbf{u}\,dx)^2 + H_i\|\mathbf{u}\|_{0;\Omega_i}^2)\\
&\le C\frac{1}{H_i^4}(\|\chi_1\|_{0;\Omega_i}^2\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + \|\nabla\chi_1\|_{0;\Omega_i}^2\|\mathbf{u}\|_{0;\Omega_i}^2 + H_i\|\mathbf{u}\|_{0;\Omega_i}^2)\\
&\le C\frac{1}{H_i^4}(H_i^2 d_1\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + (1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}\|\mathbf{u}\|_{0;\Omega_i}^2 + H_i\|\mathbf{u}\|_{0;\Omega_i}^2) \quad (5.7)\\
&\le C\frac{1}{H_i^4}(H_i^2 d_1\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + (1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}\|\mathbf{u}\|_{0;\Omega_i}^2) \quad (5.8)\\
&\le C\frac{1}{H_i^2}(d_1\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + (1+\log\frac{H_i}{h_i})\frac{1}{d_1}\|\mathbf{u}\|_{0;\Omega_i}^2)\\
&\le C(1+\log\frac{H_i}{h_i})\frac{1}{H_i^2}(d_1\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 + \frac{1}{d_1}\|\mathbf{u}\|_{0;\Omega_i}^2). \quad (5.9)
\end{aligned}
$$

We note that due to the fact that $\frac{H_i^2}{d_1} \ge H_i$, the last term of (5.7) can be absorbed into the L^2 term of (5.8).

By Lemma 5.4 and estimates for the basis functions, we have

$$
\begin{aligned}
\|\mathrm{div}\,(\Pi_h^{RT}(\chi_2\phi_{ij}))\|_{0;\Omega_i}^2 &\le C\|\mathrm{div}\,(\chi_2\phi_{ij})\|_{0;\Omega_i}^2\\
&\le C\|\chi_2\|_\infty^2\|\mathrm{div}\,\phi_{ij}\|_{0;\Omega_{i,d2}}^2 + C\|\nabla\chi_2\|_\infty^2\|\phi_{ij}\|_{0;\Omega_{i,d2}}^2\\
&\le C(d_2 + \frac{1}{d_2^2}H_i^2 d_2) \le C(d_2 + \frac{H_i^2}{d_2}) \quad (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Pi_h^{RT}(\chi_2\phi_{ij})\|_{0;\Omega_i}^2 &\le C\|(\chi_2\phi_{ij})\|_{0;\Omega_i}^2\\
&\le C\|\chi_2\|_\infty^2\|\phi_{ij}\|_{0;\Omega_{i,d2}}^2 \le CH_i^2 d_2. \quad (5.11)
\end{aligned}
$$

By (5.9), (5.10) and (5.11), we find

$$
\begin{aligned}
&\alpha_i\|\lambda_{F_{ij}}(\mathbf{u})\mathrm{div}\,(\Pi_h^{RT}(\chi_2\phi_{ij}))\|_{0;\Omega_i}^2 + \beta_i\|\lambda_{F_{ij}}(\mathbf{u})\Pi_h^{RT}(\chi_2\phi_{ij})\|_{0;\Omega_i}^2\\
&\le C(\alpha_i|\lambda_{F_{ij}}(\mathbf{u})|^2(d_2 + \frac{H_i^2}{d_2}) + \beta_i|\lambda_{F_{ij}}(\mathbf{u})|^2 H_i^2 d_2)\\
&\le C(1+\log\frac{H_i}{h_i})((\alpha_i(\frac{d_1}{d_2} + \frac{d_1 d_2}{H_i^2}) + \beta_i d_1 d_2)\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2\\
&\quad +(\alpha_i(\frac{1}{d_1 d_2} + \frac{d_2}{d_1 H_i^2}) + \beta_i\frac{d_2}{d_1})\|\mathbf{u}\|_{0;\Omega_i}^2). \quad (5.12)
\end{aligned}
$$

Let $d_1 = \sqrt{\frac{\alpha_i}{\beta_i}}$ and $d_2 = H_i\sqrt{\frac{1}{1+\frac{\beta_i H_i^2}{\alpha_i}}}$. We note that $h_i \le d_1, d_2 \le H_i$.

