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A FETI-DP ALGORITHM FOR ELASTICITY PROBLEMS WITH MORTAR DISCRETIZATION ON GEOMETRICALLY NON-CONFORMING PARTITIONS *

HYEA HYUN KIM †

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Abstract. In this paper, a FETI-DP formulation for three dimensional elasticity on non-matching grids over geometrically non-conforming subdomain partitions is considered. To resolve the nonconformity of the finite elements, a mortar matching condition is imposed on the subdomain interfaces (faces). A FETI-DP algorithm is then built by enforcing the mortar matching condition in dual and primal ways. In order to make the FETI-DP algorithm scalable, a set of primal constraints, which include average and momentum constraints over interfaces, are selected from the mortar matching condition. A condition number bound, $C(1 + \log(H/h))^2$, is then proved for the FETI-DP formulation for the elasticity problems with discontinuous material parameters. Only some faces need to be chosen as primal faces on which the average and momentum constraints are imposed.

Key words. FETI-DP, mortar methods, preconditioner, elasticity

AMS subject classifications. 65N30, 65N55

1. Introduction.

We will develop an efficient FETI-DP algorithm for solving linear systems arising from certain non-conforming discretizations of compressible elasticity problems in three dimensions. We consider a non-conforming finite element space with triangulations that are nonmatching across subdomain interfaces. Allowing such triangulations helps make adaptivity for problems with singular points or joints, or with jumps in the material parameters easier and more economical. Moreover, we are able to triangulate each subdomain independently to save the cost for mesh generation especially for three dimensional problems.

Mortar methods have been developed as non-conforming approximations with the goal of obtaining as accurate an approximate solution as for a conforming approximation; see [3, 1, 13, 30]. For this purpose, mortar matching conditions are imposed on the subdomain solutions across the interfaces. The jumps of the solutions across the subdomain interfaces are orthogonal to a certain Lagrange multiplier space. This condition can be enforced directly on the non-conforming finite element functions to produce elements of the mortar finite element space. Another approach is to impose the condition weakly by introducing Lagrange multipliers and this leads to a saddle-point problem similar to that considered in FETI-type algorithms.

FETI-type algorithms were originally developed for second order elliptic problems with

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† Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA. (hhk2@cims.nyu.edu)

conforming discretizations. These algorithms belong to the iterative substructuring domain decomposition methods of dual type. A separate set of interface unknowns is assigned to each subdomain. Point-wise continuity of solutions across the interfaces is then enforced using dual Lagrange multipliers, leading to a saddle point problem. The subdomain unknowns are then eliminated and the resulting linear system for the dual variables is solved iteratively using a preconditioner. These algorithms have evolved from the one-level FETI methods into two-level FETI, and FETI-DP methods; see [12, 11, 9]. In FETI-DP methods, a certain set of continuity constraints is enforced throughout the iteration while the remaining constraints are imposed weakly by dual Lagrange multipliers. FETI-DP algorithms have been further developed for three dimensional elliptic problems with discontinuous coefficients by Klawonn, Widlund and Dryja [23].

FETI-type algorithms have also been applied to the saddle-point problems resulting from mortar discretizations. A numerical study in [28] showed that such methods applied to these saddle-point problems are as efficient as the FETI methods for conforming discretizations. Lee and the author [17] introduced a FETI-DP algorithm for two-dimensional elliptic problems with discontinuous coefficients and showed a condition number bound, $C(1 + \log(H/h))^2$, with a constant C independent of the coefficients and mesh parameters. Numerical results show that it is the most efficient one for problems with jump coefficients; see [5]. This preconditioner is similar to previously developed FETI-DP preconditioners [7, 8] except that its weights equal zero except for the interface unknowns on the nonmortar sides. We call this preconditioner the Neumann-Dirichlet preconditioner. This algorithm has later been extended to the Stokes problem and to three-dimensional elliptic problems; see [16, 14].

The purpose of this study is to extend the FETI-DP algorithm of [17] to three-dimensional compressible elasticity problems with mortar discretizations and to improve the condition number bound on geometrically non-conforming partitions given in [15] to $C(1+\log(H/h))^2$. FETI-DP methods for three dimensional elasticity problems, with conforming discretization, have been studied extensively both theoretically and numerically; see [10, 18, 22, 26]. In [10], Farhat *et al.* introduced face average constraints in addition to vertex constraints as primal constraints and observed that these additional constraints help produce a scalable algorithm. Later Klawonn and Widlund [22] considered various primal constraints for elasticity problems with discontinuous Lamé parameters. In their work, some faces and edges were selected as fully primal faces and fully primal edges. Edge average constraints on fully primal faces, and edge average and moment constraints on fully primal edges were then enforced to get a scalable algorithm. In our FETI-DP formulation, we will introduce face average and face moment constraints related to the mortar matching conditions.

This paper is organized as follows. In Section 2, we introduce a compressible elasticity problem and Korn inequalities. In Section 3, we approximate the solution of the model problem using the mortar discretization. We then build a FETI-DP algorithm by considering the mortar matching condition similar to the point-wise continuity constraints in conforming finite element approximations. Some primal constraints are selected from the mortar matching condition. In order to make them explicit, we perform a change of unknowns. Section 4 is devoted to our condition number analysis of the FETI-DP algorithm. In the final section, we propose an algorithm which selects a quite small number of primal constraints based on the coefficient distribution.

Throughout this paper, c and C denote generic positive constants independent of the mesh size, the number of subdomains, and the problem coefficients. We will use h_i and H_i to denote the mesh size and the subdomain size of Ω_i, respectively.

2. A model problem and Korn's inequality. Let Ω be a polyhedral domain in $\mathbf{R}^3$. The Sobolev space $H^1(\Omega)$ is the set of functions in $L^2(\Omega)$ which are square integrable up to its first derivatives and equipped with the Sobolev norm;

$$\|v\|_{1,\Omega}^2 := |v|_{1,\Omega}^2 + \frac{1}{H^2}\|v\|_{0,\Omega}^2,$$

where $|v|_{1,\Omega}^2 = \int_\Omega \nabla v \cdot \nabla v \, dx$, $\|v\|_{0,\Omega} = \int_\Omega v^2 \, dx$, and H denotes the diameter of Ω.

We assume that $\partial\Omega$ is divided into two parts $\partial\Omega_D$ and $\partial\Omega_N$ on which a Dirichlet boundary condition and a natural boundary condition, respectively, are specified. The subspace $H_D^1(\Omega) \subset H^1(\Omega)$ is the set of functions having zero traces on $\partial\Omega_D$. We introduce the vector valued Sobolev spaces $[H_D^1(\Omega)]^3$ and $[H^1(\Omega)]^3$, equipped with the usual product norm.

We then consider the elasticity problem:

find $\mathbf{u} \in [H_D^1(\Omega)]^3$ such that

$$\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, d\mathbf{x} + \int_\Omega G(\mathbf{x})\beta(\mathbf{x})\nabla\cdot\mathbf{u}\,\nabla\cdot\mathbf{v}\, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in [H_D^1(\Omega)]^3, \tag{2.1}$$

where $G = E/(1+\nu)$ and $\beta = \nu/(1-2\nu)$ are material parameters which depend on the Young's modulus $E > 0$ and the Poisson ratio $\nu \in (0, 1/2)$. We assume that ν is bounded from above away from $1/2$, excluding the case of incompressible elasticity problems. The linearized strain tensor is defined by

$$\varepsilon(\mathbf{u})_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3,$$

and the tensor product and the force term are given by

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{g}\cdot\mathbf{v} d\sigma.$$

Here $\mathbf{f}$ is the body force and $\mathbf{g}$ is the surface force on the natural boundary part $\partial\Omega_N$.

The space **ker**(ε) has the following six rigid body motions as its basis elements. They are the three translations

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

and the three rotations

$$\mathbf{r}_4 = \frac{1}{H}\begin{pmatrix} x_2 - \widehat{x}_2 \\ -x_1 + \widehat{x}_1 \\ 0 \end{pmatrix}, \ \mathbf{r}_5 = \frac{1}{H}\begin{pmatrix} -x_3 + \widehat{x}_3 \\ 0 \\ x_1 - \widehat{x}_1 \end{pmatrix}, \ \mathbf{r}_6 = \frac{1}{H}\begin{pmatrix} 0 \\ x_3 - \widehat{x}_3 \\ -x_2 + \widehat{x}_2 \end{pmatrix}. \tag{2.3}$$

Here $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \widehat{\Omega}$ and H is the diameter of $\widehat{\Omega}$. This shift and the scaling make the L_2-norm of the six vectors scale in the same way with H. When Ω is partitioned into a set of subdomains, the elasticity problem given on a floating subdomain has purely natural boundary condition. The Korn inequalities in [22, Section 2] concern this case. Let $\Sigma \subset \partial\Omega$ be of positive measure. We define an L_2–inner product $(\mathbf{u}, \mathbf{r})_\Sigma$ by

$$(\mathbf{u}, \mathbf{r})_\Sigma = \int_\Sigma \mathbf{u} \cdot \mathbf{r}\, ds.$$

The following Korn inequality is provided in [22, Lemma 5]:

LEMMA 2.1. *Let Ω be a Lipschitz domain and Σ be a subset of $\partial\Omega$ with positive measure. Then there exist a constant $c > 0$, invariant under dilation, such that*

$$c|\mathbf{u}|_{1,\Omega} \leq \|\varepsilon(\mathbf{u})\|_{0,\Omega} \leq |\mathbf{u}|_{1,\Omega},$$

where $\mathbf{u} \in [H^1(\Omega)]^3$ satisfies $(\mathbf{u}, \mathbf{r})_\Sigma = 0$ for all $\mathbf{r} \in \mathbf{ker}(\varepsilon)$.

