BDDC Domain Decomposition Algorithms: Methods with Three Levels and for Flow in Porous Media

by

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ABSTRACT

Two inexact coarse solvers for Balancing Domain Decomposition by Constraints (BDDC) algorithms are introduced and analyzed. These solvers help remove a bottleneck for the two-level BDDC algorithms related to the cost of the coarse problem when the number of subdomains is large. At the same time, a good convergence rate is maintained.

BDDC algorithms are also developed for the linear systems arising from flow in porous media discretized with mixed and hybrid finite elements. Our methods are proven to be scalable and the condition numbers of the operators with our BDDC preconditioners grow only polylogarithmically with the size of the subdomain problems.

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Chapter 1

Introduction

1.1 An Overview

Usually the first step of solving an elliptic partial differential equation (PDE) numerically is its discretization. Finite difference, finite element, or other discretizations reduce the original PDE to an often huge and ill-conditioned linear or nonlinear system of algebraic equations. Limited by the memory and speed of the computers, the traditional direct solvers can often not handle such large linear systems. Moreover, iterative methods, such as Krylov space methods, may need thousands of iterations to obtain accurate solutions due to large condition numbers of such systems. Domain decomposition methods provide efficient and scalable preconditioners that can be accelerated by Krylov space methods and have become popular in applications in computational fluid dynamics, structural engineering, electromagnetics, constrained optimization, etc. The basic idea of domain decomposition methods is to split the original huge problem into many small problems that can be handled by direct solvers, and then solve these smaller problems a number of times and accelerate the solution of the original problem with Krylov

space methods.

There are two main classes of domain decomposition methods: overlapping Schwarz methods and iterative substructuring methods. In the overlapping Schwarz methods, the domain is divided into many overlapping subdomains. In each Krylov iteration of a one-level Schwarz method we primarily need to solve a local problem in each subdomain. But the number of iterations will depend on the number of subdomains. A coarse problem is added in the more powerful two-level overlapping methods. With a generous overlap, the number of iterations for the two-level methods can be independent of the number of subdomains and the size of subdomain problems. An abstract theory of Schwarz type domain decomposition methods has also been established and successfully applied to several types of methods of this kind; see [83, Chapters 2,3] and the references therein.

In iterative substructuring methods, the domain is decomposed into nonoverlapping subdomains. The unknowns in the interior of the subdomains are first eliminated independently and we then work with the Schur complement with respect to the unknowns associated with the interface. Coarse problems are constructed using one or a few degrees of freedom for each subdomain. Among these algorithms, the Neumann-Neumann and finite element tearing and interconnecting methods (FETI) families are the best known and they have been tested in many applications. For a detailed discussion, see Chapter 2 and [83, Chapters 4,5,6] and the references therein.

Recently, a new family of iterative substructuring methods, the balancing domain decomposition by constraints (BDDC) algorithms, has been developed by Clark Dohrmann in [27]. These methods have a Neumann-Neumann flavor. How-

ever, their coarse problems are given by sets of constraints enforced on the interface, which are similar to those of the dual-primal FETI (FETI-DP) methods. It has been proved recently that the preconditioned operators for BDDC and FETI-DP have identical nontrivial eigenvalues except possibly for 0 and 1, see [63, 59, 20].

However, a shortcoming of both BDDC, FETI-DP, and all other domain decomposition methods is that the coarse problem needs to be assembled and the resulting matrix needs to be factored by a direct solver at the beginning of the computation. Usually the size of the coarse problem is proportional to the number of subdomains. Nowadays some computer systems have more than 100,000 powerful processors, which allow very large and detailed simulations. The coarse component can therefore be a bottleneck if the number of subdomains is very large. Motivated by this fact, we will, in this dissertation, develop two three-level BDDC algorithms to remove this difficulty. We also successfully adapt the two-level BDDC methods to flow in porous media.

We will first review some basic function analysis tools in the following sections.

1.2 Functional Analysis Tools

1.2.1 Sobolev Spaces

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. $L^2(\Omega)$ is the space of all real measurable functions u that satisfy

$$\int_\Omega |u|^2 \, dx < \infty.$$

It is a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u \, v \, dx$$

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and an induced norm

$$\|u\|_{L^2(\Omega)}^2 = (u, u)_{L^2(\Omega)} = \int_\Omega |u|^2 \, dx.$$

The space $L_0^2(\Omega)$ is a subspace of $L^2(\Omega)$ of functions with zero average over Ω.

The space $H^1(\Omega)$ is a space of functions such that

$$\int_\Omega |u|^2 dx < \infty, \text{ and } \int_\Omega \nabla u \cdot \nabla u \, dx < \infty, \quad \forall u \in H^1(\Omega),$$

with the scaled norm

$$\|u\|_{H^1(\Omega)}^2 = \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{1}{H_\Omega^2} \int_\Omega |u|^2 dx,$$

where H_Ω is the diameter of Ω; this scaling factor is obtained by dilation from a region of unit diameter. The corresponding H^1-seminorm is defined by

$$|u|_{H^1(\Omega)}^2 = \int_\Omega \nabla u \cdot \nabla u \, dx.$$

The subspace of $H_0^1(\Omega)$ is a closure of the $C_0^\infty(\Omega)$ functions in $H^1(\Omega)$.

We also define the divergence operator for a vector function $\mathbf{u} \in \mathbb{R}^n$, $n = 2, 3$ as

$$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}, \tag{1.1}$$

where u_i is the i-th component of $\mathbf{u}$. The space $H(\text{div}; \Omega)$ is a Hilbert space with the scalar product and graph norm defined by

$$(\mathbf{u}, \mathbf{v})_{\text{div};\Omega} = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx + \int_\Omega \text{div } \mathbf{u} \, \text{div } \mathbf{v} \, dx, \quad \|\mathbf{u}\|_{\text{div};\Omega}^2 = (\mathbf{u}, \mathbf{u})_{\text{div};\Omega}.$$

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1.2.2 Trace and Extension Theorems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. We define some Sobolev spaces on $\Gamma \subseteq \partial\Omega$ and two extension theorems in this subsection. For a more detailed discussion, see [45, Section 1.5].

The trace space of $H^1(\Omega)$ is $H^{1/2}(\partial\Omega)$, and, for $\Gamma \subseteq \partial\Omega$, the corresponding semi-norm and norm are given by

$$|u|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^d} dx dy, \qquad (1.2)$$

and

$$\|u\|^2_{H^{1/2}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \frac{1}{H_\Gamma} \|u\|^2_{L^2(\Gamma)}, \qquad (1.3)$$

where H_Γ is the diameter of Γ and d is the dimension of Ω. We also define the subspace $H_{00}^{1/2}(\Gamma)$ of $H^{1/2}(\Gamma)$, of functions which can be extended by zero to $\partial\Omega\backslash\Gamma$ and still belong to $H^{1/2}(\partial\Omega)$. $H_{00}^{1/2}(\Gamma)$ is a proper subspace of $H^{1/2}(\Gamma)$. The norm of $H_{00}^{1/2}(\Gamma)$ can be defined as

$$\|u\|^2_{H_{00}^{1/2}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{u^2(x)}{d(x, \partial\Gamma)} dx, \qquad (1.4)$$

where $d(x, \partial\Gamma)$ is the distance from x to the boundary $\partial\Gamma$.

Lemma 1.1 (Trace theorem) *Let Ω be a Lipschitz region. Then, there is a bounded linear operator $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ such that $\gamma_0 u = u|_{\partial\Omega}$ if u is continuous in $\bar{\Omega}$.*

The dual spaces of $H^{1/2}(\partial\Omega)$ and $H_{00}^{1/2}(\Gamma)$ are denoted by $H^{-1/2}(\partial\Omega)$ and $H_{00}^{-1/2}(\Gamma)$, respectively.

Given a vector $\mathbf{u} \in H(\mathrm{div}\,, \Omega)$, we can define its normal component $\mathbf{u} \cdot \mathbf{n}$ on $\partial\Omega$ as an element of $H^{-1/2}(\partial\Omega)$, and the following inequality holds

$$\|\mathbf{u} \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} \leq C \left(\|\mathbf{u}\|^2_{L^2(\Omega)} + H^2_\Omega \|\mathrm{div}\,\mathbf{u}\|^2_{L^2(\Omega)} \right), \tag{1.5}$$

with a constant C that is independent of H_Ω, the diameter of Ω. The operator that maps a vector in $H(\mathrm{div}\,, \Omega)$ into its normal component in $H^{-1/2}(\Omega)$ is thus continuous, and it can be shown to be surjective; see [39, Ch. I, Th. 2.5 and Cor. 2.8]. The subspace $H_0(\mathrm{div}\,; \Omega)$ consists of all functions in $H(\mathrm{div}\,; \Omega)$ with $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

We also have the following two extension theorems.

Lemma 1.2 *Let Ω be a Lipschitz domain. Then, there exists a continuous lifting operator $\mathcal{R}_0 : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that $\gamma_0(\mathcal{R}_0 u) = u$, $u \in H^{1/2}(\partial\Omega)$.*

Lemma 1.3 *For any $\mu \in H^{-1/2}(\partial\Omega)$ with mean value zero on $\partial\Omega$, there exists an extension operator $\tilde{\mathcal{H}} : H^{1/2}(\partial\Omega) \longrightarrow L^2(\Omega)$ such that*

$$\mathrm{div}\,\tilde{\mathcal{H}}\mu = 0,$$

and

$$\|\tilde{\mathcal{H}}\mu\|_{L^2(\Omega)} \leq C \|\mu\|_{H^{-1/2}(\partial\Omega)}.$$

Here C is independent of H_Ω and μ.

Proof: Consider a Neumann problem

$$\begin{aligned}
\triangle\phi &= 0, && \text{in } \Omega, \\
\frac{\partial\phi}{\partial\mathbf{n}} &= \mu, && \text{on } \partial\Omega,
\end{aligned}$$

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where $\mathbf{n}$ is the outward normal direction of $\partial\Omega$. Since we assume that μ has mean value zero over $\partial\Omega$, this problem is solvable and we can select the solution ϕ with mean value zero over Ω. We then define $\tilde{\mathcal{H}}\mu := \triangledown\phi$ and have

$$\|\tilde{\mathcal{H}}\mu\|_{L^2(\Omega)} = |\phi|_{H^1(\Omega)} \le C\|\mu\|_{H^{-1/2}(\partial\Omega)},$$

where C is independent of the diameter of Ω.

$\square$

1.2.3 Poincaré and Friedrichs' Inequalities

In the domain decomposition theory, Poincaré and Friedrichs type inequalities are powerful tools. We formulate them and some corollaries related to domain decomposition methods in this subsection. For detailed proofs, see [72].

Lemma 1.4 (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, there exist constants C_1 and C_2, depending only on Ω, such that*

$$\|u\|_{L^2(\Omega)}^2 \le C_1|u|_{H^1(\Omega)}^2 + C_2\left(\int_\Omega u\,dx\right)^2, \quad \forall u \in H^1(\Omega).$$

Lemma 1.5 (Friedrichs' inequality) *Let Ω be the same as in Lemma 1.4 and let $\Gamma \subseteq \partial\Omega$ have nonzero $(n-1)$-dimensional measure. Then, there exist constants C_1 and C_2, depending only on Ω and Γ, such that*

$$\|u\|_{L^2(\Omega)}^2 \le C_1|u|_{H^1(\Omega)}^2 + C_2\|u\|_{L^2(\Gamma)}^2, \quad \forall u \in H^1(\Omega).$$

In particular, if $u = 0$ on Γ,

$$\|u\|_{L^2(\Omega)}^2 \le C_1|u|_{H^1(\Omega)}^2$$

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and thus

$$|u|^2_{H^1(\Omega)} \le \|u\|^2_{H^1(\Omega)} \le (C_1 + 1)\, |u|^2_{H^1(\Omega)}.$$

By simple *scaling arguments*, we obtain the following corollary, see [83, Corollary A.15].

Corollary 1.6 *Let Ω be Lipschitz continuous with diameter H_Ω and $\Gamma \subset \partial\Omega$ be defined as in Lemma 1.5 with a diameter of order H_Ω. Then, there exist constants C_1, C_2, and C_3, that depend only on the shape of Ω but are independent of H_Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \le C_1\, H_\Omega^2\, |u|^2_{H^1(\Omega)} + C_2\, H_\Omega\, \|u\|^2_{L^2(\Gamma)}, \quad \forall u \in H^1(\Omega).$$

If $u \in H^1(\Omega)$ has a vanishing mean value over Ω, then

$$\|u\|^2_{L^2(\Omega)} \le C_3\, H_\Omega^2\, |u|^2_{H^1(\Omega)}.$$

Using the operators γ_0 and $\mathcal{R}_0$ of Lemmas 1.1 and 1.2, we can obtain the following lemma which is useful in the analysis of some iterative substructuring methods; see [83, Lemma A.17].

Lemma 1.7 *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz continuous polyhedron. If $u \in H^{1/2}(\partial\Omega)$ either has a vanishing mean value on $\partial\Omega$ or belongs to the closure of the space of $C^\infty(\partial\Omega)$ functions that vanish on a face of Ω, there exists a constant C_1, that depends only on the shape of Ω and the face, such that*

$$\|u\|^2_{L^2(\partial\Omega)} \le C_1\, H_\Omega\, |u|^2_{H^{1/2}(\partial\Omega)}.$$

Similarly, if $\mathcal{F} \subset \partial\Omega$ is one of the faces of Ω of diameter on the order H_Ω and $u \in H^{1/2}(\mathcal{F})$ either has vanishing mean value on $\mathcal{F}$ or belongs to $H_{00}^{1/2}(\mathcal{F})$, then

there exists a constant C_2, that depends only on the shape of $\mathcal{F}$ but not on its size, such that

$$\|u\|^2_{L^2(\mathcal{F})} \leq C_2 \, H_\Omega \, |u|^2_{H^{1/2}(\mathcal{F})}.$$

1.3   Variational Formulations of Second Order Elliptic Boundary-Value Problems and Finite Element Spaces

We consider the following elliptic problem on a bounded polygonal domain Ω in two or three dimensions with boundary conditions:

$$\begin{cases} -\nabla \cdot (a\nabla p) & = & f & \text{in} & \Omega, \\ p & = & g_D & \text{in} & \partial\Omega_D, \\ \mathbf{n} \cdot (a\nabla p) & = & g_N & \text{in} & \partial\Omega_N, \end{cases} \qquad (1.6)$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$ and a is a positive definite matrix function with entries in $L^\infty(\Omega)$ and satisfying

$$\xi^T a(\mathbf{x})\xi \geq \alpha\|\xi\|^2, \quad \text{for a.e.} \quad \mathbf{x} \in \Omega, \qquad (1.7)$$

for some positive constant α.

We assume that the functions $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\partial\Omega_D)$, and $g_N \in H^{-1/2}(\partial\Omega_N)$. Moreover, if $\partial\Omega_N = \partial\Omega$, f and g_N should satisfy the compatibility condition

$$\int_\Omega f d\mathbf{x} + \int_{\partial\Omega_N} g_N d\mathbf{s} = 0.$$

The equation (1.6) has a unique solution p if $\partial\Omega_D$ has nonzero measure. Otherwise p is unique up to a constant. Without loss of generality, we assume that $g_D = 0$.

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1.3.1 A Standard Variational Formulation and Finite Element Spaces

Let

$$H_{0,D}^1(\Omega) = \{v | v \in H^1(\Omega), v|_{\partial\Omega_D} = 0\},$$

and standard variational formulation of (1.6) is: find $p \in H_{0,D}^1$ such that

$$\int_\Omega a\nabla p \cdot \nabla v d\mathbf{x} = \int_\Omega f v d\mathbf{x} + \int_{\partial\Omega_N} g_N v ds, \quad \forall v \in H_{0,D}^1. \tag{1.8}$$

We can use a conforming continuous piecewise linear finite element function space $\widehat{\mathbf{W}}$, of functions which vanish on $\partial\Omega_D$, to approximate $H_{0,D}^1$. Then the finite element discrete problem of (1.8) is: find $p \in \widehat{\mathbf{W}}$ such that

$$\int_\Omega a\nabla p_h \cdot \nabla v_h d\mathbf{x} = \int_\Omega f v_h d\mathbf{x} + \int_{\partial\Omega_N} g_N v_h ds, \quad \forall v_h \in \widehat{\mathbf{W}}, \tag{1.9}$$

and the matrix form is

$$A p_h = F_h, \tag{1.10}$$

where A is symmetric and positive definite if $\partial\Omega_D$ has nonzero measure, otherwise it is symmetric and positive semidefinite.

1.3.2 A Mixed Formulation

Assume that we are interested in computing $-a\nabla p$ as is often required in flow in porous media. If we use the standard formulation (1.8), we could first compute p and then use finite differences or the gradient to approximate $-a\nabla p$. This approach will introduce additional error for $-a\nabla p$. Moreover, $-a\nabla p$ can be continuous even if a has large jumps and then ∇p has large jumps too. Therefore, we introduce the velocity $\mathbf{u}$:

$$\mathbf{u} = -a\nabla p,$$

and call p the pressure. We compute $\mathbf{u}$ directly by rewriting (1.6) for the velocity $\mathbf{u}$ and the pressure p as follows:

$$\begin{cases} \mathbf{u} = -a\nabla p & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = f & \text{in} \quad \Omega, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{in} \quad \partial\Omega_N, \\ p = 0 & \text{in} \quad \partial\Omega_D. \end{cases} \tag{1.11}$$

Let $c(\mathbf{x}) = a(\mathbf{x})^{-1}$. We now introduce a mixed and a hybrid finite element methods.

We assume $\partial\Omega_N = \partial\Omega$, $g_N = 0$, and f has mean value zero, in this subsection for convenience.

The weak form of (1.11) is as follows: find $\mathbf{u} \in H_0(\text{div}, \Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = \quad 0, & \forall \mathbf{v} \in H_0(\text{div}, \Omega), \\ b(\mathbf{u}, q) & = \quad -\int_\Omega fq d\mathbf{x}, & \forall q \in L_0^2(\Omega), \end{cases} \tag{1.12}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}^T c(\mathbf{x})\mathbf{v} d\mathbf{x}$ and $b(\mathbf{u}, q) = -\int_\Omega (\nabla \cdot \mathbf{u})q d\mathbf{x}$.

We can then use the Raviart-Thomas finite element spaces to approximate $H(\text{div})$. These spaces are conforming in $H(\text{div})$ and were introduced in [75] for two dimensional cases and extended to three dimensions in [71]. See also [21, III.3.1].

Let K be a triangle or tetrahedral element of a triangulation $\mathcal{T}_h$ of Ω. Then we define

$$RT_k(K) := (P_k(K))^n + \mathbf{x}\tilde{P}_{k-1}(K), \quad k \geq 1,$$

where $\mathbf{x}$ is the position vector, $P_k(K)$ is the set of polynomials defined on K with degree at most k, $\tilde{P}_{k-1}$ is the subspace of P_{k-1} of homogeneous polynomials, and n is the dimension of K.

A function $\mathbf{u} \in RT_k(K)$ is uniquely defined by degrees of freedom associated

with each edge $(n = 2)$ or face f $(n = 3)$

$$\int_f \mathbf{u} \cdot \mathbf{n} p, \quad p \in P_{k-1}(f).$$

We add the following degrees of freedom for $k > 1$

$$\int_K \mathbf{u} \cdot \mathbf{p}, \quad \mathbf{p} \in P_{k-2}(K)^n.$$

Then we define

$$RT_k^h(\Omega) := \{\mathbf{u} \in H(\mathrm{div}\,, \Omega) | \mathbf{u}_K \in RT_k(K), K \in \mathcal{T}_h\},$$

$$RT_{k;0}^h(\Omega) := \{\mathbf{u} \in H_0(\mathrm{div}\,, \Omega) | \mathbf{u}_K \in RT_k(K), K \in \mathcal{T}_h\}.$$

Let $\widehat{\mathbf{W}} = RT_{1,0}^h(\Omega)$ and let Q be the space of piecewise constants with a zero mean value, which are finite dimensional subspaces of $H_0(\mathrm{div}\,, \Omega)$ and $L_0^2(\Omega)$, respectively. The pair $\widehat{\mathbf{W}}$, Q satisfies a uniform inf-sup condition, see [21, Chapter IV. 1.2]. The finite element discrete problem is: find $\mathbf{u}_h \in \widehat{\mathbf{W}}$ and $p_h \in Q$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= 0, & \forall \mathbf{v}_h \in \widehat{\mathbf{W}}, \\ b(\mathbf{u}_h, q_h) &= -\int_\Omega f q_h d\mathbf{x}, & \forall q_h \in Q, \end{cases} \tag{1.13}$$

and the matrix form is:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}. \tag{1.14}$$

The system (1.14) is symmetric indefinite with the matrix A symmetric, positive definite. For details on the range of its negative and positive eigenvalues of (1.14), see [76].

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1.3.3 A Hybrid Formulation

We assume $\partial\Omega_D = \partial\Omega$ in this subsection for convenience.

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$.

Let

$$\widehat{\mathbf{W}} = RT_1^h(\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}_h\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$, $c_T \in \mathbb{R}$, and the normal component of $\mathbf{v}$ is continuous across the inter-element boundary.

Let

$$\widehat{\mathbf{W}}^{(i)} = RT_1^h(\Omega_i) = \{\mathbf{v} \in L^2(\Omega_i)^2 \text{ or } L^2(\Omega_i)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}_h\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$, $c_T \in \mathbb{R}$, and the normal component of $\mathbf{v}$ is continuous across the inter-element boundaries.

We also define $\mathbf{W}$ and $\mathbf{W}^{(\mathbf{i})}$ which are similar to $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{W}}^{(i)}$, respectively. However, they do not have any continuity constraints on the normal components of the functions, i.e.,

$$\mathbf{W} = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}_h\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$, $c_T \in \mathbb{R}$; and

$$\mathbf{W}^{(i)} = \{\mathbf{v} \in L^2(\Omega_i)^2 \text{ or } L^2(\Omega_i)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}_h\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$ and $c_T \in \mathbb{R}$.

We thus relax the continuity of the normal components on the element interface in $\mathbf{W}$ and $\mathbf{W}^{(i)}$. Instead, we will introduce Lagrange multipliers to enforce the

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continuity of the Raviart-Thomas space. In an implementation, as in [42, 22], we only need to use inter-element Lagrange multiplier on the subdomain interfaces.

Let $\mathcal{F}$ denote the set of edges/faces in $\mathcal{T}_h$ and denote by $\mathcal{F}^\partial$ the subset of $\mathcal{F}$ which contains the edges/faces on $\partial\Omega$. Then the Lagrange multiplier space $\widehat{\Lambda}$ is the set of functions on $\mathcal{F}\backslash\mathcal{F}^\partial$ which take constant values on individual edges/faces of $\mathcal{F}$ and vanish on $\mathcal{F}^\partial$; see [21, Section V1.2].

We can then reformulate the mixed problem (1.13) as follows: find $(\mathbf{u}, p, \lambda) \in \mathbf{W} \times Q \times \widehat{\Lambda}$ such that for all $(\mathbf{v}, q, \mu) \in \mathbf{W} \times Q \times \widehat{\Lambda}$

$$\begin{cases} \sum_{T\in\mathcal{T}} \left(\int_T \mathbf{u}^T c\mathbf{v} - \int_T \nabla\cdot\mathbf{v}pd\mathbf{x} + \int_{\partial T} \lambda\mathbf{v}\cdot\mathbf{n}_T d\mathbf{s} \right) &= 0, \\ -\sum_{T\in\mathcal{T}} \int_T q\nabla\cdot\mathbf{u} &= -\int_\Omega fqd\mathbf{x}, \\ \sum_{T\in\mathcal{T}} \int_{\partial T} \mu\mathbf{u}\cdot\mathbf{n}_T d\mathbf{s} &= 0. \end{cases} \qquad (1.15)$$

The additional function λ is naturally interpreted as an approximation to the trace of p on the boundary of the elements. A proof of the equivalence of (1.13) and (1.15) can be found in [2, 15].

Correspondingly, the matrix form of (1.15) is

$$\begin{bmatrix} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}. \qquad (1.16)$$

1.4 Preconditioned Iterative Methods

As we mentioned before in Section 1.1, the discretization of boundary value problems of elliptic partial differential equations leads to huge sparse linear systems to solve. Denote by N, the size of the linear system obtained from the discretization of the Laplacian. The optimal flop bounds for the Cholesky factorization of such systems are $O(N^{3/2})$ in two dimensions and $O(N^2)$ in three dimensions and the cost of the forward and backward substitutions are $O(N \log N)$ and $O(N^{4/3})$ in

two and three dimensions, respectively; see [38, Section 8.1] for two dimensional and [31] for three dimensional cases. So usually iterative methods are used for solving such huge sparse systems when N is very large.

Classical iterative methods are based on splitting the matrix, as in Jacobi, Gauss-Seidel, and SOR methods. Unfortunately, they usually converge very slowly. The same is true for conjugate gradient methods without preconditioning if the condition numbers of the systems are very large.

Multigrid methods provide optimal order algorithms for solving elliptic boundary value problems. In order to obtain errors comparable to the discretization error of the finite element methods, the number of operations using the full multigrid algorithms, for solving the linear systems obtained from the finite element discretization, only depends linearly on the number of the unknowns. The optimal convergence of multigrid is independent of the number of levels. Several parallel multilevel additive Schwartz preconditioners have also been developed and analyzed. Bramble, Pasciak and Xu [14] established that the condition number of their multilevel algorithm (the BPX algorithm) grows at most quadratically with the number of the levels, see also Xu [90]. Dryja and Widlund [29] obtained similar results with multilevel additive Schwarz preconditioners using abstract Schwarz theory. Peter Oswald [73] proved that the condition number of the BPX algorithm in fact is independent of the mesh sizes and the number of levels using Besov space theory. Bramble and Pasciak [8], Xu [91], and Bornemann and Yserentant [4] provided alternative proofs of Oswald's result. A class of multilevel methods was studied by Zhang, using the Schwarz framework, in [92, 93, 94]. All these results establish that the condition number of multilevel additive Schwarz operators can

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be independent of the mesh size and the number of levels. For the general theory of multigrid algorithms, see the books [48, 70, 6] and the references therein.

Geometric multigrid methods operate on predefined grid hierarchies. It is often very difficult to use structured geometric grids for large applications with complicated geometries. Domain decomposition methods do not depend on the construction of such grid hierarchies and can easily be implemented on unstructured meshes. Algebraic multigrid methods are being developed for problems without a grid hierarchy. They are based on the observation that reasonable interpolation and Galerkin-operator can be obtained from the matrices, instead of the grid hierarchies. These multigrid methods fix the smoothers such as Gauss-Seidel iterations and coarsen in the directions where the smoothers work best in smoothing the error. This process can be performed based only on the matrix and therefore these methods avoid complex geometric meshes. However, effective parallelization of multigrid methods with Gauss-Seidel smoothers is not an easy task, see [1]. The recursive smoothers can be implemented in parallel, only in a block sense, but this can lead to bad performance. Additionally, the communication especially on coarse levels can dominate the total CPU time for multigrid methods; these methods have a much lower computation to communication ratio compared with domain decomposition algorithms. In iterative substructuring algorithms, the communication is only needed for the coarse problem and the interface nodes. In [47, 46, 80, 81], and more recently in [60], we have learned to replace the local solvers of iterative substructuring algorithms by inexact multigrid methods and still keep the good convergence. This strategy therefore can significantly decrease the communication needed compared with multigrid methods and, at the same time, such methods can

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take advantage of the fast convergence of the underlying domain decomposition methods.

In this dissertation, we will mainly use two iterative methods. One is the conjugate gradient method, which is one of the Krylov space methods for symmetric and positive definite problems. The other is the Chebyshev iteration method. We introduce these two methods and their error analysis in the following subsections.

1.4.1 The Conjugate Gradient Method

The iterates of Krylov methods are constructed from the Krylov subspace and have optimal properties in different norms for different Krylov methods. For a symmetric and positive definite problem

$$Au = b, \tag{1.17}$$

where A is symmetric and positive definite, the conjugate gradient method is defined as follows, see [83, Fig C.4].

1. Initialize: $r_0 = b - Au_0$,

2. Iterate $k = 1, 2, \cdots$ until convergence

$$
\begin{aligned}
\beta_k &= <r_{k-1}, r_{k-1}> / <r_{k-2}, r_{k-2}> \quad (\beta_1 = 0) \\
p_k &= r_{k-1} + \beta_k p_{k-1} \quad (p_1 = r_0) \\
\alpha_k &= <r_{k-1}, r_{k-1}> / <p_k, Ap_k> \\
u_k &= u_{k-1} + \alpha_k p_k \\
r_k &= r_{k-1} - \alpha_k Ap_k
\end{aligned}
$$

From these formulas, we can see that the matrix A is used only for matrix-vector products. It is not necessary to form A explicitly.

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We have the following lemma on the convergence of the conjugate gradient method for a symmetric and positive definite system.

Lemma 1.8 *Let A be symmetric and positive definite. Then the conjugate gradient method satisfies the error bound*

$$\|e^k\|_A \leq 2\eta_A^k \|e^0\|_A.$$

Here the convergence factor is

$$\eta_A = \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1},$$

where $\kappa(A)$, the condition number of A, is the ratio of the largest and smallest eigenvalues of A, see [83, Lemma C.9].

Usually the linear systems obtained from discretizing PDEs have huge condition numbers. According to Lemma 1.8, many conjugate gradient iterations may be required for a certain accuracy. We therefore try to work with preconditioned systems to improve the condition number. We solve $M^{-1}Au = M^{-1}b$ instead of $Au = b$, where M^{-1} is an approximation of A^{-1}. $M^{-1}A$ should have a much better condition number than A and M^{-1} should also be symmetric, positive definite, and easy to apply to a vector. M^{-1} is called the preconditioner.

The preconditioned conjugate gradient method is given as follows, see [83, Fig C.5].

1. Initialize: $r_0 = b - Au_0$,

2. Iterate $k = 1, 2, \cdots$ until convergence

Precondition: $z_{k-1} = M^{-1}r_{k-1}$

$$\beta_k \quad = \quad <z_{k-1}, r_{k-1}> / <z_{k-2}, r_{k-2}> \quad (\beta_1 = 0)$$

$$\begin{aligned}
p_k &= z_{k-1} + \beta_k p_{k-1} \quad (p_1 = r_0) \\
\alpha_k &= <z_{k-1}, r_{k-1}> / <p_k, Ap_k> \\
u_k &= u_{k-1} + \alpha_k p_k \\
r_k &= r_{k-1} - \alpha_k Ap_k
\end{aligned}$$

We have the following lemma:

Lemma 1.9 *Let A and M be symmetric and positive definite. Then the preconditioned conjugate gradient method satisfies the error bound*

$$\|e^k\|_A \leq 2\eta_{A,M}^k \|e^0\|_A.$$

Here the convergence factor is

$$\eta_{A,M} = \frac{\sqrt{\kappa(M^{-1}A)} - 1}{\sqrt{\kappa(M^{-1}A)} + 1}.$$

1.4.2   The Chebyshev Iteration Method

The Chebyshev iteration method is a classical iterative method for solving symmetric and nonsymmetric problems. It does not need the computation of inner products that are necessary, in each iteration, for the conjugate gradient method. This property will save communication between different processors in parallel computing. However, in order to find the parameters which are needed for the algorithm, we need to know upper and lower bounds of the spectrum of the system. This is a disadvantage of the Chebyshev iteration method. We will use the Chebyshev iteration method inside the Conjugate Gradient method in one of our three-level BDDC methods since it makes analysis possible. See Chapter 3 for details.

Algorithm

We can also use Chebyshev iteration with a preconditioner M^{-1}. We need two input parameters l and u for this method, where l and u are estimates for the smallest and largest eigenvalues of $M^{-1}A$, respectively.