We then obtain

$$
\begin{aligned}
&\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0)\\
&\le \sum \alpha_i\|\lambda_{F_{ij}}(\mathbf{u})\mathrm{div}\,(\Pi_h^{RT}(\chi_2\phi_{ij}))\|_{0;\Omega_i}^2 + \beta_i\|\lambda_{F_{ij}}(\mathbf{u})\Pi_h^{RT}(\chi_2\phi_{ij})\|_{0;\Omega_i}^2\\
&\le C(1+\log\frac{H_i}{h_i})\sqrt{1+\frac{\alpha_i}{\beta_i H_i^2}}\mathbf{a}_i(\mathbf{u},\mathbf{u}).
\end{aligned}
$$

Since $H_i^2\beta_i \geq \alpha_i$, $\sqrt{1+\frac{\alpha_i}{\beta_i H_i^2}}$ is bounded by a constant.

Hence,

$$\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0) \leq C(1+\log\frac{H_i}{h_i})\mathbf{a}_i(\mathbf{u},\mathbf{u}). \tag{5.13}$$

In all cases, we obtain the same result (5.6) and (5.13). We can conclude that (5.5) holds by summing over all subdomains. ☐

Remark 5.2. *In [28, Chap 2.2], the constant depends on* $\max_i(1+\frac{\beta_i H_i^2}{\alpha_i})$. *As we see from the numerical experiments in [28,29] and this paper, the results do not appear to depend on* α_i*'s and* β_i*'s at all. We have improved the previous results in [28,29] by using Lemma 5.7.*

We now consider the local components. Consider $\mathbf{v}=\mathbf{u}-\mathbf{u}_0$. We know that $\mathbf{v}\in X_h$. By Lemma 3.2, we can find $\mathbf{w}^\perp \in N_h^\perp$ and $\mathbf{v}^\perp \in X_h^\perp$ such that $\mathbf{v}=\mathbf{curl}\,\mathbf{w}^\perp+\mathbf{v}^\perp$.

Let θ_i be a piecewise linear function associated with the subdomain Ω_i. Each θ_i is constructed in a similar way as in [25, Lemma 3.4]. We define

$$\tilde{\theta}_i(x) = \begin{cases} 1, & \mathrm{dist}\,(x,\partial\Omega_i)\geq \delta_i, \\ 0, & x\in\partial\Omega_i, \\ \text{decays linearly}, & \text{otherwise}. \end{cases}$$

and set

$$\theta_i = I^h(\tilde{\theta}_i).$$

Lemma 5.8. *Let* $\mathbf{v}_i=\Pi_h^{RT}(\theta_i\mathbf{v}^\perp)$ *and* $\mathbf{w}_i=\Pi_h^{ND}(\theta_i\mathbf{w}^\perp)$. *Then,*

$$\sum_{i=1}^N \widetilde{\mathbf{a}}_i(\mathbf{v}_i,\mathbf{v}_i) \leq C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{v}^\perp,\mathbf{v}^\perp) \tag{5.14}$$

and

$$\sum_{i=1}^N \widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i,\mathbf{curl}\,\mathbf{w}_i) \leq C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{curl}\,\mathbf{w}^\perp,\mathbf{curl}\,\mathbf{w}^\perp), \tag{5.15}$$

with C *independent of* H, h *and jumps of the coefficients.*

Proof. We note that θ_i is supported in $\overline{\Omega}_i$. By Lemma 5.4,

$$\begin{aligned} \widetilde{\mathbf{a}}_i(\mathbf{v}_i,\mathbf{v}_i) = \mathbf{a}_i(\mathbf{v}_i,\mathbf{v}_i) &= \alpha_i\|\mathrm{div}\,\mathbf{v}_i\|_{0;\Omega_i}^2 + \beta_i\|\mathbf{v}_i\|_{0;\Omega_i}^2 \\ &= \alpha_i\|\mathrm{div}\,(\Pi_h^{RT}(\theta_i\mathbf{v}^\perp))\|_{0;\Omega_i}^2 + \beta_i\|\Pi_h^{RT}(\theta_i\mathbf{v}^\perp)\|_{0;\Omega_i}^2 \\ &\leq C(\alpha_i\|\mathrm{div}\,(\theta_i\mathbf{v}^\perp)\|_{0;\Omega_i}^2 + \beta_i\|\theta_i\mathbf{v}^\perp\|_{0;\Omega_i}^2). \end{aligned}$$