Furthermore, we have similar inequalities for semi-norms defined in the space $[H^{1/2}(\Sigma)]^3$ which is the trace space of $[H^1(\Omega)]^3$ on $\Sigma \subset \partial\Omega$. For $\mathbf{u} \in [H^{1/2}(\Sigma)]^3$, we define two semi-norms by

$$|\mathbf{u}|_{1/2,\Sigma} := \inf_{\substack{\mathbf{v} \in [H^1(\Omega)]^3 \\ \mathbf{v}|_\Sigma = \mathbf{u}}} |\mathbf{v}|_{1,\Omega}, \quad |\mathbf{u}|_{E(\Sigma)} := \inf_{\substack{\mathbf{v} \in [H^1(\Omega)]^3 \\ \mathbf{v}|_\Sigma = \mathbf{u}}} \|\varepsilon(\mathbf{v})\|_{0,\Omega}. \tag{2.4}$$

LEMMA 2.2. *Let Ω be a Lipschitz domain and Σ be a subset of $\partial\Omega$ with positive measure. Then there exists a constant $c > 0$, invariant under dilation, such that*

$$c|\mathbf{u}|_{1/2,\Sigma} \leq |\mathbf{u}|_{E(\Sigma)} \leq |\mathbf{u}|_{1/2,\Sigma},$$

for $\mathbf{u} \in [H^{1/2}(\Sigma)]^3$ satisfying $(\mathbf{u}, \mathbf{r})_\Sigma = 0$ for all $\mathbf{r} \in \mathbf{ker}(\varepsilon)$.

This lemma can be found in [22, Lemma 6]. Another important inequality, which follows from this inequality, is given in [22, Lemma 7]:

LEMMA 2.3. *Let Ω be a Lipschitz domain of diameter H and $\Sigma \subset \partial\Omega$ be an open subset with positive measure. Then there exists a constant $C > 0$ such that*

$$\inf_{\mathbf{r} \in \mathbf{ker}(\varepsilon)} \|\mathbf{u} - \mathbf{r}\|_{0,\Sigma}^2 \leq CH|\mathbf{u}|_{E(\Sigma)}^2 \quad \forall \mathbf{u} \in [H^{1/2}(\Sigma)]^3.$$

Here the infimum occurs when $\mathbf{r}$ satisfies $(\mathbf{u} - \mathbf{r}, \mathbf{q})_\Sigma = 0$ for all $\mathbf{q} \in \ker(\varepsilon)$.

3. FETI-DP formulation.

3.1. Mortar discretization. We divide the domain Ω into a geometrically non-conforming partition $\{\Omega_i\}_{i=1}^N$, where Ω_i are polyhedral and equipped with quasi-uniform triangulations T_i.

ASSUMPTION 3.1. *The subdomain partition $\{\Omega_i\}$ is locally quasi uniform, i.e., neighboring subdomains have comparable diameters.*

We consider a model compressible elasticity problem (2.1) with coefficients $G(\mathbf{x})$ and $\beta(\mathbf{x})$ positive constants in each subdomain

$$G(\mathbf{x})|_{\Omega_i} = G_i, \quad \beta(\mathbf{x})|_{\Omega_i} = \beta_i.$$

The conforming P_1–finite element space $\mathbf{X}_i$ is associated to the triangulation T_i. In addition, functions in the space $\mathbf{X}_i$ satisfy the Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega_D$. We define the union of the subdomain boundaries by

$$\Gamma = \left(\bigcup_{i=1}^N \partial\Omega_i \right) \setminus \partial\Omega.$$

The triangulations $\{T_i\}_{i=1}^N$ may not match across the subdomain boundaries. We further introduce the finite element space $\mathbf{W}_i$ that is the trace space of $\mathbf{X}_i$ on $\partial\Omega_i \cap \Gamma$. We note that the nodal unknowns corresponding to nodes at $\partial\Omega_N$ are considered as unknowns at the interior of the subdomains.

We denote the interface of two subdomains Ω_i and Ω_j by F_{ij}, that can be only part of a face of Ω_i and Ω_j. Among the subdomain faces, we select nonmortar faces F_l such that

$$\bigcup_l \overline{F}_l = \bigcup_{i,j} \overline{F}_{ij}, \quad F_l \cap F_k = \emptyset, \; l \neq k.$$

Here each F_l is a full face of a subdomain that we call the nonmortar subdomain of F_l. We call the subdomains on the other part across F_l as the mortar subdomains.

Such nonmortar faces always exist even for the geometrically non-conforming partitions; see [27, Section 4.1]. We define the collection of interfaces by

$$\mathcal{S} = \bigcup_{ij} \{F_{ij}\}.$$

For an interface F_{lk}, we select the set of interfaces $\{F_{sm}\}$ from $\mathcal{S}$ such that their union produces the largest connected component that contains F_{lk} and lies in the plane defined by F_{lk}. The connected component is, in fact, a union of full faces of subdomains, otherwise a larger component could be found. We select such full faces as nonmortar faces and the other part of the interfaces as mortar faces, that can often be only part of a subdomain face. After having made a selection, we do the same for the collection $\mathcal{S} \setminus \{F_{sm}\}$ recursively, and we can find the union of the nonmortar faces, which is equal to $\mathcal{S}$.

Since the subdomain partition can be geometrically non-conforming, a single nonmortar face $F_l \subset \partial\Omega_i$ may intersect several subdomain boundaries $\partial\Omega_j$. This provides F_l with a partition,

$$\overline{F}_l = \bigcup_j \overline{F}_{ij}, \quad F_{ij} = \partial\Omega_i \cap \partial\Omega_j.$$

Our bound for the condition number will depend on the relative diameters of faces and subdomains.

DEFINITION 3.2. *A face $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is substantial if its diameter is comparable to H_i and H_j.*

DEFINITION 3.3. *Let TOL_F be a given constant and Ω_i be the nonmortar side of F_{ij}. The face F_{ij} is weakly substantial if its diameter H_{ij} satisfies*

$$\left(1 + \mathit{log}\frac{H_{ij}}{h_i}\right)^2 \frac{\max\{H_i, H_j\}}{H_{ij}} \leq TOL_F \left(1 + \mathit{log}\frac{H_i}{h_i}\right)^2.$$

ASSUMPTION 3.4. *Let TOL_G be a given constant. On each interface $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$, the nonmortar side Ω_i and the mortar side Ω_j are selected so that*

$$\frac{G_i}{G_j} \leq TOL_G.$$

REMARK 3.5. *It is clearly possible to construct a set of subdomains such that Assumption 3.4 is not satisfied in geometrically non-conforming cases. We note that this problem arises from the specification of the mortar method itself.*

We now introduce the finite element space

$$\overset{\circ}{\mathbf{W}}(F_l) = \left\{ \mathbf{w} \in \mathbf{H}_0^1(F_l) \ : \ \mathbf{w} = \mathbf{v}|_{F_l} \text{ for } \mathbf{v} \in \mathbf{X}_i \right\}, \tag{3.1}$$

where i denotes the index of the nonmortar subdomain of the face F_l and $\mathbf{v}|_{F_l}$ denotes the trace of $\mathbf{v}$ on the face F_l. This space is spanned by the nodal basis $\{\boldsymbol{\phi}_k\}_{k=1}^n$ of the nodes in F_l given by the triangulation T_i. Based on the space $\overset{\circ}{\mathbf{W}}(F_l)$, we construct a dual Lagrange multiplier space $\mathbf{M}(F_l)$ with a basis $\{\boldsymbol{\psi}_k\}_{k=1}^n$ satisfying

$$\int_{F_l} \boldsymbol{\phi}_m \cdot \boldsymbol{\psi}_k \, ds = \delta_{mk} \int_{F_l} \boldsymbol{\phi}_m \, ds \quad \forall\, m, k = 1, \cdots, n.$$

We refer to [13, 31] for a detailed construction of such a dual Lagrange multiplier space. The result of our paper is also applicable to the standard Lagrange multiplier space, that was introduced in [2] for three dimensions. However, the dual Lagrange multiplier space leads to a computationally more efficient algorithm and also makes the implementation easier than with the older version. We note that both spaces contain the constant functions. The mortar matching condition is then written as

$$\int_{F_l} (\mathbf{v}_i - \boldsymbol{\phi}) \cdot \boldsymbol{\lambda} \, ds = 0 \quad \forall \boldsymbol{\lambda} \in \mathbf{M}(F_l), \ \forall F_l, \tag{3.2}$$

where $\mathbf{v}_i$ is a function from the nonmortar side and $\boldsymbol{\phi}$ is a function from the corresponding mortar parts. More precisely, $\boldsymbol{\phi} = \mathbf{v}_j$ on each $F_{ij} \subset F_l$.