Let $\alpha = \frac{2}{l+u}$ and $\mu = \frac{u+l}{u-l}$. Let c_k be the value of the kth Chebyshev polynomial evaluated at μ, i.e.,

$$c_{k+1} = 2\mu c_k - c_{k-1}, \quad k = 1, 2, \cdots, \tag{1.18}$$

with

$$c_0 = 1, \text{ and } c_1 = \mu. \tag{1.19}$$

Without loss of generality, we set the initial guess:

$$u_0 = 0. \tag{1.20}$$

The Chebyshev acceleration is defined by, see [44],

$$u_1 = u_0 + \alpha z_0, \tag{1.21}$$

$$u_{k+1} = u_{k-1} + \omega_{k+1}(\alpha z_k + u_k - u_{k-1}), \quad k = 1, 2, \cdots, \tag{1.22}$$

where

$$r_k = b - Au_k, \tag{1.23}$$

$$z_k = M^{-1}r_k, \tag{1.24}$$

and

$$\omega_{k+1} = 2\mu \frac{c_k}{c_{k+1}}. \tag{1.25}$$

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Error Analysis

Denote the exact solution of (1.17) by u^*. Let $e_k = u^* - u_k$. Using (1.20), (1.21), and (1.22), we obtain

$$e_{k+1} = \omega_{k+1} Q e_k + (1 - \omega_{k+1}) e_{k-1}, \tag{1.26}$$

with

$$e_0 = u^*, \text{ and } e_1 = Q e_0, \tag{1.27}$$

where

$$Q = I - \alpha M^{-1} A. \tag{1.28}$$

The symmetrized operator $M^{-\frac{1}{2}} A M^{-\frac{1}{2}}$ has the following eigenvalue decomposition:

$$M^{-\frac{1}{2}} A M^{-\frac{1}{2}} = P \Lambda P^T, \tag{1.29}$$

where Λ is a diagonal matrix and the eigenvalues $\{\lambda_j\}$ of $M^{-\frac{1}{2}} A M^{-\frac{1}{2}}$ are its diagonal entries. P is an orthogonal matrix, and P^T is its transpose.

Let

$$P_1 = M^{-\frac{1}{2}} P. \tag{1.30}$$

We note that

$$M^{-1} A = P_1 \Lambda P_1^{-1}. \tag{1.31}$$

Then, we have,

$$Q = P_1 \Sigma P_1^{-1}, \tag{1.32}$$

where Σ is a diagonal matrix with the eigenvalues $\{\sigma_j\}$ of Q on the diagonal and

$$\sigma_j = 1 - \alpha \lambda_j. \tag{1.33}$$

Let

$$f_k = c_k P_1^{-1} e_k. \tag{1.34}$$

If we substitute (1.34) into (1.26) and (1.27), we then obtain a diagonal system of difference equations by using (1.18), (1.19), (1.25), and (1.32):

$$f_{k+1} = 2\mu\Sigma f_k - f_{k-1}, \quad k = 1, 2, \cdots, \tag{1.35}$$

with

$$f_1 = \mu\Sigma f_0, \text{ and } f_0 = P_1^{-1} u^*. \tag{1.36}$$

Solving this system, see [44], we obtain

$$f_k = \Theta P_1^{-1} u^*, \quad k = 1, 2, \cdots, \tag{1.37}$$

where Θ is a diagonal matrix with $\cosh\left(k \cosh^{-1}(\mu\sigma_j)\right)$ on its diagonal.

Using (1.34), we obtain:

$$e_k = \left(P_1\Theta P_1^{-1}\right)\frac{u^*}{c_k}, \quad k = 1, 2, \cdots. \tag{1.38}$$

Using the definition of e_k, our approximate solution after k Chebyshev iterations is given by

$$u_k = P_1 J P_1^{-1} u^*, \tag{1.39}$$

where J is a diagonal matrix with $1 - \cosh\left(k \cosh^{-1}(\mu\sigma_j)\right)/c_k$ on its diagonal.

Using (1.18) and (1.19), we obtain

$$c_k = \cosh\left(k \cosh^{-1}(\mu)\right).$$

Therefore, we have $1 - \cosh\left(k \cosh^{-1}(\mu\sigma_j)\right)/\cosh\left(k \cosh^{-1}(\mu)\right)$ as the diagonal entries of the matrix J.

From (1.38), we see that the Chebyshev iteration method converges if and only if $|\sigma_j| < 1$, i.e., $0 < \lambda_j < l + u$.

1.5 Organization of the Dissertation

The rest of the dissertation is organized as follows. We discuss several iterative substructuring methods in Chapter 2. In Chapter 3, we present our three-level BDDC methods, the corresponding theory, and numerical results, which is based on our papers [84, 87]. We then extend the two-level BDDC algorithms to flow in porous media in Chapter 4 and Chapter 5 with a mixed and a hybrid finite element discretization, which are based on our papers [85, 86], respectively.

Chapter 2

Iterative Substructuring Methods

2.1 Introduction

It is known that overlapping Schwarz domain decomposition methods with generous overlap can have a rate of convergence that is independent of the number of subdomains and the size of subdomain problems, see [83, Chapters 2 and 3]. They have been successfully applied in many fields. However, we cannot just use standard coarse space for problems with coefficients with jumps across the subdomain interfaces for these methods. With proper scaling techniques, it is possible to remove this difficulty by using iterative substructuring methods. Nonstandard coarse spaces were introduced in [77, 79] for overlapping methods to remove this difficulty, which are similar to the coarse problems for iterative substructuring. Bramble, Pasciak, and Schwarz started the mathematical development of iterative substructuring methods, for the crucial case when there are cross points of the interface between the subdomains, with a series of papers [9, 10, 11, 12] in the mid-1980s. Dryja, Smith, and Widlund in [28] introduced a large class of primal iterative substructuring methods and analyzed them by using the abstract Schwarz

theory. Among the iterative substructuring algorithms, Neumann-Neumann and FETI families are the best known and those methods have been tested in many applications.

Glowinski and Wheeler [41, 43] first used the Neumann-Neumann idea for a mixed formulation for elliptic problems. Bourgat, De Roeck, Glowinski, Le Tallec, and Vidrascu [5, 24, 25] then introduced the Neumann-Neumann family in [5, 24, 25] for the standard formulation for elliptic problems without coarse spaces. Mandel and Brezina [61], Dryja and Widlund [30], and later Le Tallec [57], then introduced coarse levels to the Neumann-Neumann methods, which gave us two-level Balancing Neumann-Neumann (BNN) methods. BNN is a hybrid Schwarz algorithm (see Section 2.5 for more details) and the second level considerably improves the performance. The condition number of a well-designed BNN preconditioned operator can be estimated by:

$$\kappa \leq C \left(1 + \log \frac{H}{h} \right)^2, \tag{2.1}$$

where H is the diameter and h is the typical mesh size of the subdomains and C is constant independent of H, h, and the coefficient a in (1.6), if it varies moderately in each subdomain. Thus, the rate of convergence can be independent of the number of subdomains and can grow only slowly with the size of the subdomain problems.

One-level FETI methods were introduced by Farhat and Roux [36] and the *Dirichlet preconditioners* were later introduced by Farhat, Mandel, and Roux in [35], which makes the number of iterations less sensitive to the number of unknowns in the local problems. Theoretical work was first carried out by Mandel and Tezaur in [65], see also [16, 17] and [52]. The condition number of the preconditioned

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FETI operator can also be bounded as in (2.1). Later, Farhat, Lesoinne, Le Tallec, Pierson, and Rixen introduced the dual-primal FETI (FETI-DP) in [33] with vertex constraints and a theoretical analysis was carried out by Mandel and Tezaur [66] for two dimensions with the same condition number bound (2.1) for the preconditioned FETI-DP operators.

An advantage of FETI-DP, compared with FETI, is that we never need to solve singular problems which makes the algorithms more robust. However, in three dimensions, vertex constraints alone are not enough to obtain the good polylogarithmic condition number bound of (2.1) due to a much weaker interpolation estimate and constraints on the averages over edges or faces are needed. Farhat, Lesoinne, and Pierson [34] provided a scalable FETI-DP algorithm for three dimensional cases using optional admissible constraints of this type. Klawonn, Widlund, and Dryja [55, 56] provided several different three-dimensional FETI-DP algorithms and established the condition number bound (2.1).

The BDDC (Balancing Domain Decomposition by Constraints) methods were first introduced by Dohrmann in [27]. Mandel and Dohrmann in [62] proved that the preconditioned BDDC operators have the condition number bound (2.1) by using the abstract Schwarz framework. The BDDC methods are similar to the balancing Neumann-Neumann algorithms. However, the coarse problem, in a BDDC algorithm, is given in terms of a set of primal constraints which is similar to that of a FETI-DP algorithm. Mandel, Dohrmann, and Tezaur in [63] established an important connection between FETI-DP and BDDC, namely that the preconditioned FETI-DP and BDDC operators have the same eigenvalues except possibly for 0 and 1. Fragakis and Papadrakakis observed this fact experimentally in [37]

for certain BNN algorithms and one-level FETI methods. Klawonn and Widlund also established connections between the one-level FETI and the BNN algorithms in [53]. Later, Li and Widlund rederived the FETI-DP and BDDC algorithms in [59] and provided a much shorter proof of the main result in [63] using a change of variables technique. Brenner and Sung also provided a proof for this result in [20]. Our presentation of the FETI-DP and BDDC algorithms will be based on the work by Li and Widlund in [59].

There is a limitation of the BNN, FETI-DP, BDDC algorithms, and other iterative substructuring algorithms, namely that the matrices of the local and coarse problems need to be factored by a direct solver at beginning of the computation. When the number of subdomains or the size of subdomain problem is large, then these direct solvers will be a bottleneck of these algorithms. The inexact solvers for iterative substructuring algorithms have been discussed in [3, 47, 46, 80, 13]. Klawonn and Widlund considered inexact solver for the one-level FETI algorithms in [51]. In [84, 87], we introduced a additional level for the BDDC algorithms to solve the coarse problem approximately while at the same time maintaining a good convergence, see also Chapter 3 for details. In [60], Li and Widlund considered solving the local problems in the BDDC algorithms by multigrid methods. Dohrmann has also developed several versions of approximate BDDC preconditioners in [26]. Klawonn and Rheinbach also recently provided and analyzed approximate FETI-DP preconditioners in [50]. Dryja also provided some inexact versions of BNN in his talk at the 16th domain decomposition conference.

The rest of the chapter is organized as follows: we first describe the problem setting in Section 2.2. Some useful operators are introduced in Section 2.3 and

Schur complement systems and discrete harmonic extensions are introduced in Section 2.4. In Sections 2.5, 2.6, and 2.7, BNN, FETI-DP, and BDDC algorithms and condition number estimates are discussed. Finally, we discuss some iterative substructuring methods using inexact solvers in Section 2.8.

2.2 Problem Setting

We again consider the second order scalar elliptic problem (1.6) in a two or three dimensional region Ω. We assume that $\partial\Omega_D = \partial\Omega$ and, for convenience, we use u instead of p as the unknown variable in this and the next chapters. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We then introduce a triangulation of all the subdomains. In our analysis, we also assume that each subdomain is a union of shape-regular coarse triangles or tetrahedra and that the number of such triangles forming an individual subdomain is uniformly bounded. Moreover, when developing theory, we assume that the fine triangulation of each subdomain is quasi uniform.

Definition 2.1 *A substructure* Ω_i *is* floating *if the intersection of its boundary with* $\partial\Omega$ *is empty.*

See also [83, Definition 4.1].

Let Γ be the interface between the subdomains and let the set of interface nodes Γ_h be defined by $\Gamma_h = (\cup_i \partial\Omega_{i,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$.

Let $\mathbf{W}^{(i)}$ be the standard finite element space of continuous, piecewise linear functions on Ω_i. We assume that these functions vanish on $\partial\Omega$. Each $\mathbf{W}^{(i)}$ can

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be decomposed into a subdomain interior part $\mathbf{W}_I^{(i)}$ and a subdomain interface part $\mathbf{W}_\Gamma^{(i)}$. The subdomain interface part $\mathbf{W}_\Gamma^{(i)}$ will be further decomposed into a primal subspace $\mathbf{W}_\Pi^{(i)}$ and a dual subspace $\mathbf{W}_\Delta^{(i)}$, i.e., $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Pi^{(i)} \bigoplus \mathbf{W}_\Delta^{(i)}$.

We denote the associated product spaces by

$$\mathbf{W} := \prod_{i=1}^N \mathbf{W}^{(i)}, \quad \mathbf{W}_\Gamma := \prod_{i=1}^N \mathbf{W}_\Gamma^{(i)}$$

$$\mathbf{W}_\Delta := \prod_{i=1}^N \mathbf{W}_\Delta^{(i)}, \quad \mathbf{W}_\Pi := \prod_{i=1}^N \mathbf{W}_\Pi^{(i)},$$

and

$$\mathbf{W}_I := \prod_{i=1}^N \mathbf{W}_I^{(i)}.$$

Correspondingly, we have

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma \text{ and } \mathbf{W}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta.$$

We will often consider elements of a product space which are discontinuous across the interface. However, the finite element approximations of the elliptic problem are continuous across Γ; we denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$.

We further introduce an interface subspace $\widetilde{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$, for which certain primal constraints are enforced. The resulting subspace of continuous functions is denoted by $\widehat{\mathbf{W}}_\Pi$. The space $\widetilde{\mathbf{W}}_\Gamma$ can be decomposed into $\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta$.

The global problem is: find $(u_I, u_\Delta, u_\Pi) \in (\mathbf{W}_I, \widehat{\mathbf{W}}_\Delta, \widehat{\mathbf{W}}_\Pi)$, such that

$$A \begin{bmatrix} u_I \\ u_\Delta \\ u_\Pi \end{bmatrix} = \begin{bmatrix} A_{II} & A_{\Delta I}^T & A_{\Pi I}^T \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Pi\Delta}^T \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Delta \\ u_\Pi \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Delta \\ f_\Pi \end{bmatrix}. \tag{2.2}$$

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This problem is assembled from subdomain problems

$$A^{(i)} \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \\ u_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \\ u_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix}. \qquad (2.3)$$

Let $u_r^{(i)} = \begin{bmatrix} u_I^{(i)} & u_\Delta^{(i)} \end{bmatrix}^T$ and let

$$A_{rr}^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}. \qquad (2.4)$$

$A^{(i)}$ can then be written as

$$\begin{bmatrix} A_{rr}^{(i)} & A_{\Pi r}^{(i)^T} \\ A_{\Pi r}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}.$$

We also denote by $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$, the right-hand side spaces corresponding to $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$, respectively.

2.3 Some Useful Operators

2.3.1 Restriction, Extension, and Scaling Operators

In order to describe the iterative substructuring algorithms, we need to introduce several restriction, extension, and scaling operators between different spaces. The restriction operator $R_\Gamma^{(i)}$ maps a vector of the space $\widehat{\mathbf{W}}_\Gamma$ to its restriction to the subdomain subspace $\mathbf{W}_\Gamma^{(i)}$. Each column of $R_\Gamma^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega_{i,h} \cap \Gamma_h$, shared by the subdomain Ω_i and its next neighbors. $\overline{R}_\Gamma^{(i)}$ is similar to $R_\Gamma^{(i)}$, and represents the restriction from $\widetilde{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \mathbf{W}_\Delta \to \mathbf{W}_\Delta^{(i)}$, is the restriction matrix which extracts the subdomain part, in the space $\mathbf{W}_\Delta^{(i)}$, of the functions in the space $\mathbf{W}_\Delta$. $R_\Pi^{(i)}$ is the restriction operator from the space $\widehat{\mathbf{W}}_\Pi$ to $\mathbf{W}_\Pi^{(i)}$. Multiplying each such element of $R_\Gamma^{(i)}$, $\overline{R}_\Gamma^{(i)}$, and $R_\Delta^{(i)}$

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with $\delta_i^\dagger(x)$ gives us $R_{D,\Gamma}^{(i)}$, $\overline{R}_{D,\Gamma}^{(i)}$, and $R_{D,\Delta}^{(i)}$, respectively. Here, we define $\delta_i^\dagger(x)$ as follows: for some $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{a_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} a_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h, \tag{2.5}$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$ and $a_j(x)$ is the coefficient of (1.6) at x in the subdomain Ω_j. They provide a partition of unity:

$$\sum_i R_\Gamma^{(i)} \delta_i^\dagger(x) \equiv 1, \quad x \in \Gamma_h. \tag{2.6}$$

Also let the scaling operator $D^{(i)}$ be a diagonal matrix with $\delta_i^\dagger(x)$ on its diagonal, where the component corresponds to an $x \in \partial\Omega_{i,h}$. In this dissertation, we assume that the coefficient a_i varies moderately in each subdomain.

Furthermore, $R_{\Gamma\Delta}$ and $R_{\Gamma\Pi}$ are the restriction operators from the space $\widetilde{\mathbf{W}}_\Gamma$ onto its subspace $\mathbf{W}_\Delta$ and $\mathbf{W}_\Pi$, respectively. $R_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ and $\overline{R}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ are the direct sums of $R_\Gamma^{(i)}$ and $\overline{R}_\Gamma^{(i)}$, respectively. $\widetilde{R}_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{W}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_\Delta^{(i)} R_{\Gamma\Delta}$. The scaled operators $R_{D,\Gamma}$ and $R_{D,\Delta}$ are the direct sums of $R_{D,\Gamma}^{(i)}$ and $R_{D,\Delta}^{(i)}$, respectively. $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta} R_{\Gamma\Delta}$.

We also use the same restriction, extension, and scaled restriction operators for the right hand side spaces $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$.

We have several important properties for the restriction, extension, and scaling operators, namely

$$R_\Gamma^T R_{D,\Gamma} = R_{D,\Gamma}^T R_\Gamma = I, \quad \widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = \widetilde{R}_{D,\Gamma}^T \widetilde{R}_\Gamma = I. \tag{2.7}$$

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2.3.2   Average and Jump Operators

We define two average operators $E_{D,1} : \mathbf{W}_\Gamma \longrightarrow \widehat{\mathbf{W}}_\Gamma$ by $E_{D,1} = R_\Gamma R_{D,\Gamma}^T$ and $E_{D,2} : \widetilde{\mathbf{W}}_\Gamma \longrightarrow \widehat{\mathbf{W}}_\Gamma$ by $E_{D,2} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$, which computes the weighted averages across the subdomain interface Γ and then distributes the averages to the boundary points of each subdomain.

We define the matrix

$$B_\Delta = [B_\Delta^{(1)}, B_\Delta^{(2)}, \ldots, B_\Delta^{(N)}], \tag{2.8}$$

which is constructed from $\{0, 1, -1\}$ such that the values of the function $u_\Gamma = [u_\Delta, u_\Pi]^T \in \mathbf{W}_\Gamma$ associated with more than one subdomain coincide when $B_\Delta u_\Delta = 0$. Multiplying the entries in B_Δ, each of which corresponds to a node x on $\partial\Omega_i$, by the scaling constant $\delta_i^\dagger(x)$ in (2.5) gives us $B_{D,\Delta}$.

Let $B_\Gamma = B_\Delta R_{\Gamma\Delta}$ and $B_{D,\Gamma} = B_{D,\Delta} R_{\Gamma\Delta}$. We define a jump operator by

$$P_D := B_{D\Gamma}^T B_\Gamma, \tag{2.9}$$

which maps $\widetilde{\mathbf{W}}_\Gamma$ into itself and computes the difference of the values in different subdomain at the interface nodes.

The average and jump operators satisfy the following important relations:

Lemma 2.1

$$E_{D,2} + P_D = I; \quad E_{D,2}^2 = E_{D,2}, P_D^2 = P_D; \quad E_{D,2}P_D = P_D E_{D,2} = 0.$$

Proof: See [59, Lemma 1].

$\square$

We also have the following property of the average operators E_D ($E_{D,1}$ and $E_{D,2}$):

Lemma 2.2

$$E_D u_\Gamma = u_\Gamma, \quad \forall u_\Gamma \in \widehat{\mathbf{W}}_\Gamma.$$

Proof: This follows the definition of E_D and (2.7).

$\square$

2.4 Schur Complement Systems and Discrete Harmonic Extensions

In a first step of many iterative substructuring algorithms, the interior unknowns $u_I^{(i)}$ in each subdomain are eliminated by direct solvers. In this step, the Schur complements with respect to $u_\Gamma^{(i)} = \begin{bmatrix} u_\Delta^{(i)} \\ u_\Pi^{(i)} \end{bmatrix}$ are formed. The resulting local Schur complement can be written as

$$S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{I\Gamma}^{(i)},$$

see (2.3). They are not always be formed, in practice.

We define $\widehat{S}_\Gamma = \sum_{i=1}^{N} R_\Gamma^{(i)^T} S^{(i)} R_\Gamma^{(i)}$ and the reduced global interface problem is given by

$$\widehat{S}_\Gamma u_\Gamma = g_\Gamma, \tag{2.10}$$

where

$$g_\Gamma = \sum_{i=1}^{N} R_\Gamma^{(i)^T} \left\{ \begin{bmatrix} f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{bmatrix} A_{II}^{(i)^{-1}} f_I^{(i)} \right\}.$$

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In practice, we never form the matrix $\widehat{S}_\Gamma$ explicitly. Instead, we store the $A_{I\Gamma}^{(i)}$ and the Cholesky factors of $A_{II}^{(i)}$. Whenever we need to do a matrix-vector multiplication with $\widehat{S}_\Gamma$, we just solve a Dirichlet problem in each subdomain in addition to doing some sparse matrix-vector multiplications.

The important subspace, which is directly related to the Schur complements, is the discrete harmonic function space. We say that a function $u^{(i)}$, defined on Ω_i, is discrete harmonic on Ω_i if

$$A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0. \tag{2.11}$$

From the definition, we can see that $u^{(i)}$ is fully determined by $u_\Gamma^{(i)}$, the value of $u^{(i)}$ on $\partial\Omega_i$. We use the notation $u^{(i)} := \mathcal{H}_i(u_\Gamma^{(i)})$ and call $\mathcal{H}_i$ the discrete harmonic extension operator on Ω_i. We denote the piecewise harmonic extension operator to all of Ω by $\mathcal{H}(u_\Gamma)$.

We have the following properties of the discrete harmonic extension and discrete harmonic functions, see [83, Lemma 4.9 and Lemma 4.10].

Lemma 2.3 *Let $u_\Gamma^{(i)}$ be the restriction of a finite element function u to $\partial\Omega_i \cap \Gamma$. Then, the discrete harmonic extension $w^{(i)} = \mathcal{H}_i(u_\Gamma^{(i)})$ of $u_\Gamma^{(i)}$ into Ω_i satisfies*

$$w^{(i)^T} A^{(i)} w^{(i)} = \min_{v^{(i)}|_{\partial\Omega_i \cap \Gamma} = u_\Gamma^{(i)}} v^{(i)^T} A^{(i)} v^{(i)}$$

and

$$u_\Gamma^{(i)^T} S^{(i)} u_\Gamma^{(i)} = w^{(i)^T} A^{(i)} w^{(i)}.$$

Analogously, if u_Γ is the restriction of a finite element function u to Γ, the piecewise discrete harmonic extension $w = \mathcal{H}(u_\Gamma)$ of u_Γ into the interior of the subdomains

satisfies

$$w^T A w = \min_{v|_\Gamma = u_\Gamma} v^T A v$$

and

$$w^T A w = u_\Gamma^T \widehat{S}_\Gamma u_\Gamma.$$

Lemma 2.4 *Let u be discrete harmonic. Then, for a floating subdomain Ω_i, there exist positive constants c and C, independent of h and H, such that*

$$c \, \|u_\Gamma\|^2_{H^{1/2}(\partial\Omega_i)} \leq \|u\|^2_{H^1(\Omega_i)} \leq C \, \|u_\Gamma\|^2_{H^{1/2}(\partial\Omega_i)},$$

$$c \, |u_\Gamma|^2_{H^{1/2}(\partial\Omega_i)} \leq |u|^2_{H^1(\Omega_i)} \leq C \, |u_\Gamma|^2_{H^{1/2}(\partial\Omega_i)}.$$

Consequently,

$$c \, a_i |u_\Gamma|^2_{H^{1/2}(\partial\Omega_i)} \leq u_\Gamma^{(i)^T} S^{(i)} u_\Gamma^{(i)} \leq C \, a_i |u_\Gamma|^2_{H^{1/2}(\partial\Omega_i)},$$

with $u_\Gamma^{(i)}$ the restriction of u to $\partial\Omega_i \cap \Gamma$ and the constants independent of h, H, and the a_i. For a subdomain such that $\partial\Omega_i \bigcap \partial\Omega_D$ has a nonzero measure, we have, by Friedrichs' inequality,

$$c \, \|u\|^2_{H^1(\Omega_i)} \leq |u|^2_{H^1(\Omega_i)} \leq C \, \|u\|^2_{H^1(\Omega_i)},$$

$$c \, \|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i \cap \Gamma)} \leq \|u\|^2_{H^1(\Omega_i)} \leq C \, \|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i \cap \Gamma)},$$

and

$$c \, a_i \|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i \cap \Gamma)} \leq u_\Gamma^{(i)^T} S^{(i)} u_\Gamma^{(i)} \leq C \, a_i \|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i \cap \Gamma)}.$$

These two lemmas ensure that we can work equivalently with functions defined on Γ and with the corresponding discrete harmonic extensions defined on Ω.

2.5 Balancing Neumann-Neumann Methods

2.5.1 The Algorithm

Let $\mathbf{W}_0 \subset \widehat{\mathbf{W}}_\Gamma$ be the coarse space for a Neumann-Neumann algorithm and let R_0^T be the matrix with columns representing the basis functions of $\mathbf{W}_0$. R_0^T provides a map from $\widehat{\mathbf{W}}_\Gamma$ to $\mathbf{W}_0$. For example, we can choose a minimal coarse space $\mathbf{W}_0$ as

$$\mathbf{W}_0 = \operatorname{span}\{R_i^T \delta_i^\dagger, \ \partial\Omega_i \cap \partial\Omega = \emptyset\}. \tag{2.12}$$

Let

$$S_0 := R_0 \widehat{S}_\Gamma R_0^T$$

and the interface problem (2.10) with the Balancing Neumann-Neumann preconditioner can be written as follows:

$$M_{BNN}^{-1} \widehat{S}_\Gamma u_\Gamma = M_{BNN}^{-1} g_\Gamma, \tag{2.13}$$

where M_{BNN}^{-1} is of the form

$$R_0^T S_0^{-1} R_0 + \left(I - R_0^T S_0^{-1} R_0 \widehat{S}_\Gamma\right) \left(\sum_{i=1}^N R_{D,\Gamma}^{(i)T} S^{(i)\dagger} R_{D,\Gamma}^{(i)}\right) \left(I - \widehat{S}_\Gamma R_0^T S_0^{-1} R_0\right).$$

Let

$$P_i = R_{D,\Gamma}^{(i)T} S^{(i)\dagger} R_{D,\Gamma}^{(i)} \widehat{S}_\Gamma, \quad P_0 = R_0^T S_0^{-1} R_0 \widehat{S}_\Gamma.$$

We can then write the preconditioned operator as a hybrid Schwarz operator

$$M_{BNN}^{-1} \widehat{S}_\Gamma = P_0 + (1 - P_0) \sum_{i=1}^N P_i(I - P_0). \tag{2.14}$$

We note that, for floating subdomains, $S^{(i)}$ is not invertible. But after the operation $I - P_0$, the right hand sides of the relevant linear systems always are balanced which means that the local Neumann problems are solvable, see [83, Section 6.2].

2.5.2 Condition Number Bound

Since the preconditioned BNN operator can be written as a hybrid Schwarz operator, we can follow [83, Section 6.2] and use the abstract theory of Schwarz methods to estimate the condition number.

For $u_\Gamma \in \widehat{\mathbf{W}}_\Gamma$, by Lemma 2.2, we have a splitting of u_Γ:

$$u_\Gamma = \sum_{i=1}^{N} R_\Gamma^{(i)^T} v_i, \quad v_i := I^h(\delta_i^\dagger u_i) \in \mathbf{W}_\Gamma^{(i)}.$$

By (2.7), we have the constant $C_0 = 1$ in [83, Assumption 2.12]. Therefore the lower bound of the eigenvalues is 1.

For the upper bound, we set $w \in \mathbf{W}_\Gamma$ with $w_i = D^{(i)^{-1}} R_\Gamma^{(i)} P_i u_\Gamma$. Then $w \in range(\widehat{S}_\Gamma)$ and

$$E_{D,1} w = \sum_{i=1}^{N} P_i u_\Gamma.$$

We have the following lemma for the average operator $E_{D,1}$:

Lemma 2.5

$$|E_{D,1} w|_{\widehat{S}_\Gamma}^2 \le C(1 + \log(H/h))^2 |w|_{\widehat{S}_\Gamma}^2, \quad w \in range(\widehat{S}_\Gamma). \tag{2.15}$$

Proof. See [83, Lemma 6.3].

$\square$

By using Lemma 2.5 and some algebra, we can obtain

$$|\sum_{i=1}^{N} P_i|_{\widehat{S}_\Gamma} \le C(1 + \log(H/h))^2;$$

for details, see [83, Section 6.2.3].

Therefore the following theorem follows by using [83, Theorem 2.13].

Theorem 2.6 *For any $u_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,*

$$u_\Gamma^T M_{BNN} u_\Gamma \leq u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \leq C \left(1 + \log(H/h)\right)^2 u_\Gamma^T M_{BNN} u_\Gamma,$$

where C is independent not only of the mesh size and the number of subdomains, but also of the values a_i of the coefficient of (1.6).

2.6 FETI-DP Methods

2.6.1 The Algorithm

We present the FETI-DP algorithm as in [59].

We form a partially assembled operator $\widetilde{A}$ by assembling at the primal variables. $\widetilde{A}$ is then of the form:

$$\begin{bmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)^T} & \cdots & \cdots & \cdots & A_{\Pi I}^{(1)^T} R_\Pi^{(1)} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & \cdots & \cdots & \cdots & A_{\Pi\Delta}^{(1)^T} R_\Pi^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & A_{II}^{(N)} & A_{\Delta I}^{(N)^T} & A_{\Pi I}^{(N)^T} R_\Pi^{(N)} \\ \cdots & \cdots & \cdots & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & A_{\Pi\Delta}^{(i)^T} R_\Pi^{(i)} \\ R_\Pi^{(1)^T} A_{\Pi I}^{(1)} & R_\Pi^{(1)^T} A_{\Pi\Delta}^{(1)} & \cdots & R_\Pi^{(N)^T} A_{\Pi I}^{(N)} & R_\Pi^{(N)^T} A_{\Pi\Delta}^{(N)} & \sum_{i=1}^N R_\Pi^{(i)^T} A_{\Pi\Pi I}^{(i)} R_\Pi^{(i)} \end{bmatrix}. \tag{2.16}$$

Also we can write $\widetilde{A}$ as

$$\begin{bmatrix} A_{rr} & \widetilde{A}_{\Pi r}^T \\ \widetilde{A}_{\Pi r} & \widetilde{A}_{\Pi\Pi} \end{bmatrix}.$$

Our global system (2.2) can be written as: find $(u_I, u_\Delta, u_\Pi) \in (\mathbf{W}_I, \mathbf{W}_\Delta, \widehat{\mathbf{W}}_\Pi)$, such that

$$\widetilde{A} \begin{bmatrix} u_I^{(1)} \\ u_\Delta^{(1)} \\ \vdots \\ u_I^{(N)} \\ u_\Delta^{(N)} \\ u_\Pi \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ f_\Delta^{(1)} \\ \vdots \\ f_I^{(N)} \\ f_\Delta^{(N)} \\ \sum_{i=1}^N R_\Pi^{(i)} f_\Pi^{(i)} \end{bmatrix}, \tag{2.17}$$

under the constraints:

$$B_\Delta u_\Delta = 0,$$

where $u_I = \begin{bmatrix} u_I^{(1)} \\ \vdots \\ u_I^{(N)} \end{bmatrix}$ and $u_\Delta = \begin{bmatrix} u_\Delta^{(1)} \\ \vdots \\ u_\Delta^{(N)} \end{bmatrix}$.

