NYU-CS-TR2005-871

On the Use of Inexact Subdomain Solvers for BDDC Algorithms

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Abstract

The standard BDDC (balancing domain decomposition by constraints) preconditioner is shown to be equivalent to a preconditioner built from a partially subassembled finite element model. This results in a system of linear algebraic equations which is much easier to solve in parallel than the fully assembled model; the cost is then often dominated by that of the problems on the subdomains. An important role is also played, both in theory and practice, by an average operator and in addition exact Dirichlet solvers are used on the subdomains in order to eliminate the residual in the interior of the subdomains. The use of inexact solvers for these problems and even the replacement of the Dirichlet solvers by a trivial extension are considered. It is established that one of the resulting algorithms has the same eigenvalues as the standard BDDC algorithm, and the connection of another with the FETI–DP algorithm with a lumped preconditioner is also considered. Multigrid methods are used in the experimental work and under certain assumptions, it can be established that the iteration count essentially remains the same as when exact solvers are used, while considerable gains in the speed of the algorithm can be realized since the cost of the exact solvers grows superlinearly with the size of the subdomain problems while the multigrid methods are linear.

Key words: domain decomposition, non-overlapping, FETI–DP, BDDC, partial subassembly, multigrid, multi-level, inexact solvers

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[1] This work was supported in part by the US Department of Energy under Contract DE-FC02-01ER25482.

1 Introduction

Domain decomposition methods based on nonoverlapping subdomains have been widely used and studied for solving large symmetric positive definite linear systems arising from the discretization of elliptic partial differential equations; see [27, Chapters 4–6] and the references therein. Three of the main families of these iterative substructuring methods are the balancing Neumann-Neumann methods [21], [23], [6], the FETI methods [9], [8], and the BDDC methods [5], [24], [25]. The BDDC algorithms represent an interesting redesign of the Neumann-Neumann algorithms with the coarse, global component expressed in terms of a set of primal constraints. All these methods are closely related. Thus, common tools were developed in [18] for the study of the one-level FETI and the classical balancing Neumann–Neumann algorithms; cf. also [27, Chapter 6]. Fragakis and Papadrakakis [10] found experimentally that pairs of such methods have essentially identical spectra; they also discussed primal iterative substructuring methods which are close counterparts to various FETI algorithms. In an important contribution to the theory, Mandel, Dohrmann, and Tezaur [25] established that the preconditioned operators of a pair of BDDC and FETI–DP algorithms, with the same primal constraints, have the same nonzero eigenvalues. In a recent paper, the authors [22] rederived the BDDC and FETI–DP algorithms and also gave a short proof of the main result in [25]. A key to these simplifications is a change of variables so that a primal constraint on the average over an interface edge or face is represented by a single primal variable in the new coordinate system.

In the current standard BDDC, Neumann-Neumann, and FETI algorithms, the subdomain problems are always solved by direct solvers. For an approach to solving the coarse level problem approximately in the BDDC algorithm, see Tu [28,29]. We believe that her work illustrates the fact that it is easier to modify BDDC than FETI–DP algorithms. We also note that the computational work of a direct sparse solver grows faster than linearly with the number of unknowns, and that it can become quite expensive for problems with subdomains with many degrees of freedom. Storage considerations can also, in practice, limit the size of the subdomain problems when direct solvers are used.

The computing cost of some inexact solvers, e.g., a multigrid V-cycle iteration, grows only linearly with the problem size. If the subdomain (and coarse level) problems can be solved approximately, and the resulting algorithm retains a good condition number, substantial gains in efficiency can be realized. We note that inexact solvers have been considered in domain decomposition preconditioners in [1], [14], [15], [4], [26] and for one-level FETI methods in [17]. In some of these methods, subdomain Dirichlet problems are solved by multigrid V-cycles and a well-chosen preconditioner for the subdomain interface Schur

operator is required. It is shown that, with several strategies, there is very little deterioration of convergence rates when replacing the exact solutions of the subdomain problems by multigrid approximations, and the convergence rates of the algorithms are primarily determined by the performance of the preconditioner chosen for the interface Schur operator.

In this paper, we consider the effects of using inexact subdomain solvers in BDDC algorithms and we also consider a related question concerning FETI–DP algorithms. The standard BDDC preconditioner is shown to be equivalent to a preconditioner defined in terms of a partially subassembled problem, which is obtained from subdomain problems by assembling only with respect to a few select primal interface degrees of freedom for each subdomain. Compared with the original discrete problem, the partially subassembled one has much fewer connections between neighboring subdomains and therefore its solution is more suitable for parallel implementation. An average operator, which involves discrete harmonic extensions of the interface jump, is used to connect the solution of the partially subassembled problem, which in general is discontinuous across the subdomain interface, with the original problem defined on a space of continuous finite element functions. We also establish that one of the proposed algorithms, using a different average operator, has the same non-zero eigenvalues as a FETI–DP algorithm with a lumped preconditioner.

This paper is organized as follows. The preconditioners using a partially subassembled problem are described in Sections 2 and 3. Their connections with the standard BDDC operator and a FETP-DP algorithm with a lumped preconditioner are shown in Section 4. The condition number bounds of several preconditioned operators are established in Section 5. Different choices of the coarse level primal set of degrees of freedom are discussed in Section 6 for both two- and three-dimensional problems. The use of multigrid V-cycles for solving the subdomain problems and the partially subassembled problem is discussed in Section 7. In Section 8, numerical experiments of solving a two-dimensional Poisson equation further demonstrate the connections between the related algorithms as well as the effects of using multigrid methods in the algorithms.