Consider the L^2 term:

$$\|\theta_i\mathbf{v}^\perp\|_{0;\Omega_i}^2 \leq \|\theta_i\|_\infty^2\|\mathbf{v}^\perp\|_{0;\Omega_i}^2 \leq \|\mathbf{v}^\perp\|_{0;\Omega_i}^2.$$

We now consider the divergence term and find

$$
\begin{aligned}
\|\mathrm{div}\,(\theta_i \mathbf{v}^\perp)\|^2_{0;\Omega_i} &\le C(\|\nabla\theta_i \cdot \mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\theta_i \mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \\
&\le C(\|\nabla\theta_i\|^2_\infty \|\mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\theta_i\|^2_\infty \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \\
&\le C(\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \qquad (5.16)\\
&\le C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}. \qquad (5.17)
\end{aligned}
$$

We obtain (5.17) from (5.16) by using Lemma 5.6.

Therefore,

$$
\widetilde{\mathbf{a}}_i(\mathbf{v}_i, \mathbf{v}_i) \le C(1+\frac{H_i}{\delta_i})\mathbf{a}_i(\mathbf{v}^\perp, \mathbf{v}^\perp).
$$

We now consider (5.15). We note that $\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) = \mathbf{a}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i)$. By Lemma 5.4,

$$
\begin{aligned}
\mathbf{a}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) &= \beta_i \|\mathbf{curl}\,\mathbf{w}_i\|^2_{0;\Omega_i} \\
&= \beta_i \|\mathbf{curl}\,(\Pi_h^{ND}(\theta_i \mathbf{w}^\perp))\|^2_{0;\Omega_i} \\
&\le C\beta_i \|\mathbf{curl}\,(\theta_i \mathbf{w}^\perp)\|^2_{0;\Omega_i} \\
&\le C\beta_i (\|\nabla\theta_i \times \mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\theta_i \mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}) \\
&\le C\beta_i (\|\nabla\theta_i\|^2_\infty \|\mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\theta_i\|^2_\infty \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}) \\
&\le C\beta_i (\frac{1}{\delta_i^2}\|\mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}).
\end{aligned}
$$

By Lemma 5.6, the following inequality holds:

$$
\begin{aligned}
\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) &\le C\beta_i(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i} \\
&= C(1+\frac{H_i}{\delta_i})\mathbf{a}_i(\mathbf{curl}\,\mathbf{w}^\perp, \mathbf{curl}\,\mathbf{w}^\perp).
\end{aligned}
$$

We obtain (5.15) by summing over the subdomains. □

We next build another cut-off function $\theta_{F_{ij}}$, which is supported in the set

$$
\Xi_{ij} := (\Omega_i' \cap \Omega_j) \cup (\Omega_i \cap \Omega_j') \cup (F_{ij});
$$

cf. [9, Sect. 4 and 5]. $\theta_{F_{ij}}$ satisfies the following conditions:

$$
0 \le \theta_{F_{ij}} \le 1,
$$

$$
\theta_{F_{ij}}|_{\partial\Xi_{ij}} = 0,
$$

and

$$
\|\nabla\theta_{F_{ij}}\|_\infty \le \frac{C}{\delta_i};
$$

see [25, Lemma 3.4] for details.