We introduce a set of Lagrange multipliers $\lambda \in \mathbf{V} := range B_\Delta$ to enforce the constraints and obtain a saddle point formulation of (2.17) as: find $(u_I, u_\Delta, u_\Pi, \lambda) \in (\mathbf{W}_I, \mathbf{W}_\Delta, \widehat{\mathbf{W}}_\Pi, \mathbf{V})$, such that

$$\begin{cases} \widetilde{A} \begin{bmatrix} u_I^{(1)} \\ u_\Delta^{(1)} \\ \vdots \\ u_I^{(N)} \\ u_\Delta^{(N)} \\ u_\Pi \end{bmatrix} + B_\Delta^T \lambda = \begin{bmatrix} f_I^{(1)} \\ f_\Delta^{(1)} \\ \vdots \\ f_I^{(N)} \\ f_\Delta^{(N)} \\ \sum_{i=1}^{N} R_\Pi^{(i)} f_\Pi^{(i)} \end{bmatrix}, \\ \\ B_\Delta u_\Delta = 0. \end{cases} \tag{2.18}$$

After eliminating the variables u_I, u_Δ, and u_Π from (2.18), we obtain the FETI-DP system for the Lagrangian multipliers λ as

$$F_{FETI-DP}\lambda = B_\Gamma \widetilde{S}_\Gamma^{-1} B_\Gamma^T \lambda = -\left(d_\Lambda - \widetilde{B}_{\Lambda\Pi} S_\Pi^{-1} g_\Pi \right),$$

where

$$\widetilde{S}_\Gamma^{-1} = R_{\Gamma\Delta}^T \left(\sum_{i=1}^{N} \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T, \quad (2.19)$$

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^{N} \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{bmatrix} R_\Pi^{(i)}, \tag{2.20}$$

$$S_\Pi = \sum_{i=1}^{N} R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{bmatrix} \right\} R_\Pi^{(i)}, \tag{2.21}$$

$$d_\Lambda = -\sum_{i=1}^{N} \begin{bmatrix} 0 & B_\Delta^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \end{bmatrix},$$

and

$$\widetilde{B}_{\Lambda,\Pi} = -\sum_{i=1}^{N} \begin{bmatrix} 0 & B_\Delta^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} R_\Pi^{(i)}.$$

We note that S_Π plays the role of a coarse component and is defined by a set of constraints.

The preconditioned FETI-DP system with a Dirichlet preconditioner is of the form

$$M_{FETI-DP}^{-1} F_{FETI-DP} = -B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T \left(d_\Lambda - \widetilde{B}_{\Lambda\Pi} S_\Pi^{-1} g_\Pi \right),$$

where $M_{FETI-DP}^{-1} = B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$. See [59] for details.

2.6.2 Condition Number Bound

The lower bound of preconditioned FETI-DP operator is 1, also by a simple algebra argument. The upper bound is obtained by using the upper bound of the jump operator P_D, defined in (2.9), see [63] for two dimensions and [55, 56] for three dimensions. See also [83, Section 4].

2.7 BDDC Methods

2.7.1 The Algorithm

We define an operator $\widetilde{S}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{F}}_\Gamma$, which is of the form: given $\mathbf{u}_\Gamma = \mathbf{u}_\Pi \oplus \mathbf{u}_\Delta \in \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widetilde{\mathbf{W}}_\Gamma$, find $\widetilde{S}_\Gamma \mathbf{u}_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ by eliminating the interior variables of the

system:

$$\widetilde{A} \begin{bmatrix} u_I^{(1)} \\ u_\Delta^{(1)} \\ \vdots \\ u_I^{(N)} \\ u_\Delta^{(N)} \\ u_\Pi \end{bmatrix} = \begin{bmatrix} 0 \\ R_\Delta^{(1)} R_{\Gamma,\Delta} \widetilde{S}_\Gamma u_\Gamma \\ \vdots \\ 0 \\ R_\Delta^{(N)} R_{\Gamma,\Delta} \widetilde{S}_\Gamma u_\Gamma \\ R_{\Gamma\Pi} \widetilde{S}_\Gamma u_\Gamma \end{bmatrix}. \tag{2.22}$$

It can be proved that $\widetilde{S}_\Gamma$ defined in (2.22) is the inverse of $\widetilde{S}_\Gamma^{-1}$ which is defined in (2.19) and which appears in the FETI-DP operator, see [59]. Our definition therefore makes sense.

Since $\widetilde{S}_\Gamma$ is a partial assembled Schur complement, we can obtain the fully assembled Schur complement $\widehat{S}_\Gamma$ by a further assembly, i.e., $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma$.

Therefore, the reduced interface problem (2.10) can be written as: find $u_\Gamma \in \widehat{\mathbf{W}}_\Gamma$ such that

$$\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma = g_\Gamma.$$

The two-level preconditioned BDDC method is of the form

$$M_{BDDC}^{-1} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma = M_{BDDC}^{-1} g_\Gamma,$$

where the preconditioner $M_{BDDC}^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}$ has the following form:

$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left(\sum_{i=1}^N \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma}. \tag{2.23}$$

Here Φ and S_Π are defined in (2.20) and (2.21), respectively.

2.7.2   Condition Number Bound

We assume that the coefficient $a(x)$ of (1.6) varies moderately in each subdomain. We also assume that each subdomain is a union of shape-regular coarse triangles

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and that the number of such triangles forming an individual subdomain is uniformly bounded. Moreover, we assume that the triangulation of each subdomain is quasi uniform. We then have a good upper bound of the average operator $E_{D,2}$:

Lemma 2.7

$$|E_{D,2}u_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C \left(1 + \log\frac{H}{h}\right)^2 |u_\Gamma|^2_{\widetilde{S}_\Gamma},$$

for any $u_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$, where C is a positive constant independent of H, h, and the coefficients of (1.6).

Proof. We can use the upper bound for the jump operator P_D; see [66] for the two dimensional case with vertex constraints and [83, Lemma 6.36] for three dimensional case with vertex and edge constraints. Then apply Lemma 2.1.

$\square$

Theorem 2.8 *For any $u_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,*

$$u_\Gamma^T M_{BDDC} u_\Gamma \leq u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \leq C \left(1 + \log(H/h)\right)^2 u_\Gamma^T M_{BDDC} u_\Gamma,$$
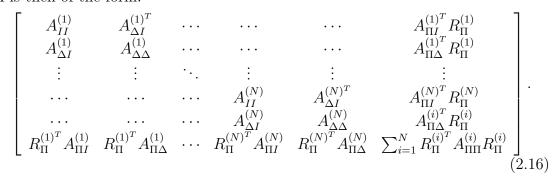
where C is independent not only of the mesh size and the number of subdomains, but also of the values a_i of the coefficient of (1.6).

Proof. We follow the proofs of [58, Theorem 1] and [87, Lemma 4.7].

Lower bound: Let

$$w_\Gamma = \left(\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}\right)^{-1} u_\Gamma \in \widehat{\mathbf{W}}_\Gamma. \tag{2.24}$$

Using the properties (2.7) and (2.24), we have,

$$u_\Gamma^T M_{BDDC} u_\Gamma = u_\Gamma^T \left(\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}\right)^{-1} u_\Gamma = u_\Gamma^T w_\Gamma$$

$$
\begin{aligned}
&= u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma = <\widetilde{R}_\Gamma u_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma} \\
&\leq \ <\widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} <\widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} \\
&= \ \left(u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma \right)^{1/2} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right)^{1/2} \\
&= \ \left(u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma \right)^{1/2} \left(u_\Gamma^T M_{BDDC} u_\Gamma \right)^{1/2}.
\end{aligned}
$$

We obtain

$$
u_\Gamma^T M_{BDDC} u_\Gamma \leq u_\Gamma^T \widehat{S}_\Gamma u_\Gamma,
$$

by canceling a common factor and squaring.

Upper bound: Using the definition of w_Γ, the Cauchy-Schwarz inequality, and Lemma 2.7, we obtain the upper bound:

$$
\begin{aligned}
u_\Gamma^T \widehat{S}_\Gamma u_\Gamma &= u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \\
&= \ <\widetilde{R}_\Gamma u_\Gamma, E_{D,2} \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma} \\
&\leq \ <\widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} <E_{D,2} \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, E_{D,2} \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} \\
&\leq \ C <\widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} (1 + \log(H/h)) |\widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma|_{\widetilde{S}_\Gamma} \\
&= \ C (1 + \log(H/h)) \left(u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma \right)^{1/2} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right)^{1/2} \\
&= \ C (1 + \log(H/h)) \left(u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \right)^{1/2} \left(u_\Gamma^T M_{BDDC} u_\Gamma \right)^{1/2}.
\end{aligned}
$$

Thus,

$$
u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \leq C \left(1 + \log(H/h) \right)^2 u_\Gamma^T M_{BDDC} u_\Gamma.
$$

$\square$

We have the following relation between the preconditioned FETI-DP and BDDC operators:

Theorem 2.9 *The preconditioned FETI-DP operator and the BDDC operator have the same eigenvalues except possibly for* 0 *and* 1.

Proof. See [63, Section 7], [59, Theorem 1], and [20, Theorem 3.9].

□

2.8 Iterative Substructuring Methods Using Inexact Solvers

The coarse or the local problems will be bottlenecks of the BNN, FETI-DP, and BDDC algorithms if the number of subdomains or the size of the subdomain problems are large. One technique for removing the difficulty related to the coarse problem is trying to choose small numbers of the primal constraints, but at the same time keep the good condition number bound. Klawonn, Widlund, and Dryja in [55] and Klawonn and Widlund in [54] proposed some ways of choosing a small number of primal constraints while maintaining a good bound for the condition number. Mandel and Sousedik [64] proposed an approach to choose a small set of constraints based on solving certain local eigenvalue problems. Another technique for removing these difficulties is by using inexact solvers for these problems. We will describe some algorithms for solving coarse or local problems approximately and still maintain a good rate of convergence.

2.8.1 The Coarse Problems

In the BDDC preconditioners and FETI-DP operators, a coarse problem is to be solved in each iteration. The size of the coarse problem is proportional to the number of subdomains and the number of primal constraints we choose. Here we

discuss approaches which use inexact solvers. It is easier to deal with the coarse problem for the BDDC algorithms than the FETI-DP algorithms since it is a part of the preconditioner. For the FETI-DP algorithms, the coarse problem appears in the operator. If we solve it inexactly, then we end up solving a different problem. A successful approach for inexact FETI-DP methods involves a return to the original saddle point problem for the primal variables and the Lagrangian multipliers and then consider preconditioners for a saddle point problem, see [50] and earlier work by Klawonn and Widlund [51] for details. Here we focus on approaches for the BDDC methods.

One approach is to introduce an additional level (or several additional levels) and to apply BDDC idea recursively to the coarse problem. This approach, the three-level BDDC method, was introduced and analyzed in [84, 87]; see also Chapter 3 for details. In that work, we assume that all the local Dirichlet and Neumann problems are solved exactly and the coarse problem is formed using exact local Neumann solvers.

We proceed as follows: we group several subdomains together to form a subregion. We could first reduce the original coarse problem to a subregion interface problem by eliminating independently the subregion interior variables, which are the primal variables on the Γ and interior to the subregions. In one of the three-level BDDC algorithms, we do not solve the subregion interface problem exactly, but replace it by one iteration of the BDDC preconditioner; Dohrmann also suggested this approach in [27]. This means that we only need to solve several subregion local problems and one coarse problem on the subregion level in each iteration. We assume that all these problems are small enough to be solved by direct solvers.

We will show in Chapter 3 that the condition number estimate for the resulting three-level preconditioned BDDC operator is bounded by

$$\kappa \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 \left(1 + \log \frac{H}{h} \right)^2, \tag{2.25}$$

where $\hat{H}$, H, and h are the typical diameters of the subregions, subdomains, and mesh of subdomains, respectively. C is constant independent of $\hat{H}$, H, h, and the coefficients of (1.6), provided that a_i varies moderately in each subregion.

In order to remove the additional factor $\left(1 + \log \frac{\hat{H}}{H} \right)^2$ in (2.25), we can use a Chebyshev iteration method to accelerate the three-level BDDC algorithms. With this device, the condition number bound is

$$\kappa \leq CC(k) \left(1 + \log \frac{H}{h} \right)^2, \tag{2.26}$$

where $C(k)$ is a constant which depends on the eigenvalues of the preconditioned coarse problem, the two parameters chosen for the Chebyshev iteration, and k, the number of the Chebyshev iterations. $C(k)$ goes to 1 as k goes to ∞. H and h are the same as before.

2.8.2 The Local Solvers

There are one Dirichlet and two Neumann local solvers (if we store the coarse basis functions, then only one Neumann local solver) in each BDDC iteration. The Dirichlet solver are used in the BDDC operator to obtain the Schur complement for the interface unknowns. We also need local Neumann solvers to assemble the coarse matrix. We call the Neumann solver for constructing the coarse matrix Neumann I and the others Neumann II, respectively. They play different roles. Here we follow Li and Widlund's work in [60].

For the Dirichlet solver, like the coarse problem for FETI-DP, we will end up solving different problems if we just replace it by an inexact solver. The way to resolve this difficulty is to use the original matrix A defined in (2.2) as the operator instead of the Schur complement for the interface. We then need to construct a BDDC preconditioner for the operator A. Similarly, we can use $\widetilde{A}^{-1}$, the inverse of the partial assembled global matrix defined in (2.16), as a part of the preconditioner. It has the form:

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{\Pi r}^T \\ I \end{bmatrix} S_\Pi^{-1} \begin{bmatrix} -\widetilde{A}_{\Pi r}A_{rr}^{-1} & I \end{bmatrix}, \qquad (2.27)$$

where S_Π is defined in (2.21).

It turns out that this preconditioner extends the interface jump by zero to the interior nodes. The condition number of the operator with this preconditioner cannot be as good as (2.1). Instead, while it is independent of the number of the subdomains, it grows linearly in H/h. If a discrete harmonic extension of the interface jump to the interior nodes is added, we recover the exact original BDDC preconditioner; see [60] for more details. Using inexact Dirichlet solvers will provide inexact discrete harmonic extensions, which can provide a good bound for the corresponding average operators. In [60], Li and Widlund use multigrid solvers to approximate the harmonic extension, which gives good approximation of the exact Dirichlet solvers.

The Neumann I problems is related to forming the coarse problem. Inexact solvers should be chosen to guarantee the positive definiteness of the corresponding coarse matrix. The inexact solver will not only effect the construction of the coarse matrix, but also the scalability of the preconditioners.

For the Neumann II problems, one can replace them by another inexact solver which is spectrally equivalent to the exact solvers, and the good condition number bound can then be retained.

2.8.3 Inexact Coarse and Local Solvers

Dohrmann proposed a preconditioner, called M_1^{-1} in [26], which assumes that the local Dirichlet solvers are exact and that

$$\alpha_1 g^T K g \leq g^T \widetilde{K} g \leq \alpha_2 g^T K g \quad \forall g, \tag{2.28}$$

where $K = diag\{A^{(i)}\}$ and $\widetilde{K} = diag\{\widetilde{A}^{(i)}\}$. Here $A^{(i)}$ is defined in (2.3) and $\widetilde{A}^{(i)}$ is similar to $A^{(i)}$ except we replace $A_{rr}^{(i)}$ by an approximation which corresponds to inexact local Neumann solvers.

We then denote by $\widetilde{K}_c$ the coarse problem, which is constructed by using inexact Neumann solvers. If the approximate coarse problem P_c satisfies:

$$\beta_1 p^T \widetilde{K}_c^{-1} p \leq p^T P_c p \leq \beta_2 p^T \widetilde{K}_c^{-1} p \quad \forall p, \tag{2.29}$$

then the condition number of the preconditioned BDDC operator with the precondictioner M_1^{-1} has the following bound:

$$\kappa \leq C \frac{\alpha_2 \max(1, \beta_2)}{\alpha_1 \min(1, \beta_1)} \left(1 + \log \frac{H}{h}\right)^2. \tag{2.30}$$

We note that if we use exact Neumann solvers and P_c is provided by the three-level approach, we then have $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 1$, and $\beta_2 = C \left(1 + \log \frac{\hat{H}}{H}\right)^2$. We then obtain the bound (2.25).

We also point out that usually the constant α_1 and α_2 are not easy to obtain directly since we can only get estimates for the inexact Neumann solvers and not for $\widetilde{K}$. In [26], Dohrmann suggests obtaining these constants from a Lanczos iteration.

Dohrmann also provides two preconditioners M_2^{-1} and M_3^{-1} that use inexact Dirichlet solvers and keep the inexact Neumann and coarse solvers the same as in M_1^{-1}.

He uses an assumption for M_2^{-1} and M_3^{-1}, namely,

$$\gamma_1 g^T K g \leq g^T \widehat{K} g \leq \gamma_2 g^T K g \quad \forall g, \tag{2.31}$$

where K is the same as in (2.28) and $\widehat{K} = diag\{\widehat{A}^{(i)}\}$. Here $\widehat{A}^{(i)}$ is similar to $A^{(i)}$ except that we replace $A_{II}^{(i)}$ by an approximation which corresponds to the inexact local Dirichlet solvers. In M_2^{-1}, the residual update in the preconditioner also use the approximate operator $\widehat{A}$ corresponding to the inexact Dirichlet solvers. Then the condition number of the preconditioned operator with the preconditioner M_2^{-1} is

$$\kappa \leq C \frac{\alpha_2 \gamma_2^3 \max(1, \beta_2)}{\alpha_1 \gamma_1^3 \min(1, \beta_1)} \left(1 + \log \frac{H}{h} \right)^2, \tag{2.32}$$

see [26]. The constants γ_1 and γ_2 are in practice estimated in the same way as α_1 and α_2. However, it is hard to obtain a condition number estimate for the preconditioned operator with the preconditioner M_3^{-1}, where the residual update in the preconditioner uses A directly, by the approach in [26]. This preconditioner is quite similar to one in [60] and in that paper the condition number of the operator, with this preconditioner, is obtained by estimating the bound for the corresponding average operators.

Dohrmann also points out that all the local inexact solvers should satisfy the null space property, see [26] for more details.

Chapter 3

Three-level BDDC

In this chapter, we introduce two three-level BDDC methods. The BDDC algorithms, previously developed for two levels [27, 62, 63], are similar to the balancing Neumann-Neumann algorithms. However, their coarse problems, in BDDC, are given in terms of sets of primal constraints and they are generated and factored by a direct solver at the beginning of the computation. The coarse components of the preconditioners can ultimately become a bottleneck if the number of subdomains is very large. We will try to remove this problem by using one or several additional levels. We introduce two three-level BDDC methods in two and three dimensional cases for vertex and edge average constraints, respectively. We also provide estimates of the condition numbers of the system with these two new preconditioners. Our presentation follows [84, 87].

The rest of the chapter is organized as follows. We introduce our first three-level BDDC method and the corresponding preconditioner $\widetilde{M}^{-1}$ in Section 3.1. We give some auxiliary results in Section 3.2. In Section 3.3, we provide an estimate of the condition number for the system with the preconditioner $\widetilde{M}^{-1}$ which is of the form $C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \left(1 + \log \frac{H}{h}\right)^2$, where $\hat{H}$, H, and h are the diameters of the

subregions, subdomains, and elements, respectively. In Section 3.4, we introduce a second three-level BDDC method which uses Chebyshev iterations. We denote the corresponding preconditioner by $\widehat{M}^{-1}$. We show that the condition number of the system with the preconditioner $\widehat{M}^{-1}$ is of the form $CC(k)\left(1 + \log\frac{H}{h}\right)^2$, where $C(k)$ is a constant depending on the eigenvalues of the preconditioned coarse problem, the two parameters chosen for the Chebyshev iteration, and k, the number of Chebyshev iterations. $C(k)$ goes to 1 as k goes to ∞. Finally, some computational results are presented in Section 3.5.

3.1 The Algorithm

For the three-level cases, we will not factor the coarse problem matrix S_Π, defined in (2.21), by a direct solver. Instead, we will solve the coarse problem approximately using ideas similar to those for the two-level preconditioners.

We decompose Ω into N subregions $\Omega^{(j)}$ with diameters $\hat{H}^{(j)}$, $j = 1, \cdots, N$. Each subregion $\Omega^{(j)}$ is the union of N_j subdomains $\Omega_i^{(j)}$ with diameters $H_i^{(j)}$. Let $\hat{H} = \max_j \hat{H}^{(j)}$ and $H = \max_{i,j} H_i^{(j)}$, for $j = 1, \cdots, N$, and $i = 1, \cdots, N_j$. We introduce the subregional Schur complement

$$S_\Pi^{(j)} = \sum_{i=1}^{N_j} R_\Pi^{(i)T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} \right\} R_\Pi^{(i)},$$

and note that the coarse problem matrix S_Π can be assembled from the $S_\Pi^{(j)}$.

Let $\widehat{\Gamma}$ be the interface between the subregions; $\widehat{\Gamma} \subset \Gamma$. We denote the vector space corresponding to the subdomain primal variables (subdomain corners for two dimensions or subdomain edge average variables for three dimensions) in $\Omega^{(i)}$, by $\mathbf{W}_c^{(i)}$. Each $\mathbf{W}_c^{(i)}$ can be decomposed into a subregion interior part $\mathbf{W}_{c,\widehat{I}}^{(i)}$ and

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a subregion interface part $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, i.e., $\mathbf{W}_{c}^{(i)} = \mathbf{W}_{c,\widehat{I}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, where the subregion interface part $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$ can be further decomposed into a primal subspace $\mathbf{W}_{c,\widehat{\Pi}}^{(i)}$ and a dual subspace $\mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, i.e., $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)} = \mathbf{W}_{c,\widehat{\Pi}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}^{(i)}$. We denote the associated product spaces by $\mathbf{W}_{c} := \prod_{i=1}^{N} \mathbf{W}_{c}^{(i)}$, $\mathbf{W}_{c,\widehat{\Gamma}} := \prod_{i=1}^{N} \mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Delta}} := \prod_{i=1}^{N} \mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Pi}} := \prod_{i=1}^{N} \mathbf{W}_{c,\widehat{\Pi}}^{(i)}$, and $\mathbf{W}_{c,\widehat{I}} := \prod_{i=1}^{N} \mathbf{W}_{c,\widehat{I}}^{(i)}$. Correspondingly, we have $\mathbf{W}_{c} = \mathbf{W}_{c,\widehat{I}} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}}$, and $\mathbf{W}_{c,\widehat{\Gamma}} = \mathbf{W}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}$. We denote by $\widehat{\mathbf{W}}_{c}$ the subspace of $\mathbf{W}_{c}$ of functions that are continuous across $\widehat{\Gamma}$.

We next introduce an interface subspace $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \subset \mathbf{W}_{c,\widehat{\Gamma}}$, for which primal constraints are enforced. Here, we only consider vertex constraints for two dimensions and edge average constraints for three dimensions. For the three dimensional case, we need to change the variables for all the local coarse matrices corresponding to the edge average constraints. From now on, we assume all the matrices and variables have been changed if it is necessary. The continuous primal subspace is denoted by $\widehat{\mathbf{W}}_{c,\widehat{\Pi}}$. The space $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ can be decomposed into $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} = \widehat{\mathbf{W}}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}$.

In the three-level BDDC algorithm, we need to introduce several restriction, extension, and scaling operators between different subregion spaces. The restriction operator $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ maps a vector of the space $\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$ to a vector of the subdomain subspace $\mathbf{W}_{c,\Gamma}^{(i)}$. Each column of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega^{(i)} \cap \Omega^{(j)}$, shared by the subregion Ω_{i} and certain neighboring subregions. $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$ is similar to $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ which represents the restriction from $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ to $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$. $\widehat{R}_{\widehat{\Delta}}^{(i)}$ is the restriction matrix which extracts the subregion part, in the space $\mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, of the functions in the space $\mathbf{W}_{c,\widehat{\Delta}}$. Multiplying each such element of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$, $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$, and $\widehat{R}_{\widehat{\Delta}}^{(i)}$ with $\widehat{\delta}_{i}^{\dagger}(x)$ gives us $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, $\widehat{\overline{R}}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, and $\widehat{R}_{\widehat{D},\widehat{\Delta}}^{(i)}$, respectively. Here, we define $\widehat{\delta}_{i}^{\dagger}(x)$ as follows: for $\gamma \in [1/2,\infty)$, $\widehat{\delta}_{i}^{\dagger}(x) = \frac{a_{i}^{\gamma}(x)}{\sum_{j \in \mathcal{N}_{x}} a_{j}^{\gamma}(x)}$, $x \in \partial\Omega_{H}^{(i)} \cap \widehat{\Gamma}_{H}$,

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_H^{(j)}$ and $a_j(x)$ is the coefficient of (1.6) at x in the subregion $\Omega^{(j)}$. (In our theory, we assume the a_i are constant in each subregion.) Furthermore, $\widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$ and $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ are the restriction operators from the space $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ onto its subspace $\mathbf{W}_{c,\widehat{\Delta}}$ and $\mathbf{W}_{c,\widehat{\Pi}}$ respectively. $\widehat{R}_{\widehat{\Gamma}} : \widehat{\mathbf{W}}_{c,\widehat{\Gamma}} \to \mathbf{W}_{c,\widehat{\Gamma}}$ and $\widehat{\overline{R}}_{\widehat{\Gamma}} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \to \mathbf{W}_{c,\widehat{\Gamma}}$ are the direct sum of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ and $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$, respectively. $\widetilde{\widehat{R}}_{\widehat{\Gamma}} : \widehat{\mathbf{W}}_{c,\Gamma} \to \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ is the direct sum of $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ and the $\widehat{R}_{\widehat{\Delta}}^{(i)}\widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$. The scaled operators $\widehat{R}_{\widehat{D},\widehat{\Gamma}}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}}$ are the direct sums of $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}}^{(i)}$. $\widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}$ is the direct sum of $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}}\widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$.

We also denote by $\mathbf{F}_{c,\widehat{\Gamma}}$, $\widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, and $\widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$, the right-hand side spaces corresponding to $\mathbf{W}_{c,\Gamma}$, $\widehat{\mathbf{W}}_{c,\Gamma}$, and $\widetilde{\mathbf{W}}_{c,\Gamma}$, respectively, and will use the same restriction, extension, and scaled restriction operators for $\mathbf{F}_{c,\widehat{\Gamma}}$, $\widehat{\mathbf{F}}_{\Gamma}$, and $\widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$.

We define our three-level preconditioner $\widetilde{M}^{-1}$ by

$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left(\sum_{i=1}^{N} \begin{bmatrix} \mathbf{0} & R_{\Delta}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_{\Delta}^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi\widetilde{S}_{\Pi}^{-1}\Phi^T \right\} \widetilde{R}_{D,\Gamma},$$

(3.1)

cf. (2.23), where $\widetilde{S}_{\Pi}^{-1}$ is an approximation of S_{Π}^{-1} and is defined as follows: given $\Psi \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, let $\mathbf{y} = S_{\Pi}^{-1}\Psi$ and $\widetilde{\mathbf{y}} = \widetilde{S}_{\Pi}^{-1}\Psi$. Here $\Psi = \begin{bmatrix} \Psi_{\widehat{I}}^{(1)}, \cdots, \Psi_{\widehat{I}}^{(N)}, \Psi_{\widehat{\Gamma}} \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} \mathbf{y}_{\widehat{I}}^{(1)}, \cdots, \mathbf{y}_{\widehat{I}}^{(N)}, \mathbf{y}_{\widehat{\Gamma}} \end{bmatrix}^T$, and $\widetilde{\mathbf{y}} = \begin{bmatrix} \widetilde{\mathbf{y}}_{\widehat{I}}^{(1)}, \cdots, \widetilde{\mathbf{y}}_{\widehat{I}}^{(N)}, \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \end{bmatrix}^T$.

To solve $S_{\Pi}\mathbf{y} = \Psi$ by block factorization in the two-level case, we can write

$$\begin{bmatrix} S_{\Pi_{\widehat{I}\widehat{I}}}^{(1)} & \cdots & \cdots & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)^T}\widehat{R}_{\widehat{\Gamma}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & S_{\Pi_{\widehat{I}\widehat{I}}}^{(N)} & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(N)^T}\widehat{R}_{\widehat{\Gamma}}^{(N)} \\ \widehat{R}_{\widehat{\Gamma}}^{(1)^T}S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)} & \cdots & \widehat{R}_{\widehat{\Gamma}}^{(N)^T}S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(N)} & \sum_{i=1}^{N}\widehat{R}_{\widehat{\Gamma}}^{(i)^T}S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)} \end{bmatrix}\begin{bmatrix} \mathbf{y}_{\widehat{I}}^{(1)} \\ \vdots \\ \mathbf{y}_{\widehat{I}}^{(N)} \\ \mathbf{y}_{\widehat{\Gamma}} \end{bmatrix} = \begin{bmatrix} \Psi_{\widehat{I}}^{(1)} \\ \vdots \\ \Psi_{\widehat{I}}^{(N)} \\ \Psi_{\widehat{\Gamma}} \end{bmatrix}.$$

(3.2)

We have

$$\mathbf{y}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \mathbf{y}_{\widehat{\Gamma}} \right), \tag{3.3}$$

and

$$\left(\sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} (S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}) \widehat{R}_{\widehat{\Gamma}}^{(i)} \right) \mathbf{y}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}.$$

Let $T^{(i)} = S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}$ and $T = diag(T^{(1)}, \cdots, T^{(N)})$. We then introduce a partially assembled Schur complement of S_Π, $\widetilde{T} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \to \widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$ by

$$\widetilde{T} = \widehat{\overline{R}}_{\widehat{\Gamma}}^{T} T \widehat{\overline{R}}_{\widehat{\Gamma}}, \tag{3.4}$$

and define $\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, by

$$\mathbf{h}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}. \tag{3.5}$$

The reduced subregion interface problem can be written as: find $\mathbf{y}_{\widehat{\Gamma}} \in \widehat{W}_{c,\widehat{\Gamma}}$, such that

$$\widehat{\widetilde{R}}_{\widehat{\Gamma}}^{T} \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{y}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}. \tag{3.6}$$

When using the three-level preconditioner $\widetilde{M}^{-1}$, we do not solve (3.6) exactly. Instead, we replace $\mathbf{y}_{\widehat{\Gamma}}$ by

$$\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^{T} \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}}. \tag{3.7}$$

We will maintain the same relation between $\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)}$ and $\widetilde{\mathbf{y}}_{\widehat{\Gamma}}^{(i)}$, i.e.,

$$\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \right). \tag{3.8}$$

3.2 Technical Tools

In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants c and C, independent of any parameters, such that $cA \leq B \leq CA$, where $C < \infty$ and $c > 0$.

3.2.1 Two Dimensions

Lemma 3.1 *Let $\mathcal{D}$ be a square with vertices $A = (0,0)$, $B = (H,0)$, $C = (H,H)$, and $D = (0,H)$, with a quasi-uniform triangulation of mesh size h. Then, there exists a discrete harmonic function v defined on $\mathcal{D}$ such that $\|v\|_{L^\infty(\mathcal{D})} = v(A) \approx 1 + \log \frac{H}{h}$, $v(B) = v(C) = v(D) = 0$ and $|v|^2_{H^1(\mathcal{D})} \approx 1 + \log \frac{H}{h}$.*

Proof: This lemma follows from a result by Brenner and Sung [19, Lemma 4.2]. Let N be an integer and G_N be the function defined on $(0,1)$ by

$$G_N(x) = \sum_{n=1}^{N} \left(\frac{1}{4n-3} \sin\left((4n-3)\pi x \right) \right).$$

G_N is symmetric with respect to the midpoint of $(0,1)$, where it attains its maximum in absolute value. Moreover, we have:

$$|G_N|^2_{H^{1/2}_{00}(0,1)} \approx 1 + \log N,$$

and

$$\|G_N\|_{L^\infty(0,1)} = G_N(1/2) \approx 1 + \log N;$$

see [19, Lemma 3.2].

Let $[-H, 0]$ and $[0, H]$ have the mesh inherited from the quasi-uniform mesh on DA and AB respectively and let $g_h(x)$ be the nodal interpolation of $G_N(\frac{x+H}{2H})$. Then we have

$$|g_h|^2_{H_{00}^{1/2}(-H,H)} \approx 1 + \log \frac{H}{h} \text{ and } \|g_h\|_{L^\infty(0,1)} \approx 1 + \log \frac{H}{h}.$$

See [19, Corollary 3.6]. We point out that in [19, Corollary 3.6], a uniform mesh is used. But in the proof of the bound for $|\cdot|_{H_{00}^{1/2}(-H,H)}$, we only need the interpolation error estimate theorem and the fact that $H_{00}^{1/2}(-H, H)$ is the interpolation space halfway between $L^2(-H, H)$ and $H_0^1(-H, H)$. Therefore the result is still valid for a quasi-uniform mesh.

We can define v as 0 on the line segments CD and CB and by

$$v(x, 0) = g_h(x), \qquad \text{for} \quad 0 \le x \le H,$$

and

$$v(0, y) = g_h(-y), \qquad \text{for} \quad 0 \le y \le H.$$

Since v is a discrete harmonic function in $\mathcal{D}$, we have,

$$|v|^2_{H^1(\mathcal{D})} = |v|^2_{H^{1/2}(\partial \mathcal{D})} \approx |g_h|^2_{H_{00}^{1/2}(-H,H)} \approx 1 + \log \frac{H}{h}.$$

$\square$

Remark: In Lemma 3.1, we have constructed the function v for the square $\mathcal{D}$. By using similar ideas, we can easily construct a function v for other shape-regular polygons which satisfy similar properties.