2 Discretization and decomposition

Let us consider Poisson's equation on a bounded, polyhedral domain Ω, in two or three dimensions, with homogeneous Dirichlet boundary conditions. The equivalent variational problem is: find $u \in (H_0^1(\Omega))^d = \{w \in (H^1(\Omega))^d \mid w = 0 \text{ on } \partial\Omega\}$, $d = 2$ or 3, such that,

$$a(u, v) = (f, v), \quad \forall v \in (H_0^1(\Omega))^d, \tag{1}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$.

We denote by $\widehat{W}$ the conforming finite element space of functions which are continuous across the element boundaries. The finite element solution $u \in \widehat{W}$, of the problem (1), can be written as

$$Au = f, \tag{2}$$

where the stiffness matrix A is symmetric positive definite.

The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1, 2, ..., N$, of a characteristic diameter H. Each subdomain is a union of shape regular elements and the nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\cup\partial\Omega_i)\backslash\partial\Omega$. The interface Γ is composed of the subdomain faces and/or edges, which are regarded as open sets, and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subregions, and the edges are shared by more than two subregions; in two dimensions, the edges are shared by two subregions. The interface of the subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We assume, as is customary in domain decomposition theory, that the triangulation of each subdomain is quasi uniform. A characteristic diameter of the elements of the underlying triangulation is denoted by h. We also denote the set of nodes on Ω_i and Γ_i by $\Omega_{i,h}$ and $\Gamma_{i,h}$, respectively.

The discrete solution space $\widehat{W}$ is decomposed into subspaces of subdomain interior type and of subdomain interface type, i.e.,

$$\widehat{W} = W_I \bigoplus \widehat{W}_\Gamma = \left(\prod_{i=1}^{N} W_I^{(i)}\right) \bigoplus \widehat{W}_\Gamma,$$

where W_I is the product of subdomain interior variable spaces $W_I^{(i)}$. The elements of $W_I^{(i)}$ are supported in the subdomain Ω_i and vanish on the subdomain interface Γ_i. $\widehat{W}_\Gamma$ is the space of traces on Γ of functions in $\widehat{W}$. For any function in the space $\widehat{W}$, the neighboring subdomains share the same degrees of freedom on the common subdomain interface Γ. We also denote the space of interface variables on the subdomain Ω_i by $W_\Gamma^{(i)}$, and the associated product space by $W_\Gamma = \prod_{i=1}^{N} W_\Gamma^{(i)}$. We will often consider functions in this space are discontinuous across the interface.

The subdomain problems with Neumann boundary conditions on the subdomain interface can be written as

$$A^{(i)}u^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)^T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{bmatrix} = f^{(i)}, \quad i = 1, 2, ..., N, \tag{3}$$

where $u^{(i)} = (u_I^{(i)}, u_\Gamma^{(i)}) \in (W_I^{(i)}, W_\Gamma^{(i)})$. The global problem (2) can then be assembled from the subdomain problems (3), i.e., it can be represented as

$$A = \sum_{i=1}^{N} R^{(i)^T} A^{(i)} R^{(i)}, \quad \text{and} \quad f = \sum_{i=1}^{N} R^{(i)^T} f^{(i)},$$

where $R^{(i)}$ is the restriction operator from the global vector space $\widehat{W}$ to the subdomain vector space $W^{(i)}$, i.e., it maps vectors in the space $\widehat{W}$ to their components on the subdomain Ω_i.

3 Preconditioners using partially subassembled problems

We introduce a space of partially subassembled variables as

$$\widetilde{W} = W_r \bigoplus \widehat{W}_\Pi = \left(\prod_{i=1}^{N} W_r^{(i)} \right) \bigoplus \widehat{W}_\Pi.$$

The space $\widehat{W}_\Pi$ corresponds to a few select subdomain interface degrees of freedom, for each subdomain, and is typically spanned by subdomain vertex nodal basis functions, and/or by interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions correspond to the primal interface continuity constraints enforced in the BDDC algorithm. These interface degrees of freedom are shared by neighboring subdomains, and they are called the coarse level, primal degrees of freedom. The remaining interface degrees of freedom are the dual interface variables. We will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints on edges or faces; see Remark 1 in Section 6. The complementary space W_r is the product of the subdomain spaces $W_r^{(i)}$, which corresponds to the subdomain interior and dual interface degrees of freedom and is spanned by all the basis functions which vanish at the primal degrees of freedom. Thus, the functions in the space $\widetilde{W}$ are only continuous at the coarse level, primal degrees of freedom and are typically discontinuous elsewhere across the subdomain interface.

The partially subassembled problem matrix, corresponding to the variables in the space $\widetilde{W}$, is obtained by assembling the subdomain matrices (3) only with respect to the coarse level primal variables, and we have

$$\widetilde{A} = \sum_{i=1}^{N} \overline{R}^{(i)^T} A^{(i)} \overline{R}^{(i)},$$

where $\overline{R}^{(i)}$ is the restriction operator from the space $\widetilde{W}$ to $W^{(i)}$. Denoting the injection of $\widehat{W}$ into $\widetilde{W}$ by $\widetilde{R}$, we have $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, i.e., the matrix A can

be formed from $\widetilde{A}$ by assembling with respect to the dual interface variables on the subdomain interface. Thus, we can obtain the fully assembled stiffness matrices by two stages of subassembly.