We next introduce several finite element spaces,

$$\begin{aligned} \mathbf{X} &= \prod_{i=1}^{N} \mathbf{X}_i, \\ \mathbf{W} &= \prod_{i=1}^{N} \mathbf{W}_i, \\ \mathbf{W}_n &= \prod_{l,\text{nonmortar}} \mathring{\mathbf{W}}(F_l), \\ \mathbf{M} &= \prod_{l,\text{nonmortar}} \mathbf{M}(F_l). \end{aligned} \tag{3.3}$$

Here the spaces $\mathbf{W}_n$ and $\mathbf{M}$ consist of functions defined on the nonmortar faces, while the spaces $\mathbf{W}$ and $\mathbf{X}$ consist of functions, defined on both mortar and nonmortar faces, with elements that can be discontinuous across the interfaces. The space $\mathbf{W}$ can be considered as the trace space of the space $\mathbf{X}$ on the subdomain boundaries. In addition, we define the mortar finite element space,

$$\widehat{\mathbf{X}} = \{ \mathbf{v} \in \mathbf{X} \, : \, \mathbf{v} \text{ satisfies (3.2)} \} \, .$$

The mortar discretization provides (2.1) an approximate solution $\mathbf{u}$ in the space $\widehat{\mathbf{X}}$. In what follows, we derive a FETI-DP algorithm that solves the system of equations of this mortar discretization.

3.2. Primal constraints in the FETI-DP formulation. We will build a FETI-DP algorithm that solves the model problem (2.1) in the space X defined in (3.3) by enforcing the mortar matching condition (3.2) across subdomain interfaces in dual and primal ways. We select some constraints from the mortar condition and enforce them strongly for the functions in the space X. We call them the primal constraints. The other constraints will be imposed weakly using Lagrange multipliers.

A proper selection of primal constraints is important for obtaining a scalable FETI-DP algorithm. In the work by Klawonn and Widlund [22] on elasticity problems, edge average and edge moment constraints, and vertex constraints are selected. Furthermore the concepts of an acceptable face path and an acceptable vertex path are introduced in an attempt to reduce the number of primal constraints.

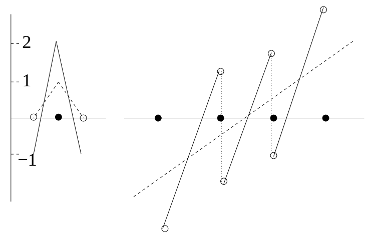

FIG. 1. *A dual Lagrange basis element (left, the solid line) and the nodal interpolant $I_{M(F_{ij})}(v)$ (right, the solid lines) to the linear function v (the dashed line)*

We now select primal constraints from the mortar matching condition. We consider a nonmortar face $F \subset \partial\Omega_i$ and its partition $\{F_{ij}\}_j$ by its mortar neighbors, i.e., $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$. We define the space $\mathbf{M}(F_{ij})$ as a subspace of $\mathbf{M}(F)$ of functions that are supported in $\overline{F}_{ij}$. Six primal constraints are now introduced for each F_{ij} in the following way.

On each face F_{ij}, we first consider the six rigid body motions $\{\mathbf{r}_l\}_{l=1}^6$ as in (2.2) and (2.3), with H the diameter of the face F_{ij} and $\widehat{\mathbf{x}}$ a point on F_{ij}. We define a nodal interpolant $I_{M(F_{ij})} : [C(\overline{F}_{ij})]^3 \to \mathbf{M}(F_{ij})$ by

$$I_{M(F_{ij})}(\mathbf{v})(x) = \mathbf{v}(x), \quad x \in \mathbf{M}_{ij}^h,$$

where $\mathbf{M}_{ij}^h$ is the nodal set corresponding to the Lagrange multiplier space $\mathbf{M}(F_{ij})$ and $C(\overline{F}_{ij})$ denotes the set of continuous functions on F_{ij}; see Fig 1 for a two-dimensional case. We now select six primal constraints using the interpolated rigid body motions,

$$\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot I_{M(F_{ij})}(\mathbf{r}_l)\, ds = 0, \quad \forall l = 1, \cdots, 6.$$

REMARK 3.6. *When F_{ij} is the whole nonmortar face F, $\mathbf{M}(F_{ij})$ contains constant functions. The constraints with $\{I_{M(F_{ij})}(\mathbf{r}_l)\}_{l=1}^3$ are then nothing but the average matching condition across F_{ij} because $I_{M(F_{ij})}(\mathbf{r}_l) = \mathbf{r}_l$, for $l = 1, 2, 3$. The remaining constraints with $\{I_{M(F_{ij})}(\mathbf{r}_l)\}_{l=4}^6$, are similar to the moment matching constraints which were introduced for fully primal edges in [22] except that our constraints use the interpolated rotational rigid body motions and are imposed on faces. We call the constraints based on $\{I_{M(F_{ij})}(\mathbf{r}_l)\}_{l=4}^6$ the moment constraints.*

Even though we have introduced a set of primal constraints to make the FETI-DP method more efficient, the enlarged coarse problem can be a bottleneck for the computation. To reduce the size of the coarse problem, we will select some interfaces as primal and impose the six constraints only over them. For the remaining, the non-primal faces, we assume that they each satisfy an acceptable face path condition. This assumption leads to a FETI-DP method, with primal constraints only for the primal interfaces, that has a condition number bound comparable to one obtained when the six primal constraints are imposed on all interfaces.

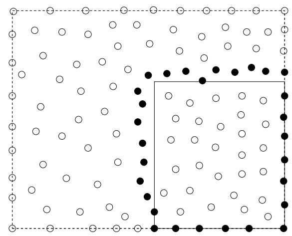

FIG. 2. *A primal interface F (solid line rectangle) of a face (dashed line rectangle) of subdomain Ω_i. The supports of the basis elements corresponding to the black nodes and the white nodes inside F intersect F; the values at the black nodes are unaffected by the transform T_F.*

We now define an acceptable face path. Here, H_{ij} denotes the diameter of an interface F_{ij}.

DEFINITION 3.7. **(Acceptable face path)** *Let L and TOL_P be given constants. For a pair of subdomains (Ω_i, Ω_j) having a common interface F_{ij}, an acceptable face path is a path*

$$\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$$

from Ω_i to Ω_j such that the coefficient G_{k_l} of Ω_{k_l} satisfy the condition

$$TOL_P * (1 + \mathit{log}(H_{ij}/h_i))^{-1} (1 + \mathit{log}(H_i/h_i))^2 * G_{k_l} \geq \min(G_i, G_j).$$

Moreover, the path from one subdomain to another must always be through a primal face and the number of subdomains appearing in the path is bounded by the constant L.

3.3. The FETI-DP formulation with a change of basis. Let A_i be the stiffness matrix obtained from the finite element discretization of the bilinear form,

$$a_i(\mathbf{u}_i, \mathbf{v}_i) := G_i \int_{\Omega_i} \varepsilon(\mathbf{u}_i) : \varepsilon(\mathbf{v}_i)\, dx + G_i \beta_i \int_{\Omega_i} \nabla \cdot \mathbf{u}_i \nabla \cdot \mathbf{v}_i \, dx,$$

using the space $\mathbf{X}_i$. Let S_i be the Schur complement of the matrix A_i, that is obtained by eliminating the interior unknowns. We then write the mortar matching condition for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in W$ as

$$\sum_{i=1}^{N} B_i \mathbf{w}_i = 0.$$

We now express the matrices S_i and B_i in a new set of unknowns after a change of basis (unknowns). This idea was first presented in [21, 24] and algorithmic details were later described by Klawonn and Widlund [22]. The change of unknowns leads to a much simpler presentation of the algorithm as well as a more robust implementation; see [19, 20].

Let $\mathbf{w}_i$ denote the unknowns (or a function) in the space $\mathbf{W}_i$. We consider a primal interface $F \subset \partial\Omega_i$. Since the partition can be geometrically nonconforming, F might be only part of a face of Ω_i. Let $\mathbf{w}_F$ denote the restriction of a function $\mathbf{w}$ to a primal face $F \subset \Omega_i$. In other words, $\mathbf{w}_F$ is the vector of unknowns corresponding to the nodal basis elements with supports that intersect F; see Figure 2. For each primal face $F \subset \partial\Omega_i$, we define a transformation T_F by

$$\mathbf{w}_F = T_F \begin{pmatrix} \mathbf{w}_{F,\Pi} \\ \mathbf{w}_{F,\Delta} \end{pmatrix},$$

where the six components of $\mathbf{w}_{F,\Pi}$ are given by

$$(\mathbf{w}_{F,\Pi})_l = \frac{\int_F \mathbf{w}_F \cdot I_{M(F)}(\mathbf{r}_l)\, ds}{H_F^2},\ l = 1, \cdots, 6,$$

and $\mathbf{v} = T_F \begin{pmatrix} 0 \\ \mathbf{w}_{F,\Delta} \end{pmatrix}$ has the six zero components, i.e.,

$$\int_F \mathbf{v} \cdot I_{M(F)}(\mathbf{r}_l)\, ds = 0,\ l = 1, \cdots, 6.$$

Here H_F denotes the diameter of F. In addition, the transformation T_F retains the unknowns at the nodes other than those inside F. Such a transform can be built just like in [15, Section 2.2]; we omit the details. Since only the unknowns at the nodes inside F has been changed, each transform corresponding to an individual primal face F can be applied independently.