Lemma 3.2 *Let V_i^H be the standard continuous piecewise linear finite element function space for a subregion $\Omega^{(i)}$ with a quasi-uniform coarse mesh with mesh*

size H. And let $V_{i,j}^h$, $j = 1, \cdots, N_i$, be the space for a subdomain $\Omega_j^{(i)}$ with a quasi-uniform fine mesh with mesh size h. Moreover, each subdomain is a union of coarse triangles with vertices on the boundary of the subdomain. Given $u \in V_i^H$, let $\hat{u}$ interpolate u at each coarse node and be the discrete $V_{i,j}^h$-harmonic extension in each subdomain $\Omega_j^{(i)}$ constrained only at the vertices of $\Omega_j^{(i)}$, $j = 1, \cdots, N_i$. Then, there exist two positive constants C_1 and C_2, which are independent of $\hat{H}$, H, and h, such that

$$C_1(1 + \log \frac{H}{h}) \left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega_j^{(i)})} \right) \leq |u|^2_{H^1(\Omega^{(i)})} \leq C_2(1 + \log \frac{H}{h}) \left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega_j^{(i)})} \right).$$

Proof: Without loss of generality, we assume that the subdomains are quadrilaterals. Denote the vertices of the subdomain $\Omega_j^{(i)}$ by a_j, b_j, c_j, and d_j, and denote the nodal values of u at these four crosspoints by $u(a_j)$, $u(b_j)$, $u(c_j)$, and $u(d_j)$, respectively. Since u is a piecewise linear function, we have,

$$|u|^2_{H^1(\Omega^{(i)})} = \sum_{j=1}^{N_i} |u|^2_{H^1(\Omega_j^{(i)})},$$

and

$$|u|^2_{H^1(\Omega_j^{(i)})} = |u - u(a_j)|^2_{H^1(\Omega_j^{(i)})} \approx C \left(\sum_{m=b,c,d} ((u(m_j) - u(a_j))^2) \right). \tag{3.9}$$

According to Lemma 3.1, we can construct three discrete harmonic functions ϕ_b, ϕ_c, and ϕ_d on $\Omega_j^{(i)}$ such that

$$\phi_b(b_j) = (u(b_j) - u(a_j)) \left(1 + \log \frac{H}{h}\right), \quad \phi_b(a_j) = \phi_b(c_j) = \phi_b(d_j) = 0,$$

$$\phi_c(c_j) = (u(c_j) - u(a_j)) \left(1 + \log \frac{H}{h}\right), \quad \phi_c(a_j) = \phi_c(b_j) = \phi_c(d_j) = 0,$$

$$\phi_d(d_j) = (u(d_j) - u(a_j)) \left(1 + \log \frac{H}{h}\right), \quad \phi_d(a_j) = \phi_d(b_j) = \phi_d(c_j) = 0,$$

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and with

$$|\phi_m|^2_{H^1(\Omega_j^{(i)})} \approx (u(m_j) - u(a_j))^2 \left(1 + \log \frac{H}{h}\right), \quad m = b, c, d. \qquad (3.10)$$

Let $v_j = \frac{1}{1+\log \frac{H}{h}}(\phi_b + \phi_c + \phi_d) + u(a_j)$; we then have $v_j(m_j) = u(m_j)$, $m = a, b, c, d$, and

$$
\begin{aligned}
|v_j|^2_{H^1(\Omega_j^{(i)})} &= \left|\frac{1}{1+\log\frac{H}{h}}(\phi_b + \phi_c + \phi_d) + u(a_j)\right|^2_{H^1(\Omega_j^{(i)})} \\
&= \left(\frac{1}{1+\log\frac{H}{h}}\right)^2 |\phi_b + \phi_c + \phi_d|^2_{H^1(\Omega_j^{(i)})} \\
&\leq 3\left(\frac{1}{1+\log\frac{H}{h}}\right)^2 \sum_{m=b,c,d} |\phi_m|^2_{H^1(\Omega_j^{(i)})} \\
&\leq \left(\frac{1}{c^{1/2}(1+\log\frac{H}{h})}\right)^2 \left(1 + \log\frac{H}{h}\right) \sum_{m=b,c,d}(u(m_j) - u(a_j))^2 \\
&\leq \frac{1}{C_1(1+\log\frac{H}{h})}|u|^2_{H^1(\Omega_j^{(i)})}. \qquad (3.11)
\end{aligned}
$$

Here, we have used (3.9) and (3.10) for the last two inequalities.

By the definition of $\hat{u}$, we have,

$$|\hat{u}|^2_{H^1(\Omega_j^{(i)})} \leq |v|^2_{H^1(\Omega_j^{(i)})} \leq \frac{1}{C_1(1+\log\frac{H}{h})}|u|^2_{H^1(\Omega_j^{(i)})}.$$

Summing over all the subdomains in the subregion $\Omega^{(i)}$, we have,

$$C_1\left(1 + \log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|\hat{u}|^2_{H^1(\Omega_j^{(i)})}\right) \leq \sum_{j=1}^{N_i}|u|^2_{H^1(\Omega_j^{(i)})} = |u|^2_{H^1(\Omega^{(i)})}.$$

This proves the first inequality.

We prove the second inequality as follows:

$$|u|^2_{H^1(\Omega^{(i)})} = \sum_{j=1}^{N_i}|u|^2_{H^1(\Omega_j^{(i)})} = \sum_{j=1}^{N_i}|u - u(a_j)|^2_{H^1(\Omega_j^{(i)})}$$

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$$\leq \; C_2 \left(\sum_{j=1}^{N_i} \max_{m=b,c,d} \left((u(m_j) - u(a_j))^2 \right) \right) \leq C_2 \left(\sum_{j=1}^{N_i} \| \hat{u} - u(a_j) \|^2_{L^\infty(\Omega_j^{(i)})} \right)$$

$$\leq \; C_2 \left(1 + \log \frac{H}{h} \right) \left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega_j^{(i)})} \right).$$

Here, we have used a standard finite element Sobolev inequality [83, Lemma 4.15].

$\square$

We next list several results for the two-level BDDC methods. To be fully rigorous, we assume that each subregion is a union of shape-regular coarse triangles and the number of such triangles forming an individual subregion is uniformly bounded. Thus, there is a quasi-uniform coarse triangulation of each subregion. Similarly, each subdomain is a union of shape-regular coarse triangles with the vertices on the boundary of the subdomain. Moreover the fine triangulation of each subdomain is quasi uniform. We can then get uniform constants C_1 and C_2 in Lemma 3.2, which work for all the subregions.

We define the interface averages operator $\widehat{E}_{\widehat{D}}$ on $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ as

$$\widehat{E}_{\widehat{D}} = \widehat{\widetilde{R}}_{\widehat{\Gamma}} \widehat{\widetilde{R}}^T_{\widehat{D},\widehat{\Gamma}}, \tag{3.12}$$

which computes the averages across the subregion interface $\widehat{\Gamma}$ and then distributes the averages to the boundary points of the subregions.

We have the following estimate for $\widehat{E}_{\widehat{D}}$:

Lemma 3.3 *Consider the two-level BDDC, we have*

$$|\widehat{E}_{\widehat{D}} \mathbf{u}_{\widehat{\Gamma}}|^2_{\widetilde{S}_{\widehat{\Gamma}}} \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 |\mathbf{u}_{\widehat{\Gamma}}|^2_{\widetilde{S}_{\widehat{\Gamma}}}, \quad \forall \mathbf{u}_{\widehat{\Gamma}} \in \widetilde{W}_{\widehat{\Gamma}},$$

where $\widetilde{S}_{\widehat{\Gamma}}$ and $\widetilde{W}_{\widehat{\Gamma}}$, which corresponds to a mesh with size H, are analogous to $\widetilde{S}_{\Gamma}$ and $\widetilde{W}_{\Gamma}$, which corresponds to a mesh with size h, respectively.

Proof: See Lemma 2.7.

$\square$

In addition, we have:

Lemma 3.4

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}},$$

for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, *where C is a positive constant independent of $\hat{H}$, H, and h. Recall that $\widetilde{T}$ is defined in (3.4).*

Proof: Denote by $\mathcal{H}^{(i)}$ the discrete harmonic extension in the subregion $\Omega^{(i)}$ with respect to $S_{\Pi}^{(i)}$, given by the values on the boundary of $\Omega^{(i)}$, i.e., $\mathcal{H}^{(i)}(w) \in \widetilde{\mathbf{W}}_c^{(i)}$ satisfies:

$$|\mathcal{H}^{(i)}(w)|_{S_{\Pi}^{(i)}} = \min_{v \in \widetilde{\mathbf{W}}_c^{(i)}, v=w \text{ on } \partial\Omega^i} a_i |v|_{S_{\Pi}^{(i)}}, \quad w \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}^{(i)}.$$

For $w \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}^{(i)}$, let $\hat{\mathcal{H}}^{(i)}(w) \in \widetilde{\mathbf{W}}_c^{(i)}$ satisfy:

$$|I_H \left(\hat{\mathcal{H}}^{(i)}(w) \right)|_{H^1(\Omega^{(i)})} = \min_{v \in \widetilde{\mathbf{W}}_c^{(i)}, v=w \text{ on } \partial\Omega^{(i)}} |I_H(v)|_{H^1(\Omega^{(i)})}, \quad w \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}^{(i)},$$

where $I_H(\cdot)$ is the nodal interpolation onto V_i^H and V_i^H is defined in Lemma 3.2.

Denote by $\tilde{\mathcal{H}}_j^{(i)}$ the discrete harmonic extension in each subdomain $\Omega_j^{(i)}$, with respect to the fine mesh with mesh size h, given the crosspoint nodal values, where $i = 1, \cdots, N$, and $j = 1, \cdots, N_i$.

We have

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} = \sum_{i=1}^{N} |\mathcal{H}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})|^2_{S^{(i)}_{\Pi}} \le \sum_{i=1}^{N} |\hat{\mathcal{H}}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})|^2_{S^{(i)}_{\Pi}}$$

$$= \sum_{i=1}^{N} a_i \left(\sum_{j=1}^{N_i} |\tilde{\mathcal{H}}^{(i)}_j \left(\hat{\mathcal{H}}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^{(i)}_j)} \right).$$

Here we have used the definitions of $\mathcal{H}$, $\hat{\mathcal{H}}$, $\tilde{\mathcal{H}}$, and $S^{(i)}_{\Pi}$.

By Lemma 3.2,

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \le \sum_{i=1}^{N} a_i \left(\sum_{j=1}^{N_i} |\tilde{\mathcal{H}}^{(i)}_j \left(\hat{\mathcal{H}}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^{(i)}_j)} \right)$$

$$\le \frac{1}{C_1(1 + \log\frac{H}{h})} \sum_{i=1}^{N} a_i \left(|I_H \left(\hat{\mathcal{H}}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^i)} \right)$$

$$= \frac{1}{C_1(1 + \log\frac{H}{h})} |\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{S}_{\widehat{\Gamma}}}.$$

Using Lemma 3.3, we obtain

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \le \frac{1}{C_1(1 + \log\frac{H}{h})} |\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{S}_{\widehat{\Gamma}}}$$

$$\le \frac{C}{C_1(1 + \log\frac{H}{h})} \left(1 + \log\frac{\hat{H}}{H} \right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{S}_{\widehat{\Gamma}}}$$

$$= \frac{C}{C_1(1 + \log\frac{H}{h})} \left(1 + \log\frac{\hat{H}}{H} \right)^2 \left(\sum_{i=1}^{N} a_i |I_H \left(\hat{\mathcal{H}}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^{(i)})} \right)$$

$$\le \frac{C}{C_1(1 + \log\frac{H}{h})} \left(1 + \log\frac{\hat{H}}{H} \right)^2 \left(\sum_{i=1}^{N} a_i |I_H \left(\mathcal{H}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^{(i)})} \right).$$

Here we have used the definition of $\mathcal{H}$ and $\hat{\mathcal{H}}$ again.

By Lemma 3.2 and the definition of $\mathcal{H}$, we have

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\Gamma}|^2_{\widetilde{T}} \le \frac{C}{C_1(1 + \log\frac{H}{h})} \left(1 + \log\frac{\hat{H}}{H} \right)^2 \left(\sum_{i=1}^{N} a_i |I_H \left(\mathcal{H}^{(i)}(\widehat{\overline{R}}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}) \right) |^2_{H^1(\Omega^{(i)})} \right)$$

$$\leq \frac{C}{C_1(1 + \log \frac{H}{h})} \left(1 + \log \frac{\hat{H}}{H}\right)^2 C_2 \left(1 + \log \frac{H}{h}\right) \cdot$$

$$\left(\sum_{i=1}^{N} a_i \sum_{j=1}^{N_i} \left(|\tilde{\mathcal{H}}_j^{(i)} \left(\mathcal{H}^{(i)}(\widehat{\overline{R}}_{\hat{\Gamma}}^{(i)} \mathbf{w}_{\hat{\Gamma}})\right)|_{H^1(\Omega_j^{(i)})}^2\right)\right)$$

$$= \frac{CC_2}{C_1} \left(1 + \log \frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N} |\mathcal{H}^{(i)}(\widehat{\overline{R}}_{\hat{\Gamma}}^{(i)} \mathbf{w}_{\hat{\Gamma}})|_{S_\Pi^{(i)}}^2\right)$$

$$= \frac{CC_2}{C_1} \left(1 + \log \frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\hat{\Gamma}}|_{\widetilde{T}}^2.$$

$\square$

3.2.2 Three Dimensions

Lemma 3.5 *Let $\mathcal{D}$ be a cube with vertices $A_1 = (0, 0, 0)$, $B_1 = (H, 0, 0)$, $C_1 = (H, H, 0)$, $D_1 = (0, H, 0)$, $A_2 = (0, 0, H)$, $B_2 = (H, 0, H)$, $C_2 = (H, H, H)$ and $D_2 = (0, H, H)$ with a quasi-uniform triangulation of mesh size h. Then, there exists a discrete harmonic function v defined in $\mathcal{D}$ such that $\bar{v}_{A_1 B_1} \approx 1 + \log \frac{H}{h}$, where $\bar{v}_{A_1 B_1}$ is the average of v over the edge $A_1 B_1$, $|v|_{H^1(\mathcal{D})}^2 \approx H \left(1 + \log \frac{H}{h}\right)$, and v has a zero average over the other edges.*

Proof: Again we will use a result by Brenner and He [18, Lemma 4.2]. Let N be an integer and G_N the function defined on $(0, 1)$ by

$$G_N(x) = \sum_{n=1}^{N} \left(\frac{1}{4n - 3} \sin((4n - 3)\pi x)\right).$$

$G_N(x)$ is even with respect to the midpoint of $(0, 1)$, where it attains its maximum in absolute value. Moreover, we have:

$$|G_N|_{H_{00}^{1/2}(0,1)}^2 \approx 1 + \log N \text{ and } \|G_N\|_{L^2(0,1)} \approx 1;$$

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see [18, Lemma 3.7].

Let $[-H, 0]$ and $[0, H]$ have a mesh inherited from the quasi-uniform meshes on D_1A_1 and A_1B_1, respectively, and let $g_h(x)$ be the nodal interpolation of $G_N(\frac{x+H}{2H})$. Then, we have $\|g_h\|_{L^\infty(-H,H)} \approx 1 + \log\frac{H}{h}$,

$$|g_h|^2_{H_{00}^{1/2}(-H,H)} \approx 1 + \log\frac{H}{h} \quad \text{and} \quad \|g_h\|_{L^2(-H,H)} \approx H; \tag{3.13}$$

see [18, Lemma 3.7] or [84, Lemma 1].

Let $\tau_h(x)$ be a function on $[0, H]$ defined as follows:

$$\tau_h(x) = \begin{cases} \frac{x}{h_1} & 0 \le x \le h_1, \\ 1 & h_1 \le x \le H - h_2, \\ \frac{H-x}{h_2} & H - h_2 \le x \le H, \end{cases}$$

where h_1 and h_2 are the lengths of the two end intervals.

Then the following estimates hold:

$$\|\tau_h\|^2_{L^2(0,H)} \approx H \quad \text{and} \quad |\tau_h|^2_{H_{00}^{1/2}(0,H)} \approx 1 + \log\frac{H}{h}; \tag{3.14}$$

see [18, Lemma 3.6].

Define the discrete harmonic function v as 0 on the boundary of $\mathcal{D}$ except two open faces $A_1B_1C_1D_1$ and $A_1B_1B_2A_2$. It is defined on these two faces by

$$v(x_1, x_2, 0) = g_h(x_2)\tau_h(x_1), \quad \text{for} \quad (x_1, x_2) \in A_1B_1C_1D_1,$$

$$v(x_1, 0, x_3) = g_h(-x_3)\tau_h(x_1), \quad \text{for} \quad (x_1, x_3) \in A_1B_1B_2A_2.$$

It is clear that $\bar{v}_{A_1B_1} \approx 1 + \log\frac{H}{h}$ and that v has a zero average over the other edges. Since v is discrete harmonic in $\mathcal{D}$, we have,

$$\begin{aligned} |v|^2_{H^1(\mathcal{D})} &= |v|^2_{H^{1/2}(\partial\mathcal{D})} \\ &\approx |g_h|^2_{H_{00}^{1/2}(-H,H)}\|\tau_h\|^2_{L^2(0,H)} + |\tau_h|^2_{H_{00}^{1/2}(0,H)}\|g_h\|^2_{L^2(-H,H)} \\ &\approx H\left(1 + \log\frac{H}{h}\right), \end{aligned}$$

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where we have used (3.13), (3.14), and [18, Corollary 3.5].

$\square$

Remark: In Lemma 3.5, we have constructed the function v for a cube $\mathcal{D}$. By using similar ideas, we can construct functions v for other shape-regular polyhedra which satisfy similar properties and bounds.

Lemma 3.6 *Let $\Omega_j^{(i)}$ be the subdomains in a subregion $\Omega^{(i)}$, $j = 1, \cdots, N_i$, and $V_{i,j}^h$ be the standard continuous piecewise trilinear finite element function space in the subdomain $\Omega_j^{(i)}$ with a quasi-uniform fine mesh with mesh size h. Denote by $\mathcal{E}_k$, $k = 1 \cdots K_j$, the edges of the subdomain $\Omega_j^{(i)}$. Given the average values of u, $\bar{u}_{\mathcal{E}_k}$, over each edge, let $u \in V_{i,j}^h$ be the discrete $V_{i,j}^h$-harmonic extension in each subdomain $\Omega_j^{(i)}$ with the average values given on the edges of $\Omega_j^{(i)}$, $j = 1, \cdots, N_i$. Then, there exist two positive constants C_1 and C_2, which are independent of $\hat{H}$, H, and h, such that*

$$C_1 \left(1 + \log \frac{H}{h}\right) \left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega_j^{(i)})}\right) \leq \sum_{j=1}^{N_i} \sum_{k_1,k_2=1}^{K_j} H|\bar{u}_{\mathcal{E}_{k_1}} - \bar{u}_{\mathcal{E}_{k_2}}|^2$$

$$\leq C_2 \left(1 + \log \frac{H}{h}\right) \left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega_j^{(i)})}\right).$$

Proof: Without loss of generality, we assume that the subdomains are hexahedral. Denote the edges of the subdomain $\Omega_j^{(i)}$ by $\mathcal{E}_k$, $k = 1, \cdots, 12$, and denote the average values of u over these twelve edges by $\bar{u}_{\mathcal{E}_k}$, $k = 1, \cdots, 12$, respectively.

According to Lemma 3.5, we can construct eleven discrete harmonic functions ϕ_m, $m = 2, \cdots, 12$, on $\Omega_j^{(i)}$ such that

$$(\bar{\phi}_m)_{\mathcal{E}_k} = \begin{cases} (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})(1 + \log \frac{H}{h}) & m = k, \\ 0 & m \neq k, \end{cases}$$

and with

$$|\phi_m|^2_{H^1(\Omega^{(i)}_j)} \approx (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2 H(1 + \log \frac{H}{h}), \quad m = 2, \cdots, 12. \qquad (3.15)$$

Let $v_j = \frac{1}{1+\log \frac{H}{h}} \left(\sum_{m=2}^{12} \phi_m \right) + \bar{u}_{\mathcal{E}_1}$; we then have $(\bar{v}_j)_{\mathcal{E}_k} = \bar{u}_{\mathcal{E}_k}$, for $k = 1, \cdots, 12$, and

$$
\begin{aligned}
|v_j|^2_{H^1(\Omega^{(i)}_j)} &= |\frac{1}{1 + \log \frac{H}{h}} \left(\sum_{m=2}^{12} \phi_m \right) + \bar{u}_{\mathcal{E}_1}|^2_{H^1(\Omega^{(i)}_j)} \\
&= \left(\frac{1}{1 + \log \frac{H}{h}} \right)^2 |\sum_{m=2}^{12} \phi_m|^2_{H^1(\Omega^{(i)}_j)} \leq 11 \left(\frac{1}{1 + \log \frac{H}{h}} \right)^2 \sum_{m=2}^{12} |\phi_m|^2_{H^1(\Omega^{(i)}_j)} \\
&\leq \left(\frac{1}{C_1^{1/2}(1 + \log \frac{H}{h})} \right)^2 H \left(1 + \log \frac{H}{h} \right) \sum_{m=2}^{12} (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2 \\
&\leq \frac{1}{C_1(1 + \log \frac{H}{h})} \sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.
\end{aligned}
$$

Here, we have used (3.15) for the penultimate inequality.

By the definition of u, we have,

$$|u|^2_{H^1(\Omega^{(i)}_j)} \leq |v_j|^2_{H^1(\Omega^{(i)}_j)} \leq \frac{1}{C_1(1 + \log \frac{H}{h})} \sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.$$

Summing over all the subdomains in the subregion $\Omega^{(i)}$, we have,

$$C_1 \left(1 + \log \frac{H}{h} \right) \left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^{(i)}_j)} \right) \leq \sum_{j=1}^{N_i} \sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.$$

This proves the first inequality.

We prove the second inequality as follows:

$$
\begin{aligned}
\sum_{j=1}^{N_i} \sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2 &= \sum_{j=1}^{N_i} \sum_{k=1}^{12} H|\overline{(u - \bar{u}_{\mathcal{E}_1})}_{\mathcal{E}_k}|^2 \\
&\leq C_2 \left(\sum_{j=1}^{N_i} H \frac{1}{H} \|u - \bar{u}_{\mathcal{E}_1}\|^2_{L^2(\mathcal{E}_k)} \right) \leq C_2 \left(\sum_{j=1}^{N_i} (1 + \log \frac{H}{h}) |u|^2_{H^1(\Omega^{(i)}_j)} \right) \\
&\leq C_2 \left(1 + \log \frac{H}{h} \right) \left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^{(i)}_j)} \right).
\end{aligned}
$$

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Here, we have used a standard finite element Sobolev inequality [83, Lemma 4.30] for the second inequality and [83, Lemma 4.16] for the penultimate inequality. We complete the proof of the second inequality by using the triangle inequality.

$$\square$$

We now introduce a new mesh on each subregion; we follow [22, 79]. The purpose for introducing this mesh is to relate the quadratic form of Lemma 3.6 to one for a more conventional finite element space.

Given a subregion $\Omega^{(i)}$ and subdomains $\Omega_j^{(i)}$, $j = 1, \cdots, N_i$, let $\mathcal{T}$ be a quasi-uniform sub-triangulation of $\Omega^{(i)}$ such that its set of the vertices include the vertices and the midpoints of edges of $\Omega_j^{(i)}$. For the hexahedral case, we decompose each hexahedron into 8 hexahedra by connecting the midpoints of edges. We then partition the vertices in the new mesh $\mathcal{T}$ into two sets. The midpoints of edges are called primal and the others are called secondary. We call two vertices in the triangulation $\mathcal{T}$ adjacent if there is an edge of $\mathcal{T}$ between them, as in the standard finite element context.

Let $U_H(\Omega)$ be the continuous piecewise trilinear finite element function space with respect to the new triangulation $\mathcal{T}$. For a subregion $\Omega^{(i)}$, $U_H(\Omega^{(i)})$ and $U_H(\partial\Omega^{(i)})$ are defined as restrictions:

$$U_H(\Omega^{(i)}) = \{u|_{\Omega^{(i)}} : \ u \in U_H(\Omega)\}, \qquad U_H(\partial\Omega^{(i)}) = \{u|_{\partial\Omega^{(i)}} : \ u \in U_H(\Omega)\}.$$

We define a mapping $I_H^{\Omega^{(i)}}$ of any function ϕ, defined at the primal vertices in $\Omega^{(i)}$, to $U_H(\Omega^{(i)})$ by

$$
I_H^{\Omega^{(i)}}\phi(x) = \begin{cases}
\phi(x), \text{ if } x \text{ is a primal node;} \\[2ex]
\text{the average of the values at all adjacent primal nodes} \\
\text{on the edges of } \Omega^{(i)}, \text{ if } x \text{ is a vertex of } \Omega^{(i)}; \\[2ex]
\text{the average of the values at two adjacent primal nodes} \\
\text{on the same edge of } \Omega^{(i)}, \text{ if } x \text{ is an edge secondary node of } \Omega^{(i)}; \\[2ex]
\text{the average of the values at all adjacent primal nodes on the} \\
\text{ boundary of } \Omega^{(i)}, \text{ if } x \text{ is a face secondary boundary node of } \Omega^{(i)}; \\[2ex]
\text{the average of the values at all adjacent primal nodes} \\
\text{ if } x \text{ is a interior secondary node of } \Omega^{(i)}; \\[2ex]
\text{the result of trilinear interpolation using the vertex values,} \\
\text{ if } x \text{ is not a vertex of } \mathcal{T}.
\end{cases}
$$

$$(3.16)$$

We recall that $W_c^{(i)}$ is the discrete space of the values at the primal nodes given by the subdomain edge average values. $I_H^{\Omega^{(i)}}$ can be considered as a map from $W_c^{(i)}$ to $U_H(\Omega^{(i)})$ or as a map from $U_H(\Omega^{(i)})$ to $U_H(\Omega^{(i)})$.

Let $I_H^{\partial\Omega^{(i)}}$ be the mapping of a function ϕ defined at the primal vertices on the boundary of $\Omega^{(i)}$ to $U_H(\partial\Omega^{(i)})$ and defined by $I_H^{\partial\Omega^{(i)}}\phi = (I_H^{\Omega^{(i)}}\phi_e)|_{\partial\Omega^{(i)}}$, where ϕ_e is any function in $W_c^{(i)}$ such that $\phi_e|_{\partial\Omega^{(i)}} = \phi$. The map is well defined since the boundary values of $I_H^{\Omega^{(i)}}\phi_e$ only depend on the boundary values of ϕ_e.

Finally, let

$$
\widetilde{U}_H(\Omega^{(i)}) = \{\psi = I_H^{\Omega^{(i)}}\phi, \ \phi \in U_H(\Omega^{(i)})\}, \qquad \widetilde{U}_H(\partial\Omega^{(i)}) = \{\psi|_{\partial\Omega^{(i)}}, \ \psi \in \widetilde{U}_H(\Omega^{(i)})\}.
$$

$I_H^{\partial\Omega^{(i)}}$ also can be considered as a map from $W_{c,\widehat{\Gamma}}^{(i)}$ to $\widetilde{U}_H(\partial\Omega^{(i)})$.

Remark: We carefully define the $I_H^{\Omega^{(i)}}$ and $I_H^{\partial\Omega^{(i)}}$ so that, if the edge averages of $w_i \in W_{c,\widehat{\Gamma}}^{(i)}$ and $w_j \in W_{c,\widehat{\Gamma}}^{(j)}$ over an edge $\mathcal{E}$ are the same, we have $\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_\mathcal{E} = \overline{(I_H^{\partial\Omega^{(j)}}w_j)}_\mathcal{E}$. Here we need to use a weighted average which has a smaller weight at the two end points. But this will not effect our analysis. We could also define a weighted edge average of w_i and w_j and obtain $\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_\mathcal{E} = \overline{(I_H^{\partial\Omega^{(j)}}w_j)}_\mathcal{E}$ for the usual average.

We list some useful lemmas from [22]. For the proofs of Lemma 3.7 and Lemma 3.8, see [22, Lemma 6.1 and Lemma 6.2], respectively.

Lemma 3.7 *There exists a constant $C > 0$, independent of H and $|\Omega^{(i)}|$, the volume of $\Omega^{(i)}$, such that*

$$|I_H^{\Omega^{(i)}}\phi|_{H^1(\Omega^{(i)})} \le C|\phi|_{H^1(\Omega^{(i)})} \text{ and } \|I_H^{\Omega^{(i)}}\phi\|_{L^2(\Omega^{(i)})} \le C\|\phi\|_{L^2(\Omega^{(i)})}, \quad \forall\phi \in U_H(\Omega^{(i)}).$$

Lemma 3.8 *For $\hat{\phi} \in \widetilde{U}_H(\partial\Omega^{(i)})$,*

$$\inf_{\phi\in\widetilde{U}_H(\Omega^{(i)}),\phi|_{\partial\Omega^{(i)}}=\hat{\phi}} \|\phi\|_{H^1(\Omega^{(i)})} \approx \|\hat{\phi}\|_{H^{1/2}(\partial\Omega^{(i)})},$$

$$\inf_{\phi\in\widetilde{U}_H(\Omega^{(i)}),\phi|_{\partial\Omega^{(i)}}=\hat{\phi}} |\phi|_{H^1(\Omega^{(i)})} \approx |\hat{\phi}|_{H^{1/2}(\partial\Omega^{(i)})}.$$

Lemma 3.9 *There exist constants C_1 and $C_2 > 0$, independent of $\hat{H}$, H, h, and the coefficient of (1.6) such that for all $w_i \in W_{c,\widehat{\Gamma}}^{(i)}$,*

$$a_i C_1 |I_H^{\partial\Omega^{(i)}}w_i|_{H^{1/2}(\partial\Omega^{(i)})}^2 \le \left(1 + \log\frac{H}{h}\right)(T^{(i)}w_i, w_i) \le a_i C_2 |I_H^{\partial\Omega^{(i)}}w_i|_{H^{1/2}(\partial\Omega^{(i)})}^2,$$

where $(T^{(i)}w_i, w_i) = w_i^T T^{(i)} w_i = |w_i|_{T^{(i)}}^2$ and $T^{(i)} = S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}$.

Proof: By the definition of $T^{(i)}$, we have

$$\left(1 + \log \frac{H}{h}\right)(T^{(i)}w_i, w_i) = \left(1 + \log \frac{H}{h}\right) \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^{(i)}}=w_i} |v|^2_{S_{\Pi}^{(i)}}$$

$$= \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^{(i)}}=w_i} a_i \left(1 + \log \frac{H}{h}\right) \left(\sum_{j=1}^{N_i} \inf_{u \in V_{i,j}^h, \bar{u}_{\mathcal{E}}=v, \mathcal{E}\subset\partial\Omega_j^{(i)}} |u|^2_{H^1(\Omega_j^{(i)})}\right)$$

$$\approx \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^{(i)}}=w_i} a_i \sum_{j=1}^{N_i} \sum_{k_1,k_2=1}^{K_j} H|\bar{v}_{\mathcal{E}_{k_1}} - \bar{v}_{\mathcal{E}_{k_2}}|^2$$

$$\approx \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^{(i)}}=w_i} a_i |I_H^{\Omega^{(i)}} v|^2_{H^1(\Omega^{(i)})} \approx \inf_{\phi \in \widetilde{U}_H(\Omega^{(i)}), \phi|_{\partial\Omega^{(i)}}=I_H^{\partial\Omega^{(i)}} w_i} a_i |\phi|^2_{H^1(\Omega^{(i)})}$$

$$\approx a_i |I_H^{\partial\Omega^{(i)}} w_i|_{H^{1/2}(\partial\Omega^{(i)})}.$$

We use Lemma 3.6 for the third bound, the definitions of $I_H^{\Omega^{(i)}}$ and $I_H^{\partial\Omega^{(i)}}$ for the fourth and fifth bounds, and Lemma 3.8 for the final one.

$\square$

To be fully rigorous, we assume that there is a quasi-uniform coarse triangulation of each subregion. We can then obtain uniform constants C_1 and C_2 in Lemma 3.9, which work for all the subregions.