In order to define certain scaling operators, we need to introduce a positive scaling factor $\delta_i^\dagger(x)$, for each node x on the interface Γ_i of the subdomain Ω_i. In applications, these scaling factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see [20], [19]. Here, with $\mathcal{N}_x$ the set of indices of the subdomains which have x on their boundaries, we will only need to use inverse counting functions defined by $\delta_i^\dagger(x) = 1/card(\mathcal{N}_x)$, where $card(\mathcal{N}_x)$ is the number of the subdomains in the set $\mathcal{N}_x$. It is easy to see that $\sum_{j\in\mathcal{N}_x} \delta_j^\dagger(x) = 1$. Given the scaling factors at the subdomain interface nodes, we can define the scaled injection operator $\widetilde{R}_D$; each row of $\widetilde{R}$ corresponds to a degree of freedom of the space $\widetilde{W}$, and multiplying each such row which corresponds to a dual interface degree of freedom with the scaling factor $\delta_i^\dagger(x)$, where $x \in \Gamma_{i,h}$ is the corresponding interface node, gives us $\widetilde{R}_D$.

The first preconditioner introduced in this paper for solving problem (2) is

$$M_1^{-1} = \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D.$$

To multiply $\widetilde{A}^{-1}$ with a vector $\widetilde{g}$, which belongs to the space of right hand sides corresponding to $\widetilde{W}$, we need to solve the following partially subassembled problem, with a leading block diagonal submatrix,

$$\widetilde{A}\widetilde{u} = \begin{bmatrix} A_{rr}^{(1)} & & \widetilde{A}_{\Pi r}^{(1)^T} \\ & \ddots & \vdots \\ \widetilde{A}_{\Pi r}^{(1)} & \dots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_r^{(1)} \\ \vdots \\ u_\Pi \end{bmatrix} = \begin{bmatrix} g_r^{(1)} \\ \vdots \\ g_\Pi \end{bmatrix} = \widetilde{g}, \tag{4}$$

where $\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}$ and $\widetilde{A}_{\Pi r}^{(i)} = R_\Pi^{(i)^T} A_{\Pi r}^{(i)}$. Here $R_\Pi^{(i)}$ is the restriction operator which maps functions in the space $\widetilde{W}_\Pi$ onto their components on the subdomain Ω_i. The inverse of $\widetilde{A}$ can be written as, cf. [22],

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{\Pi r}^T \\ I \end{bmatrix} \widetilde{S}_\Pi^{-1} \begin{bmatrix} -\widetilde{A}_{\Pi r} A_{rr}^{-1} & I \end{bmatrix}, \tag{5}$$

where the coarse problem matrix $\widetilde{S}_\Pi = \sum_{i=1}^N R_\Pi^{(i)^T} (A_{\Pi\Pi}^{(i)} - A_{\Pi r}^{(i)} A_{rr}^{(i)^{-1}} A_{\Pi r}^{(i)^T}) R_\Pi^{(i)}$. It is formed by solving subdomain Neumann problems with given values of the primal degrees of freedom. We see from equation (5) that to multiply $\widetilde{A}^{-1}$ by a vector, subdomain Neumann problems with given primal values and a coarse problem need to be solved.

Since $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, the preconditioned operator $M_1^{-1}A$ can be written as $\widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D \widetilde{R}^T \widetilde{A} \widetilde{R}$, which has the same non-zero eigenvalues as the operator $\widetilde{R}\widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D \widetilde{R}^T \widetilde{A}$. We define an average operator

$$E_{D,1} = \widetilde{R}\widetilde{R}_D^T \ : \ \widetilde{W} \rightarrow \widetilde{W},$$

which preserves the values of the coarse level primal component and the subdomain interior component of the vectors, and provides an average of the dual subdomain interface values. Given any $w \in \widetilde{W}$, we denote its subdomain interior components by $w_I^{(i)}$, its coarse level component on subdomain Ω_i by $w_\Pi^{(i)}$, and its dual interface components by $w_\Delta^{(i)}$. For two-dimensional problems where the coarse level primal set of variables contain all the subdomain corner variables, the component of $E_{D,1}w$ on subdomain Ω_i can be written as,

$$(E_{D,1}w)^{(i)}(x) = w_I^{(i)}(x) + w_\Pi^{(i)}(x) + \sum_{\mathcal{E}^{ij} \subset \partial\Omega_i} (\delta_i^\dagger(x) w_{\Delta,\mathcal{E}^{ij}}^{(i)}(x) + \delta_j^\dagger(x) w_{\Delta,\mathcal{E}^{ij}}^{(j)}(x)),$$

at any node $x \in \overline{\Omega}_{i,h}$. Here $\mathcal{E}^{ij}$ represents an edge of the subdomain Ω_i, which is shared with the subdomain Ω_j, and $w_{\Delta,\mathcal{E}^{ij}}^{(i)}$ represents the restriction of $w_\Delta^{(i)}$ to the edge $\mathcal{E}^{ij}$. From the fact that $\sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) = 1$, for any interface node $x \in \Gamma_{i,h}$, i.e., $\delta_i^\dagger(x) = 1 - \delta_j^\dagger(x)$, in this case, we have

$$(E_{D,1}w)^{(i)} = w^{(i)} - \sum_{\mathcal{E}^{ij} \subset \partial\Omega_i} \delta_j^\dagger(\mathcal{E}^{ij})(w^{(i)} \mid_{\mathcal{E}^{ij}} - w^{(j)} \mid_{\mathcal{E}^{ij}}), \tag{6}$$

where we have also used that fact that $\delta_j^\dagger(x)$ have the same value, $\delta_j^\dagger(\mathcal{E}^{ij})$, at all the nodes of $\mathcal{E}^{ij}$. We can see from equation (6) that the interface jump $w^{(i)} - w^{(j)}$ is extended by zero to the interior of subdomain Ω_i in $E_{D,1}w$.