After the change of unknowns, we order the unknowns $\mathbf{w}_i$ and the matrices S_i and B_i as

$$\mathbf{w}_i = \begin{pmatrix} \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{pmatrix}, \quad S_i = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix}, \quad B_i = \begin{pmatrix} B_\Delta^{(i)} & B_\Pi^{(i)} \end{pmatrix}, \tag{3.4}$$

where Π denotes the primal unknowns and Δ denotes the others, which we call the dual displacement unknowns. Any interface of a pair of subdomains is either a primal face or has an acceptable face path, so that each subdomain is connected to at least one neighbor through a primal face. This fact ensures that the matrix $S_{\Delta\Delta}^{(i)}$ is invertible, since the six primal constraints are linearly independent; a proof will be provided in Lemma 4.2.

According to the separation of the unknowns, the space $\mathbf{W}_i$ is decomposed into

$$\mathbf{W}_i = \mathbf{W}_\Delta^{(i)} \times \mathbf{W}_\Pi^{(i)},$$

where $\mathbf{W}_\Delta^{(i)}$ and $\mathbf{W}_\Pi^{(i)}$ contains the dual unknowns and the primal unknowns, respectively. We further decompose the space $\mathbf{W}_\Delta^{(i)}$ of dual unknowns into

$$\mathbf{w}_\Delta^{(i)} = (\mathbf{w}_{\Delta,n}^{(i)}, \mathbf{w}_{\Delta,m}^{(i)}) \in \mathbf{W}_{\Delta,n}^{(i)} \times \mathbf{W}_{\Delta,m}^{(i)},$$

where n denotes the unknowns of the nonmortar faces and m denotes the remaining unknowns.

After enforcing the primal constraints on each primal face, we define the following space,

$$\widetilde{\mathbf{W}} = \{\mathbf{w} \in \mathbf{W} \,:\, \mathbf{w} \text{ satisfies the primal constraints across the primal faces}\}.$$

It can be decomposed into

$$\widetilde{\mathbf{W}} = \mathbf{W}_\Delta \times \mathbf{W}_\Pi,$$

where $\mathbf{W}_\Delta = \prod_{i=1}^N \mathbf{W}_\Delta^{(i)}$ and $\mathbf{W}_\Pi$ is the space of global primal unknowns. The space $\mathbf{W}_\Delta$ is further decomposed into

$$\mathbf{W}_\Delta = \mathbf{W}_{\Delta,n} \times \mathbf{W}_{\Delta,m},$$

where n and m refer to the nonmortar and the remaining part of the interfaces.

Throughout this paper, we use the same notation for a function and the corresponding vector of unknowns representing the function. Thus, $\mathbf{w}_\Delta^{(i)}$ denotes a vector of dual unknowns of $\mathbf{w}^{(i)}$ or the corresponding finite element function. The same convention applies to the spaces $\mathbf{W}_{\Delta,n}$, $\mathbf{W}$, $\mathbf{M}$, etc.

We now consider the mortar matching matrices B_i. When the mortar matching constraints are imposed on $\mathbf{w}$ in $\widetilde{\mathbf{W}}$, they are redundant, since the six primal constraints on each primal face already have been enforced on $\mathbf{w}$. We therefore eliminate six equations from the rows of B_i for each primal face $F \subset \partial\Omega_i$ and obtain the mortar matching constraints for the space $\widetilde{\mathbf{W}}$ that are nonredundant. We will use the same notation B_i after that we have made them nonredundant.

By introducing Lagrange multipliers $\boldsymbol{\lambda}$ for the mortar matching constraints, we obtain the following mixed formulation of the problem (2.1):

$$\begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} & B_\Delta^t \\ S_{\Pi\Delta} & S_{\Pi\Pi} & B_\Pi^t \\ B_\Delta & B_\Pi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_\Delta \\ \mathbf{w}_\Pi \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_\Delta \\ \mathbf{g}_\Pi \\ 0 \end{pmatrix}, \tag{3.5}$$

where each block matrices are obtained from subassembly of blocks of S_i and B_i in (3.4) at the global primal unknowns and at the dual unknowns.

The FETI-DP algorithm solves this mixed problem iteratively after eliminating all unknowns other than $\boldsymbol{\lambda}$. The elimination of the unknowns $\mathbf{w}_\Delta$ and $\mathbf{w}_\Pi$ leads to

$$F_{DP}\boldsymbol{\lambda} = \boldsymbol{d}. \tag{3.6}$$

We note that the matrix F_{DP} is symmetric and positive definite and it satisfies the well-known relation, see [25, Lemma 4.3],

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle = \max_{\mathbf{w}\in\widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w}, \mathbf{w}\rangle}, \tag{3.7}$$

where

$$S = \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}, \quad B = \begin{pmatrix} B_{\Delta} & B_{\Pi} \end{pmatrix}.$$

The FETI-DP algorithm solves (3.6) for $\boldsymbol{\lambda}$ using a preconditioned conjugate gradient method with an appropriate preconditioner.

We now introduce a preconditioner $\widehat{M}^{-1}$ given by

$$\langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_{\Delta,n} \in \mathbf{W}_{\Delta,n}} \frac{\langle BE(\mathbf{w}_{\Delta,n}), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_{\Delta,n}), E(\mathbf{w}_{\Delta,n}) \rangle}, \tag{3.8}$$

where $E(\mathbf{w}_{\Delta,n})$ is the zero extension of $\mathbf{w}_{\Delta,n}$ into the space $\widetilde{\mathbf{W}}$. To be more precise,

$$E(\mathbf{w}_{\Delta,n}) = (\mathbf{w}_{\Delta,n}, 0, 0) \in \mathbf{W}_{\Delta,n} \times \mathbf{W}_{\Delta,m} \times \mathbf{W}_{\Pi} = \widetilde{\mathbf{W}}.$$

We then obtain

$$\langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_{\Delta,n} \in \mathbf{W}_{\Delta,n}} \frac{\langle BE(\mathbf{w}_{\Delta,n}), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_{\Delta,n}), E(\mathbf{w}_{\Delta,n}) \rangle} \leq \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle} = \langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \tag{3.9}$$

Therefore the lower bound of the FETI-DP algorithm is bounded from below by 1.

The explicit form of the preconditioner

$$\widehat{M}^{-1} = \sum_{i=1}^{N} B_i D_i S_i D_i B_i^t \tag{3.10}$$

is similar to other FETI-DP preconditioners except that the weight matrix D_i is given differently. The weight matrix D_i to the preconditioner $\widehat{M}^{-1}$ in (3.8) is expressed by

$$D_i = \begin{pmatrix} ((B_{\Delta,n}^{(i)})^t B_{\Delta,n}^{(i)})^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $B_{\Delta,n}^{(i)}$ is the matrix with the columns of $B_{\Delta}^{(i)}$ corresponding to the unknowns in the space $\mathbf{W}_{\Delta,n}^{(i)}$; see (3.4). We note that $B_{\Delta,n}^{(i)}$ is square and invertible and that the weight matrix D_i is applied to the unknowns in the space $\mathbf{W}_i = \mathbf{W}_{\Delta,n}^{(i)} \times \mathbf{W}_{\Delta,m}^{(i)} \times \mathbf{W}_{\Pi}^{(i)}$. We can further express the preconditioner in a much simpler form,

$$\widehat{M}^{-1} = \sum_{i=1}^{N} \begin{pmatrix} ({B_{\Delta,n}^{(i)}}^t)^{-1} & 0 & 0 \end{pmatrix} S_i \begin{pmatrix} {B_{\Delta,n}^{(i)}}^{-1} \\ 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

This form shows that multiplying $\widehat{M}^{-1}$ by a vector involves solving local elasticity problems with natural boundary condition on the nonmortar faces and zero Dirichlet boundary condition on the other part. We call $\widehat{M}^{-1}$ the Neumann-Dirichlet preconditioner.

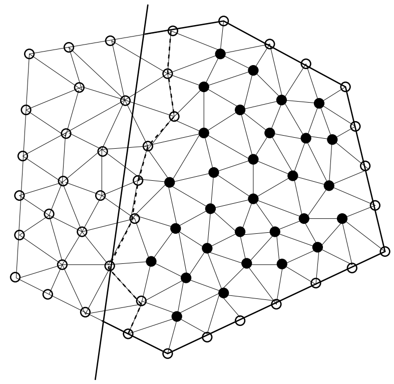

FIG. 3. *The face F, to the right of the solid line, is part of a nonmortar face. The Lagrange multiplier basis of the black nodes are supported in F; $M(F)$ is the space spanned by these basis elements. The union of their supports is F_I, a subset of F that is to the right of the dashed lines.*

4. Condition number analysis. In this section, we will analyze the condition number bound of the proposed FETI-DP algorithm. First, we will construct functionals $\{f_l\}_{l=1}^6$, dual to the space **ker**(ε), that satisfy

$$f_m(\mathbf{r}_k) = \delta_{mk}, \quad m, k = 1, \cdots, 6,$$
$$|f_m(\mathbf{w})|^2 \leq C \frac{\|\mathbf{w}\|_{0,F}^2}{H^2} \quad \text{for } \mathbf{w} \in [L^2(F)]^3. \tag{4.1}$$

Here $\{\mathbf{r}_k\}_{k=1}^6$ is a basis of **ker** (ε) with six rigid body motions scaled with respect to a face $F \subset \partial\Omega_i$; this means that we take $\widehat{\mathbf{x}} \in F$ and $H = \text{diam}(F)$ in (2.2) and (2.3). Such a dual basis was introduced by Klawonn and Widlund [22].