We define the interface averages operator $\widehat{E}_{\widehat{D}}$ as in two dimensional case (3.12).

The interface average operator $\widehat{E}_{\widehat{D}}$ in three dimensions also has the following property:

Lemma 3.10

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \leq C \left(1 + \log \frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}},$$

for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, *where* C *is a positive constant independent of* $\hat{H}$, H, h, *and the coefficients of (1.6). Here* $\widetilde{T}$ *is define in (3.4).*

Proof: Let $w_i = \widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)} \mathbf{w}_{\widehat{\Gamma}} \in W_{c,\widehat{\Gamma}}^{(i)}$. We rewrite the formula for $v := w_{\widehat{\Gamma}} - \widehat{E}_{\widehat{\Gamma}} w_{\widehat{\Gamma}}$ for an arbitrary element $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, and find that for $i = 1, \cdots, N$,

$$v_i(x) := (w_{\widehat{\Gamma}}(x) - \widehat{E}_{\widehat{\Gamma}} w_{\widehat{\Gamma}}(x))_i = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(w_i(x) - w_j(x)), \quad x \in \partial\Omega^{(i)} \cap \widehat{\Gamma}. \quad (3.17)$$

Here $\mathcal{N}_x$ is the set of indices of the subregions that have x on their boundaries.

We have

$$|\widehat{E}_{\widehat{D}} \mathbf{w}_{\widehat{\Gamma}}|_{\widetilde{T}}^2 = \sum_{i=1}^{N} |w_i - v_i|_{T^{(i)}}^2 \leq 2 \sum_{i=1}^{N} |w_i|_{T^{(i)}}^2 + 2 \sum_{i=1}^{N} |v_i|_{T^{(i)}}^2 \text{ and } |\mathbf{w}_{\widehat{\Gamma}}|_{\widetilde{T}}^2 = \sum_{i=1}^{N} |w_i|_{T^{(i)}}^2.$$

We can therefore focus on the estimate of the contribution from a single subregion $\Omega^{(i)}$ and proceed as in the proof of [83, Lemma 6.36].

We will also use the simple inequality

$$a_i \delta_j^{\dagger^2} \leq \min(a_i, a_j), \text{ for } \gamma \in [1/2, \infty). \quad (3.18)$$

By Lemma 3.9,

$$(T^{(i)} v_i, v_i) \leq C_2 \frac{1}{(1 + \log \frac{H}{h})} a_i |I_H^{\partial\Omega^{(i)}}(v_i)|_{H^{1/2}(\partial\Omega^{(i)})}^2. \quad (3.19)$$

Let $L_i = I_H^{\partial\Omega^{(i)}}(v_i)$. We have, by using a partition of unity as in [83, Lemma 6.36],

$$L_i = \sum_{\mathcal{F} \subset \partial\Omega_i} I^H(\theta_{\mathcal{F}} L_i) + \sum_{\mathcal{E} \subset \partial\Omega_i} I^H(\theta_{\mathcal{E}} L_i) + \sum_{\mathcal{V} \in \partial\Omega_i} \theta_{\mathcal{V}} L_i(\mathcal{V}),$$

where I^H is the nodal piecewise linear interpolant on the coarse mesh $\mathcal{T}$. We note that the analysis for face and edge terms is almost identical to that in [83, Lemma 6.36]. But the vertex terms are different because of $I_H^{\partial\Omega^{(i)}}$. We only need to consider the vertex term when two subregion share at least an edge. This make the analysis simpler than in the proof of [83, Lemma 6.36].

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Face Terms. First consider,

$$I^H(\theta_{\mathcal{F}} L_i) = I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^{(i)}}(\delta_j^\dagger(w_i - w_j))).$$

Similar to [83, Lemma 6.36], we obtain, by using (3.18),

$$(3.20)$$

$$a_i |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^{(i)}}(\delta_j^\dagger(w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^{(i)})}$$

$$= a_i \delta_j^{\dagger 2} |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^{(i)}}(w_i - w_j))|^2_{H^{1/2}(\partial\Omega^{(i)})}$$

$$\leq \min(a_i, a_j)|I^H(\theta_{\mathcal{F}}((I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}}) - (I_H^{\partial\Omega^{(i)}} w_j - (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}}) +$$

$$((\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}})))|^2_{H^{1/2}(\partial\Omega^{(i)})}$$

$$\leq 3 \min(a_i, a_j) \left(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})} + \right.$$

$$|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_j - (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})} +$$

$$\left. |\theta_{\mathcal{F}}((\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^{(i)})} \right).$$

By the definition of $I_H^{\partial\Omega^{(i)}}$,

$$I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_j)) = I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(j)}} w_j)) \quad \text{and} \quad (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}} = (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{F}}.$$

By [83, Lemma 4.26], the first and second terms in (3.20) can be estimated as follows:

$$\min(a_i, a_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})} +$$

$$|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_j - (\overline{I_H^{\partial\Omega^{(i)}} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})})$$

$$= \min(a_i, a_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})} +$$

$$|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^{(j)}} w_j - (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^{(i)})})$$

$$\leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 \left(a_i |I_H^{\partial\Omega^{(i)}} w_i|^2_{H^{1/2}(\partial\Omega^{(i)})} + a_j |I_H^{\partial\Omega^{(j)}} w_j|^2_{H^{1/2}(\partial\Omega^{(j)})} \right).$$

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Let $\mathcal{E} \subset \partial\mathcal{F}$. Since the edge averages of w_i and w_j are the same, by the definition of $I_H^{\partial\Omega^{(i)}}$ and $I_H^{\partial\Omega^{(j)}}$, we have $\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{E}} = \overline{(I_H^{\partial\Omega^{(j)}}w_j)}_{\mathcal{E}}$. As we have pointed out before, we use the weighted average which has a smaller weight at the two end points.

We then have

$$
\begin{aligned}
&|\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}} - \overline{(I_H^{\partial\Omega^{(j)}}w_j)}_{\mathcal{F}}|^2 \\
\leq\ & 2\left(|\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{E}} - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}}|^2 + |\overline{(I_H^{\partial\Omega^{(j)}}w_j)}_{\mathcal{E}} - \overline{(I_H^{\partial\Omega^{(j)}}w_j)}_{\mathcal{F}}|^2\right).
\end{aligned}
$$

$$(3.21)$$

It is sufficient to consider the first term on the right hand side. Using [83, Lemma 4.30], we find

$$
\begin{aligned}
&|\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{E}} - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}}|^2 \\
=\ & |\overline{(I_H^{\partial\Omega^{(i)}}w_i - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}})}_{\mathcal{E}}|^2 \leq C/\hat{H}_i \|I_H^{\partial\Omega^{(i)}}w_i - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}}\|^2_{L^2(\mathcal{E})},
\end{aligned}
$$

and, by using [83, Lemma 4.17] and the Poincaré inequality given as [83, Lemma A.17],

$$
|\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{E}} - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}}|^2 \leq C/\hat{H}_i\left(1 + \log\frac{\hat{H}}{H}\right)|I_H^{\partial\Omega^{(i)}}w_i - \overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}}|^2_{H^{1/2}(\mathcal{F})}.
$$

Combining this with the bound for $\theta_{\mathcal{F}}$ in [83, Lemma 4.26], we have:

$$
\begin{aligned}
&\min(a_i, a_j)|\theta_{\mathcal{F}}(\overline{(I_H^{\partial\Omega^{(i)}}w_i)}_{\mathcal{F}} - \overline{(I_H^{\partial\Omega^{(i)}}w_j)}_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^{(i)})} \\
\leq\ & C\left(1 + \log\frac{\hat{H}}{H}\right)^2\left(a_i|I_H^{\partial\Omega^{(i)}}w_i|^2_{H^{1/2}(\partial\Omega^{(i)})} + a_j|I_H^{\partial\Omega^{(j)}}w_j|^2_{H^{1/2}(\partial\Omega^{(j)})}\right).
\end{aligned}
$$

Edge Terms. We can develop the same estimate as in [83, Lemma 6.34]. For simplicity, we only consider an edge $\mathcal{E}$ common to four subregions $\Omega^{(i)}$, $\Omega^{(j)}$, Ω^k,

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and Ω^l.

$$a_i |I^H(\theta_{\mathcal{E}} L_i)|^2_{H^{1/2}(\partial\Omega^{(i)})}$$

$$\leq \ a_i \left(|I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^{(i)}}(\delta_j^\dagger(w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^{(i)})} + \right.$$

$$\left. |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^{(i)}}(\delta_k^\dagger(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^{(i)})} + |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^{(i)}}(\delta_l^\dagger(w_i - w_l)))|^2_{H^{1/2}(\partial\Omega^{(i)})} \right) \tag{3.22}$$

We recall that $\delta_j^\dagger$, $\delta_k^\dagger$, and $\delta_l^\dagger$ are constants.

By the definition of $I_H^{\partial\Omega^{(i)}}$, $I_H^{\partial\Omega^{(j)}}$, $I_H^{\partial\Omega^{(k)}}$, and $I_H^{\partial\Omega^{(l)}}$, we have

$$\theta_{\mathcal{E}}(I_H^{\partial\Omega^{(i)}} w_j) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(j)}} w_j), \ \ \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(i)}} w_k) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(k)}} w_k), \ \ \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(i)}} w_l) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(l)}} w_l),$$

and,

$$(\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^{(k)}} w_k})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^{(l)}} w_l})_{\mathcal{E}}.$$

We assume that $\Omega^{(i)}$ shares a face with $\Omega^{(j)}$ as well as Ω^l, and shares an edge only with Ω^k.

We consider the second term in (3.22) first. By [83, Lemma 4.19] and [83, Lemma 4.17], and (3.18), we have

$$a_i |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^{(i)}}(\delta_k^\dagger(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^{(i)})}$$

$$\leq \ Ca_i \delta_k^{\dagger 2} \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}}) - \theta_{\mathcal{E}}(I_H^{\partial\Omega^{(k)}} w_k - (\overline{I_H^{\partial\Omega^{(k)}} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})}$$

$$\leq \ 2C \left(a_i \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} + \right.$$

$$\left. a_k \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^{(k)}} w_k - (\overline{I_H^{\partial\Omega^{(k)}} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} \right)$$

$$\leq \ 2C \left(a_i \|I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})} + a_k \|I_H^{\partial\Omega^{(k)}} w_k - (\overline{I_H^{\partial\Omega^{(k)}} w_k})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})} \right)$$

$$\leq \ 2C \left(1 + \log \frac{\hat{H}}{H} \right) \left(a_i |I_H^{\partial\Omega^{(i)}} w_i|^2_{H^{1/2}(\mathcal{F}^{(i)})} + a_k |I_H^{\partial\Omega^{(k)}} w_k|^2_{H^{1/2}(\mathcal{F}^k)} \right)$$

$$\leq \ 2C \left(1 + \log \frac{\hat{H}}{H} \right) \left(a_i |I_H^{\partial\Omega^{(i)}} w_i|^2_{H^{1/2}(\partial\Omega^{(i)})} + a_k |I_H^{\partial\Omega^{(k)}} w_k|^2_{H^{1/2}(\partial\Omega^k)} \right),$$

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where $\mathcal{F}^{(i)}$ is a face of $\Omega^{(i)}$ and $\mathcal{F}^k$ is a face of Ω^k, and $\mathcal{F}^{(i)}$ and $\mathcal{F}^k$ share the edge $\mathcal{E}$.

The first term and the third term can be estimated similarly.

Vertex Terms. We can do the estimate similarly to that of the proof of [83, Lemma 6.36]. We have

$$a_i|\theta_{\mathcal{V}} L_i(\mathcal{V})|^2_{H^{1/2}(\partial\Omega_i)} = a_i|\theta_{\mathcal{V}}(I_H^{\partial\Omega^{(i)}} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega_i)}. \tag{3.23}$$

By (3.17) and the definition of $I_H^{\partial\Omega^{(i)}}$, we see that $(I_H^{\partial\Omega^{(i)}} v_i)(\mathcal{V})$ is nonzero only when two subregions share one or several edges with a common vertex $\mathcal{V}$.

In the definition of $I_H^{\Omega^{(i)}}$, we denote by $\mathcal{E}_{i,m}$, $m = 1, 2, 3 \cdots$, the edges in $\Omega^{(i)}$ which share $\mathcal{V}$. Denote by $v_{i,m}$ the primal nodes on the edges $\mathcal{E}_{i,m}$ which are adjacent to $\mathcal{V}$.

By the definition of $I_H^{\Omega^{(i)}}$, (3.23), and $|\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^{(i)})} \leq CH_i$, we have,

$$\begin{aligned}
a_i|\theta_{\mathcal{V}}(I_H^{\partial\Omega^{(i)}} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega^{(i)})} &\leq Ca_i|\sum_m v_i(v_{i,m})|^2 |\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^{(i)})} \\
&\leq Ca_i H_i \sum_m |v_i(v_{i,m})|^2. \tag{3.24}
\end{aligned}$$

Let us look at the first term in (3.24), the other terms can be estimated in the same way.

$$\begin{aligned}
&a_i H_i |v_i(v_{i,1})|^2 \\
={}& a_i H_i |\sum_{j,\mathcal{E}_{i,1}\subset\Omega_j} \delta_j^{\dagger}(w_i(v_{i,1}) - w_j(v_{i,1}))|^2 \\
\leq{}& C \sum_{j,\mathcal{E}_{i,1}\subset\Omega^{(j)}} \min(a_i, a_j) H_i |w_i(v_{i,1}) - w_j(v_{i,1}))|^2
\end{aligned}$$

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$$
= C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^{(j)}} \min(a_i, a_j) H_i |I_H^{\partial\Omega^{(i)}} w_i(v_{i,1}) - I_H^{\partial\Omega^{(j)}} w_j(v_{i,1})|^2
$$

$$
\leq C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^{(j)}} \min(a_i, a_j) H_i \left(|I_H^{\partial\Omega^{(i)}} w_i(v_{i,1}) - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}_{i,1}}|^2 + \right.
$$

$$
\left. |I_H^{\partial\Omega^{(j)}} w_j(v_{i,1}) - (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{E}_{i,1}}|^2 \right)
$$

$$
\leq C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^{(j)}} \min(a_i, a_j) \left(H_i| \left(I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}_{i,1}} \right) (v_{i,1})|^2 + \right.
$$

$$
\left. H_i| \left(I_H^{\partial\Omega^{(j)}} w_j - (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{E}_{i,1}} \right) (v_{i,1})|^2 \right)
$$

$$
\leq C \sum_{j, \mathcal{E}i,1 \subset \Omega^{(j)}} \min(a_i, a_j) \left(\|I_H^{\partial\Omega^{(i)}} w_i - (\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}_{i,1}}\|_{L^2(\mathcal{E}_{i,1})}^2 + \right.
$$

$$
\left. \|I_H^{\partial\Omega^{(j)}} w_j - (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{E}_{i,1}}\|_{L^2(\mathcal{E}_{i,1})}^2 \right)
$$

$$
\leq C \sum_{j, \mathcal{E}i,1 \subset \Omega^{(j)}} \left(1 + \log \frac{\hat{H}}{H} \right) \left(a_i|I_H^{\partial\Omega^{(i)}} w_i|_{H^{1/2}(\partial\Omega^{(i)})}^2 + a_j|I_H^{\partial\Omega^{(j)}} w_j|_{H^{1/2}(\partial\Omega^{(i)})}^2 \right).
$$

For the third equality, we use here that $v_{i,1}$ is a primary node. For the fourth inequality, we use that $(\overline{I_H^{\partial\Omega^{(i)}} w_i})_{\mathcal{E}_{i,1}} = (\overline{I_H^{\partial\Omega^{(j)}} w_j})_{\mathcal{E}_{i,1}}$. We use [83, Lemmas B.5] for the sixth inequality and [83, Lemma 4.17] for the last inequality.

Combining all face, edge, and vertex terms, we obtain

$$
a_i|I_H^{\partial\Omega^{(i)}}(v_i)|_{H^{1/2}(\partial\Omega^{(i)})}^2 \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j: \partial\Omega^{(j)} \cap \partial\Omega^{(i)} \neq \emptyset} a_j|I_H^{\partial\Omega^{(i)}}(w_j)|_{H^{1/2}(\partial\Omega^{(j)})}^2. \tag{3.25}
$$

Using (3.25), Lemma 3.9 and (3.19), we obtain

$$
(T^{(i)} v_i, v_i) = |v_i|_{T^{(i)}}^2 \leq C_2 \frac{1}{(1 + \log \frac{H}{h})} a_i|I_H^{\partial\Omega^{(i)}}(v_i)|_{H^{1/2}(\partial\Omega^{(i)})}^2
$$

$$
\leq C C_2 \frac{\left(1 + \log \frac{\hat{H}}{H} \right)^2}{(1 + \log \frac{H}{h})} \sum_{j: \partial\Omega^{(j)} \cap \partial\Omega^{(i)} \neq \emptyset} a_j|I_H^{\partial\Omega^{(i)}}(w_j)|_{H^{1/2}(\partial\Omega^{(j)})}^2
$$

$$
\leq C \frac{C_2}{C_1} \left(1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j: \partial\Omega^{(j)} \cap \partial\Omega^{(i)} \neq \emptyset} (T^{(j)} w_j, w_j)
$$

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$$= C\frac{C_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2 \sum_{j:\partial\Omega^{(j)}\cap\partial\Omega^{(i)}\neq\emptyset} |w_j|^2_{T^{(j)}}.$$

$\square$

3.3 Condition Number Estimate for the Preconditioned Operator with the Preconditioner $\widetilde{M}^{-1}$

We start with a lemma:

Lemma 3.11 *Given any* $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$, *let* $\mathbf{\Psi} = \Phi^T \widetilde{R}_{D,\Gamma}\mathbf{u}_\Gamma$. *We have,*

$$\mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi} \leq \mathbf{\Psi}^T \widetilde{S}_\Pi^{-1}\mathbf{\Psi}^T \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 \mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi}.$$

Proof: Using (3.3), (3.5), and (3.6), we have

(3.26)

$$\mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi} = \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\mathbf{y}_{\widehat{I}}^{(i)} + \mathbf{\Psi}_{\widehat{\Gamma}}^T\mathbf{y}_{\widehat{\Gamma}}$$

$$= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)}\mathbf{y}_{\widehat{\Gamma}})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^N \widehat{R}_{\widehat{\Gamma}}^{(i)^T}S_{\Pi_{\widehat{\Gamma}I}}^{(i)}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}^{(i)}\right)^T \mathbf{y}_{\widehat{\Gamma}}$$

$$= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\mathbf{y}_{\widehat{\Gamma}} = \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\left(\widetilde{R}_{\widehat{\Gamma}}^T\widetilde{T}\widetilde{\widehat{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}.$$

Using (3.8), (3.5), and (3.7), we also have

$$\mathbf{\Psi}^T \widetilde{S}_\Pi^{-1}\mathbf{\Psi} = \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\widetilde{\mathbf{y}}_{\widehat{\Gamma}^{(i)}} + \mathbf{\Psi}_{\widehat{\Gamma}}^T\widetilde{\mathbf{y}}_{\widehat{\Gamma}}$$

$$= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)}\widetilde{\mathbf{y}}_{\widehat{\Gamma}})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^N \widehat{R}_{\widehat{\Gamma}}^{(i)^T}S_{\Pi_{\widehat{\Gamma}I}}^{(i)}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}^{(i)}\right)^T \widetilde{\mathbf{y}}_{\widehat{\Gamma}}$$

$$= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\left(\widetilde{R}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}\right)\mathbf{h}_{\widehat{\Gamma}}.$$

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We only need to compare $\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}}$ and $\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}$ for any $\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$. The following estimate is established as [58, Theorem 1]. Let

$$\mathbf{w}_{\widehat{\Gamma}} = \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}} \text{ and } \mathbf{v}_{\widehat{\Gamma}} = \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}. \qquad (3.27)$$

Noting the fact that $\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} = \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widehat{\widetilde{R}}_{\widehat{\Gamma}} = I$ and using (3.27), we have,

$$\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}^T \mathbf{w}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}^T \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}}$$

$$= \mathbf{h}_{\widehat{\Gamma}}^T \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} = \left(\widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} \right)^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}}$$

$$= \mathbf{v}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} = < \mathbf{v}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} >_{\widetilde{T}} \leq < \mathbf{v}_{\widehat{\Gamma}}, \mathbf{v}_{\widehat{\Gamma}} >_{\widetilde{T}}^{1/2} < \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} >_{\widetilde{T}}^{1/2}$$

$$= \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2}.$$

We obtain that

$$\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \leq \mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}.$$

On the other hand,

$$\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} = \mathbf{w}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}$$

$$= < \mathbf{w}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \left(\widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} \right) >_{\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)} = < \mathbf{w}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >_{\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)}$$

$$\leq < \mathbf{w}_{\widehat{\Gamma}}, \mathbf{w}_{\widehat{\Gamma}} >_{\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)}^{1/2} < \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >_{\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)}^{1/2}$$

$$= \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} < \widehat{\widetilde{R}}_{\widehat{\Gamma}} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{\Gamma}} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >_{\widetilde{T}}^{1/2}$$

$$= \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} |\widehat{E}_{\widehat{D}} \mathbf{v}_{\widehat{\Gamma}}|_{\widetilde{T}}$$

$$\leq C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1} \mathbf{h}_{\widehat{\Gamma}}\right)^{1/2} |\mathbf{v}_{\widehat{\Gamma}}|_{\widetilde{T}}$$

$$= C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1} \mathbf{h}_{\widehat{\Gamma}}\right)^{1/2} \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right) \mathbf{h}_{\widehat{\Gamma}}\right)^{1/2},$$

where we use Lemma 3.10 for the penultimate inequality.

Finally we obtain that

$$\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right) \mathbf{h}_{\widehat{\Gamma}} \leq C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \left(\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1} \mathbf{h}_{\widehat{\Gamma}}\right).$$

$\square$

In order to estimate the condition number for the system with the new preconditioner $\widetilde{M}^{-1}$, we compare it to the system with the preconditioner M_{BDDC}^{-1} defined in (2.23).

Lemma 3.12 *Given any* $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M_{BDDC}^{-1} \mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma \leq C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \mathbf{u}_\Gamma^T M_{BDDC}^{-1} \mathbf{u}_\Gamma. \qquad (3.28)$$

Proof: We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M_{BDDC}^{-1} \mathbf{u}_\Gamma$$

$$= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma$$

$$+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi S_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma,$$

and

$$\mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma$$

$$= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma$$

$$+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi \widetilde{S}_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma.$$

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We obtain our result by using Lemma 3.11.

$\square$

Theorem 3.13 *The condition number for the system with the three-level precon-ditioner $\widetilde{M}^{-1}$ is bounded by $C(1 + \log \frac{\hat{H}}{H})^2(1 + \log \frac{H}{h})^2$.*

Proof: Combining the condition number bound, given in Theorem 2.8, for the two-level BDDC method, and Lemma 3.12, we find that the condition number for the three-level method is bounded by $C(1 + \log \frac{\hat{H}}{H})^2(1 + \log \frac{H}{h})^2$.

$\square$

3.4 Acceleration by the Chebyshev Iteration Method

Another approach to the three-level BDDC methods is to use an iterative method with a preconditioner to solve (3.6). Here, we consider a Chebyshev method with a fixed number of iterations and use $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)$ as a preconditioner.

Thus, we do not solve (3.6) directly. Instead, we replace $\mathbf{y}_{\widehat{\Gamma}}$ by $\mathbf{y}_{\widehat{\Gamma},k}$, where $\mathbf{y}_{\widehat{\Gamma},k}$ is the approximation of $\mathbf{y}_{\widehat{\Gamma}}$ given by a k-step Chebyshev iteration with zero initial guess.

We will maintain the same relation between $\mathbf{y}_{\widehat{I},k}^{(i)}$ and $\mathbf{y}_{\widehat{\Gamma},k}^{(i)}$ as in (3.3), i.e.,

$$\mathbf{y}_{\widehat{I},k}^{(i)} = S_{\Pi_{\widehat{II}}}^{(i)^{-1}} \left(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{II}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \mathbf{y}_{\widehat{\Gamma},k} \right). \tag{3.29}$$

Let $\mathbf{y}_k = \left[\mathbf{y}_{\widehat{I},k}^{(1)}, \cdots, \mathbf{y}_{\widehat{I},k}^{(N)}, \mathbf{y}_{\widehat{\Gamma},k} \right]^T$, and denote the corresponding new coarse prob-lem matrix by $\widehat{S}_\Pi$. Then,

$$\widehat{S}_\Pi \mathbf{y}_k = \Psi, \tag{3.30}$$

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and the new preconditioner $\widehat{M}^{-1}$ is defined by:

$$\widetilde{R}_{D,\Gamma}^T \left\{ \sum_{i=1}^{N} R_{\Gamma\Delta}^T \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} R_{\Gamma\Delta} + \Phi \widehat{S}_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma}. \tag{3.31}$$

From Subsection 1.4.2, we know that we need to estimate the smallest and largest eigenvalues of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$ to set up the Chebyshev itera-

tions. From our analysis above, we know that $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$ has a

smallest eigenvalue 1 and a largest eigenvalue bounded by $C(1+\log \frac{\hat{H}}{H})^2(1+\log \frac{H}{h})^2$.

We can use the conjugate gradient method to get an estimate for the largest eigen-

value at the beginning of the computation and to choose a proper u to guarantee

that $\lambda_j < l + u$, where λ_j are the eigenvalues of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$.

We apply the error analysis for Chebyshev iterations in Subsection 1.4.2 for our

preconditioned system $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$. We know that the Chebyshev

iteration converges and have

$$\mathbf{y}_{\widehat{\Gamma},k} = P_1 J P_1^{-1} \mathbf{y}_{\widehat{\Gamma}}, \tag{3.32}$$

which is from (1.39).

Since we choose u such that $\lambda_j < l + u$, we find that $1 - \frac{\cosh\left(k \cosh^{-1}(\mu \sigma_j)\right)}{\cosh\left(k \cosh^{-1}(\mu)\right)} > 0$,

i.e., J has positive diagonal elements.

We begin with a lemma for the condition number estimate for the second new

preconditioner $\widehat{M}^{-1}$.

Lemma 3.14 *Given any* $\mathbf{u_\Gamma} \in \widehat{\mathbf{W}}_\Gamma$, *let* $\mathbf{\Psi} = \Phi^T D_\Gamma R_\Gamma \mathbf{u_\Gamma}$. *If we choose* u *such*

that $\lambda_j < u + l$, *then there exist two constants* $C_1(k)$ *and* $C_2(k)$ *that*

$$C_1(k)\mathbf{\Psi}^T S_\Pi^{-1} \mathbf{\Psi} \leq \mathbf{\Psi}^T \widehat{S}_\Pi^{-1} \mathbf{\Psi}^T \leq C_2(k)\mathbf{\Psi}^T S_\Pi^{-1} \mathbf{\Psi}, \tag{3.33}$$

where

$$C_1(k) = \min_j \left(1 - \frac{\cosh(k \cosh^{-1}(\mu\sigma_j))}{\cosh(k \cosh^{-1}(\mu))} \right), \tag{3.34}$$

and

$$C_2(k) = \max_j \left(1 - \frac{\cosh(k \cosh^{-1}(\mu\sigma_j))}{\cosh(k \cosh^{-1}(\mu))} \right). \tag{3.35}$$

Proof: Using (3.29), (3.5), and (3.6), we have

$$
\begin{aligned}
\boldsymbol{\Psi}^T \widehat{S}_\Pi^{-1} \boldsymbol{\Psi} &= \sum_{i=1}^{N} \Psi_{\widehat{I}}^{(i)^T} \mathbf{y}_{\widehat{I},k}^{(i)} + \boldsymbol{\Psi}_{\widehat{\Gamma}}^T \mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^{N} \boldsymbol{\Psi}_{\widehat{I}}^{(i)^T} \left(S_{\Pi_{\widetilde{I}\widehat{I}}}^{(i)^{-1}} (\boldsymbol{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \mathbf{y}_{\widehat{\Gamma},k}) \right) \\
&\quad + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}I}}^{(i)} S_{\Pi_{\widetilde{I}\widehat{I}}}^{(i)^{-1}} \boldsymbol{\Psi}^{(i)} \right)^T \mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^{N} \boldsymbol{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widetilde{I}\widehat{I}}}^{(i)^{-1}} \boldsymbol{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T \mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^{N} \boldsymbol{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widetilde{I}\widehat{I}}}^{(i)^{-1}} \boldsymbol{\Psi}_{\widehat{I}}^{(i)} + \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k}. \tag{3.36}
\end{aligned}
$$

Comparing (3.36) with (3.26), we only need to compare $\mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}}$ and $\mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k}$.

Using (3.32) and (1.30), we obtain

$$
\begin{aligned}
&\mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k} \\
&= \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) P_1 J P_1^{-1} \mathbf{y}_{\widehat{\Gamma}} \\
&= \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} P J P^{-1} \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{-\frac{1}{2}} \mathbf{y}_{\widehat{\Gamma}}. \tag{3.37}
\end{aligned}
$$

Let $\mathbf{Y}_{\widehat{\Gamma}} = P^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{-\frac{1}{2}} \mathbf{y}_{\widehat{\Gamma}}$. Using (1.29), we have

$$
\mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T P^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} P \mathbf{Y}_{\widehat{\Gamma}}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T P^T P \Lambda P^T P \mathbf{Y}_{\widehat{\Gamma}}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T \Lambda \mathbf{Y}_{\widehat{\Gamma}}, \tag{3.38}
$$

and, using (3.37), we have

$$
\mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T P^T \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} PJ \mathbf{Y}_{\widehat{\Gamma}}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T P^T P \Lambda P^T PJ \mathbf{Y}_{\widehat{\Gamma}}
$$

$$
= \mathbf{Y}_{\widehat{\Gamma}}^T \Lambda J \mathbf{Y}_{\widehat{\Gamma}}. \tag{3.39}
$$

Under our assumption, J is a diagonal matrix with positive diagonal entries $\left(1 - \frac{\cosh(k \cosh^{-1}(\mu \sigma_j))}{\cosh(k \cosh^{-1}(\mu))} \right)$. Thus, we have,

$$
C_1(k) \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}} \leq \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k} \leq C_2(k) \mathbf{y}_{\widehat{\Gamma}}^T \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}}. \tag{3.40}
$$

$\square$

Lemma 3.15 *Given any* $\mathbf{u}_{\boldsymbol{\Gamma}} \in \widehat{\mathbf{W}}_{\Gamma}$,

$$
C_1(k) \mathbf{u}_{\Gamma}^T M^{-1} \mathbf{u}_{\Gamma} \leq \mathbf{u}_{\Gamma}^T \widehat{M}^{-1} \mathbf{u}_{\Gamma} \leq C_2(k) \mathbf{u}_{\Gamma}^T M^{-1} \mathbf{u}_{\Gamma}, \tag{3.41}
$$

where $C_1(k)$ *and* $C_2(k)$ *are defined in (3.34) and (3.35), respectively.*

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Proof: We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$
\begin{aligned}
&\mathbf{u}_\Gamma^T \widehat{M}^{-1} \mathbf{u}_\Gamma \\
=\ &\mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ \sum_{i=1}^N R_{\Gamma\Delta}^T \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} R_{\Gamma\Delta} + \Phi \widehat{S}_\Pi^{-1} \Phi^T \right\} \\
&\widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma \\
=\ &\mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ \sum_{i=1}^N R_{\Gamma\Delta}^T \begin{bmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{bmatrix} R_{\Gamma\Delta} \right\} \mathbf{u}_\Gamma \\
&+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi \widehat{S}_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma.
\end{aligned}
\tag{3.42}
$$

Comparing this expression with (3.29), we obtain the result by using Lemma 3.14.