In Section 6, we will establish a result on the stability of this average operator, i.e., we will bound the jump of the interface variables in equation (6) by the norms of $w^{(i)}$ and $w^{(j)}$. Since the discrete harmonic extension of the interface values to the subdomain interiors gives the minimum in the energy norm, cf. [27, Section 4.4], a better stability bound can be obtained if, in the above example, the component of the averaged vector on the subdomain Ω_i would be

$$(E_{D,2}w)^{(i)} = w^{(i)} - (I \oplus \mathcal{H}^{(i)}) \sum_{\mathcal{E}^{ij} \subset \partial\Omega_i} \delta_j^\dagger(\mathcal{E}^{ij})(w^{(i)} \mid_{\mathcal{E}^{ij}} - w^{(j)} \mid_{\mathcal{E}^{ij}}), \tag{7}$$

where $\mathcal{H}^{(i)} = -{A_{II}^{(i)}}^{-1} {A_{\Gamma I}^{(i)}}^T$ maps the subdomain interface variables to the interior of the subdomain Ω_i. Correspondingly, we define our second average operator by

$$E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathcal{H} J_D),$$

where $J_D : \widetilde{W} \rightarrow \widetilde{W}$, is a jump operator for the dual interface variables across the subdomain interface. For any $w \in \widetilde{W}$, the component on subdomain Ω_i of $J_D w$ is defined by

$$(J_D w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x)(w^{(i)}(x) - w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

We note that $J_D w$ always vanishes in the interior of the subdomain and for the coarse level primal component. For a matrix form of the jump operator, see [27, Section 6.3.3]. $\mathcal{H} : \widetilde{W} \rightarrow \widetilde{W}$, is the direct sum of the subdomain discrete harmonic extension operators $\mathcal{H}^{(i)}$.

The preconditioner corresponding to the use of the average operator $E_{D,2}$ in the algorithm is defined by

$$M_2^{-1} = (\widetilde{R}_D^T - \mathcal{H} J_D)\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}^T),$$

where $\mathcal{H}^T$ is the direct sum of subdomain operators $-A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1}$. The component on subdomain Ω_i of $J_D^T w$ is given by

$$(J_D^T w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} (\delta_j^\dagger(x) w^{(i)}(x) - \delta_i^\dagger(x) w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

The subdomain interior and the coarse level primal components of $J_D^T w$ always vanish.

4 Connections between FETI–DP and BDDC algorithms

We know that the preconditioned operator $M_1^{-1}A$ has the same non-zero eigenvalues as the operator $E_{D,1}\widetilde{A}^{-1}E_{D,1}^T\widetilde{A}$. It can then be shown, as in [22], that the operators $E_{D,1}\widetilde{A}^{-1}E_{D,1}^T\widetilde{A}$ and $(I - E_{D,1}^T)\widetilde{A}(I - E_{D,1})\widetilde{A}^{-1}$ have the same non-zero eigenvalues. From the fact that $I - E_{D,1}$ equals the jump operator J_D defined in Section 3, it can then further be shown that the operator $(I - E_{D,1}^T)\widetilde{A}(I - E_{D,1})\widetilde{A}^{-1}$ has the same nonzero eigenvalues as the FETI–DP algorithm with a lumped preconditioner, cf. [22]. In a recent paper by Fragakis and Papadrakakis [11], it is shown that the performances of FETI algorithms with lumped preconditioners and their primal versions are very similar.

We next show that the preconditioned operator $M_2^{-1}A$ has the same eigenvalues as those of the standard BDDC operator. Here we write the BDDC operator as $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, for solving the reduced interface problem, which has been shown in [22] to be equivalent to the original BDDC operator introduced in [5], [24]. S_Γ and $\widetilde{S}_\Gamma$ are the subdomain interface Schur complements of the matrices in problems (2) and (4), respectively. $\widetilde{R}_{D,\Gamma}$ is formed in the

same way as $\widetilde{R}_D$ and maps the space of continuous interface variables to the space of partially subassembled interface variables with the appropriate scaling. We also denote the restriction of the operators $\widetilde{R}$ and J_D to the space of the partially subassembled interface variables by $\widetilde{R}_\Gamma$ and $J_{D,\Gamma}$, respectively.

We first introduce the following lemma,

Lemma 1 $A_{\Gamma I}^T \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma I}^T J_{D,\Gamma} - \widetilde{A}_{\Gamma I}^T = 0.$

Proof: Since $A_{\Gamma I}$ can be obtained from $\widetilde{A}_{\Gamma I}$ by assembling with respect to the dual interface variables, we have $A_{\Gamma I} = \widetilde{R}_\Gamma^T \widetilde{A}_{\Gamma I}$. Then, from the fact that $\widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T + J_{D,\Gamma} = I$, cf. [22, Lemma 1], we have

$$A_{\Gamma I}^T \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma I}^T J_{D,\Gamma} - \widetilde{A}_{\Gamma I}^T = \widetilde{A}_{\Gamma I}^T \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma I}^T J_D - \widetilde{A}_{\Gamma I}^T = 0.$$

□

Theorem 1 *The preconditioned operator $M_2^{-1}A$ has the same eigenvalues as the BDDC operator $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, except for some eigenvalues equal to 1.*