We will now introduce six functionals which are closely related to the primal constraints applied across a primal face F,

$$g_l(\mathbf{w}) = \frac{\int_F \mathbf{w} \cdot I_{M(F)}(\mathbf{r}_l)\, ds}{H^2}, \quad \text{for } \mathbf{w} \in [L^2(F)]^3,\ l = 1, \cdots, 6. \tag{4.2}$$

Here, $I_{M(F)}(\mathbf{r}_l)$ is the nodal interpolant to the Lagrange multiplier space $\mathbf{M}(F)$ provided for the face F and H is the diameter of F.

LEMMA 4.1. *For any bounded function f and any linear function g, we have*

$$\left| \int_F f \left(g - I_{M(F)}(g) \right) ds \right| \leq ChH \|f\|_\infty \|g\|_\infty,$$

where H is the diameter of the face F.

Proof. We consider the case when F is a part of a nonmortar face as in Figure 3. The subset F_I of F is the union of the supports of the basis elements in $M(F)$. Since g is linear, we obtain, at each nodal point $x \in F_I$,

$$|g(x) - I_{M(F)}(g)(x)| \leq Ch|g'| \leq C \frac{h}{H} \|g\|_\infty. \tag{4.3}$$

The above bound also holds for any $x \in F_I$, because both g and $I_{M(F)}(g)$ are linear in each element in F_I. We note that $I_{M(F)}(g)$ vanishes outside F_I.

We now have

$$\int_F f\left(g - I_{M(F)}(g)\right)\, ds = \int_{F_I} f\left(g - I_{M(F)}(g)\right)\, ds + \int_{F \setminus F_I} fg\, ds.$$

From the above bound and (4.3), using $|F_I| \leq CH^2$ and $|F \setminus F_I| \leq ChH$, the required bound then follows. □

LEMMA 4.2. *The functionals* $\{g_l\}_{l=1}^6$ *are linearly independent in the space* $(ker(\varepsilon))'$, *when* h *is sufficiently small.*

Proof. It suffices to show that the matrix G with the following entries is invertible,

$$G_{jk} = \int_F \mathbf{r}_j \cdot I_{M(F)}(\mathbf{r}_k)\, ds, \quad j,k = 1,\cdots,6.$$

Here we select $\widehat{x}_i$ in $\{\mathbf{r}_k\}_{k=4}^6$ so that

$$\int_F \frac{1}{H}(x_i - \widehat{x}_i)\, ds = 0, \quad i = 1,2,3.$$

The rigid body motions $\{\mathbf{r}_k\}_{k=4}^6$ are then orthogonal to $\{\mathbf{r}_k\}_{k=1}^3$ with respect to the inner product,

$$(\mathbf{r}, \boldsymbol{q})_F = \int_F \mathbf{r} \cdot \boldsymbol{q}\, ds.$$

We define $\widetilde{\mathbf{r}}_5$ and $\widetilde{\mathbf{r}}_6$ by

$$\widetilde{\mathbf{r}}_5 = \mathbf{r}_5 - a_4 \mathbf{r}_4, \quad \widetilde{\mathbf{r}}_6 = \mathbf{r}_6 - b_4 \mathbf{r}_4 - b_5 \widetilde{\mathbf{r}}_5,$$

where

$$a_4 = \frac{(\mathbf{r}_4, \mathbf{r}_5)_F}{(\mathbf{r}_4, \mathbf{r}_4)_F},\ b_4 = \frac{(\mathbf{r}_4, \mathbf{r}_6)_F}{(\mathbf{r}_4, \mathbf{r}_4)_F},\ b_5 = \frac{(\widetilde{\mathbf{r}}_5, \mathbf{r}_6)_F}{(\widetilde{\mathbf{r}}_5, \widetilde{\mathbf{r}}_5)_F}.$$

The constants a_4, b_4, and b_5 are invariant to scaling. In other words, they are independent of H and h. We denote by $\{\widetilde{\mathbf{r}}_i\}_{i=1}^6$ the rigid body motions $\{\mathbf{r}_i\}_{i=1}^6$ with $\mathbf{r}_5$ and $\mathbf{r}_6$ replaced by $\widetilde{\mathbf{r}}_5$ and $\widetilde{\mathbf{r}}_6$. The rigid body motions $\{\widetilde{\mathbf{r}}_i\}_{i=1}^6$ are then orthogonal and the values of $(\widetilde{\mathbf{r}}_i, \widetilde{\mathbf{r}}_i)_F$ are scaling invariant. In other words, they are constants independent of H and h.

Using Lemma 4.1, we obtain

$$\left| \int_F \widetilde{\mathbf{r}}_j \cdot \widetilde{\mathbf{r}}_k\, ds - \int_F \widetilde{\mathbf{r}}_j \cdot I_{M(F)}(\widetilde{\mathbf{r}}_k)\, ds \right| \leq ChH, \tag{4.4}$$

where C is a constant independent of H and h. We now consider a matrix $\widetilde{G}$ with entries,

$$\widetilde{G}_{jk} = \int_F \widetilde{\mathbf{r}}_j \cdot I_{M(F)}(\widetilde{\mathbf{r}}_k), \quad j,k = 1,\cdots,6.$$

Since the $\widetilde{\mathbf{r}}_i$ are orthogonal, we see, using the bound in (4.4), that $\widetilde{G}$ is diagonally dominant when h is sufficiently small. Therefore, $\widetilde{G}$ is invertible when h is small enough. The matrix G can be obtained from the invertible matrix $\widetilde{G}$ by using certain column and row operations and is therefore invertible. □

REMARK 4.3. *The proof of Lemma 4.2 also holds for an interface F that is not a flat surface.*

Since the six functionals $\{g_l\}$ are linearly independent, they provide a basis of the dual space $(\mathbf{ker}(\varepsilon))'$. Thus there exists $\{\beta_{ml}\}_{m,l=1}^{6}$ such that

$$f_m = \sum_{l=1}^{6} \beta_{ml} g_l, \quad m = 1, \cdots, 6. \tag{4.5}$$

Using Lemma 4.1, for any linear functions f and g we have

$$\left| \int_F fg\, ds - \int_F I_{M(F)}(f) I_{M(F)}(g)\, ds \right| \leq C\|f\|_\infty \|g\|_\infty hH.$$

When h is small enough, we then find

$$(I_{M(F)}(\mathbf{r}_k), I_{M(F)}(\mathbf{r}_k))_F \leq C(\mathbf{r}_k, \mathbf{r}_k)_F \leq CH^2. \tag{4.6}$$

From (4.2), (4.6), and the Cauchy-Schwarz inequality, we obtain

$$|g_l(\mathbf{w})|^2 \leq C \frac{\|\mathbf{w}\|_{0,F}^2}{H^2}.$$

The inequality (4.1) follows from (4.5) and the above bound. We denote the dual functionals described above by $\{f_l^F\}_{l=1}^6$ for the given face F. We can then express any rigid body motion $\mathbf{r} \in \mathbf{ker}(\varepsilon)$ as a linear combination using the dual basis,

$$\mathbf{r} = \sum_{l=1}^{6} f_l^F(\mathbf{r}) \mathbf{r}_l.$$

In the following, we will provide several lemmas which will be used to provide an upper bound of the FETI-DP algorithm equipped with the preconditioner $\widehat{M}^{-1}$ given in (3.8); see also (3.11). We note that a lower bound with the constant 1 is provided in (3.9).

For a face $F \subset \partial\Omega_i$, the space $H_{00}^{1/2}(F)$ consists of the functions whose zero extension to the whole boundary $\partial\Omega_i$ belongs to the space $H^{1/2}(\partial\Omega_i)$; it is equipped with the norm,

$$\|w\|_{H_{00}^{1/2}(F)} := \left(|w|_{1/2,F}^2 + \int_F \frac{w(x)^2}{\text{dist}(x, \partial F)}\, ds(x) \right)^{1/2},$$

where

$$|w|_{1/2,F}^2 = \int_F \int_F \frac{|w(x) - w(y)|^2}{|x - y|^3}\, ds(x) ds(y).$$

In addition, a norm for the space $H^{1/2}(F)$ is defined by

$$\|w\|_{1/2,F} = \left(|w|^2_{1/2,F} + \frac{1}{H_F}\|w\|^2_{0,F} \right)^{1/2},$$

where H_F is the diameter of F.

These norms can be extended to the product spaces $[H^{1/2}_{00}(F)]^3$ and $[H^{1/2}(F)]^3$ by using the usual product norms. Similarly, we can extend the edge and face lemmas given below to product spaces. These lemmas can be found in Toselli and Widlund [29, Lemmas 4.24 and 4.25]. In the following, $W(F)$ denotes any conforming P_1–finite element space provided for a face F with its mesh size comparable to that of its nonmortar subdomain. Since the face F can be only part of a full face of its nonmortar subdomain Ω_i, the triangulation equipped for the face F can be different from that of Ω_i. The space $\mathbf{W}(F)$ is the corresponding vector valued finite element space.