$\square$

Theorem 3.16 *The condition number using the three-level preconditioner $\widehat{M}^{-1}$ is bounded by $C\frac{C_2(k)}{C_1(k)}(1 + \log\frac{H}{h})^2$, where $C_1(k)$ and $C_2(k)$ are defined in (3.34) and (3.35), respectively.*

Proof: Combining the condition number bound, given in Theorem 2.8, for the two-level BDDC method and Lemma 3.15, we find that the condition number for the system with the three-level preconditioner $\widehat{M}^{-1}$ is bounded by

$$
C\frac{C_2(k)}{C_1(k)}(1 + \log\frac{H}{h})^2.
$$

$\square$

Table 3.1: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$

Num. of Subregions	Iterations	Condition number
4×4	12	3.04
8×8	15	3.45
12×12	17	3.53
16×16	17	3.56
20×20	17	3.57

Table 3.2: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subdomains, 4×4 subregions and $\frac{H}{h} = 4$

$\frac{\hat{H}}{H}$	Iterations	Condition number
4	12	3.04
8	13	4.17
12	13	4.96
16	14	5.57
20	15	6.08

Table 3.3: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the size of subdomain problems, 4×4 subregions and 4×4 subdomains

$\frac{H}{h}$	Iterations	Condition number
4	12	3.04
8	15	4.08
12	16	4.80
16	17	5.36
20	19	5.83

Table 3.4: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$

Num. of Subregions	Iterations	Condition number
4×4	11	1.81
8×8	11	1.82
12×12	12	1.82
16×16	12	1.82
20×20	12	1.82

Table 3.5: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subdomains, 4×4 subregions and $\frac{H}{h} = 4$

$\frac{\hat{H}}{H}$	Iterations	Condition number
4	11	1.81
8	12	1.85
12	12	1.88
16	12	1.89
20	12	1.91

Table 3.6: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the size of subdomain problems, 4×4 subregions and 4×4 subdomains

$\frac{H}{h}$	Iterations	Condition number
4	11	1.81
8	14	2.50
12	16	3.00
16	17	3.35
20	18	3.65

Table 3.7: Condition number estimates and iteration counts for the operator with the preconditioner $\widehat{M}$, $u = 3.2$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{H}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	20	0.4762	0.4829	2.7110	5.6141
2	13	0.8410	0.8540	1.8820	2.2038
3	11	0.9548	0.9981	1.9061	1.9098
4	11	0.9872	1.0019	1.8663	1.8629
5	11	0.9964	1.0006	1.8551	1.8541

Table 3.8: Condition number estimates and iteration counts for the operator with the preconditioner $\widehat{M}$, $u = 4$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{H}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	22	0.4000	0.4053	2.3027	5.6821
2	14	0.7805	0.7909	1.9687	2.4892
3	12	0.9260	0.9781	1.9382	1.9816
4	11	0.9753	1.0028	1.8891	1.8837
5	11	0.9918	1.0026	1.8787	1.8739

Table 3.9: Condition number estimates and iteration counts for the operator with the preconditioner $\widehat{M}$, $u = 6$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{H}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	24	0.2857	0.2899	1.8287	6.3086
3	16	0.6575	0.6670	2.3435	3.5134
3	12	0.8524	0.9286	1.9628	3.1136
4	12	0.9377	0.9795	1.9850	2.0266
5	12	0.9738	0.9983	1.9403	1.9437

3.5   Numerical Experiments

3.5.1   Two Dimensional Cases

We have applied our two three-level BDDC algorithms to the model problem (1.6), where $\Omega = [0,1]^2$. We decompose the unit square into $\widehat{N} \times \widehat{N}$ subregions with the sidelength $\widehat{H} = 1/\widehat{N}$ and each subregion into $N \times N$ subdomains with the sidelength $H = \widehat{H}/N$. Equation (1.6) is discretized, in each subdomain, by conforming piecewise linear elements with a finite element diameter h. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of 10^{-8}.

We have carried out three different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of the experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $a \equiv 1$. Table 3.1 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 3.2 gives the results with a change of the number of subdomains. Table 3.3 gives the results with a change of the size of the subdomain problems.

In the second set of the experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $a = 1$ in one subregion and $a = 101$ in the neighboring subregions, i.e., in a checkerboard pattern. Table 3.4 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table

Table 3.10: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 3$ and $\frac{H}{h} = 3$

Num. of Subregions	Iterations	Condition number
$3 \times 3 \times 3$	9	2.66
$4 \times 4 \times 4$	10	2.87
$5 \times 5 \times 5$	11	2.97
$6 \times 6 \times 6$	11	3.02

3.5 gives the results with a change of number of subdomains. Table 3.6 gives the results with a change of the size of the subdomain problems.

In the third set of the experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $a \equiv 1$. We use the PCG to estimate the largest eigenvalue of $\left(\widehat{\widetilde{R}}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$ which is approximately 3.2867. And if we have 64×64 subdomains and $\frac{H}{h} = 4$ for the two-level BDDC, we have a condition number estimate of 1.8380. We select different values of u and k to see how the condition number changes. We take $u = 3.2$ in Table 3.7. We also give an estimate for $C_1(k)$ for $k = 1, 2, 3, 4, 5$. From Table 3.7, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and the condition number estimate becomes closer to 1.8380, the value in the two-level case, as k increases.

We take $u = 4$ in Table 3.8 and $u = 6$ in Table 3.9. From these two tables, we see that if we can get more precise estimate for the largest eigenvalue of $\left(\widehat{\widetilde{R}}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$, we need fewer Chebyshev iterations to get a condition number, similar to that of the two-level case. However, the iteration count is not very sensitive to the choice of u.

Table 3.11: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the number of subdomains, $3 \times 3 \times 3$ subregions and $\frac{H}{h} = 3$

$\frac{\dot{H}}{H}$	Iterations	Condition number
3	9	2.66
4	9	3.04
5	10	3.36
6	10	3.64

Table 3.12: Condition number estimates and iteration counts for the operator with the preconditioner $\widetilde{M}$ with a change of the size of subdomain problems, $3 \times 3 \times 3$ subregions and $3 \times 3 \times 3$ subdomains

$\frac{H}{h}$	Iterations	Condition number
3	9	2.66
4	9	2.73
5	10	2.84
6	10	2.97

Table 3.13: Condition number estimates and iteration counts for the operator with the preconditioner $\widehat{M}$, $u = 2.3$, $3 \times 3 \times 3$ subregions, $\frac{\hat{H}}{H} = 6$ and $\frac{H}{h} = 3$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	13	0.6061	0.6167	2.3309	3.7797
2	9	0.9159	0.9255	1.8968	2.0496
3	8	0.9827	1.0000	1.8835	1.8836
4	8	0.9964	1.0016	1.8854	1.8825
5	8	0.9993	1.0009	1.8797	1.8780

Table 3.14: Condition number estimates and iteration counts for the operator with the preconditioner $\widehat{M}$, $u = 3$, $3 \times 3 \times 3$ subregions, $\frac{\hat{H}}{H} = 6$ and $\frac{H}{h} = 3$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	15	0.5000	0.5093	2.0150	3.9562
2	10	0.8571	0.8678	1.9744	2.2753
3	8	0.9615	0.9900	1.8821	1.9012
4	8	0.9897	1.0015	1.8955	1.8927
5	8	0.9972	1.0020	1.8903	1.8866

3.5.2   Three Dimensional Cases

We have applied our two three-level BDDC algorithms to the model problem (1.6), where $\Omega = [0, 1]^3$. We decompose the unit cube into $\widehat{N} \times \widehat{N} \times \widehat{N}$ subregions with the sidelength $\hat{H} = 1/\widehat{N}$ and each subregion into $N \times N \times N$ subdomains with the sidelength $H = \hat{H}/N$. Equation (1.6) is discretized, in each subdomain, by conforming piecewise trilinear elements with a finite element diameter h. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of the experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $a \equiv 1$. Table 3.10 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 3.11 gives results with a change of the number of subdomains. Table 3.12 gives results with a change of the size of the subdomain problems.

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In the second set of the experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $a \equiv 1$. We use the Preconditioned Conjugate Gradient (PCG) to estimate the largest eigenvalue of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$, which is approximately 2.3249, and if we have $18 \times 18 \times 18$ subdomains and $\frac{H}{h} = 3$ for the two-level BDDC, we have a condition number estimate of 1.8767. We select different values of u, the upper bound eigenvalue estimate of the preconditioned system, and k to see how the condition number changes. We take $u = 2.3$ in Table 3.13. We also give an estimate for $C_1(k)$ for $k = 1, 2, 3, 4, 5$. From Table 3.13, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and the condition number estimate becomes closer to 1.8767, the value for the two-level case, as k increases.

We take $u = 3$ in Table 3.14. From the tables, we see that if we can get more precise estimate for the largest eigenvalue of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \right)$, we need fewer Chebyshev iterations to get a condition number, close to that of the two-level case. However, again the iteration count is not very sensitive to the choice of u.

3.6   Conclusion

As we have noted before, the coarse components of the two-level BDDC preconditioners can ultimately become a bottleneck if the number of subdomains is very large. In three-level algorithms, we alleviate this difficulty by using one additional level.

Let us look at the cost of the two-level and three-level BDDC methods closely. We assume that the diameter of the computational domain is $O(1)$. $\hat{H}$, H, and h are the typical diameters of the subregions, subdomains, and elements of sub-

domains, respectively. The numbers of degrees of freedom of the original system, the subdomain local problems, and the coarse problem are $O(\frac{1}{h^2})$ $(O(\frac{1}{h^3}))$, $O(\frac{H^2}{h^2})$ $(O(\frac{H^3}{h^3}))$, and $O(\frac{1}{H^2})$ $(O(\frac{1}{H^3}))$, for two (three) dimensions respectively. Therefore the optimal flop bounds for the Cholesky factorization of a local matrix and the coarse matrix are $O(\frac{H^3}{h^3})$ $(O(\frac{H^6}{h^6}))$ and $O(\frac{1}{H^3})$ $(O(\frac{1}{H^6}))$, respectively; see Section 1.4. In each iteration, the cost of the forward and backward substitutions are $O(\frac{H^2}{h^2}\log(\frac{H^2}{h^2}))$ $(O(\frac{H^4}{h^4}))$ and $O(\frac{1}{H^2}\log(\frac{1}{H^2})$ $(O(\frac{1}{H^4}))$. In a parallel computation, we can assign one subdomain to one processor and the coarse problem to one processor also or assign the coarse problem to each processor. We focus on the first approach here. If we store the coarse basis functions at the beginning of the computation, then the BDDC preconditioner fits an additive framework. This means that we can solve a few coarse and local problems simultaneously. We denote the number of the processors by $N+1$. Then $N \sim \frac{1}{H^2}(\frac{1}{H^3})$. When N is large, then the size of the coarse matrix is large and all the processors for the local problems are waiting for the processor for the coarse problem and the coarse problem becomes a bottleneck of the two-level BDDC methods.

For the three-level BDDC methods, we could first reduce the original coarse problem to a subregion interface problem by eliminating independently the subregion interior variables, which are the primal variables on the subdomain interface and interior to the subregions. However, in our three-level BDDC algorithms, we do not solve the subregion interface problem exactly, but replace it by one iteration of the BDDC preconditioner. This means that we only need to solve several subregion local problems and one coarse problem on the subregion level in each iteration. Then the sizes of the subregion local problems and the subregion coarse problem

are $O(\frac{\hat{H}^2}{H^2})$ $(O(\frac{\hat{H}^3}{H^3}))$ and $O(\frac{1}{\hat{H}^2})$ $(O(\frac{1}{\hat{H}^3}))$. Therefore, the optimal flop bounds for the Cholesky factorization for a subdomain local matrix, a subregion local matrix, and the coarse matrix are $O(\frac{H^3}{h^3})$ $(O(\frac{H^6}{h^6}))$, $O(\frac{\hat{H}^3}{H^3})$ $(O(\frac{\hat{H}^6}{H^6}))$, and $O(\frac{1}{\hat{H}^3})$ $(O(\frac{1}{\hat{H}^6}))$, respectively. In each iteration, the cost of the forward and backward substitutions are $O(\frac{H^2}{h^2}\log(\frac{H^2}{h^2}))$ $(O(\frac{H^4}{h^4}))$, $O(\frac{\hat{H}^2}{H^2}\log(\frac{\hat{H}^2}{H^2}))$ $(O(\frac{\hat{H}^4}{H^4}))$, and $O(\frac{1}{\hat{H}^2}\log(\frac{1}{\hat{H}^2}))$ $(O(\frac{1}{\hat{H}^4}))$. We can assign the subregion local problems to the processors for the subdomain local problems and the subregion coarse problem to the processor for the two-level coarse problem. Usually $\hat{H}$ is much larger than H, therefore the size of the subregion coarse problem is much smaller than that of the original coarse problem.

3.7   Future Work

BDDC has been extended to incompressible Stokes equations [58], flow in porous media [85, 86], and to mortar finite element discretization [49]. I will continue to work on extending our three-level BDDC algorithms to these problems, successfully extending the earlier work on two-level BDDC methods.

Chapter 4

A BDDC Algorithm for a Mixed Formulation of Flow in Porous Media

4.1 Introduction

Mixed formulations of elliptic problems, see Section 1.3 and [21], lead to large, sparse, symmetric, indefinite linear systems. Such methods have extensive applications, as in flow in porous media, where a good approximation to the velocity is required.

Overlapping domain decomposition methods for this kind of problem were developed in [32, 67, 68, 69]. These additive or multiplicative overlapping Schwartz alternating methods reduce the problem to a symmetric positive definite problem for a vector, divergence free in a finite element sense. Then two-level overlapping methods are applied to the reduced positive definite problem in a benign, divergence free subspace. The algorithms converge at a rate independent of the mesh parameters and the coefficients of the original equation.

In [42], two non-overlapping domain decomposition algorithms were proposed.

They are unpreconditioned conjugate gradient methods for certain interface variables and are, to the best of our knowledge, the first iterative substructuring methods. The rate of convergence is independent of the coefficients of the original equations, but depends mildly on the mesh parameters. The consequence of the singular local Neumann problems that arise was addressed in [42]. Other non-overlapping domain decomposition methods were proposed in [40] and [23] with improved rates of convergence. A BNN version of the Method II of [42] was proposed in [22], see also [79]. The same rate of convergence is obtained as for simple elliptic cases.

Using mixed formulations of flow in porous media, we will obtain a saddle point problem which is closely related to that arising from the incompressible Stokes equations. We note that, in a recent paper [58], the BDDC algorithms have been applied to the incompressible Stokes equation, where the constraints enforced across the interface satisfy two assumptions. One assumption forces the iterates into the benign subspace in which the operator is positive definite and the other ensures a good bound for the average operator. In general, both these assumptions are required for a good bound for the condition number.

In this chapter, we extend the BDDC algorithms to mixed formulations of flow in porous media. This work is directly related to [58], but our situation is also different. First of all, our problem is not originally formulated in the benign, divergence free subspace, and it will therefore be reduced to the benign subspace, as in [32, 67, 68, 69], at the beginning of the computation. In addition, only edge/face constraints are needed to force the iterates into the benign subspace and to ensure a good bound for the condition number, since Raviart-Thomas finite elements, see [21, Chapter III], are utilized. These elements have no degrees of freedom asso-

ciated with vertices/edges in two/three dimensions. Also, the condition number estimate for the Stokes case can be simplified since the Stokes extension is equivalent to the harmonic extension, see [7]. However, this is not the case here, and different technical tools are required. We also note that our BDDC method is closely related to edge/face-based iterative substructuring methods, see [77, 79] and [83, Chapter 5]. We will give a detailed description later.

An iterative substructuring method for Raviart-Thomas finite elements for vector field problems was proposed in [89, 82]. We will borrow some technical tools from those papers in our analysis of the BDDC algorithms.

The rest of the chapter is organized as follows. We reduce our system to an interface problem in Section 4.2. In Section 4.3, we introduce the BDDC methods for our mixed methods. We give some auxiliary results in Section 4.4. In Section 4.5, we provide an estimate of the form $C\left(1 + \log \frac{H}{h}\right)^2$ of the condition number for the system with the BDDC preconditioner; H and h are the diameters of the subdomains and elements, respectively. We also compare the BDDC methods with an edge/face-based algorithm in Section 4.6. Finally, some computational results are given in Section 4.7.

Our presentation here is based on [85] and the mixed formulation is based on Subsection 1.3.2.

4.2   Reduction to an Interface Problem

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and with $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse rectangles/hexahedra and that the number of such rectan-

gles/hexahedra forming an individual subdomain is uniformly bounded. We then introduce quasi-uniform triangulations of each subdomain. The global problem (1.14) is assembled from the subdomain problems

$$
\begin{bmatrix} A^{(i)} & B^{(i)^T} \\ B^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h^{(i)} \\ p_h^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \end{bmatrix}. \tag{4.1}
$$

The degrees of freedom of the Raviart-Thomas finite elements are the normal components on the boundary of each element only.

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i \neq j} \partial \Omega_{i,h} \cap \partial \Omega_{j,h}) \setminus \partial \Omega_h$, where $\partial \Omega_{i,h}$ is the set of nodes on $\partial \Omega_i$ and $\partial \Omega_h$ is the set of nodes on $\partial \Omega$. We decompose the discrete velocity and pressure spaces $\widehat{\mathbf{W}}$ and Q into

$$
\widehat{\mathbf{W}} = \mathbf{W}_I \bigoplus \widehat{\mathbf{W}}_\Gamma, \quad Q = Q_I \bigoplus Q_0. \tag{4.2}
$$

$\widehat{\mathbf{W}}_\Gamma$ is the space of traces on Γ of functions of $\widehat{\mathbf{W}}$. $\mathbf{W}_I$ and Q_I are direct sums of subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$, and subdomain interior pressure spaces $Q_I^{(i)}$, i.e.,

$$
\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_I^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}.
$$

The elements of $\mathbf{W}_I^{(i)}$ are supported in the subdomain Ω_i and their normal components vanish on the subdomain interface $\Gamma_i = \Gamma \cap \partial \Omega_i$, while the elements of $Q_I^{(i)}$ are restrictions of elements in Q to Ω_i which satisfy $\int_{\Omega_i} q_I^{(i)} = 0$. Q_0 is the subspace of Q with constant values $q_0^{(i)}$ in the subdomain Ω_i that satisfy

$$
\sum_{i=1}^{N} q_0^{(i)} \, m(\Omega_i) = 0, \tag{4.3}
$$

where $m(\Omega_i)$ is the measure of the subdomain Ω_i. $R_0^{(i)}$ is the operator which maps functions in the space Q_0 to its constant component of the subdomain Ω_i.

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We denote the subdomain velocity space by $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma$, the space of the interface velocity variables by $\mathbf{W}_\Gamma^{(i)}$, and the associate product space by $\mathbf{W}_\Gamma = \prod_{i=1}^N \mathbf{W}_\Gamma^{(i)}$.

The subdomain saddle point problems (4.1) can be written as

$$
\begin{bmatrix}
A_{II}^{(i)} & B_{II}^{(i)T} & A_{\Gamma I}^{(i)T} & 0 \\
B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\
A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)T} \\
0 & 0 & B_{0\Gamma}^{(i)} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{h,I}^{(i)} \\
p_{h,I}^{(i)} \\
\mathbf{u}_{h,\Gamma}^{(i)} \\
p_{h,0}^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
F_{h,I}^{(i)} \\
0 \\
F_{h,\Gamma}^{(i)}
\end{bmatrix},
\tag{4.4}
$$

where $(\mathbf{u}_{h,I}^{(i)}, p_{h,I}^{(i)}, \mathbf{u}_{h,\Gamma}^{(i)}, p_{h,0}^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Gamma^{(i)}, Q_0^{(i)})$. We note that, by the divergence theorem, the lower left block of the matrix of (4.4) is zero since the bilinear form $b(\mathbf{v}_I^{(i)}, q_0^{(i)})$ always vanishes for any $\mathbf{v}_I^{(i)} \in \mathbf{W}_I^{(i)}$ and a constant $q_0^{(i)}$ in the subdomain Ω_i.

4.2.1 Obtaining a Divergence Free Correction

First of all, we seek a discrete velocity $\mathbf{u}_h^* \in \widehat{\mathbf{W}}$ such that

$$
B\mathbf{u}_h^* = F_h. \tag{4.5}
$$

Let $\widehat{\mathbf{W}}^H$ be the lowest order Raviart-Thomas finite element space on the coarse triangulation, associated with the subdomains, with zero normal components on $\partial\Omega$ and let Q^H be the space of piecewise constants with vanishing mean value. R_0^T is the natural interpolation operator from $\widehat{\mathbf{W}}^H \times Q^H$ to $\widehat{\mathbf{W}} \times Q$. We also use the same interpolation operator on the corresponding right hand side space. Let

$$
\begin{bmatrix}
A_0 & B_0^T \\
B_0 & 0
\end{bmatrix}
= R_0
\begin{bmatrix}
A & B^T \\
B & 0
\end{bmatrix}
R_0^T,
\tag{4.6}
$$

and

$$
\begin{bmatrix}
\mathbf{u}_0^* \\
p_0^*
\end{bmatrix}
= R_0^T
\begin{bmatrix}
A_0 & B_0^T \\
B_0 & 0
\end{bmatrix}^{-1}
R_0
\begin{bmatrix}
\mathbf{0} \\
F_h
\end{bmatrix}.
$$

We note that the coarse grid solution $\mathbf{u}_0^*$ does not necessarily satisfy (4.5), but that $B\mathbf{u}_0^* - F_h$ has mean value zero over each subdomain Ω_i, see [68, 32]. Then the local Neumann problems, with $\mathbf{u}_{h,\Gamma}^{(i)} = 0$ and the right hand sides

$$\begin{bmatrix} -A^{(i)}\mathbf{u}_0^{*,(i)} \\ F_h^{(i)} - B^{(i)}\mathbf{u}_0^{*,(i)} \end{bmatrix}, i = 1, \cdots, N,$$

are all well-posed. We can solve

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ p_{h,I}^{(i)} \end{bmatrix} = \begin{bmatrix} -(A^{(i)}\mathbf{u}_0^{*,(i)})_I \\ (F_h^{(i)} - B^{(i)}\mathbf{u}_0^{*,(i)})_I \end{bmatrix}, \quad i = 1, \cdots, N, \tag{4.7}$$

and set

$$\mathbf{u}_i^* = \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \cdots, N.$$

Let $\mathbf{u}_h^* = \mathbf{u}_0^* + \mathbf{u}_1^* + \cdots + \mathbf{u}_N^*$ which satisfies (4.5). We then write the solution of

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}$$

as

$$\begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{u}_h^* \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix},$$

where the correction $(\mathbf{u}, p)^T$ satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} -A\mathbf{u}_h^* \\ 0 \end{bmatrix}. \tag{4.8}$$

This problem can be assembled from the subdomain problems:

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\ 0 & 0 & B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(i)} \\ 0 \\ \mathbf{f}_\Gamma^{(i)} \\ 0 \end{bmatrix}, \tag{4.9}$$

where $(\mathbf{u}_I^{(i)}, p_I^{(i)}, \mathbf{u}_\Gamma^{(i)}, p_0^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Gamma^{(i)}, Q_0^{(i)})$ and $\mathbf{f}_I^{(i)} = -(A^{(i)}\mathbf{u}^{*(i)})_I$ and $\mathbf{f}_\Gamma^{(i)} = -(A^{(i)}\mathbf{u}^{*(i)})_\Gamma$.

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4.2.2   A Reduced Interface Problem

We now reduce the global problem (4.8) to an interface problem.

We define the subdomain Schur complements $S_\Gamma^{(i)}$ as: given $\mathbf{w}_\Gamma^{(i)} \in \mathbf{W}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} \mathbf{w}_\Gamma^{(i)}$ such that

$$
\begin{bmatrix}
A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} \\
B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} \\
A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)}
\end{bmatrix}
\begin{bmatrix}
\mathbf{w}_I^{(i)} \\
p_I^{(i)} \\
\mathbf{w}_\Gamma^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
S_\Gamma^{(i)} \mathbf{w}_\Gamma^{(i)}
\end{bmatrix} .
\tag{4.10}
$$

We know from the definition in (4.10), that the action of $S_\Gamma^{(i)}$ can be evaluated by solving a Neumann problem on the subdomain Ω_i. We note that these Neumann problems are always well-posed, even without any constraints on the normal component of the velocity since we have removed the constant pressure component constraints. Furthermore, since the local matrices

$$
\begin{bmatrix}
A_{II}^{(i)} & A_{\Gamma I}^{(i)^T} \\
A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)}
\end{bmatrix}
$$

are symmetric, positive definite, we have, by an inertia argument,

Lemma 4.1 *The subdomain Schur complements $S_\Gamma^{(i)}$ defined in (4.10) are symmetric, positive definite.*

Given the definition of $S_\Gamma^{(i)}$, the subdomain problems (4.9) are reduced to the subdomain interface problems

$$
\begin{bmatrix}
S_\Gamma^{(i)} & B_{0\Gamma}^{(i)^T} \\
B_{0\Gamma}^{(i)} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_\Gamma^{(i)} \\
p_0^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{g}_\Gamma^{(i)} \\
0
\end{bmatrix} , \quad i = 1, 2, ..., N,
$$

where

$$
\mathbf{g}_\Gamma^{(i)} = \mathbf{f}_\Gamma^{(i)} - \begin{bmatrix} A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} \end{bmatrix}
\begin{bmatrix}
A_{II}^{(i)} & B_{II}^{(i)^T} \\
B_{II}^{(i)} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
\mathbf{f}_I^{(i)} \\
0
\end{bmatrix} .
$$

We denote the direct sum of $S_\Gamma^{(i)}$ by S_Γ. Let $R_\Gamma^{(i)}$ be the operator which maps functions in the continuous interface velocity space $\widehat{\mathbf{W}}_\Gamma$ to the subdomain components in the space $\mathbf{W}_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ. Then the global interface problem, assembled from the subdomain interface problems, can be written as: find $(\mathbf{u}_\Gamma, p_0) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$\widehat{S} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}, \tag{4.11}$$

where $\mathbf{g}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} \mathbf{g}_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} R_\Gamma^{(i)}$, and

$$\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}. \tag{4.12}$$

Thus, $\widehat{S}$ is an interface saddle point operator defined on the space $\widehat{\mathbf{W}}_\Gamma \times Q_0$. But by Lemma 4.1, this operator is symmetric positive definite on the benign subspace where $\widehat{B}_{0\Gamma}\mathbf{u}_\Gamma = 0$. From (4.8), we know that the correction $(\mathbf{u}_\Gamma, p)^T$ lies in this benign subspace. We will propose a preconditioner for (4.11) which keeps all the iterates in this benign subspace. Therefore, the iterates remain in the benign subspace in which the preconditioned operator is positive definite and a preconditioned conjugate gradient method can be applied.

4.3 The BDDC Methods

We follow [58, Section 4] closely in this section. We introduce a partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ by

$$\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widehat{\mathbf{W}}_\Pi \bigoplus \left(\prod_{i=1}^N \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{W}}_\Pi$ is the coarse level, primal interface velocity space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes

of the edge/face for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see [59] and [54]. The space $\mathbf{W}_\Delta$ is the direct sum of the $\mathbf{W}_\Delta^{(i)}$, which is spanned by the remaining interface velocity degrees of freedom with a zero average over each edge/face. In the space $\widetilde{W}_\Gamma$, we have relaxed most continuity constraints on the velocity across the interface but retained all primal continuity constraints, which has the important advantage that all the linear systems are nonsingular in the computation. This is the main difference from an edge/face-based iterative substructuring domain decomposition method, where we will encounter singular local problems; see Section 4.6.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to the components $\mathbf{W}_\Gamma^{(i)}$ related to the subdomain Ω_i. $R_\Delta^{(i)}$ maps functions from $\widehat{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Delta^{(i)}$, its dual subdomain component. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\mathbf{W}}_\Gamma$ to its subspaces $\widehat{\mathbf{W}}_\Pi$ and $R_\Pi^{(i)}$ is the operator which maps vectors in $\widehat{\mathbf{W}}_\Pi$ into their components in $W_\Pi^{(i)}$. $\overline{R}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{W}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{c_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} c_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We assume that a, the coefficient of (1.6), is a constant in each subdomain. So are the $c_i(x)$. We then note that $\delta_i^\dagger(x)$ is constant on each edge/face since the nodes on each edge/face are shared by the same pair of subdomains. Multiplying each row of $R_\Delta^{(i)}$ with the scaling factor $\delta_i^\dagger(x)$ gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the

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direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. We also use the notation

$$\widetilde{R} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \quad \text{and} \quad \widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}.$$

We also denote by $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$, the right hand side spaces corresponding to $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$, respectively. We will use the same restriction, extension, and scaled restriction operators for the spaces $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$ as for $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$.

We define the partially assembled interface velocity Schur complement $\widetilde{S}_\Gamma$: $\widetilde{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{F}}_\Gamma$ by

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{4.13}$$

$\widetilde{S}_\Gamma$ can also be defined by: for any given $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$, $\widetilde{S}_\Gamma \mathbf{w}_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ satisfies

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)^T} & A_{\Delta I}^{(1)^T} & & \widetilde{A}_{\Pi I}^{(1)^T} \\ B_{II}^{(1)} & 0 & B_{I\Delta}^{(1)} & & \widetilde{B}_{I\Pi}^{(1)} \\ A_{\Delta I}^{(1)} & B_{I\Delta}^{(1)^T} & A_{\Delta\Delta}^{(1)} & & \widetilde{A}_{\Pi\Delta}^{(1)^T} \\ & & & \ddots & \vdots \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{I\Pi}^{(1)^T} & \widetilde{A}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(1)} \\ p_I^{(1)} \\ \mathbf{w}_\Delta^{(1)} \\ \vdots \\ \mathbf{w}_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Delta^{(1)} \\ \vdots \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Pi \end{bmatrix}. \tag{4.14}$$

Here,

$$\widetilde{A}_{\Pi I}^{(i)} = R_\Pi^{(i)^T} A_{\Pi I}^{(i)}, \ \ \widetilde{A}_{\Pi\Delta}^{(i)} = R_\Pi^{(i)^T} A_{\Pi\Delta}^{(i)}, \ \ \widetilde{A}_{\Pi\Pi} = \sum_{i=1}^{N} R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}, \ \ \widetilde{B}_{I\Pi}^{(i)} = B_{I\Pi}^{(i)} R_\Pi^{(i)}.$$

Given the definition $\widetilde{S}_\Gamma$ on the partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$, we can also obtain $\widehat{S}_\Gamma$ from $\widetilde{S}_\Gamma$ by assembling the dual interface velocity part on the subdomain interface, i.e.,

$$\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma. \tag{4.15}$$

We can also define the operator $\widetilde{B}_{0\Gamma}$, partially assembled from the subdomain operators $B_{0\Gamma}^{(i)}$, which maps the partially assembled interface velocity to the right

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hand sides corresponding to the subdomain constant pressures. Then $\widehat{B}_{0\Gamma}$ can also be obtained from $\widetilde{B}_{0\Gamma}$ by assembling the dual interface velocity part on the subdomain interface, i.e., $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma}\widetilde{R}_{\Gamma}$.