Proof: We know, by using Gaussian elimination, that A and $\widetilde{A}^{-1}$ can be written as

$$A = \begin{bmatrix} A_{II} & A_{\Gamma I}^T \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} I & \\ A_{\Gamma I} A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & \\ & S_\Gamma \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1} A_{\Gamma I}^T \\ & I \end{bmatrix}, \tag{8}$$

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{II} & \widetilde{A}_{\Gamma I}^T \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{II}^{-1} \widetilde{A}_{\Gamma I}^T \\ & I \end{bmatrix} \begin{bmatrix} A_{II}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -\widetilde{A}_{\Gamma I} A_{II}^{-1} & I \end{bmatrix}. \tag{9}$$

Replacing $\widetilde{A}^{-1}$ and A in the preconditioned operator $M_2^{-1}A$ by the products in equations (8) and (9), we find that $M_2^{-1}A$ has the same eigenvalues as the operator

$$\begin{bmatrix} I & A_{II}^{-1} A_{\Gamma I}^T \\ & I \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1} \widetilde{A}_{\Gamma I}^T J_D \\ & \widetilde{R}_{D,\Gamma}^T \end{bmatrix} \begin{bmatrix} I & -A_{II}^{-1} \widetilde{A}_{\Gamma I}^T \\ & I \end{bmatrix} \begin{bmatrix} A_{II}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -\widetilde{A}_{\Gamma I} A_{II}^{-1} & I \end{bmatrix}$$
$$\begin{bmatrix} I & \\ J_D^T \widetilde{A}_{\Gamma I} A_{II}^{-1} & \widetilde{R}_{D,\Gamma} \end{bmatrix} \begin{bmatrix} I & \\ A_{\Gamma I} A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & \\ & S_\Gamma \end{bmatrix}. \tag{10}$$

Here we have written the operator, $\widetilde{R}_D^T - \mathcal{H} J_D$, and its transpose in matrix form, and moved the last term of the operator to the front, which does not

change the spectrum of the operator. From Lemma 1 and a simple computation, we see that the product of the matrices in (10) equals

$$\begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma}^T \end{bmatrix} \begin{bmatrix} A_{II}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma} \end{bmatrix} \begin{bmatrix} A_{II} & \\ & S_\Gamma \end{bmatrix} = \begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma \end{bmatrix},$$

which has the same eigenvalues as the BDDC operator, $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, except for some eigenvalues equal to 1. □

5 Condition number bounds

Condition number bounds of the BDDC algorithms have been established in [24], [25], and [22], and they also apply to the preconditioned operator $M_2^{-1}A$ as just established in Section 4. In this section, we will follow the analysis in [25] and [18], and reduce the condition number bounds of $M_1^{-1}A$ and $M_2^{-1}A$ to estimates of the norm of the average operators $E_{D,1}$ and $E_{D,2}$. This will also lead to a condition number bound for the FETI-DP algorithm with a lumped preconditioner.

We define the $\widetilde{A}-$seminorm on the space $\widetilde{W}$ by

$$|w|_{\widetilde{A}} = w^T \widetilde{A} w = \sum_{i=1}^N w^T \overline{R}^{(i)^T} A^{(i)} \overline{R}^{(i)} w = \sum_{i=1}^N |\overline{R}^{(i)} w|_{A^{(i)}}.$$

We know that the seminorms $|\cdot|_{A^{(i)}}$ and $|\cdot|_{H^1(\Omega_i)}$ are the same on the space $W^{(i)}$. The following assumption on the stability of $E_{D,1}$ and $E_{D,2}$, will be made specific in Section 6 for different cases.

Assumption 1 *For the average operators $E_{D,1}$ and $E_{D,2}$,*

$$\sum_{i=1}^N |\overline{R}^{(i)}(E_{D,k} w)|^2_{H^1(\Omega_i)} \leq \Phi_k(H,h) \sum_{i=1}^N |\overline{R}^{(i)} w|^2_{H^1(\Omega_i)}, \quad \forall w \in \widetilde{W}, \quad k = 1, 2,$$

where $\Phi_k(H,h)$ are functions of the mesh sizes H and h.

With Assumption 1, we are ready to prove the following theorem.

Theorem 2 *Let Assumption 1 hold. Then the preconditioned operators $M_k^{-1}A$, $k = 1, 2$, are symmetric, positive definite with respect to the bilinear from $\langle \cdot, \cdot \rangle_A$ and*

$$\langle u, u \rangle_A \leq \left\langle M_k^{-1} A u, u \right\rangle_A \leq \Phi_k(H,h) \langle u, u \rangle_A, \ \forall u \in \widehat{W}.$$

Proof: Here we only give a proof for the preconditioned operator $M_2^{-1}A$. The same arguments are also valid for $M_1^{-1}A$.