Lemma 5.9. *Let* $\mathbf{v}_{ij} = \Pi_h^{RT}(\frac{1}{2}\theta_{F_{ij}}\mathbf{v}^\perp)$ *and* $\mathbf{w}_{ij} = \Pi_h^{ND}(\frac{1}{2}\theta_{F_{ij}}\mathbf{w}^\perp)$. *Then, we have*

$$\sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{v}_{ij},\mathbf{v}_{ij}) \le C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{v}^\perp,\mathbf{v}^\perp) \tag{5.18}$$

$$\sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij},\mathbf{curl}\,\mathbf{w}_{ij}) \le C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{curl}\,\mathbf{w}^\perp,\mathbf{curl}\,\mathbf{w}^\perp), \tag{5.19}$$

with C *which is independent of* H, h *and the jumps of the coefficients.*

Proof. Because $\theta_{F_{ij}}$ is supported in $\overline{\Xi}_{ij}$, we have

$$\begin{aligned}
\widetilde{\mathbf{a}}_i(\mathbf{v}_{ij},\mathbf{v}_{ij}) &= \int_{\Xi_{ij}}\alpha\mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Xi_{ij}}\beta\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx\\
&= \int_{\Omega_i\cap\Omega_j'}\alpha_i\mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Omega_i\cap\Omega_j'}\beta_i\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx\\
&\quad + \int_{\Omega_i'\cap\Omega_j}\alpha_j\mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Omega_i'\cap\Omega_j}\beta_j\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx.
\end{aligned}$$

By Lemma 5.4,

$$\alpha_j\|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \le C\alpha_j\|\mathrm{div}\,(\theta_{F_{ij}}\mathbf{v}^\perp)\|^2_{0;\Omega_i'\cap\Omega_j}$$

and

$$\beta_j\|\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \le C\beta_j\|\theta_{F_{ij}}\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}.$$

Moreover,

$$\begin{aligned}
\|\mathrm{div}\,(\theta_{F_{ij}}\mathbf{v}^\perp)\|^2_{0;\Omega_i'\cap\Omega_j} &\le C(\|\nabla\theta_{F_{ij}}\cdot\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\theta_{F_{ij}}\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j})\\
&\le C(\|\nabla\theta_{F_{ij}}\|^2_\infty\|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\theta_{F_{ij}}\|^2_\infty\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j})\\
&\le C(\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}) \qquad (5.20)\\
&\le C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_j} \qquad (5.21)
\end{aligned}$$

and

$$\|\theta_{F_{ij}}\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \le \|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \le \|\mathbf{v}^\perp\|^2_{0;\Omega_j}.$$

We obtain (5.21) from (5.20) by using Lemma 5.6.

Hence,

$$\alpha_j\|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \le C\alpha_j(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_j}$$

and

$$\beta_j\|\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \le C\beta_j\|\mathbf{v}^\perp\|^2_{0;\Omega_j}.$$

Similarly,

$$\alpha_i \|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} \le C\alpha_i(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}$$

and

$$\beta_i\|\mathbf{v}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} \le C\beta_i\|\mathbf{v}^\perp\|^2_{0;\Omega_i}.$$

Therefore, we can obtain (5.18) by a coloring argument and summing over all the partitions Ξ_{ij}.

We now consider (5.19):

$$\begin{aligned}\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij},\mathbf{curl}\,\mathbf{w}_{ij}) &= \int_{\Xi_{ij}} \beta\mathbf{curl}\,w_{ij}\mathbf{curl}\,w_{ij}\,dx \\ &= \beta_i\|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} + \beta_j\|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega'_i\cap\Omega_j}.\end{aligned}$$

By Lemma 5.4,

$$\beta_i\|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} \le C\beta_i\|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega_i\cap\Omega'_j},$$

$$\beta_j\|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega'_i\cap\Omega_j} \le C\beta_j\|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j}.$$

Therefore,

$$\begin{aligned}C\beta_j\|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j} &\le C\beta_j(\|\nabla\theta_{F_{ij}}\times\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\theta_{F_{ij}}\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}) \\ &\le C\beta_j(\|\nabla\theta_{F_{ij}}\|^2_\infty\|\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\theta_{F_{ij}}\|^2_\infty\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}) \\ &\le C\beta_j(\frac{1}{\delta_i^2}\|\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}).\end{aligned}$$

By Lemma 5.6,

$$C\beta_j\|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j} \le C\beta_j(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_j}.$$

Similarly,

$$C\beta_i\|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega_i\cap\Omega'_j} \le C\beta_i(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}.$$