LEMMA 4.4. **(Edge lemma)** *Let E be an edge of a face $F_{ij} \subset \partial\Omega_i$ and F_{ij} has Ω_i as its nonmortar subdomain. Then, for any $\mathbf{w} \in \mathbf{W}(F_{ij})$, we have*

$$\|\mathbf{w}\|^2_{0,E} \le C \left(1 + \log \frac{H_{ij}}{h_i}\right) \|\mathbf{w}\|^2_{1/2,F_{ij}}.$$

Let $C(\overline{F})$ be the space of continuous functions defined on F. For any subset $A \subset \overline{F}$, we define an interpolant $I_A : C(\overline{F}) \to W(F)$ by

$$I_A(w)(x) = \begin{cases} w(x), & \text{for } x \in A \cap N^h, \\ 0, & \text{for the other nodes,} \end{cases} \tag{4.7}$$

where N^h is the set of nodes of the finite element space $W(F)$. We note that $I_F(w)$, when $A = F$, vanishes at the boundary of F. We can extend the interpolant to a vector valued function $\mathbf{w} \in [C(\overline{F})]^3$ and we simply denote it by $I_A(\mathbf{w})$.

LEMMA 4.5. **(Face lemma)** *Let F_{ij} be a face of $\partial\Omega_i$ with Ω_i as its nonmortar subdomain. Then, for any $\mathbf{w} \in \mathbf{W}(F_{ij})$, we have*

$$\|I_{F_{ij}}(\mathbf{w})\|^2_{\mathbf{H}^{1/2}_{00}(F_{ij})} \le C \left(1 + \log \frac{H_{ij}}{h_i}\right)^2 \|\mathbf{w}\|^2_{1/2,F_{ij}}.$$

We now derive several inequalities for the mortar projection of functions. We recall that the spaces $\overset{\circ}{\mathbf{W}}(F_l)$ and $\mathbf{M}(F_l)$ are given on the nonmortar face F_l; see Section 3.1.

DEFINITION 4.6. *(**Mortar projection**) The mortar projection $\pi_l : [L^2(F_l)]^3 \to \overset{\circ}{\mathbf{W}}(F_l)$ is defined by*

$$\int_{F_l} (\pi_l(\mathbf{w}) - \mathbf{w}) \cdot \boldsymbol{\psi} \, ds = 0, \quad \forall \boldsymbol{\psi} \in \mathbf{M}(F_l).$$

It is known that the mortar projection is bounded in both the L^2– and the $H_{00}^{1/2}$–norms; see [1, 31].

For any function w defined on $F_{ij} \subset F_l$, we define $\chi_{F_{ij}} w$ by

$$\chi_{F_{ij}} w(x) = \begin{cases} w(x), & x \in F_{ij}, \\ 0, & x \in F_l \setminus F_{ij}. \end{cases}$$

LEMMA 4.7. *Let $F_l \subset \partial\Omega_i$ be a nonmortar face with its partition $\{F_{ij}\}_j$ and let $\mathbf{w}$ be a function in $[H^{1/2}(F_{ij})]^3$. Then*

$$\|\pi_l(\chi_{F_{ij}} \mathbf{w})\|^2_{H_{00}^{1/2}(F_l)} \le C \left(1 + log \frac{H_{ij}}{h_i}\right)^2 \|\mathbf{w}\|^2_{1/2,F_{ij}}.$$

Proof. On the face F_{ij}, we consider a quasi–uniform triangulation of which mesh size is comparable to h_i, that of its nonmortar subdomain Ω_i and denote the corresponding conforming P_1–finite element space by $\mathbf{W}(F_{ij})$. We then define the L^2-projection,

$$Q : [L^2(F_{ij})]^3 \to \mathbf{W}(F_{ij}). \tag{4.8}$$

We note that

$$\|\pi_l(\chi_{F_{ij}} \mathbf{w})\|^2_{H_{00}^{1/2}(F_l)} \le 2\|\pi_l(\chi_{F_{ij}}(\mathbf{w}-Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)} + 2\|\pi_l(\chi_{F_{ij}}(Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)}. \tag{4.9}$$

The first term above is estimated by

$$\begin{aligned} \|\pi_l(\chi_{F_{ij}}(\mathbf{w} - Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)} &\le C h_i^{-1} \|\pi_l(\chi_{F_{ij}}(\mathbf{w} - Q\mathbf{w}))\|^2_{L^2(F_l)} \\ &\le C h_i^{-1} \|(\mathbf{w} - Q\mathbf{w})\|^2_{0,F_{ij}} \\ &\le C \|\mathbf{w}\|^2_{1/2,F_{ij}}. \end{aligned} \tag{4.10}$$

Here we have used an inverse inequality, the continuity of π_l in the L^2–norm, and the approximation property of the projection Q combined with an interpolation argument, see [4, Chapter II],

$$\|Q\mathbf{w} - \mathbf{w}\|_{0,F_{ij}} \le C h_i \|\mathbf{w}\|_{1,F_{ij}}, \quad \|Q\mathbf{w}\|_{1,F_{ij}} \le C\|\mathbf{w}\|_{1,F_{ij}}.$$

We now decompose $Q\mathbf{w}$ into interior and boundary parts, using the interpolant $I_A(\mathbf{w})$ defined in (4.7) to the space $\mathbf{W}(F_{ij})$,

$$Q\mathbf{w} = I_{F_{ij}}(Q\mathbf{w}) + I_{\partial F_{ij}}(Q\mathbf{w}).$$

Since $I_{F_{ij}}(Q\mathbf{w})$ is zero at the boundary of F_{ij}, we have

$$\chi_{F_{ij}}(I_{F_{ij}}(Q\mathbf{w})) \in [H_{00}^{1/2}(F_l)]^3, \quad I_{F_{ij}}(Q\mathbf{w}) \in [H_{00}^{1/2}(F_{ij})]^3.$$

The definition of the $H_{00}^{1/2}$–norm gives that

$$\|\chi_{F_{ij}}(I_{F_{ij}}(Q\mathbf{w}))\|_{H_{00}^{1/2}(F_l)} \leq \|I_{F_{ij}}(Q\mathbf{w})\|_{H_{00}^{1/2}(F_{ij})}.$$

The second term in (4.9) is then bounded by

$$\begin{aligned}
&\|\pi_l(\chi_{F_{ij}}(Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)} \\
&\leq 2\|\pi_l(\chi_{F_{ij}}(I_{F_{ij}}(Q\mathbf{w})))\|^2_{H_{00}^{1/2}(F_l)} + 2\|\pi_l(\chi_{F_{ij}}(I_{\partial F_{ij}}(Q\mathbf{w})))\|^2_{H_{00}^{1/2}(F_l)} \\
&\leq C\left(\|\chi_{F_{ij}}(I_{F_{ij}}(Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)} + h_i^{-1}\|I_{\partial F_{ij}}(Q\mathbf{w}))\|^2_{0,F_{ij}}\right) \\
&\leq C\left(\|I_{F_{ij}}(Q\mathbf{w})\|^2_{H_{00}^{1/2}(F_{ij})} + h_i^{-1}h_i\|I_{\partial F_{ij}}(Q\mathbf{w}))\|^2_{0,\partial F_{ij}}\right). \qquad (4.11)
\end{aligned}$$

Here we have used an inverse inequality and the continuity of π_l in the L^2– and $H_{00}^{1/2}$–norms.

By applying Lemmas 4.5 and 4.4 to the two terms in (4.11), we obtain

$$\begin{aligned}
\|\pi_l(\chi_{F_{ij}}(Q\mathbf{w}))\|^2_{H_{00}^{1/2}(F_l)} &\leq C\left(1+\log\frac{H_{ij}}{h_i}\right)^2 \|Q\mathbf{w}\|^2_{1/2,F_{ij}} \\
&\leq C\left(1+\log\frac{H_{ij}}{h_i}\right)^2 \|\mathbf{w}\|^2_{1/2,F_{ij}}. \qquad (4.12)
\end{aligned}$$

We obtain the required bound from (4.9)–(4.12). □

REMARK 4.8. *In our previous paper [15], also on geometrically nonconforming partitions, a slightly weaker bound,* $C(1+\log(H/h))^3$*, was proved for three dimensional elliptic problems. Here we are able to improve this result by using an additional finite element space* $\mathbf{W}(F_{ij})$ *and the* L^2*–projection* Q *in the proof.*

The following lemma is a simple modification of a result in Dryja, Smith, and Widlund [6, Lemma 4.4]:

LEMMA 4.9. *Let* $F_{ij} \subset \partial\Omega_i$ *be a face. For a linear function* ϕ*, we have*

$$\|I_{F_{ij}}(\phi)\|^2_{H_{00}^{1/2}(F_{ij})} \leq C\left(1+\log\frac{H_{ij}}{h_i}\right) H_{ij}\|\phi\|^2_{\infty,F_{ij}}.$$