Lower bound: Given $u \in \widehat{W}$, let

$$w = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au \in \widetilde{W}. \tag{11}$$

We then have, from the fact that $\widetilde{R}^T\widetilde{R}_D = I$ and that $range(J_D^T) \subset null(\widetilde{R}^T)$, that

$$\begin{aligned}\langle u,u\rangle_A &= u^TAu = \frac{1}{2}u^T(\widetilde{R}^T + \mathcal{H}^TJ_D)(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au \\ &\quad + \frac{1}{2}u^T(\widetilde{R}^T - \mathcal{H}^TJ_D)(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au \\ &= \frac{1}{2}u^T(\widetilde{R}^T + \mathcal{H}^TJ_D)\widetilde{A}w + \frac{1}{2}u^T(\widetilde{R}^T - \mathcal{H}^TJ_D)\widetilde{A}w \\ &= u^T\widetilde{R}^T\widetilde{A}w = \left\langle w, \widetilde{R}u\right\rangle_{\widetilde{A}}. \end{aligned} \tag{12}$$

From the Cauchy-Schwarz inequality and the fact that $A = \widetilde{R}^T\widetilde{A}\widetilde{R}$, we have

$$\left\langle w, \widetilde{R}u\right\rangle_{\widetilde{A}} \le \langle w,w\rangle_{\widetilde{A}}^{1/2}\left\langle \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{A}}^{1/2} = \langle w,w\rangle_{\widetilde{A}}^{1/2}\langle u,u\rangle_A^{1/2}. \tag{13}$$

Therefore, from (12) and (13), we have, $\langle u,u\rangle_A \le \langle w,w\rangle_{\widetilde{A}}$. Since

$$\begin{aligned}&\langle w,w\rangle_{\widetilde{A}} = u^TA(\widetilde{R}_D - J_D^T\mathcal{H}^T)^T\widetilde{A}^{-1}\widetilde{A}\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au \\ &= u^TA(\widetilde{R}_D^T - \mathcal{H}J_D)\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au = \left\langle u, M_2^{-1}Au\right\rangle_A, \end{aligned} \tag{14}$$

we have, $\langle u,u\rangle_A \le \left\langle u, M_2^{-1}Au\right\rangle_A$, which gives the lower bound of the theorem.

Upper bound: Given $u \in \widehat{W}$, take $w \in \widetilde{W}$ as in equation (11). We have, $(\widetilde{R}_D^T - \mathcal{H}J_D)w = M_2^{-1}Au$. Using that $A = \widetilde{R}^T\widetilde{A}\widetilde{R}$ and Assumption 1, we have

$$\begin{aligned}&\left\langle M_2^{-1}Au, M_2^{-1}Au\right\rangle_A = \left\langle (\widetilde{R}_D^T - \mathcal{H}J_D)w, (\widetilde{R}_D^T - \mathcal{H}J_D)w\right\rangle_A \\ &= \left\langle \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)w, \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)w\right\rangle_{\widetilde{A}} = |E_{D,2}w|_{\widetilde{A}}^2 \ \le\ \Phi_2(H,h)|w|_{\widetilde{A}}^2. \end{aligned}$$

Therefore, from equation (14), we have

$$\left\langle M_2^{-1}Au, M_2^{-1}Au\right\rangle_A \le \Phi_2(H,h)\left\langle u, M_2^{-1}Au\right\rangle_A. \tag{15}$$

Using the Cauchy-Schwarz inequality and equation (15), we have

$$\left\langle u, M_2^{-1}Au\right\rangle_A \le \langle u,u\rangle_A^{1/2}\left\langle M_2^{-1}Au, M_2^{-1}Au\right\rangle_A^{1/2}$$
$$\le \sqrt{\Phi_2(H,h)}\,\langle u,u\rangle_A^{1/2}\left\langle u, M_2^{-1}Au\right\rangle_A^{1/2}.$$

This gives

$$\left\langle u, M_2^{-1}Au\right\rangle_A \le \Phi_2(H,h)\,\langle u,u\rangle_A,$$

and the upper bound of the theorem. □

6 Examples related to Assumption 1

In this section, we specify the functions $\Phi_k(H,h)$ in Assumption 1 for both two and three dimensional problems, and for different choices of the coarse level primal set degrees of freedom. Here, we always denote by C a positive constant, which is independent of H, h, and the number of subdomains.

6.1 Two-dimensional problems

For a two-dimensional subdomain Ω_i, we denote its edge shared with subdomain Ω_j by $\mathcal{E}^{ij}$. For any finite element function $u^{(i)} \in W^{(i)}$, let $I^{H,\mathcal{E}^{ij}}u^{(i)}$ be the linear interpolant of $u^{(i)}$ on $\mathcal{E}^{ij}$ between its two end point, and let $\overline{u}^{(i)}_{\mathcal{E}^{ij}}$ be its average value on the edge $\mathcal{E}^{ij}$, which is defined by

$$\overline{u}^{(i)}_{\mathcal{E}^{ij}} = \frac{\int_{\mathcal{E}^{ij}} u^{(i)}\,ds}{\int_{\mathcal{E}^{ij}} 1\,ds}. \tag{16}$$

The following lemma can be found in [27, Lemma 4.15].

Lemma 2 *Let Ω_i be a two-dimensional subdomain. For any $u^{(i)} \in W^{(i)}$,*

$$\|u^{(i)}\|^2_{L^\infty(\Omega_i)} \le C\,(1+\log(H/h))\,\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

By using Lemma 2, we can prove the following lemma.

Lemma 3 *Let Ω_i be a two-dimensional subdomain. For any $u^{(i)} \in W^{(i)}$,*

$$\|u^{(i)} - I^{H,\mathcal{E}^{ij}}u^{(i)}\|^2_{L^2(\mathcal{E}^{ij})} \le CH\,(1+\log(H/h))\,|u^{(i)}|^2_{H^1(\Omega_i)},$$

$$\|u^{(i)} - \overline{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ij})} \le CH|u^{(i)}|^2_{H^1(\Omega_i)}.$$

Proof: We first use a trace theorem, cf. [27, Lemma A.6],

$$\|u^{(i)}\|^2_{L^2(\mathcal{E}^{ij})} \leq H\|u^{(i)}\|^2_{H^1(\Omega_i)}, \tag{17}$$

where the factor H results from a scaling argument; more precisely, it can be obtained by mapping the domain Ω_i, of diameter H, into a reference domain of diameter 1, by isotropic dilation as in [27, Section 3.4].