Therefore,

$$\begin{aligned}&\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij},\mathbf{curl}\,\mathbf{w}_{ij}) \\ &\le C\beta_i(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i} + C\beta_j(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_j}.\end{aligned}$$

Finally, (5.19) holds by a coloring argument and summing over all the partitions Ξ_{ij}. ☐

We finally construct the remaining parts of the partition of unity. For each edge $E_{jl} \subset \partial\Omega_i$, which is $\overline{F}_{ij} \cap \overline{F}_{il}$, consider a cut-off function $\theta_{E_{jl}}$ which is supported in the set

$$\Psi_{jl} =: \bigcap_{m \in I_{jl}} \Omega'_m,$$

where I_{jl} is the set of indices of the subdomains which have the edge E_{jl} in common with Ω_i; cf. [9, Sect. 4 and 5]. $\theta_{E_{jl}}$ satisfies following conditions:

$$0 \le \theta_{E_{jl}} \le 1,$$

$$\|\nabla \theta_{E_{jl}}\|_\infty \le \frac{C}{\delta_i},$$

and

$$\sum_{i=1}^{N} (\theta_i + \sum_{F_{ij} \subset \partial\Omega_i} \theta_{F_{ij}} + \sum_{E_{jl} \subset \partial\Omega_i} \theta_{E_{jl}}) = 1.$$

LEMMA 5.10. *Let* $\mathbf{v}_{E_{jl}} = \Pi_h^{RT}(\frac{1}{|I_{jl}|}\theta_{E_{jl}}\mathbf{v}^\perp)$ *and* $\mathbf{w}_{E_{jl}} = \Pi_h^{ND}(\frac{1}{|I_{jl}|}\theta_{E_{jl}}\mathbf{w}^\perp)$. *Then,*

$$\sum_{i=1}^{N} \sum_{E_{jl} \subset \partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}}, \mathbf{v}_{E_{jl}}) \le C(1 + \frac{H}{\delta})\mathbf{a}(\mathbf{v}^\perp, \mathbf{v}^\perp) \tag{5.22}$$

and

$$\sum_{i=1}^{N} \sum_{E_{jl} \subset \partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{E_{jl}}, \mathbf{curl}\,\mathbf{w}_{E_{jl}}) \le C(1 + \frac{H}{\delta})\mathbf{a}(\mathbf{curl}\,\mathbf{w}^\perp, \mathbf{curl}\,\mathbf{w}^\perp), \tag{5.23}$$

with C *which is independent of* H, h *and the jumps of the coefficients.*

Proof. Because $\theta_{E_{jl}}$ is supported in $\overline{\Psi}_{jl}$, we find

$$\widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}}, \mathbf{v}_{E_{jl}}) = \sum_{m \in I_{jl}} (\alpha_m \|\mathrm{div}\,\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m} + \beta_m \|\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m}).$$

We can apply the same idea to each subset $\Psi_{jl} \cap \Omega_m$. It suffices to consider just one subset.

By Lemma 5.4,

$$\|\mathrm{div}\,\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m} \le C\|\mathrm{div}\,(\theta_{E_{jl}}\mathbf{v}^\perp)\|^2_{0;\Psi_{jl}\cap\Omega_m}$$

and

$$\|\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m} \le C\|\theta_{E_{jl}}\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m} \le C\|\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m}.$$

Therefore,

$$\begin{aligned}
&C\|\mathrm{div}\,(\theta_{E_{jl}}\mathbf{v}^\perp)\|^2_{0;\Psi_{jl}\cap\Omega_m} \\
&\le C(\|\nabla\theta_{E_{jl}}\|^2_\infty \|\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m} + \|\theta_{E_{jl}}\|^2_\infty \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m}) \\
&\le C(\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m} + \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m}).
\end{aligned}$$

By Lemma 5.6,

$$C\|\mathrm{div}\,(\theta_{E_{jl}}\mathbf{v}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m} \le C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^{\perp}\|^2_{0;\Omega_m}.$$