Therefore, we can write the global interface saddle point problem operator $\widehat{S}$, introduced in Equation (4.11), as

$$\widehat{S} = \left[\begin{array}{cc} \widehat{S}_{\Gamma} & \widehat{B}_{0\Gamma}^{T} \\ \widehat{B}_{0\Gamma} & 0 \end{array} \right] = \left[\begin{array}{cc} \widetilde{R}_{\Gamma}^{T}\widetilde{S}_{\Gamma}\widetilde{R}_{\Gamma} & \widetilde{R}_{\Gamma}^{T}\widetilde{B}_{0\Gamma}^{T} \\ \widetilde{B}_{0\Gamma}\widetilde{R}_{\Gamma} & 0 \end{array} \right]. \tag{4.16}$$

The BDDC preconditioner for solving the global interface saddle point problem (4.11) is then

$$M^{-1} = \left[\begin{array}{cc} \widetilde{R}_{D,\Gamma}^{T} & \\ & I \end{array} \right] \left[\begin{array}{cc} \widetilde{S}_{\Gamma} & \widetilde{B}_{0\Gamma}^{T} \\ \widetilde{B}_{0\Gamma} & 0 \end{array} \right]^{-1} \left[\begin{array}{cc} \widetilde{R}_{D,\Gamma} & \\ & I \end{array} \right]. \tag{4.17}$$

We use the notation

$$\widetilde{S} = \left[\begin{array}{cc} \widetilde{S}_{\Gamma} & \widetilde{B}_{0\Gamma}^{T} \\ \widetilde{B}_{0\Gamma} & 0 \end{array} \right],$$

then the preconditioned BDDC algorithm is of the form: find $(\mathbf{u}_{\Gamma}, \mathbf{p}_{0}) \in \widehat{\mathbf{W}}_{\Gamma} \times Q_{0}$, such that

$$\widetilde{R}_{D}^{T}\widetilde{S}^{-1}\widetilde{R}_{D}\widehat{S} \left[\begin{array}{c} \mathbf{u}_{\Gamma} \\ p_{0} \end{array} \right] = \widetilde{R}_{D}^{T}\widetilde{S}^{-1}\widetilde{R}_{D} \left[\begin{array}{c} \mathbf{g}_{\Gamma} \\ 0 \end{array} \right]. \tag{4.18}$$

We define two subspaces $\widehat{W}_{\Gamma,B}$ and $\widetilde{W}_{\Gamma,B}$ of $\widehat{W}_{\Gamma}$ and $\widetilde{W}_{\Gamma}$, respectively, as in [58, Definition 1]:

$$\widehat{\mathbf{W}}_{\Gamma,B} = \{\mathbf{w}_{\Gamma} \in \widehat{\mathbf{W}}_{\Gamma} \mid \widehat{B}_{0\Gamma}\mathbf{w}_{\Gamma} = 0\},$$

$$\widetilde{\mathbf{W}}_{\Gamma,B} = \{\mathbf{w}_{\Gamma} \in \widetilde{\mathbf{W}}_{\Gamma} \mid \widetilde{B}_{0\Gamma}\mathbf{w}_{\Gamma} = 0\}.$$

We call $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_{0}$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_{0}$ the *benign subspaces* of $\widehat{\mathbf{W}}_{\Gamma} \times Q_{0}$ and $\widetilde{\mathbf{W}}_{\Gamma} \times Q_{0}$, respectively. With Lemma 4.1, it is easy to check that both operators $\widehat{S}_{\Gamma}$ and $\widetilde{S}_{\Gamma}$, given in (4.12) and (4.13), are symmetric, positive definite when restricted to the benign subspaces $\widehat{\mathbf{W}}_{\Gamma} \times Q_{0}$ and $\widetilde{\mathbf{W}}_{\Gamma} \times Q_{0}$, respectively and we also have

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Lemma 4.2 *For any* $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0, \quad \widetilde{R}_D^T \mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0.$

Proof: We need to show that for any $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0, \widetilde{R}_D^T \mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0.$ Given $\mathbf{w} = (\mathbf{w}_\Gamma, p_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, we have $\widetilde{B}_{0\Gamma} \mathbf{w}_\Gamma = 0$ and

$$\widetilde{R}_D^T \mathbf{w} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_\Gamma \\ p_0 \end{bmatrix} \in \widehat{\mathbf{W}}_\Gamma \times Q_0. \tag{4.19}$$

We only need to show that $\widehat{B}_{0\Gamma} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_\Gamma = 0$ and we find that

$$\widehat{B}_{0\Gamma} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_\Gamma = \widetilde{B}_{0\Gamma} \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \mathbf{w}_\Gamma = \widetilde{B}_{0\Pi} \mathbf{w}_\Pi = 0.$$

Here we use the definitions of $\widehat{B}_{0\Gamma}$ and $\widetilde{B}_{0\Gamma}$ for the first equality. For the second, we use that the Raviart-Thomas finite element functions only have degrees of freedom on edges/faces. In our BDDC algorithm, we choose the continuous primal interface velocity space $\mathbf{W}_\Pi$ and the subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$ such that if $\mathbf{u}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$, then $\mathbf{u}_\Delta^{(i)}$ has a zero edge/face average for each edge/face. In fact, $\widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$ computes the average of the dual interface velocities $\mathbf{w}_\Delta$, then distributes them back to each subdomain and leaves $\mathbf{w}_\Pi$ the same. We recall that the weights at these nodes are the same for each edge/face since these nodes are shared by the same pair of subdomains. The averaged dual interface velocity still has a zero edge/face average for each edge/face. For the third equality, we use that $\widetilde{B}_{0\Gamma} \mathbf{w}_\Gamma = \widetilde{B}_{0\Pi} \mathbf{w}_\Pi = 0$, since $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.

$\square$

Therefore, we can conclude that the preconditioned BDDC operator, defined in (4.18), is positive definite in the benign subspace $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.

4.4　Some Auxiliary Results

We first list some results for Raviart-Thomas finite element function spaces needed in our analysis. These results were originally given in [89, 82, 88].

We consider the interpolation operator Π^H_{RT} from $\widehat{\mathbf{W}}$ onto $\widehat{\mathbf{W}}^H$. Recall that $\widehat{\mathbf{W}}^H$ is the Raviart-Thomas finite element space on the coarse mesh with mesh size H, which is defined in terms of the degrees of freedom $\lambda_{\mathcal{F}}$, by

$$\lambda_{\mathcal{F}}(\Pi^H_{RT}\mathbf{u}) := \frac{1}{|\mathcal{F}|}\int_{\mathcal{F}}\mathbf{u}\cdot\mathbf{n}ds, \quad \mathcal{F}\subset\mathcal{F}_H.$$

We consider the stability of the interpolant Π^H_{RT} in the next lemma.

Lemma 4.3 *There exists a constant C, which depends only on the aspect ratios of $K\in\mathcal{T}_H$ and of the elements of $\mathcal{T}_h$, such that, for all $\mathbf{u}\in\widehat{\mathbf{W}}$,*

$$\|div(\Pi^H_{RT}\mathbf{u})\|^2_{L^2(K)} \leq \|div\mathbf{u}\|^2_{L^2(K)},$$

$$\|\Pi^H_{RT}\mathbf{u}\|^2_{L^2(K)} \leq C\left(1+\log\frac{H}{h}\right)\left(\|\mathbf{u}\|^2_{L^2(K)}+H^2_K\|div\mathbf{u}\|^2_{L^2(K)}\right).$$

Proof: See [89, Lemma 4.1].

$\square$

We define $N(\partial\Omega_i)$ as the the space of functions that are constant on each element of the edges/faces of the boundary of Ω_i and its subspace $N_0(\partial\Omega_i)$, of functions that have mean value zero on $\partial\Omega_i$. Let N^H be the space of functions μ defined on Γ, such that for each subdomain Ω_i and each edge/face $\mathcal{F}$ of Ω_i, μ is constant on $\mathcal{F}$. We note that N^H is the space of normal components on Γ of vectors in $\widehat{\mathbf{W}}^H$.

We define a divergence-free extension of boundary data given on $\partial\Omega_i$ in the next lemma.

Lemma 4.4 *There exists an extension operator $\tilde{\mathcal{H}}_i : N_0(\partial\Omega_i) \longrightarrow \mathbf{W}^{(i)}$, such that, for any $\mu \in N_0(\partial\Omega_i)$,*

$$div\tilde{\mathcal{H}}_i\mu = 0, \quad for \ x \in \Omega_i,$$

and

$$\|\tilde{\mathcal{H}}_i\mu\|_{L^2(\Omega_i)} \le C\|\mu\|_{H^{-1/2}(\partial\Omega_i)}. \tag{4.20}$$

Here C is independent of h, H, and μ.

Proof: See [89, Lemma 4.3].

$\square$

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/faces. Let $\zeta_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$, i.e., the function that is identically one on $\mathcal{F}$ and zero on $\partial\Omega_i\backslash\mathcal{F}$. We clearly have

$$\sum_{\mathcal{F}\subset\partial\Omega_i} \zeta_{\mathcal{F}}(x) = 1, \qquad \text{almost everywhere on} \ \ \partial\Omega_i\backslash\partial\Omega.$$

Given a function $\mu \in N(\partial\Omega_i)$ and a face $\mathcal{F} \subset \partial\Omega_i$, let

$$\mu_{\mathcal{F}} := \zeta_{\mathcal{F}}\mu \in N(\partial\Omega_i).$$

We have the following estimates for the edge/face components of the particular functions in $N(\partial\Omega_i)$ with a vanishing average on the subdomain edges/faces.

Lemma 4.5 *Let $\mu \in N(\partial\Omega_i)$ with $\int_{\partial\Omega_i} \mu ds = 0$, and for any $\mathcal{F} \subset \partial\Omega_i$, $\int_{\mathcal{F}} \mu ds = \int_{\mathcal{F}} \mu_{\mathcal{F}} ds = 0$. There then exists a constant C, independent of h and μ_H, such that, for any $\mu_H \in N^H$,*

$$
\begin{aligned}
&\|\mu_{\mathcal{F}}\|^2_{\mathcal{H}^{-1/2}(\partial\Omega_i)} \\
\leq\ &C \left(1 + \log\frac{H}{h}\right) \left((1 + \log\frac{H}{h})\|\mu + \mu_H\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu\|^2_{H^{-1/2}(\partial\Omega_i)}\right) (4.21)
\end{aligned}
$$

Proof: See [89, Lemma 4.4].

$\square$

The following lemma compares norms of traces on the subdomain boundaries that share an edge/face.

Lemma 4.6 *Let Ω_i and Ω_j be two subdomains with a common edge/face $\mathcal{F}$. Let $\mu_{\mathcal{F}}$ be a function in $H^{-1/2}(\partial\Omega_i)$, that vanishes outside $\mathcal{F}$. Then, there is a constant C, that depends only on the aspect ratios of Ω_i and Ω_j, such that*

$$
\|\mu_F\|_{H^{-1/2}(\partial\Omega_i)} \leq C\|\mu_F\|_{H^{-1/2}(\partial\Omega_j)}.
$$

Proof: See [82, Lemma 5.5.2].

$\square$

We next list some results for the benign subspace $\widetilde{W}_{\Gamma,B} \times Q_0$.

Let $\|\mathbf{w}\|^2_{\widetilde{S}} = \mathbf{w}^T \widetilde{S} \mathbf{w}$ and $\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = \mathbf{w}_\Gamma^T \widetilde{S}_\Gamma \mathbf{w}_\Gamma$. We then have

Lemma 4.7 *Given any $\mathbf{w} \in \widetilde{W}_{\Gamma,B} \times Q_0$, we have*

$$
\|\mathbf{w}\|^2_{\widetilde{S}} = \|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma}.
$$

Proof:

$$\|\mathbf{w}\|_{\widetilde{S}}^2 = \mathbf{w}^T \widetilde{S} \mathbf{w} = \begin{bmatrix} \mathbf{w}_\Gamma^T & q_0^T \end{bmatrix} \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \mathbf{w}_\Gamma^T \widetilde{S}_\Gamma \mathbf{w}_\Gamma = \|\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}^2.$$

$\square$

We define the *average operator* by $E_D = \widetilde{R} \widetilde{R}_D^T$. We see that for any vector $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_\Gamma \times Q_0$,

$$E_D \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} E_{D,\Gamma} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix}, \tag{4.22}$$

where $E_{D,\Gamma} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$, which computes the average of the interface velocities across the subdomain interface. Lemma 4.2 shows that after averaging a benign vector across a subdomain interface the result is still benign.

An estimate of the norm of the E_D operator restricted to the benign subspace $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ is given in the next lemma.

Lemma 4.8 *There exists a positive constant C, which is independent of H and h, and the number of subdomains, such that*

$$\|E_D \mathbf{w}\|_{\widetilde{S}}^2 \le C \left(1 + \log \frac{H}{h}\right)^2 \|\mathbf{w}\|_{\widetilde{S}}^2, \quad \forall \mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0.$$

Proof: Given any $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, we know, from Lemma 4.2, that $\widetilde{R}_D^T \mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$. Therefore, $E_D \mathbf{w} = \widetilde{R}_D \widetilde{R}_D^T \mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$. We have, by Lemma 4.7, that

$$\|E_D \mathbf{w}\|_{\widetilde{S}}^2$$
$$\le 2 \left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \|\mathbf{w} - E_D \mathbf{w}\|_{\widetilde{S}}^2\right)$$
$$\le 2 \left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \|\mathbf{w}_\Gamma - E_{D,\Gamma} \mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}^2\right)$$

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$$= 2 \left(\|\mathbf{w}\|_{\tilde{S}}^2 + \|\overline{R}_\Gamma (\mathbf{w}_\Gamma - E_{D,\Gamma} \mathbf{w}_\Gamma)\|_{S_\Gamma}^2 \right)$$

$$= 2 \left(\|\mathbf{w}\|_{\tilde{S}}^2 + \sum_{i=1}^N \|\overline{R}_\Gamma^{(i)} (\mathbf{w}_\Gamma - E_{D,\Gamma} \mathbf{w}_\Gamma)\|_{S_\Gamma^{(i)}}^2 \right). \tag{4.23}$$

Let $\mathbf{w}_i = \overline{R}^{(i)} \mathbf{w}_\Gamma$ and set

$$\mathbf{v}_i(x) := \overline{R}_\Gamma^{(i)} (\mathbf{w}_\Gamma - E_{D,\Gamma} \mathbf{w}_\Gamma)(x) = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger (\mathbf{w}_i(x) - \mathbf{w}_j(x)), \quad x \in \partial \Omega^i \cap \Gamma. \tag{4.24}$$

Here $\mathcal{N}_x$ is the set of indices of the subdomains that have x on their boundaries. Since a fine edge/face only belongs to exactly two subdomains, for an edge/face $\mathcal{F}^{ij} \subset \partial \Omega_i$ that is also shared by Ω_j, we have

$$\mathbf{v}_i = \delta_j^\dagger \mathbf{w}_i - \delta_j^\dagger \mathbf{w}_j, \text{ on } \mathcal{F}^{ij}. \tag{4.25}$$

We note that the simple inequality

$$c_i \delta_j^{\dagger^2} \leq \min(c_i, c_j), \tag{4.26}$$

holds for $\gamma \in [1/2, \infty)$.

Since $\mathbf{v}_i \cdot \mathbf{n}$ has a vanishing mean value on each face of Ω_i, we can define, by Lemma 4.4, $\mathbf{v}_i^E = \tilde{\mathcal{H}}_i (\mathbf{v}_i \cdot \mathbf{n})$. Then

$$div \mathbf{v}_i^E = 0, \quad \text{for } x \in \Omega_i, \tag{4.27}$$

and,

$$\|\mathbf{v}_i^E\|_{L^2(\Omega_i)}^2 \leq C \|\mathbf{v}_i \cdot \mathbf{n}\|_{H^{-1/2}(\partial \Omega_i)}^2. \tag{4.28}$$

We then obtain

$$\|\mathbf{v}_i\|_{S_\Gamma^{(i)}}^2 = c_i \|\mathbf{v}_i^E\|_{L^2(\Omega_i)}^2 \leq C c_i \|\mathbf{v}_i \cdot \mathbf{n}\|_{H^{-1/2}(\partial \Omega_i)}^2$$

$$\leq C c_i \sum_{\mathcal{F}^{ij} \subset \partial \Omega_i} \|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i \cdot \mathbf{n})\|_{H^{-1/2}(\partial \Omega_i)}^2. \tag{4.29}$$

Using (4.25), we have, with $\overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}}$ the average over $\mathcal{F}^{ij}$,

$$c_i \|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i \cdot \mathbf{n})\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$= c_i \|\zeta_{\mathcal{F}^{ij}}\delta_j^\dagger(\mathbf{w}_i - \mathbf{w}_j) \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$\leq 2c_i\delta_j^{\dagger^2}\left(\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}\right.$$

$$+ \left.\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_j \cdot \mathbf{n} - \overline{(\mathbf{w}_j \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}\right)$$

$$\leq 2c_i\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$+ 2c_j\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_j \cdot \mathbf{n} - \overline{(\mathbf{w}_j \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_j)}.$$

Here we use Lemma 4.6 and (4.26) for the last inequality.

We only need to estimate the first term since the second term can be estimated similarly.

Since $\mathbf{w}$ is in the benign space, $\mathbf{w}_i \cdot \mathbf{n}$ has vanishing mean value on $\partial\Omega_i$. By Lemma 5.2, we can construct

$$\mathbf{w}_i^E = \tilde{\mathcal{H}}_i(\mathbf{w}_i \cdot \mathbf{n}),$$

such that

$$div\,\mathbf{w}_i^E = 0, \quad \text{for } x \in \Omega_i.$$

Let $\mathbf{w}_0^E \in \widehat{\mathbf{W}}$ be defined by

$$\mathbf{w}_0^E = \begin{cases} \mathbf{w}_i^E & \text{in } \Omega_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{u}_H = \Pi_{RT}^H \mathbf{w}_0$ and $\bar{}_H = \mathbf{u}_H \cdot \mathbf{n}$. By the definition of Π_{RT}^H, we know that $\zeta_{\mathcal{F}^{ij}}\mu_H = \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}}$, and for any $\mathcal{F} \subset \partial\Omega_i$, $\int_{\mathcal{F}}(\mathbf{w}_i \cdot \mathbf{n} - \mu_H)ds = 0$. Using Lemma 4.5, we have

$$\tag{4.30}$$

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$$\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$= \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \mu_H)\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$\leq C\left(1 + \log\frac{H}{h}\right)\left((1 + \log\frac{H}{h})\|\mathbf{w}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mathbf{w}_i \cdot \mathbf{n} - \mu_H\|^2_{H^{-1/2}(\partial\Omega_i)}\right)$$

$$\leq C\left(1 + \log\frac{H}{h}\right)\left((1 + \log\frac{H}{h})\|\mathbf{w}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu_H\|^2_{H^{-1/2}(\partial\Omega_i)}\right),$$

where we use the triangle inequality for the last inequality.

By Lemma 4.3, we know that

$$\|div\mathbf{u}_H\|^2_{L^2(\Omega_i)} \leq \|div\mathbf{w}_0^E\|^2_{L^2(\Omega_i)} = \|div\mathbf{w}_i^E\|^2_{L^2(\Omega_i)} = 0. \tag{4.31}$$

and

$$\|\mathbf{u}_H\|^2_{L^2(\Omega_i)} \leq C\left(1 + \log\frac{H}{h}\right)\left(\|\mathbf{w}_0^E\|^2_{L^2(\Omega_i)} + H^2\|div\mathbf{w}_0^E\|^2_{L^2(\Omega_i)}\right)$$

$$= C\left(1 + \log\frac{H}{h}\right)\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}. \tag{4.32}$$

Using (4.30), (1.5), (4.31), and (4.32), we obtain:

$$\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}$$

$$\leq C\left(1 + \log\frac{H}{h}\right)\left((1 + \log\frac{H}{h})\|\mathbf{w}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu_H\|^2_{H^{-1/2}(\partial\Omega_i)}\right)$$

$$\leq C\left(1 + \log\frac{H}{h}\right)\left((1 + \log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{H(div,\Omega_i)} + \|\mathbf{u}_H\|^2_{H(div,\Omega_i)}\right)$$

$$\leq C\left(1 + \log\frac{H}{h}\right)\left((1 + \log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)} + (1 + \log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}\right)$$

$$\leq C\left(1 + \log\frac{H}{h}\right)^2\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}$$

$$\leq \frac{C}{c_i}\left(1 + \log\frac{H}{h}\right)^2\|\mathbf{w}_i\|^2_{S_\Gamma^{(i)}}.$$

Here we use that $div\mathbf{w}_i^E = 0$ for the third inequality.

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Finally, we obtain

$$c_i\|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i \cdot \mathbf{n})\|_{H^{-1/2}(\partial\Omega_i)}^2 \leq C\left(1 + \log\frac{H}{h}\right)^2 \|\mathbf{w}_i\|_{S_\Gamma^{(i)}}^2. \qquad (4.33)$$

Since $\mathbf{w}$ is benign, we have, from Lemma 4.7, that $\|\mathbf{w}\|_{\widetilde{S}} = \|\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}$; then by Equations (4.23), (4.24), (4.29), and (4.33), we have

$$\|E_D\mathbf{w}\|_{\widetilde{S}}^2 \leq C\left(1 + \log\frac{H}{h}\right)^2 \|\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}^2 = C\left(1 + \log\frac{H}{h}\right)^2 \|\mathbf{w}\|_{\widetilde{S}}^2.$$

$\square$

4.5 Condition Number Estimate for the BDDC Preconditioner

We are now ready to formulate and prove our main result; this follows directly from the proof of [58, Theorem 1] by using Lemma 4.2 and Lemma 4.8.

Theorem 4.9 *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the benign space $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and*

$$\langle\mathbf{u},\mathbf{u}\rangle_{\widehat{S}} \leq \left\langle M^{-1}\widehat{S}\mathbf{u},\mathbf{u}\right\rangle_{\widehat{S}} \leq C\left(1 + \log\frac{H}{h}\right)^2 \langle\mathbf{u},\mathbf{u}\rangle_{\widehat{S}}, \;\; \forall\mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0. \quad (4.34)$$

Here, C is a constant which is independent of h and H.

4.6 Comparison with an Edge/face-based Iterative Substructuring Domain Decomposition Method

We define an edge/face-based iterative substructuring method as a hybrid method (see [83, Section 2.5.2]). Similar to the BNN method, as defined in Section 2.5 and

[74, Section 4], the coarse problems and the local problems are treated multiplica-tively and additively, respectively, in this preconditioner. We use a different coarse component, i.e., a different choice of the matrix L_0 for the coarse problem, but the same local problems as in [74, Section 4]. Here, each column of L_0 corresponds to an edge/face on the interface of Ω and is given by the positive scaling factor $\delta_i^\dagger(x)$. It is clear that we can prove that the condition number with this preconditioner is also bounded by $C\left(1 + \log \frac{H}{h}\right)^2$. We will call this method the FBA.

The size and sparsity of the coarse problems of the BDDC and the FBA are the same. However, the two algorithms are different. The FBA is a hybrid algorithm and a coarse problem has to be solved before the rest of the iterations. In contrast, only the variables have to be changed at the beginning of computation with the BDDC, to accommodate the edge/face constraints. In addition, the FBA requires two Dirichlet local problems and one singular local Neumann problem in each iter-ation, whereas the BDDC requires one local Dirichlet problem and two nonsingular local Neumann problems. In the latter algorithm, singular problems are avoided. Numerical experiments show that FBA is somewhat slower than BDDC.

4.7   Numerical Experiments

We have applied our BDDC and FBA algorithms to the model problem (1.6), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (1.6) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure, respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the

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Table 4.1: Condition number estimates and iteration counts, for a pair of BDDC and FBA algorithms, with a change of the number of subdomains. $H/h = 8$ and $c \equiv 1$.

Num. of sub.	BDDC		FBA	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	5	1.66	5	2.43
8×8	8	2.95	8	2.90
12×12	9	3.08	7	2.75
16×16	9	3.13	7	2.72
20×20	8	3.15	7	2.71

Table 4.2: Condition number estimates and iteration counts, for a pair of BDDC and FBA algorithms, with a change of the size of subdomain problems. 8×8 subdomains and $c \equiv 1$.

$\frac{H}{h}$	BDDC		FBA	
	Iter.	Cond. Num.	Iter.	Cond. Num.
4	8	2.17	7	2.12
8	8	2.95	8	2.90
12	9	3.47	9	3.45
16	9	3.88	9	3.83
20	9	4.20	9	4.15

Table 4.3: Condition number estimates and iteration counts, for a pair of BDDC and FBA algorithms, with a change of the number of subdomains. $H/h = 8$ and c is in a checkerboard pattern.

Num. of sub.	BDDC		FBA	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	3	1.03	5	2.20
8×8	3	1.06	7	2.44
12×12	3	1.07	7	2.49
16×16	3	1.08	7	2.51
20×20	3	1.08	7	2.53

Table 4.4: Condition number estimates and iteration counts, for a pair of BDDC and FBA algorithms, with a change of the size of subdomain problems. 8×8 subdomains and c is in a checkerboard pattern.

$\frac{H}{h}$	BDDC		FBA	
	Iter.	Cond. Num.	Iter.	Cond. Num.
4	3	1.04	7	2.00
8	3	1.06	7	2.44
12	4	1.10	8	2.69
16	4	1.11	8	2.88
20	4	1.12	8	3.02

residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we take the coefficient $c \equiv 1$. Table 4.1 gives the iteration counts and the estimate of the condition numbers, with a change of the number of subdomains. We find that the condition number is independent of the number of subdomains for both algorithms. Table 4.2 gives the results with a change of the size of the subdomain problems.

In the second set of experiments, we take the coefficient $c = 1$ in half the subdomains and $c = 100$ in the neighboring subdomains, in a checkerboard pattern. Table 4.3 gives the iteration counts, and condition number estimates with a change of the number of subdomains. We find that the condition numbers are independent of the number of subdomains for both algorithms. Table 4.4 gives the results with a change of the size of the subdomain problems.

4.8 More General Subdomains

4.8.1 Obtaining a Correction in the Benign Subspace

In the first step, to obtain a divergence free correction, of the algorithm of Section 4.2, we have assumed that the subdomains form a coarse triangulation of our domain since we need to define the Raviart-Thomas finite element space at the subdomain level. This limits the algorithm. However, we only need that the corrections belong to the benign subspace for the BDDC algorithms. Moreover, we can obtain such a correction easily by acting with the BDDC precondioner on a carefully chosen vector. Therefore, the algorithm can be extended to different types of subdomains.

In a more general case, we can still define faces, regarded as open sets, that are shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains.

We define g_c as follows:

$$\begin{bmatrix} \mathbf{0} \\ F_{h,\Gamma}^{(1)} \\ \vdots \\ F_{h,\Gamma}^{(N)} \end{bmatrix}. \tag{4.35}$$

Let $\mathbf{u}_{h,\Gamma}^* = M^{-1}g_c$, i.e.,

$$\begin{bmatrix} \mathbf{u}_{h,\Gamma}^* \\ p^* \end{bmatrix} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ F_{h,\Gamma}^{(1)} \\ \vdots \\ F_{h,\Gamma}^{(N)} \end{bmatrix}. \tag{4.36}$$

Let us calculate $\widehat{B}_{0\Gamma}\mathbf{u}_{h,\Gamma}^*$. From the definition of $\mathbf{u}_{h,\Gamma}^*$, we know that $\mathbf{u}_{h,\Gamma}^* = \widetilde{R}_{D,\Gamma}^T\mathbf{w}_\Gamma$,

where $\widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = \begin{bmatrix} F_{h,\Gamma}^{(1)} \\ \vdots \\ F_{h,\Gamma}^{(N)} \end{bmatrix}$. Then, we have,

$$\widehat{B}_{0\Gamma}\mathbf{u}_{h,\Gamma}^* = \widehat{B}_{0\Gamma}\widetilde{R}_{D,\Gamma}^T\mathbf{w}_\Gamma = \widetilde{B}_{0\Gamma}\widetilde{R}_\Gamma\widetilde{R}_{D,\Gamma}^T\mathbf{w}_\Gamma = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi = \begin{bmatrix} F_{h,\Gamma}^{(1)} \\ \vdots \\ F_{h,\Gamma}^{(N)} \end{bmatrix}.$$

Here we use the same argument as for Lemma 4.2 and the definitions of $\widehat{B}_{0\Gamma}$ and $\widetilde{B}_{0\Gamma}$ for the first equality. For the second, we use that the Raviart-Thomas finite element functions only have degrees of freedom on edges/faces. In our BDDC algorithm, we choose the continuous primal interface velocity space $\widehat{\mathbf{W}}_\Pi$ and the subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$ such that if $\mathbf{u}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$, then $\mathbf{u}_\Delta^{(i)}$ has a zero edge/face average for each interface edge/face. In fact, $\widetilde{R}_\Gamma\widetilde{R}_{D,\Gamma}^T$ computes the average of the dual interface velocities $\mathbf{W}_\Delta$, and then distributes them back to each subdomain and leaves $\widehat{\mathbf{W}}_\Pi$ the same. We recall that the weights at these nodes are the same for each edge/face since these nodes are shared by the same pair of subdomains. The averaged dual interface velocity still has a zero edge/face average for each edge/face. For the third equality, we use that $\widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi$.

We obtain $\mathbf{u}_h^*$ by a harmonic extension into each subdomain.

As before, we can then write the solution of

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}$$

as

$$\begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{u}_h^* \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix},$$

where the correction $(\mathbf{u}, p)^T$ satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} -A\mathbf{u}_h^* \\ F_h - B\mathbf{u}_h^* \end{bmatrix}. \tag{4.37}$$

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Table 4.5: Condition number estimates and iteration counts, for the pair of BDDC algorithms, with a change of the number of subdomains. $H/h = 8$ and $c \equiv 1$.

Num. of sub.	Old Algorithm		New Algorithm	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	5	1.66	8	2.59
8×8	8	2.95	10	3.02
12×12	9	3.08	10	3.12
16×16	9	3.13	10	3.15
20×20	8	3.15	10	3.16

This problem can be assembled from the subdomain problems:

$$
\begin{bmatrix}
A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} & 0 \\
B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\
A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\
0 & 0 & B_{0\Gamma}^{(i)} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_I^{(i)} \\
p_I^{(i)} \\
\mathbf{u}_\Gamma^{(i)} \\
p_0^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f}_I^{(i)} \\
F_{h,I}^{(i)} \\
\mathbf{f}_\Gamma^{(i)} \\
0
\end{bmatrix},
\qquad (4.38)
$$

where $(\mathbf{u}_I^{(i)}, p_I^{(i)}, \mathbf{u}_\Gamma^{(i)}, p_0^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Gamma^{(i)}, Q_0^{(i)})$ and $\mathbf{f}_I^{(i)} = -\left(A^{(i)}\mathbf{u}_h^{*\,(i)}\right)_I$ and $\mathbf{f}_\Gamma^{(i)} = -\left(A^{(i)}\mathbf{u}_h^{*\,(i)}\right)_\Gamma$.

We note that for the correction, the divergence of $\mathbf{u}$ is not zero anymore, but that $\mathbf{u}$ is in the subspace $\widehat{W}_{\Gamma,B}$.

The rest of the algorithm is the same as before except that

$$
\mathbf{g}_\Gamma^{(i)} = \mathbf{f}_\Gamma^{(i)} - \begin{bmatrix} A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} \end{bmatrix}
\begin{bmatrix}
A_{II}^{(i)} & B_{II}^{(i)^T} \\
B_{II}^{(i)} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
\mathbf{f}_I^{(i)} \\
F_{h,I}^{(i)}
\end{bmatrix}.
$$

4.8.2 Numerical Experiments

We have tested our new algorithm and the results in Tables 4.5-4.8 illustrate that there is only small differences in the performance. We recall that the advantage of the new algorithm over the old is that it is well defined for quite general subdomains.

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Table 4.6: Condition number estimates and iteration counts, for the pair of BDDC algorithms, with a change of the size of subdomain problems. 8×8 subdomains and $c \equiv 1$.

$\frac{H}{h}$	Old Algorithm		New Algorithm	
	Iter.	Cond. Num.	Iter.	Cond. Num.
4	8	2.17	9	2.23
8	8	2.95	10	3.02
12	9	3.47	12	3.56
16	9	3.88	12	3.97
20	9	4.20	13	4.30

Table 4.7: Condition number estimates and iteration counts, for the pair of BDDC algorithms, with a change of the number of subdomains. $H/h = 8$ and c is in a checkerboard pattern.

Num. of sub.	Old Algorithm		New Algorithm	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	3	1.03	3	1.05
8×8	3	1.06	4	1.08
12×12	3	1.07	4	1.08
16×16	3	1.08	4	1.08
20×20	3	1.08	4	1.08

Table 4.8: Condition number estimates and iteration counts, for the pair of BDDC algorithms, with a change of the size of subdomain problems. 8×8 subdomains and c is in a checkerboard pattern.

$\frac{H}{h}$	Old Algorithm		New Algorithm	
	Iter.	Cond. Num.	Iter.	Cond. Num.
4	3	1.04	4	1.04
8	3	1.06	4	1.08
12	4	1.10	4	1.10
16	4	1.11	4	1.12
20	4	1.12	4	1.13

Chapter 5

A BDDC Algorithm for Flow in Porous Media with a Hybrid Finite Element Discretization

5.1   Introduction

Mixed formulations of elliptic problems, see [21], have many applications, e.g., for flow in porous media, for which a good approximation to the velocity, which involves derivatives of the solution of the differential equations, is required. These discretizations lead to large, sparse, symmetric, indefinite linear systems.

In our recent paper [85] and Chapter 4, we extended the BDDC algorithm to this mixed formulation of elliptic problems. There, the original saddle point problem is reduced to finding a correction pair which stays in the divergence-free, benign subspace, as in [32, 67, 68, 69]. Then the BDDC method, with edge/face constraints, is applied to the reduced system. It is similar to the BDDC algorithm proposed for the Stokes case in [58]. The analysis of this approach is focused on estimating the norm of the average operator. Several useful technical tools for the Raviart Thomas finite elements, originally given in [89, 82, 88], are used and the

algorithm converges at a rate similar to that of simple elliptic cases.

The hybrid finite element discretization is equivalent to a nonconforming finite element method. Two-level domain decomposition methods have been developed for a nonconforming approximation in [79, 78]. The condition number bounds are independent of the jumps in the coefficients of the original equations and grow only logarithmically with the number of degrees of freedom in each subdomain, a result which is the same as for a conforming case.