We denote the two finite element nodal basis functions on the coarse mesh associated with the two end points of the edge $\mathcal{E}^{ij}$ by ϕ_1^H and ϕ_2^H, and denote the function values of $u^{(i)}$ at the two end points by u_1 and u_2. We have

$$\|I^{H,\mathcal{E}^{ij}}u^{(i)}\|^2_{L^2(\mathcal{E}^{ij})} \leq 2|u_1|^2\|\phi_2^H\|^2_{L^2(\mathcal{E}^{ij})} + 2|u_2|^2\|\phi_2^H\|^2_{L^2(\mathcal{E}^{ij})}.$$

We know that $\|\phi_1^H\|^2_{L^2(\mathcal{E}^{ij})}$ and $\|\phi_2^H\|^2_{L^2(\mathcal{E}^{ij})}$ can be bounded by CH, cf. [27, Lemma B.5]. Then, from Lemma 2, we have

$$\|I^{H,\mathcal{E}^{ij}}u^{(i)}\|^2_{L^2(\mathcal{E}^{ij})} \leq CH\,(1+\log(H/h))\,\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

From (17) and the above inequality, we obtain the first inequality of the lemma, by using a Poincaré inequality, cf. [27, Theorem A.18], and the fact that $u^{(i)} - I^{H,\mathcal{E}^{ij}}u^{(i)}$ does not change with the addition of a constant to $u^{(i)}$.

For the second inequality, from the definition of the average value in equation (16), the Cauchy-Schwarz inequality, and inequality (17), we have

$$|\overline{u}^{(i)}_{\mathcal{E}^{ij}}|^2 = \Big|\int\limits_{\mathcal{E}^{ij}} u^{(i)}ds\Big|^2 \Big/ \Big|\int\limits_{\mathcal{E}^{ij}} 1\;ds\Big|^2 \leq C\frac{1}{H}\|u^{(i)}\|^2_{L^2(\mathcal{E}^{ij})} \leq C\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

Therefore, $\|\overline{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ij})} \leq CH\|u^{(i)}\|^2_{H^1(\Omega_i)}$. Combining with (17) and noting that $u^{(i)} - \overline{u}^{(i)}_{\mathcal{E}^{ij}}$ is unchanged with respect to a constant shift of the function $u^{(i)}$, we have the second inequality of the lemma, by using a Poincaré inequality. □

From equation (6), we know that, for any $w \in \widetilde{W}$, we can write the component of $E_{D,1}w$ on the subdomain Ω_i as

$$(E_{D,1}w)^{(i)} = w^{(i)} - \sum_{\mathcal{E}^{ij}\subset\partial\Omega_i} \delta_j^\dagger(\mathcal{E}^{ij})I^h(\vartheta_{\mathcal{E}^{ij}}(w^{(i)} - w^{(j)})), \tag{18}$$

where we assume that all the subdomain corner variables are coarse level primal variables. $\vartheta_{\mathcal{E}^{ij}}$ is the standard linear finite element cut-off functions of the edge $\mathcal{E}^{ij}$; it equals 1 at the interior nodes of the edge $\mathcal{E}^{ij}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$. The standard interpolation operator I^h brings us back to the finite element space; it acts on a piecewise quadratic, continuous

function. To bound the norm of $(E_{D,1}w)^{(i)}$, as required in Assumption 1, we only need to bound the terms with the interface jump in equation (18). It is known, see [27, Lemma 4.31], that we can ignore the I^h operator when we derive our estimates.

For the case where only the subdomain corner variables are the primal variables, $I^{H,\mathcal{E}^{ij}}w^{(i)}$ and $I^{H,\mathcal{E}^{ij}}w^{(j)}$ are the same, and we have,

$$\begin{aligned}|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-w^{(j)})|_{H^1(\Omega_i)} \leq & |\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)})|_{H^1(\Omega_i)} \\ & +|\vartheta_{\mathcal{E}^{ij}}(w^{(j)}-I^{H,\mathcal{E}^{ij}}w^{(j)})|_{H^1(\Omega_i)}.\end{aligned}$$

A bound of the first term in the right hand side will be sufficient; the same argument can be used for the second term. The support of the function $\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)})$ in the subdomain Ω_i is only in the strip of elements, with a width h, next to the edge $\mathcal{E}^{ij}$. We denote such an element by K and denote its nodes on the edge $\mathcal{E}^{ij}$ by x_s, $s=1,...,n_K$, where n_K is the number of such nodes. Then the slope of the function $\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)})$ in the element K can be bounded in terms of its values at the nodes x_s. We have

$$\int_K |\nabla(\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)}))|^2 \leq Ch^2\sum_{s=1}^{n_K}\frac{(w^{(i)}(x_s)-I^{H,\mathcal{E}^{ij}}w(x_s))^2}{h^2}.$$

Summing over the elements K next to the edge $\mathcal{E}^{ij}$, we have

$$|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)})|^2_{H^1(\Omega_i)} \leq C\frac{1}{h}\|w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)}\|^2_{L^2(\mathcal{E}^{ij})}.$$

Then from the first inequality in Lemma 3,

$$|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-I^{H,\mathcal{E}^{ij}}w^{(i)})|^2_{H^1(\Omega_i)} \leq C\frac{H}{h}\left(1+\log(H/h)\right)|w^{(i)}|^2_{H^1(\Omega_i)},$$

i.e., $\Phi_1(H,h)=C\frac{H}{h}\left(1+\log(H/h)\right)$, in Assumption 1, for the case where only subdomain corner degrees of freedom are primal.