By a coloring argument and summing over all the partitions,

$$\sum_{i=1}^{N}\sum_{E_{jl}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}},\mathbf{v}_{E_{jl}}) \le C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{v}^{\perp},\mathbf{v}^{\perp}).$$

Consider the second estimate (5.23):

$$\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{E_{jl}},\mathbf{curl}\,\mathbf{w}_{E_{jl}}) = \sum_{m\in I_{jl}}\beta_m\|\mathbf{curl}\,\mathbf{w}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m}.$$

By Lemma 5.4,

$$\|\mathbf{curl}\,\mathbf{w}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m} \le C\|\mathbf{curl}\,(\theta_{E_{jl}}\mathbf{w}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m}.$$

$$\begin{aligned}
&\|\mathbf{curl}\,(\theta_{E_{jl}}\mathbf{w}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m}\\
&\le C(\|\nabla\theta_{E_{jl}}\|^2_{\infty}\|\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m} + \|\theta_{E_{jl}}\|^2_{\infty}\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m})\\
&\le C(\frac{1}{\delta_i^2}\|\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m} + \|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}).
\end{aligned}$$

By Lemma 5.6,

$$\|\mathbf{curl}\,(\theta_{E_{jl}}\mathbf{w}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m} \le (1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Omega_m}.$$

Therefore, we obtain

$$\sum_{i=1}^{N}\sum_{E_{jl}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{E_{jl}},\mathbf{curl}\,\mathbf{w}_{E_{jl}}) \le C(1+\frac{H}{\delta})\mathbf{a}(\mathbf{curl}\,\mathbf{w}^{\perp},\mathbf{curl}\,\mathbf{w}^{\perp}),$$

by summing over all the partitions. ☐

5.3. Main Result. Let $P_i = R_i^T A_i^{-1} R_i A$ and $P_{ad} = \sum_{i=0}^{N} P_i$; see [21], [25, Chap 3].

THEOREM 5.11 (Condition number estimate). *The condition number of the preconditioned system satisfies*

$$\kappa(P_{ad}) \le C(1+\log\frac{H}{h})(1+\frac{H}{\delta}),$$

where C is a constant which does not depend on the number of subdomains, H, h and δ. C is also independent of the coefficients α_i, β_i and the jumps of the coefficients between subdomains.

Proof. We obtain this main result by using Lemmas 5.7, 5.8, 5.9, 5.10 and the triangle inequality. ☐

REMARK 5.3. *In the previous result in [13], there was a second factor of $(1+\frac{H}{\delta})$. We have improved the result by reducing the power of the $\frac{H}{\delta}$ term.*

6. Numerical Results.

6.1. The 2D case. We apply the Overlapping Schwarz method with the energy-minimizing coarse space to our model problem. We use $\Omega = [0,1]^2$ and the lowest order Raviart-Thomas elements. We decompose the domain into N^2 identical subdomains. In each subdomain, we assume that the coefficients α and β are constant. We consider cases where the coefficients have jumps across the interface between the subdomains, in particular, the checkerboard distribution pattern of Fig. 6.1. We use a fixed β for the whole domain and have different values of α for the black and white regions. We have 1 for the black regions and another specified value for the white regions.

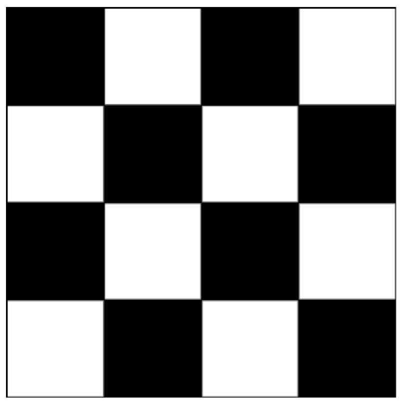

Fig. 6.1. *Checkerboard distribution of the coefficients*

Each subdomain Ω_i has side length H and each mesh triangle has diameter h. We also introduce extended subdomains whose boundaries do not cut any mesh elements. We recall Assumption 4.1. We use the preconditioned conjugate gradient method to solve the linear system of the finite element discretization. In order to estimate the condition numbers, we use the method outlined in [17]. We stop the iteration when the residual l_2-norm has been reduced by a factor of 10^{-8}.