A non-overlapping domain decomposition algorithm for the hybrid formulation, called Method II, was proposed already in [42]. It is an unpreconditioned conjugate gradient method for certain interface variables. The rate of convergence is independent of the coefficients, but depends mildly on the number of degrees of freedom in the subdomains. Problems related to singular local Neumann problems arising in the preconditioners were addressed also in [42]. In addition, other non-overlapping domain decomposition methods were proposed with improved rates of convergence in [40] and [23].

A Balancing Neumann-Neumann (BNN) method was extended and analyzed in [22] for Method II in [42], see also [79] for a nonconforming case. The same rate of convergence was obtained as for the conforming case. We will extend the BDDC algorithm to Method II of [42] in this chapter. In contrast to [22], we need not solve any singular systems with BDDC.

The method proposed here differs from the one in [85]. We reduce the original saddle-point problem to a positive definite system for the pressure by eliminating the velocity in each subdomain. Thus, we need not find a velocity that satisfies the divergence constraint at the beginning of the computation and then restrict

the iterates to the divergence free, benign subspace. Our approach is quite similar to work on the FETI-DP method, see [83, Chapter 6]. We use the BDDC preconditioner to solve the interface problem for the Lagrange multipliers, which can be interpreted as an approximation to the trace of the pressure. By enforcing a suitable set of constraints, we obtain the same convergence rate as for a conforming finite element case. Our analysis will also focus on the estimate of the norm of the average operator. However, we cannot use properties of Raviart-Thomas finite element directly since we work with the Lagrange multipliers. The technical tools, originally given in [79, 78], and followed by [22], are needed to make a connection between the hybrid finite element method and a conforming finite element method.

The rest of the chapter is organized as follows. In Section 5.2, we reduce our problem to a symmetric positive definite interface problem. We introduce the BDDC preconditioner for the interface system in Section 5.3 and give some auxiliary results in Section 5.4. In Section 5.5, we provide an estimate of the condition number for the system with the BDDC preconditioner which is of the form $C \left(1 + \log \frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively. Finally, some computational results are presented in Section 5.6.

Our presentation here is based on [86].

5.2 The Problem Reduced to the Subdomain Interface

We follow the hybrid formulation introduced in Subsection 1.3.3. We denote the discrete space of nodal values of $Q \times \widehat{\Lambda}$ by $\widehat{\mathcal{P}}$. We note that $\widehat{\mathcal{P}}$ has the natural

interpretation as the space of values of the pressure p in the interior and on the edges/faces of the element. By this definition, $\widehat{\mathcal{P}}$ is isomorphic to $Q \times \widehat{\Lambda}$; we can then write an element of $\widehat{\mathcal{P}}$ as $\hat{p} = [p, \lambda]$.

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i \neq j} \partial \Omega_{i,h} \cap \partial \Omega_{j,h}) \setminus \partial \Omega_h$, where $\partial \Omega_{i,h}$ is the set of nodes on $\partial \Omega_i$ and $\partial \Omega_h$ is the set of nodes on $\partial \Omega$.

We can write the discrete pressure spaces $\widehat{\mathcal{P}}$ as

$$\widehat{\mathcal{P}} = Q \bigoplus \widehat{\Lambda}. \tag{5.1}$$

The space Q is a direct sum of subdomain interior pressure spaces $Q^{(i)}$, i.e.,

$$Q = \bigoplus_{i=1}^{N} Q^{(i)}.$$

The elements of $Q^{(i)}$ are restrictions of elements in Q to Ω_i.

We can further decompose $\widehat{\Lambda}$ into

$$\widehat{\Lambda} = \Lambda_I \bigoplus \widehat{\Lambda}_\Gamma,$$

where $\widehat{\Lambda}_\Gamma$ denotes the set of degrees of freedom associated with Γ and Λ_I is a direct sum of subdomain interior degrees of freedom, i.e,

$$\Lambda_I = \bigoplus_{i=1}^{N} \Lambda_I^{(i)}.$$

We denote the subdomain interface pressure space by $\widehat{\Lambda}_\Gamma^{(i)}$ and the associated product space by $\widehat{\Lambda}_\Gamma = \prod_{i=1}^{N} \widehat{\Lambda}_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface pressure space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\widehat{\Lambda}_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ.

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The global saddle point problem (1.16) is assembled from subdomain problems

$$
\begin{bmatrix}
A^{(i)} & B_1^{(i)^T} & B_{2,I}^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\
B_1^{(i)} & 0 & 0 & 0 \\
B_{2,I}^{(i)} & 0 & 0 & 0 \\
B_{2,\Gamma}^{(i)} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^{(i)} \\
p^{(i)} \\
\lambda_I^{(i)} \\
\lambda_\Gamma^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{0} \\
F_h^{(i)} \\
0 \\
0
\end{bmatrix},
\tag{5.2}
$$

where $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_I^{(i)}, \lambda_\Gamma^{(i)}) \in (\mathbf{W}^{(i)}, Q^{(i)}, \Lambda_I^{(i)}, \widehat{\Lambda}_\Gamma^{(i)})$.

We define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \widehat{\Lambda}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} \lambda_\Gamma^{(i)}$ such that

$$
\begin{bmatrix}
A^{(i)} & B_1^{(i)^T} & B_{2,I}^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\
B_1^{(i)} & 0 & 0 & 0 \\
B_{2,I}^{(i)} & 0 & 0 & 0 \\
B_{2,\Gamma}^{(i)} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^{(i)} \\
p^{(i)} \\
\lambda_I^{(i)} \\
\lambda_\Gamma^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{0} \\
0 \\
0 \\
-S_\Gamma^{(i)} \lambda_\Gamma^{(i)}
\end{bmatrix}.
\tag{5.3}
$$

We note that $A^{(i)}$ is block diagonal, with each block corresponding to an element $T \subset \mathcal{T}(\Omega_i)$. We first eliminate the velocity $\mathbf{u}^{(i)}$ and obtain a system for the $p^{(i)}$, $\lambda_I^{(i)}$, and $\lambda_\Gamma^{(i)}$. We then eliminate the degrees of freedom interior to the subdomain, i.e., the $p^{(i)}$ and $\lambda_I^{(i)}$.

As we mentioned before, in practice, for each subdomain Ω_i, we only need to use the inter-element multipliers on the interface of the subdomains. Let $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_\Gamma^{(i)}) \in (\widehat{\mathbf{W}}^{(i)}, Q^{(i)}, \widehat{\Lambda}^{(i)})$ and obtain the following subdomain problems

$$
\begin{bmatrix}
\hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\
B_1^{(i)} & 0 & 0 \\
B_{2,\Gamma}^{(i)} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^{(i)} \\
p^{(i)} \\
\lambda_\Gamma^{(i)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{0} \\
F_h^{(i)} \\
0
\end{bmatrix}.
\tag{5.4}
$$

We note that $\hat{A}^{(i)}$ is no longer block diagonal by elements. We eliminate the velocity $\mathbf{u}^{(i)}$ and the pressure $p^{(i)}$ and obtain the following Schur complement for $\lambda_\Gamma^{(i)}$.

$$
\begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ -S_\Gamma^{(i)} \lambda_\Gamma^{(i)} \end{bmatrix}. \tag{5.5}
$$

Here we use the same notation $S_\Gamma^{(i)}$ since this matrix, in fact, is the same as in (5.3). This follows from the equivalence of (1.13) and (1.15). The action of $S_\Gamma^{(i)}$ can then be evaluated by solving a Dirichlet problem in the variational form: find $\{\mathbf{u}_i, p_i\} \in \widehat{\mathbf{W}}^{(i)} \times Q^{(i)}$ such that

$$
\int_{\Omega_i} \mathbf{u}_i^T c \mathbf{v}_i d\mathbf{x} - \int_{\Omega_i} \nabla \cdot \mathbf{v}_i d\mathbf{x} \; = \; -\int_{\partial\Omega_i \partial\Omega} \lambda_\Gamma^{(i)} \mathbf{v}_i \cdot \mathbf{n} ds \quad \forall\; \mathbf{v}_i \in \widehat{\mathbf{W}}^{(i)}
$$
$$
\int_{\Omega_i} \nabla \cdot \mathbf{u}_i q_i \; = \; 0 \quad \forall\; q_i \in Q^{(i)}; \tag{5.6}
$$

then set $S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = -B_{2,\Gamma}^{(i)} \mathbf{u}_i$. We note that these Dirichlet problems are always well posed and that $S_\Gamma^{(i)}$ is symmetric and positive definite. We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ.

Given the definition of $S_\Gamma^{(i)}$, the subdomain problem (5.4) corresponds to the subdomain interface problem

$$
S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = g_\Gamma^{(i)}, \quad i = 1, 2, ..., N,
$$

where

$$
g_\Gamma^{(i)} = - \begin{bmatrix} B_{2,\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} \\ B_1^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \end{bmatrix}.
$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$
\widehat{S}_\Gamma \lambda_\Gamma = g_\Gamma, \tag{5.7}
$$

where $g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} g_\Gamma^{(i)}$, and

$$
\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}. \tag{5.8}
$$

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Thus, $\widehat{S}_\Gamma$ is a symmetric positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (5.7) with a preconditioned conjugate gradient method.

5.3   The BDDC Preconditioner

We introduce a partially assembled interface pressure space $\widetilde{\boldsymbol{\Lambda}}_\Gamma$ by

$$\widetilde{\boldsymbol{\Lambda}}_\Gamma = \widehat{\boldsymbol{\Lambda}}_\Pi \bigoplus \boldsymbol{\Lambda}_\Delta = \widehat{\boldsymbol{\Lambda}}_\Pi \bigoplus \left(\prod_{i=1}^{N} \boldsymbol{\Lambda}_\Delta^{(i)} \right).$$

Here, $\widehat{\boldsymbol{\Lambda}}_\Pi$ is the coarse level, primal interface pressure space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see [59] and [54]. The space $\boldsymbol{\Lambda}_\Delta$ is the direct sum of the $\boldsymbol{\Lambda}_\Delta^{(i)}$, which are spanned by the remaining interface pressure degrees of freedom with a zero average over each edge/face. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints on the pressure across the interface but retain all primal continuity constraints, which makes all the linear systems nonsingular. This is the main difference from the BNN method in [22], where we encounter singular local problems.

   We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\boldsymbol{\Lambda}}_\Gamma$ to the components $\boldsymbol{\Lambda}_\Gamma^{(i)}$ related to the subdomain Ω_i. $R_\Delta^{(i)}$ maps functions from $\widehat{\boldsymbol{\Lambda}}_\Gamma$ to $\boldsymbol{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\boldsymbol{\Lambda}}_\Gamma$ to its subspace $\widehat{\boldsymbol{\Lambda}}_\Pi$ and $R_\Pi^{(i)}$ is the operator which maps vectors in $\widehat{\boldsymbol{\Lambda}}_\Pi$ into their components in $\Lambda_\Pi^{(i)}$. $\overline{R}_\Gamma : \widetilde{\boldsymbol{\Lambda}}_\Gamma \to \boldsymbol{\Lambda}_\Gamma$ is the direct sum of the $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\boldsymbol{\Lambda}}_\Gamma \to \widetilde{\boldsymbol{\Lambda}}_\Gamma$ is the direct

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sum of $R_{\Gamma\Pi}$ and $R_{\Delta}^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{a_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} a_j^\gamma(x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We note that $\delta_i^\dagger(x)$ is constant on each edge/face, since we assume that the $a_i(x)$ is constant in each subdomain, and the nodes on each edge/face are shared by the same subdomains. Multiplying each row of $R_{\Delta}^{(i)}$, with the scaling factor $\delta_i^\dagger(x)$, gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. Furthermore, $\widetilde{R}_{\Delta}^{(i)}$ maps functions from $\widetilde{\boldsymbol{\Lambda}}_\Gamma$ to $\boldsymbol{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $\widetilde{R}_{\Gamma\Pi}$ is a restriction operator from $\widetilde{\boldsymbol{\Lambda}}_\Gamma$ to its subspace $\widehat{\boldsymbol{\Lambda}}_\Pi$.

We also denote by $\widetilde{\mathbf{F}}_\Gamma$, the right hand side space corresponding to $\widetilde{\boldsymbol{\Lambda}}_\Gamma$. We will use the same restriction, extension, and scaled restriction operators for the space $\widetilde{\mathbf{F}}_\Gamma$ as for $\widetilde{\boldsymbol{\Lambda}}_\Gamma$.

The interface pressure Schur complement $\widetilde{S}_\Gamma$, on the partially assembled interface pressure space $\widetilde{\Lambda}_\Gamma$, is partially assembled from subdomain Schur complements $S_\Gamma^{(i)}$, i.e.,

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{5.9}$$

$\widetilde{S}_\Gamma$ can also be defined by: for any given $\lambda_\Gamma \in \widetilde{\boldsymbol{\Lambda}}_\Gamma$, $\widetilde{S}_\Gamma\lambda_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ satisfies

$$H \begin{bmatrix} \mathbf{u}^{(1)} \\ p_I^{(1)} \\ \lambda_\Delta^{(1)} \\ \vdots \\ \mathbf{u}^{(N)} \\ p_I^{(N)} \\ \lambda_\Delta^{(N)} \\ \lambda_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ (\widetilde{S}_\Gamma\lambda_\Gamma)_\Delta^{(1)} \\ \vdots \\ \mathbf{0} \\ 0 \\ (\widetilde{S}_\Gamma\lambda_\Gamma)_\Delta^{(N)} \\ (\widetilde{S}_\Gamma\lambda_\Gamma)_\Pi \end{bmatrix}, \tag{5.10}$$

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where

$$
H = \begin{bmatrix}
A^{(1)} & B_1^{(1)^T} & B_{2,\Delta}^{(1)^T} & & & & & \widetilde{B}_{2,\Pi}^{(1)^T} \\
B_1^{(1)} & 0 & 0 & & & & & 0 \\
B_{2,\Delta}^{(1)} & 0 & 0 & & & & & 0 \\
& & & \ddots & \vdots & & & \\
& & & & A^{(N)} & B_1^{(N)^T} & B_{2,\Delta}^{(N)^T} & \widetilde{B}_{2,\Pi}^{(N)^T} \\
& & & & B_1^{(N)} & 0 & 0 & 0 \\
& & & & B_{2,\Delta}^{(N)} & 0 & 0 & 0 \\
\widetilde{B}_{2,\Pi}^{(1)} & 0 & 0 & \cdots & \widetilde{B}_{2,\Pi}^{(N)} & 0 & 0 & 0
\end{bmatrix}, \qquad (5.11)
$$

and

$$
\widetilde{B}_{2,\Pi}^{(i)} = R_\Pi^{(i)^T} B_{2,\Pi}^{(i)}.
$$

Given the definition of $\widetilde{S}_\Gamma$ on the partially assembled interface pressure space $\widetilde{\Lambda}_\Gamma$, we can also obtain $\widehat{S}_\Gamma$, introduced in (5.7), from $\widetilde{S}_\Gamma$ by assembling the dual interface pressure part on the subdomain interface, i.e.,

$$
\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma. \qquad (5.12)
$$

The BDDC preconditioner for solving the global interface problem (5.7) is

$$
M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}. \qquad (5.13)
$$

Here, from a block Cholesky factorization, we have

$$
\widetilde{S}_\Gamma^{-1} = -\sum_{i=1}^N \begin{bmatrix} 0 & 0 & \widetilde{R}_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Delta}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Delta}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \widetilde{R}_\Delta^{(i)} \end{bmatrix} + \Phi S_{CC}^{-1} \Phi^T, \quad (5.14)
$$

$$
S_{CC} = \sum_{i=1}^N R_\Pi^{(i)^T} \left\{ \begin{bmatrix} B_{2,\Pi}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Delta}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Delta}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_{2,\Pi}^{(i)^T} \\ 0 \\ 0 \end{bmatrix} \right\} R_\Pi^{(i)},
$$

and the matrix Φ is defined by

$$\Phi = \widetilde{R}_{\Gamma\Pi}^T - \sum_{i=1}^{N} \begin{bmatrix} 0 & 0 & \widetilde{R}_{\Delta}^{(i)^T} \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)} & B_{2,\Delta}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Pi}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_{2,\Delta}^{(i)^T} \\ 0 \\ 0 \end{bmatrix} R_\Pi^{(i)}.$$

The preconditioned BDDC algorithm is then of the form: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma. \tag{5.15}$$

This preconditioned problem is symmetric positive definite and we can use the preconditioned conjugate gradient method to solve it.

5.4   Some Auxiliary Results

In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants c and C, independent of any parameters, such that $cA \leq B \leq CA$, where $C < \infty$ and $c > 0$.

In order to connect our hybrid finite element discretization to a conforming finite element method, we need to introduce a new mesh on each subdomain. The idea follows [78, 79, 22]. We will use similar techniques as in Chapter 3. In order to be complete and for the readers unfamiliar with these technical tools, we give the construction of the new mesh, the definitions of two important maps, and some useful lemmas, which were originally given in [22, 78, 79].

Given an element $\tau \in \mathcal{T}$, let $\hat{\tau}$ be a subtriangulation of τ which includes the vertices of τ and the nodal points in τ for the degrees of the freedom of $Q \times \Lambda$. We then obtain a quasi-uniform sub-triangulation $\widehat{\mathcal{T}}$. We partition the vertices in

the new mesh $\widehat{\mathcal{T}}$ into two sets. The nodes in $\mathcal{T}$ are called primary and the rest are called secondary. We say that two vertices in the triangulation $\widehat{\mathcal{T}}$ are adjacent if there is an edge of $\widehat{\mathcal{T}}$ between them.

Let $U_h(\Omega)$ be the continuous piecewise linear finite element function space with respect to the new triangulation $\widehat{\mathcal{T}}$. For a subdomain Ω_i, $U_h(\Omega_i)$ and $U_h(\partial\Omega_i)$ are defined by restrictions:

$$U_h(\Omega_i) = \{u|_{\Omega_i} : \quad u \in U_h(\Omega)\}, \qquad U_h(\partial\Omega_i) = \{u|_{\partial\Omega_i} : \quad u \in U_h(\Omega)\}.$$

Define a mapping $I_h^{\Omega_i}$ from any function ϕ defined at the primary vertices in Ω_i to $U_h(\Omega_i)$ by

$$I_h^{\Omega_i}\phi(x) = \begin{cases} \phi(x), \text{ if } x \text{ is a primary node;} \\[1em] \text{the average of all adjacent primary vertices on } \partial\Omega_i, \\ \quad \text{if } x \text{ is a secondary vertex on } \partial\Omega_i; \\[1em] \text{the average of all adjacent primary vertices,} \\ \quad \text{if } x \text{ is a secondary vertex in the interior of } \Omega_i; \\[1em] \text{the linear interpolation of the vertex values,} \\ \quad \text{if } x \text{ is not a vertex of } \mathcal{T}. \end{cases} \qquad (5.16)$$

We note that $I_H^{\Omega^{(i)}}$ defines a map from $Q(\Omega_i) \times \Lambda(\Omega_i)$ to $U_h(\Omega_i)$ and also a map from $U_h(\Omega_i)$ to $U_h(\Omega_i)$.

Let $I_H^{\partial\Omega^{(i)}}$ be the mapping from a function ϕ, defined at the primary vertices on $\partial\Omega_i$, to $U_h(\Omega_i)$ and defined by $I_H^{\partial\Omega^{(i)}}\phi = (I_H^{\Omega^{(i)}}\hat{p})|_{\partial\Omega_i}$, where $\hat{p}$ is any functions in $Q(\Omega_i) \times \Lambda(\Omega_i)$ such that $\hat{p}|_{\partial\Omega_i} = \phi$. The map is well defined since the boundary values of $I_H^{\Omega^{(i)}}\hat{p}$ only depend on the boundary values of $\hat{p}$.

Let

$$\widetilde{U}_h(\Omega_i) = \{\psi = I_H^{\Omega^{(i)}}\phi, \quad \phi \in U_h(\Omega_i)\} \quad \text{and} \quad \widetilde{U}_h(\partial\Omega_i) = \{\psi|_{\partial\Omega}, \quad \psi \in \widetilde{U}_h(\Omega_i)\}.$$

We list some useful lemmas from [22].

Lemma 5.1 *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$|I_H^{\Omega^{(i)}}\phi|_{H^1(\Omega_i)} \le C|\phi|_{H^1(\Omega_i)}, \quad \forall \phi \in U_h(\Omega_i), \tag{5.17}$$

$$\|I_H^{\Omega^{(i)}}\phi\|_{L^2(\Omega_i)} \le C\|\phi\|_{L^2(\Omega_i)}, \quad \forall \phi \in U_h(\Omega_i). \tag{5.18}$$

Proof: See [22, Lemms 6.1].

$\square$

Lemma 5.2 *For $\hat{\phi} \in \widetilde{U}_h(\partial\Omega_i)$,*

$$\inf_{\phi \in \widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i} = \hat{\phi}} \|\phi\|_{H^1(\Omega_i)} \approx \|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)}, \tag{5.19}$$

$$\inf_{\phi \in \widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i} = \hat{\phi}} |\phi|_{H^1(\Omega_i)} \approx |\hat{\phi}|_{H^{1/2}(\partial\Omega_i)}. \tag{5.20}$$

Proof: See [22, Lemms 6.2].

$\square$

Lemma 5.3 *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$\|I_H^{\partial\Omega^{(i)}}\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \le C\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \quad \forall\, \hat{\phi} \in U_h(\partial\Omega_i). \tag{5.21}$$

Proof: See [22, Lemms 6.3].

$\square$

Lemma 5.4 *There exist constants $C_1, C_2 > 0$ independent of H, h, and the coefficient of (1.6), such that for all $\lambda_i \in \Lambda_\Gamma^{(i)}$,*

$$a_i C_1 |I_H^{\partial\Omega^{(i)}}\lambda_i|_{H^{1/2}(\partial\Omega_i)}^2 \le |\lambda_i|_{S_\Gamma^{(i)}}^2 \le a_i C_2 |I_H^{\partial\Omega^{(i)}}\lambda_i|_{H^{1/2}(\partial\Omega_i)}^2. \tag{5.22}$$

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Proof: See [22, Theorem 6.5].

$\square$

We define the interface averages operator E_D by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T, \qquad (5.23)$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the boundary points of the subdomain.

The interface average operator E_D has the following property:

Lemma 5.5

$$|E_D \lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 \leq C \left(1 + \log \frac{H}{h} \right)^2 |\lambda_\Gamma|_{\widetilde{S}_\Gamma}^2,$$

for any $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$, where C is a positive constant independent of H, h, and the coefficient of (1.6),

Proof: Given any $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$, we have

$$\begin{aligned}
&|E_D \lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 \\
\leq\ & 2 \left(|\lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 + |\lambda_\Gamma - E_D \lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 \right) \\
\leq\ & 2 \left(|\lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 + |\overline{R}_\Gamma (\lambda_\Gamma - E_D \lambda_\Gamma)|_{S_\Gamma}^2 \right) \\
=\ & 2 \left(|\lambda_\Gamma|_{\widetilde{S}_\Gamma}^2 + \sum_{i=1}^N |\overline{R}_\Gamma^{(i)} (\lambda_\Gamma - E_D \lambda_\Gamma)|_{S_\Gamma^{(i)}}^2 \right).
\end{aligned} \qquad (5.24)$$

Let $\lambda_i = \overline{R}^{(i)} \lambda_\Gamma$ and set

$$v_i(x) := \overline{R}_\Gamma^{(i)} (\lambda_\Gamma - E_D \lambda_\Gamma)(x) = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger (\lambda_i(x) - \lambda_j(x)), \quad x \in \partial\Omega^i \cap \Gamma. \qquad (5.25)$$

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Here $\mathcal{N}_x$ is the set of indices of the subdomains that have x on their boundaries. Since a fine edge/face only belongs to exactly two subdomains, we have, for an edge/face $\mathcal{F}^{ij} \subset \partial\Omega_i$ that is also shared by Ω_j,

$$v_i = \delta_j^\dagger \lambda_i - \delta_j^\dagger \lambda_j, \ \text{on } \mathcal{F}^{ij}. \tag{5.26}$$

We note again that the simple inequality

$$a_i \delta_j^{\dagger^2} \leq \min(a_i, a_j), \tag{5.27}$$

holds for $\gamma \in [1/2, \infty)$.

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/faces. Let $\zeta_\mathcal{F}$ be the characteristic function of $\mathcal{F}$, i.e., the function that is identically one on $\mathcal{F}$ and zero on $\partial\Omega_i \backslash \mathcal{F}$. We clearly have

$$\sum_{\mathcal{F} \subset \partial\Omega_i} \zeta_\mathcal{F}(x) = 1, \qquad \text{almost everywhere on } \ \partial\Omega_i \backslash \partial\Omega.$$

Let $\vartheta_\mathcal{F}$ be the partition of unity functions associated with the edges/faces for a function in the space $U_h(\Omega_i)$, which is defined in [83, Lemma 4.23].

We have

$$|v_i|^2_{S_\Gamma^{(i)}} \leq C \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} |\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}}. \tag{5.28}$$

By Lemma 5.4, with $\overline{\lambda}_{i,\mathcal{F}^{ij}}$ the average over $\mathcal{F}^{ij}$,

$$|\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}}$$
$$\leq \ C_2 a_i |I_H^{\partial\Omega^{(i)}}(\zeta_{\mathcal{F}^{ij}} v_i)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$= \ C_2 a_i |I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}} \delta_j^\dagger (\lambda_i - \lambda_j) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$= \ C_2 a_i \delta_j^{\dagger^2} |I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}} (\lambda_i - \lambda_j) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$

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$$\leq\ 2C_2 a_i \delta_j^{\dagger 2} \left(|I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}}(\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)} \right.$$
$$+\ \left. |I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)} \right). \tag{5.29}$$

We estimate these two terms in (5.29) separately.

The first term is estimated as follows:

$$a_i \delta_j^{\dagger 2} |I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}}(\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ a_i |I_H^{\partial\Omega^{(i)}} \left(\vartheta_{\mathcal{F}^{ij}} I_H^{\partial\Omega^{(i)}}(\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ a_i |\vartheta_{\mathcal{F}^{ij}} I_H^{\partial\Omega^{(i)}}(\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}})|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ a_i |\vartheta_{\mathcal{F}^{ij}} (I_H^{\partial\Omega^{(i)}}\lambda_i - (\overline{I_H^{\partial\Omega^{(i)}}\lambda_i})_{\mathcal{F}^{ij}})|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ C a_i \left(1 + \log \frac{H}{h} \right)^2 |I_H^{\partial\Omega^{(i)}}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)}, \tag{5.30}$$

where we use (5.26) and the definition of $I_H^{\partial\Omega^{(i)}}$ for the first inequality. Using Lemma 5.3, we obtain the second inequality. We use $I_H^{\partial\Omega^{(i)}}(\overline{\lambda}_{i,\mathcal{F}^{ij}}) = (\overline{I_H^{\partial\Omega^{(i)}}\lambda_i})_{\mathcal{F}^{ij}}$ and [83, Lemma 4.26] for the penultimate and final inequalities.

For the second term in (5.29), similarly as for the first term. We have,

$$a_i \delta_j^{\dagger 2} |I_H^{\partial\Omega^{(i)}} \left(\zeta_{\mathcal{F}^{ij}}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ a_j |I_H^{\partial\Omega^{(i)}} \left(\vartheta_{\mathcal{F}^{ij}} I_H^{\partial\Omega^{(j)}}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}}) \right)|^2_{H^{1/2}(\partial\Omega_i)}$$
$$\leq\ a_j |\vartheta_{\mathcal{F}^{ij}} I_H^{\partial\Omega^{(j)}}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}})|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})}$$
$$\leq\ a_j |\vartheta_{\mathcal{F}^{ij}} (I_H^{\partial\Omega^{(j)}}\lambda_j - (\overline{I_H^{\partial\Omega^{(j)}}\lambda_j})_{\mathcal{F}^{ij}})|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})}$$
$$\leq\ C a_j \left(1 + \log \frac{H}{h} \right)^2 |I_H^{\partial\Omega^{(j)}}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)}, \tag{5.31}$$

where we use (5.26) and the definition of $I_H^{\partial\Omega^{(i)}}$ and $I_H^{\partial\Omega^{(j)}}$ for the first inequality. Using Lemma 5.3, we obtain the second inequality. We use $I_H^{\partial\Omega^{(j)}}(\overline{\lambda}_{j,\mathcal{F}^{ij}}) = (\overline{I_H^{\partial\Omega^{(j)}}\lambda_j})_{\mathcal{F}^{ij}}$ and [83, Lemma 4.26] for the penultimate and final inequalities.

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Combining (5.30), (5.31), (5.29), and (5.28), we have

$$
\begin{aligned}
&|v_i|^2_{S_\Gamma^{(i)}} \\
&\leq\ CC_2\left(1+\log\frac{H}{h}\right)^2\left(a_i|I_H^{\partial\Omega^{(i)}}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)}+a_j|I_H^{\partial\Omega^{(j)}}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)}\right) \\
&\leq\ C\frac{C_2}{C_1}\left(1+\log\frac{H}{h}\right)^2\left(|\lambda_i|^2_{S_\Gamma^{(i)}}+|\lambda_j|^2_{S_\Gamma^{(j)}}\right), \qquad\qquad (5.32)
\end{aligned}
$$

where we use Lemma 5.4 again for the final inequality.

Using (5.24), (5.25), and (5.32), we obtain

$$
|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}\leq C\left(1+\log\frac{H}{h}\right)^2|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}.
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.5 Condition Number Estimate for the BDDC Preconditioner

We are now ready to formulate and prove our main result; it follows exactly in the same way as the proof of [58, Theorem 1] by using Lemma 5.5.

Theorem 5.6 *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear from $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the space $\widehat{\Lambda}$ and*

$$
\langle\lambda,\lambda\rangle_{\widehat{S}_\Gamma}\leq\left\langle M^{-1}\widehat{S}_\Gamma\lambda,\lambda\right\rangle_{\widehat{S}_\Gamma}\leq C\left(1+\log\frac{H}{h}\right)^2\langle\lambda,\lambda\rangle_{\widehat{S}_\Gamma},\ \ \forall\lambda\in\widehat{\Lambda}_\Gamma. \qquad (5.33)
$$

Here, C is a constant which is independent of h and H.

5.6 Numerical Experiments

We have applied our BDDC algorithms to the model problem (1.6), where $\Omega = [0,1]^2$. We decompose the unit square into $N\times N$ subdomains with the sidelength

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Table 5.1: Condition number estimates and iteration counts for the BDDC pre-conditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ and $a \equiv 1$.

Number of Subdomains	Iterations	Condition number
4×4	7	2.53
8×8	10	3.01
12×12	10	3.06
16×16	10	3.06
20×20	10	3.06

Table 5.2: Condition number estimates and iteration counts for the BDDC precon-ditioner with a change of the size of the subdomain problems. 8×8 subdomains and $a \equiv 1$.

$\frac{H}{h}$	Iterations	Condition number
4	8	2.23
8	10	3.01
12	11	3.54
16	11	3.95
20	11	4.29

Table 5.3: Condition number estimates and iteration counts for the BDDC pre-conditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ and a is in a checkerboard pattern.

Number of Subdomains	Iterations	Condition number
4×4	8	2.98
8×8	10	2.97
12×12	11	2.98
16×16	11	2.98
20×20	10	2.98

Table 5.4: Eigenvalue estimates and iteration counts for the BDDC preconditioner with a change of the size of the subdomain problems. 8×8 subdomains and a is in a checkerboard pattern.

$\frac{\hat{H}}{h}$	Iterations	Condition number
4	9	2.19
8	10	2.97
12	11	3.51
16	12	3.92
20	13	4.26

$H = 1/N$. Equation (1.6) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure, respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we take the coefficient $a \equiv 1$. Table 5.1 gives the iteration counts and the estimate of the condition numbers, with a change of the number of subdomains. We find that the condition number is independent of the number of subdomains for both algorithms. Table 5.2 gives results with a change of the size of the subdomain problems.

In the second set of experiments, we take the coefficient $a = 1$ in half the subdomains and $a = 100$ in the neighboring subdomains, in a checkerboard pattern. Table 5.3 gives the iteration counts, and condition number estimates with a change of the number of subdomains. We find that the condition numbers are independent

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of the number of subdomains. Table 5.4 gives results with a change of the size of the subdomain problems.

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