LEMMA 4.10. *Let* $F_{ij} \subset F_l$ *where* F_l *is a nonmortar face of* $\partial\Omega_i$*. For a linear function* ϕ*, we have*

$$\|\pi_l(\chi_{F_{ij}}\phi)\|^2_{H_{00}^{1/2}(F_l)} \leq C\left(1+\log\frac{H_{ij}}{h_i}\right) H_{ij}\|\phi\|^2_{\infty,F_{ij}}.$$

Proof. By applying the estimate in (4.11) of Lemma 4.7 to ϕ, we have

$$\|\pi_l(\chi_{F_{ij}}(Q\phi))\|^2_{H_{00}^{1/2}(F_l)} \leq C\left(\|I_{F_{ij}}(Q\phi)\|^2_{H_{00}^{1/2}(F_{ij})} + \|I_{\partial F_{ij}}(Q\phi)\|^2_{0,\partial F_{ij}}\right),$$

where Q is the L^2–projection described in (4.8). Since $Q\phi = \phi$, the required bound follows by using Lemma 4.9. □

LEMMA 4.11. *Let $F_l \subset \partial\Omega_i$ be a nonmortar face with the partition $\{F_{ij}\}_j$ by its mortar neighbors. We assume that each interface F_{ij} is either a primal face or has an acceptable face path, and assume that the primal faces are substantial and the others, non-primal faces, are weakly substantial. In addition, the subdomain partition satisfies Assumption 3.1. Then, for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$, we have*

$$\begin{aligned} G_i \|\pi_l(\mathbf{w}_i - \boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)} \le C \Bigg\{ & \left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i} \\ & + \sum_j L(F_{ij}) * \sum_{k \in A(i,j)} \left(1 + log\frac{H_{ij}}{h_i}\right) \frac{G_i}{G_k} |\mathbf{w}_k|^2_{S_k} \\ & + \sum_j \frac{G_i}{G_j} \left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_j|^2_{S_j} \Bigg\}. \end{aligned}$$

Here the constant C depends on TOL_F (see Definition 3.3), the restriction of ϕ to F_{ij} is $\mathbf{w}_j$, $A(i,j)$ is the set of subdomain indices of the acceptable face path of F_{ij}, and the constant $L(F_{ij})$ is the number of the subdomains in the path.

Proof. Since $\{F_{ij}\}_j$ is a partition of the nonmortar face F_l, we can write

$$\mathbf{w}_i - \boldsymbol{\phi} = \sum_j \chi_{F_{ij}} (\mathbf{w}_i - \mathbf{w}_j)$$

and it suffices to estimate each term in the above expression.

Let $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$ be the related acceptable face path of F_{ij} which passes through the primal faces $\{F_{ik_1}, F_{k_1k_2}, \cdots, F_{k_nj}\}$. When F_{ij} is a primal face, the path is simply $\{\Omega_i, \Omega_j\}$. Let $\{\mathbf{r}_m^{ik_1}\}$, $\{\mathbf{r}_m^{k_1k_2}\}$, $\cdots$, $\{\mathbf{r}_m^{k_nj}\}$ be bases of **ker**(ε) scaled with respect to the primal faces F_{ik_1}, $F_{k_1k_2}$, $\cdots$, F_{k_nj}, respectively. We denote the dual basis to $\{\mathbf{r}_m^{lk}\}_m$ by $\{f_m^{lk}\}_m$. We introduce the notation,

$$f_{lk}(\mathbf{w}) = \sum_{m=1}^{6} f_m^{lk}(\mathbf{w}) \mathbf{r}_m^{lk}.$$

We note that $f_{lk}(\mathbf{r}) = \mathbf{r}$ for any rigid body motion $\mathbf{r}$ and that $f_{lk}(\mathbf{w}_l) = f_{lk}(\mathbf{w}_k)$, since $\mathbf{w}$ satisfies the primal constraints on the primal face F_{lk}.

We then have

$$\begin{aligned} \mathbf{w}_i - \mathbf{w}_j = & (\mathbf{w}_i - \mathbf{r}_i) - f_{ik_1}(\mathbf{w}_i - \mathbf{r}_i) + f_{ik_1}(\mathbf{w}_{k_1} - \mathbf{r}_{k_1}) - f_{k_1k_2}(\mathbf{w}_{k_1} - \mathbf{r}_{k_1}) \\ & + f_{k_1k_2}(\mathbf{w}_{k_2} - \mathbf{r}_{k_2}) - f_{k_2k_3}(\mathbf{w}_{k_2} - \mathbf{r}_{k_2}) + \cdots \\ & + f_{k_nj}(\mathbf{w}_j - \mathbf{r}_j) - (\mathbf{w}_j - \mathbf{r}_j), \end{aligned} \tag{4.13}$$

where the $\mathbf{r}_k$ denote any rigid body motions.

We apply Lemma 4.7 to the first and the last terms of the above equation and obtain

$$\begin{aligned}
&G_i\left(\|\pi_l(\chi_{F_{ij}}(\mathbf{w}_i-\mathbf{r}_i))\|^2_{H^{1/2}_{00}(F_l)}+\|\pi_l(\chi_{F_{ij}}(\mathbf{w}_j-\mathbf{r}_j))\|^2_{H^{1/2}_{00}(F_l)}\right)\\
&\leq C\left(1+\log\frac{H_{ij}}{h_i}\right)^2\left(G_i\|\mathbf{w}_i-\mathbf{r}_i\|^2_{1/2,F_{ij}}+G_i\|\mathbf{w}_j-\mathbf{r}_j\|^2_{1/2,F_{ij}}\right)\\
&\leq C\left(1+\log\frac{H_{ij}}{h_i}\right)^2\left(\frac{H_i}{H_{ij}}|\mathbf{w}_i|^2_{S_i}+\frac{H_j}{H_{ij}}\frac{G_i}{G_j}|\mathbf{w}_j|^2_{S_j}\right)\\
&\leq C(TOL_F)\left(1+\log\frac{H_i}{h_i}\right)^2\left(|\mathbf{w}_i|^2_{S_i}+\frac{G_i}{G_j}|\mathbf{w}_j|^2_{S_j}\right),
\end{aligned}$$

where $C(TOL_F)$ denotes a constant which depends on TOL_F. Here we have also used Lemmas 2.2 and 2.3, and the assumption that the face F_{ij} is weakly substantial; see Definition 3.3.

We next consider the other terms in (4.13). From (4.1) and Lemmas 4.10 and 2.3, each term can be estimated as

$$\begin{aligned}
&G_i|f_m^{k_1k_2}(\mathbf{w}_{k_2}-\mathbf{r}_{k_2})|^2\ \|\pi_l(\chi_{F_{ij}}\mathbf{r}_m^{k_1k_2})\|^2_{H^{1/2}_{00}(F_l)}\\
&\leq CG_i\frac{\|\mathbf{w}_{k_2}-\mathbf{r}_{k_2}\|^2_{0,\partial\Omega_{k_2}}}{H^2_{k_1k_2}}\left(1+\log\frac{H_{ij}}{h_i}\right)H_{ij}\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F_{ij}}\\
&\leq C*\frac{G_i}{G_{k_2}}\left(1+\log\frac{H_{ij}}{h_i}\right)|\mathbf{w}_{k_2}|^2_{S_{k_2}}\frac{H_{k_2}H_{ij}}{H^2_{k_1k_2}}\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F_{ij}}.
\end{aligned}$$

We will show that the factor $(H_{k_2}H_{ij}/H^2_{k_1k_2})\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F_{ij}}$ is bounded by a constant that depends only on $L(F_{ij})$. The assumption of the acceptable face path gives that $L(F_{ij})\leq L$ for a given constant L. Since the length of the face path is less than or equal to L and the subdomain partition is locally quasi uniform, the diameter of the subdomains in the path are comparable to H_i. In addition, the diameter of any primal face of the path is comparable to H_i, because primal faces are substantial. From these observations, we obtain

$$\|\mathbf{r}_m^{k_1k_2}\|_{\infty,F_{ij}}\leq CL(F_{ij}),\quad \frac{H_{k_2}H_{ij}}{H^2_{k_1k_2}}\leq C\frac{H_{ij}}{H_i},$$

and have proved the bound

$$\frac{H_{k_2}H_{ij}}{H^2_{k_1k_2}}\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F_{ij}}\leq CL(F_{ij})^2. \tag{4.14}$$

The remaining terms in (4.13) can be bounded in a similar way leading to the required bound of $G_i\|\pi_l(\mathbf{w}_i-\boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)}$. □

REMARK 4.12. *The proof for the bound in* (4.14) *suggests that it is beneficial that the diameter of the non-primal interface F_{ij} is smaller than those of the primal interfaces in the acceptable face path as long as it is weakly substantial with respect to the given constant TOL_F. In addition, the definition of an acceptable face path shows that a smaller diameter of F_{ij} provides more chances of finding an acceptable path of F_{ij} for a given TOL_P and L.*

We note that the acceptable face path assumption and Assumption 3.4 give

$$\left(1+\log\frac{H_{ij}}{h_i}\right)\frac{G_i}{G_{k_l}} \le TOL_P * \left(1+\log\frac{H_i}{h_i}\right)^2, \quad L(F_{ij}) \le L, \quad \frac{G_i}{G_j} \le TOL_G.$$

Therefore the bound in Lemma 4.11 is reduced to

$$G_i\|\pi_l(\mathbf{w}_i-\boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)} \le C\left(1+\log\frac{H_i}{h_i}\right)^2 \sum_{l\in N_l} |\mathbf{w}_l|^2_{S_l}, \tag{4.15}$$

where N_l is the set of subdomain indices, that appear on the acceptable face path of $F_{ij} \subset F_l$. The constant C depends on TOL_F, TOL_G, TOL_P, and L but does not depend on any mesh parameters nor on the coefficients G_i.