We perform two different kinds of experiments. We first fix the overlap $\frac{H}{\delta}$ and vary $\frac{H}{h}$. We second fix the size of $\frac{H}{h}$ and use various size of $\frac{H}{\delta}$. Table 6.1 and Table 6.2 show the first results and Table 6.3 and Table 6.4 show the second results.

In the first set of experiments, we see that the condition numbers and iteration counts do not depend on the size of $\frac{H}{h}$. In the second set, we can conclude that the condition numbers and iteration counts grow linearly on $\frac{H}{\delta}$. For both cases, the condition numbers and iteration counts are also quite independent of the jumps of coefficients between the subdomains. Fig. 6.2 shows that the estimated condition number depends on linearly with $\frac{H}{\delta}$. Even though these results are independent of $\frac{H}{h}$, our numerical results are compatible with our main result.

6.2. The 3D case. For the 3D case, we use $\Omega = [0,1]^3$ and hexahedral instead of tetrahedral elements. In a way similar to the 2D case, we decompose the domain into N^3 subdomains. We again assume that we have constant coefficients in each subdomain. Other general settings are also quite similar to the 2D case. We use the preconditioned conjugate method with the stopping criteria of reducing the residual l_2-norm by a factor of 10^{-6}. We complete the same kinds of experiments as in the 2D case. For a fixed $\frac{H}{\delta}$, see the results in Table 6.5 and Table 6.7. For a fixed $\frac{H}{h}$, Table 6.6 and Table 6.8 show the results.

Table 6.1

Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{\delta} = 8$ *(2D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	7.35	19	10.98	23	13.96	22	14.76	23	14.84	23
16	7.32	19	10.95	23	13.91	22	14.70	23	14.79	23
32	7.31	19	10.95	23	13.85	22	14.69	23	14.77	23
64	7.31	19	10.95	23	12.87	22	14.69	24	14.77	23

Table 6.2

Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{\delta} = 4$ *(2D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	5.44	17	7.46	20	9.17	19	9.50	21	9.53	20
8	5.38	17	7.41	20	9.07	19	9.38	21	9.42	20
16	5.36	17	7.39	20	9.01	19	9.36	21	9.39	20
32	5.35	17	7.38	20	8.45	19	9.35	21	9.38	20
64	5.35	17	7.38	20	6.34	17	9.35	21	9.38	20

Table 6.3

Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{h} = 16$ *(2D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
16	11.62	23	18.04	28	23.25	26	25.14	29	25.36	27
8	7.32	19	10.95	23	13.91	22	14.70	23	14.79	23
4	5.36	17	7.39	20	9.01	19	9.36	21	9.39	20
2	5.09	15	5.49	17	5.18	17	6.37	17	5.66	15

Table 6.4

Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{h} = 32$ *(2D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
32	19.97	29	31.30	36	38.91	34	44.50	38	45.24	33
16	11.61	23	18.03	28	23.22	27	25.11	29	25.33	27
8	7.31	19	10.95	23	13.85	22	14.69	23	14.77	23
4	5.36	17	7.39	20	8.45	19	9.35	21	9.38	20
2	5.05	15	5.48	17	5.18	16	6.32	17	5.55	15

We find that the 3D case is very similar to the 2D case. This means that the condition numbers and iteration counts are independent of $\frac{H}{h}$ and depend linearly on the value of $\frac{H}{\delta}$. Moreover, they appear to be independent of the jumps of coefficients between subdomains. We see that the estimated condition number depends on linearly

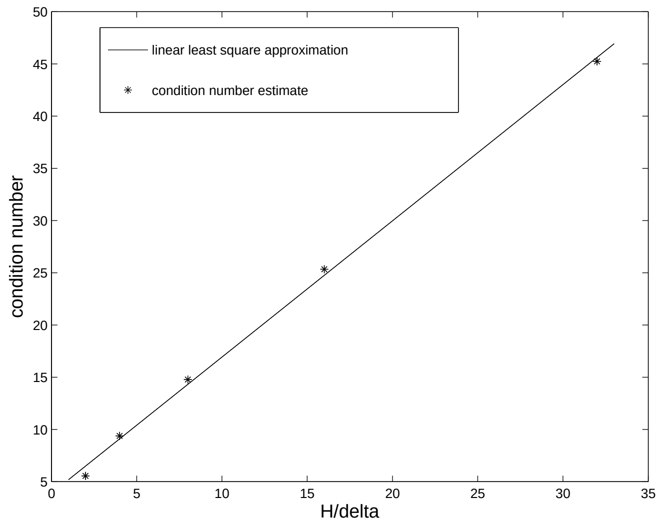

FIG. 6.2. *Estimated condition number and linear least square fitting, versus* $\frac{H}{\delta}$; $\alpha_i = 1$ *and* $\alpha_i = 100$ *in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{h} = 32$ *(2D case)*

with $\frac{H}{\delta}$ in Fig. 6.3. Our numerical results for the 3D case are compatible with our main result as well.

TABLE 6.5
Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{\delta} = 3$ *(3D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.37	19	8.70	19	9.47	20	9.68	20	9.71	20
6	8.44	19	8.70	20	9.51	20	9.73	21	9.76	23
12	8.46	20	8.67	21	9.52	21	9.74	22	9.73	23

TABLE 6.6
Condition numbers and iteration counts. $\alpha_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{h} = 12$ *(3D case)*

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
12	13.61	23	19.05	27	27.33	28	29.30	28	29.53	28
6	9.69	21	12.21	23	15.91	23	16.66	26	16.75	26
3	8.46	20	8.67	21	9.52	21	9.74	22	9.73	23

Acknowledgment. The author would like to thank Prof. Olof Widlund for his suggestions and assistance. The author is also grateful to Dr. Clark Dohrmann for his useful comments.

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Table 6.7

Condition numbers and iteration counts. $\beta_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\alpha_i \equiv 1$ *and* $\frac{H}{\delta} = 3$ *(3D case)*

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.47	21	9.02	20	9.47	20	8.85	20	8.38	20
6	8.38	21	9.06	21	9.51	20	8.84	21	8.39	20
12	8.34	21	9.08	21	9.52	21	8.81	21	8.39	20

Table 6.8

Condition numbers and iteration counts. $\beta_i = 1$ *or specified values as indicated in a checkerboard pattern,* $\alpha_i \equiv 1$ *and* $\frac{H}{h} = 12$ *(3D case)*

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
12	15.31	23	27.22	27	27.33	28	24.95	26	14.14	23
6	10.14	23	15.17	23	15.91	23	14.21	22	9.65	21
3	8.34	21	9.08	21	9.52	21	8.81	21	8.39	20

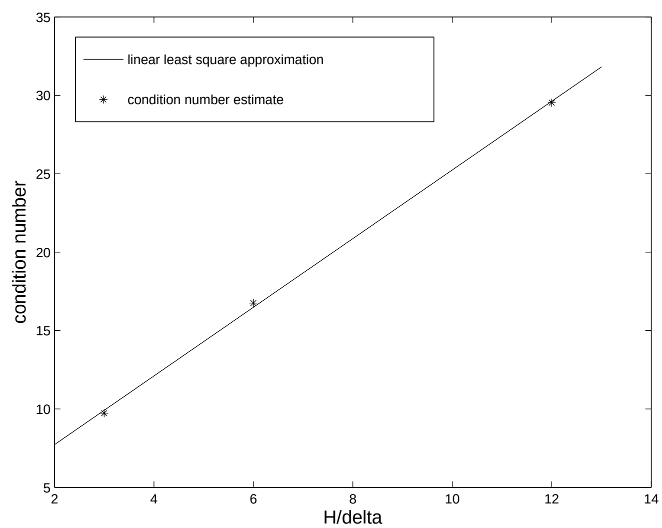

Fig. 6.3. *Estimated condition number and linear least square fitting, versus* $\frac{H}{\delta}$*;* $\alpha_i = 1$ *and* $\alpha_i = 100$ *in a checkerboard pattern,* $\beta_i \equiv 1$ *and* $\frac{H}{h} = 12$ *(3D case)*

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