LEMMA 4.13. *Assume that every non-primal face satisfies the acceptable face path condition with given TOL_P, L, and coefficients G_i and that every primal face is substantial and every non–primal face is weakly substantial with a given TOL_F. In addition, Assumptions 3.1 and 3.4 hold with a given TOL_G. We then obtain*

$$\max_{\mathbf{w}\in\widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w},\boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w},\mathbf{w}\rangle} \le C \max_{i=1,\cdots,N}\left\{\left(1+log\frac{H_i}{h_i}\right)^2\right\}\langle \widehat{M}\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle,$$

where the constant C depends on the TOL_F, TOL_G, TOL_P, and L but not on any mesh parameters and coefficient G_i.

Proof. We consider

$$\langle B\mathbf{w},\boldsymbol{\lambda}\rangle^2 = \left(\sum_{l,\text{nonmortar}}\int_{F_l}(\mathbf{w}_i-\boldsymbol{\phi})\cdot\boldsymbol{\lambda}\,ds\right)^2 = \left(\sum_{l,\text{ nonmortar}}\int_{F_l}\pi_l(\mathbf{w}_i-\boldsymbol{\phi})\cdot\boldsymbol{\lambda}\,ds\right)^2.$$

Since $\mathbf{w}\in\widetilde{\mathbf{W}}$, $\mathbf{w}$ satisfies the primal constraints on any primal face F_{ij}, i.e.,

$$\int_{F_{ij}}(\mathbf{w}_i-\boldsymbol{\phi})\cdot I_{M(F_{ij})}(\mathbf{r}_m)\,ds = 0, \quad m=1,\cdots,6.$$

This implies that $\pi_l(\mathbf{w}_i-\boldsymbol{\phi})$ also satisfies the primal constraints,

$$\int_{F_{ij}}\pi_l(\mathbf{w}_i-\boldsymbol{\phi})\cdot I_{M(F_{ij})}(\mathbf{r}_m)\,ds = 0, \quad m=1,\cdots,6,$$

because $I_{M(F_{ij})}(\mathbf{r}_m)$ belong to the Lagrange multiplier space $\mathbf{M}(F_l)$. Therefore, $\mathbf{z}_n$, defined on each nonmortar face F_l by

$$\mathbf{z}_n|_{F_l} = \pi_l(\mathbf{w}_i-\boldsymbol{\phi}),$$

belongs to $\mathbf{W}_{\Delta,n}$. In other words, $\mathbf{z}_n$ has all its six primal components zero on each primal face. We then define by $\mathbf{z}=E(\mathbf{z}_n)$ the extension of $\mathbf{z}_n$ to $\widetilde{\mathbf{W}}(=\mathbf{W}_{\Delta,n}\times\mathbf{W}_{\Delta,m}\times\mathbf{W}_\Pi)$ by zero.

By the definition of $\widehat{M}$ in (3.8), we find that

$$\begin{aligned}\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle^2 &= \langle B\mathbf{z}, \boldsymbol{\lambda}\rangle^2 \\ &\leq \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \langle S\mathbf{z}, \mathbf{z}\rangle.\end{aligned}$$

It suffices to show that

$$\langle S\mathbf{z}, \mathbf{z}\rangle \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \langle S\mathbf{w}, \mathbf{w}\rangle. \tag{4.16}$$

We now consider

$$\begin{aligned}\langle S\mathbf{z}, \mathbf{z}\rangle &= \sum_{i=1}^{N} \langle S_i \mathbf{z}_i, \mathbf{z}_i\rangle \\ &\leq C \sum_{i=1}^{N} \sum_{l,\ \text{nonmortar}} G_i |\mathcal{H}^i\left(\pi_l(\mathbf{w}_i - \boldsymbol{\phi})\right)|^2_{1,\Omega_i} \\ &\leq C \sum_{i=1}^{N} \sum_{l,\ \text{nonmortar}} G_i \|\pi_l(\mathbf{w}_i - \boldsymbol{\phi})\|^2_{H^{1/2}_{00}(F_l)}, \end{aligned} \tag{4.17}$$

where $\mathcal{H}^i$ is the discrete harmonic extension into X_i. Here we have used that, see Lemma 2.2 and (2.4),

$$\langle S_i \mathbf{z}_i, \mathbf{z}_i\rangle \leq C G_i |\mathcal{H}^i(\mathbf{z}_i)|^2_{1,\Omega_i}.$$

From the bound (4.15) and (4.17), we obtain (4.16) with a constant C which depends on TOL_F, TOL_G, TOL_P, and L. □

The lower bound in (3.9) and the bound in Lemma 4.13, combined with (3.7), lead to the following condition number bound.

THEOREM 4.14. *We assume that the assumptions in Lemma 4.13 hold. We then obtain the condition number bound,*

$$\kappa(\widehat{M}^{-1} F_{DP}) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \mathit{log} \frac{H_i}{h_i}\right)^2 \right\}$$

Here the constant C is independent of the mesh parameters and the coefficients G_i, but depends on the given constants TOL_F, TOL_G, TOL_P, and L.

5. An algorithm for selecting primal faces. We now introduce an algorithm which selects a quite small number of primal faces for an arbitrary distribution of $\{G_i\}_{i=1}^N$. We first choose constants TOL_P and L that will be used in the selection. Next we select an initial set of primal faces and put them in the set P of primal faces. We then determine non-primal faces based on the set P and the given constants TOL_P and L. We then visit the remaining undetermined faces in a certain order and add some of them, one by one, to the set P. Whenever we add an undetermined face to the set P, we determine the current set of non-primal faces based on the updated primal set P. We repeat this process until every face is determined. In order to choose a small initial primal set P, we introduce the concept of an essentially primal face.

DEFINITION 5.1. *(**Essentially primal face**) A face $F = \partial\Omega_i \cap \partial\Omega_j$ is essentially primal, if there is no acceptable face path for (Ω_i, Ω_j) for the given TOL_P and L, when all faces except F are chosen to be primal.*

We will now explain the algorithm in detail. For the given constants TOL_P and L, we determine the essentially primal faces and add them to the set P of primal faces. Based on this set P, we determine the non-primal faces. For the remaining undetermined faces, we order them with respect to decreasing ratios of the coefficients between the two subdomain Ω_i and Ω_j. If we have more than one face having the same coefficient ratio, we then select the one with most neighbors. We then add an undetermined face to the set P and determine the non-primal faces of this updated set P. We repeat this until every face is determined. The ordering of the undetermined faces increases our chances that there will exist acceptable face paths for other faces which are previously undetermined.

Algorithm $(TOL_P,\ L,\ \{G_i\},\ \{H_i\}, \{h_i\}$ given $)$

Step 1. Determine essentially primal faces F and add them to the primal face set P.

Step 2. Determine non-primal faces based on the set P.

Step 3. For the remaining undetermined faces F, order them in decreasing order of the ratio of the coefficients. If there are more than two faces with the same ratio then order them in decreasing order of the number of neighbors of the two subdomains which intersect the current face F.

Step 4. **Do** until every undetermined face F determined

- Add a current undetermined face F to the primal face set P
- Determine the non-primal faces based on the updated primal face set P

End

We have tested the algorithm for both constant and variable coefficient cases. The domain $\Omega = [0,1]^3$ is partitioned into N^3 hexagonal subdomains. For the case of constant coefficients, we take $G(x) = 1$, and for the case of discontinuous coefficient we distribute the values 1, 10, 10^2 and 10^3 randomly over the subdomain partition.

In Table 1, we present the number of primal faces when $TOL_P = 10$, $L = 6$, and the number of nodes (H_i/h_i) are the same for all subdomains. Here **Total** means the total number of faces in the subdomain partition, **Min** denotes the number of primal faces what we obtain from the algorithm without any limit on TOL_P and L. For this case, our algorithm gives exactly the minimum number of primal faces, $N^3 - 1$, that are required to resolve the rigid body motions generated by the N^3 subdomains. The columns **Const** and **Random** show the number of primal faces for the constant coefficient case and the discontinuous coefficient

N^3	**Total**	**Min**	**Const**	**Random**
2^3	12	7	7	8
4^3	144	63	68	89
6^3	540	215	246	322
8^3	1344	511	646	804
10^3	2700	999	1300	1598

TABLE 1

The number of primal faces from the algorithm: N^3 (the number of subdomains), **Total** *(the number of faces over the subdomain partition),* **Min** *(the number of primal faces without any limit on TOL_P and L),* **Const** *(the number of primal faces for the constant coefficient case with $TOL_P = 10$ and $L = 6$),* **Random** *(the number of primal faces for the discontinuous coefficient case with $TOL_P = 10$ and $L = 6$)*

case, respectively. Comparing these two columns, we see that this algorithm gives a quite small number of primal faces even for the case with the discontinuous coefficients.

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