For the case where the coarse level primal variables also include averages over the subdomain edges, we can bound the terms with the interface jump in equation (18) by

$$|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-w^{(j)})|_{H^1(\Omega_i)} \leq |\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ij}})|_{H^1(\Omega_i)} + |\vartheta_{\mathcal{E}^{ij}}(w^{(j)}-\overline{w}_{\mathcal{E}^{ij}})|_{H^1(\Omega_i)},$$

where $\overline{w}_{\mathcal{E}^{ij}}$ represents the common edge average value of $w^{(i)}$ and $w^{(j)}$ on $\mathcal{E}^{ij}$. Following the same argument as above and using the second inequality in Lemma 3, we can bound the first term in the right hand side by

$$|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ij}})|^2_{H^1(\Omega_i)} \leq C\frac{1}{h}\|w^{(i)}-\overline{w}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ij})} \leq C\frac{H}{h}|w^{(i)}|^2_{H^1(\Omega_i)},$$

i.e., $\Phi_1(H,h) = CH/h$, in Assumption 1, for the case where the set of coarse level primal variables contain both the subdomain corner and the edge average degrees of freedom.

Remark 1 *To enforce the continuity of the edge integrals (16) across the subdomain interface, a change of basis is implemented such that there is an explicit degree of freedom for each edge corresponding to a finite element basis function with weights at the nodes of the edge determined by the integral (16). Since subdomain corner variables, i.e., the variables at the end points of the edges, are also primal variables and therefore continuous, the weights at the corner points can be taken as zero. Then only the interior points of each edge are involved in the edge average basis function, which makes changing basis completely independent on different edges. For an discussion of how to implement the change of variables, see [22], [19, Section 6], and [16].*

6.2 Three-dimensional problems

For a three-dimensional subdomain Ω_i, we denote its face shared with the subdomain Ω_j by $\mathcal{F}^{ij}$. $\vartheta_{\mathcal{F}^{ij}}$ is the face finite element cut-off function, which equals 1 at the interior nodes of $\mathcal{F}^{ij}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$. For each edge $\mathcal{E}^{ik}$ of the subdomain Ω_i, we denote by $\mathcal{M}_{\mathcal{E}^{ik}}$ the index set of all neighboring subdomains which have $\mathcal{E}^{ik}$ as an edge. The edge cut-off function $\vartheta_{\mathcal{E}^{ik}}$ equals 1 at the interior nodes of $\mathcal{E}^{ik}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$.

The following results can be found in Lemmas 4.16 and 4.21 of [27].

Lemma 4 *Let Ω_i be a three-dimensional subdomain. For any $u^{(i)} \in W^{(i)}$,*

$$\|u^{(i)} - \overline{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ik})} \leq C\,(1+\log(H/h))\,|u^{(i)}|^2_{H^1(\Omega_i)},$$

$$\|u^{(i)} - \overline{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{F}^{ij})} \leq CH\,(1+\log(H/h))\,|u^{(i)}|^2_{H^1(\Omega_i)}.$$

In this section, we only specify the bound for $\Phi_1(H,h)$ in Assumption 1 for the case where the coarse level primal variables consist of both the subdomain vertex and edge average degrees of freedom; for other options, see [20] or [27, Section 6.4]. Very much as for the two-dimensional problems, we can write the component on the subdomain Ω_i of $E_{D,1}w$ as

$$\begin{aligned}(E_{D,1}w)^{(i)} = w^{(i)} &- \sum_{\mathcal{F}^{ij}\subset\partial\Omega_i} \delta_j^\dagger(\mathcal{F}^{ij}) I^h(\vartheta_{\mathcal{F}^{ij}}(w^{(i)} - w^{(j)})) \\ &- \sum_{\mathcal{E}^{ik}\subset\partial\Omega_i} \sum_{l\in\mathcal{M}_{\mathcal{E}^{ik}}} \delta_l^\dagger(\mathcal{E}^{ik}) I^h(\vartheta_{\mathcal{E}^{ik}}(w^{(i)} - w^{(l)})). \end{aligned} \tag{19}$$

In the following, we give bounds for the edge and face terms.

Edge terms: Denote the common edge average value of $w^{(i)}$ and $w^{(l)}$ on $\mathcal{E}^{ik}$ by $\overline{w}_{\mathcal{E}^{ik}}$, and we have,

$$|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-w^{(l)})|^2_{H^1(\Omega_i)} \leq 2|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} + 2|\vartheta_{\mathcal{E}^{ik}}(w^{(l)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)}.$$

To bound the first term in the right hand side, we only need to look at the elements next to the edge $\mathcal{E}^{ik}$ in the subdomain Ω_i. For each such element K, we denote its nodes on $\mathcal{E}^{ik}$ by x_s, $s=1,...,n_K$, and we have

$$\int_K \left|\nabla(\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}}))\right|^2 \leq Ch^3 \sum_{s=1}^{n_K} \frac{(w^{(i)}(x_s)-\overline{w}_{\mathcal{E}^{ik}})^2}{h^2}.$$

Summing over the elements K, we have

$$|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} \leq C\|w^{(i)}-\overline{w}_{\mathcal{E}^{ik}}\|^2_{L^2(\mathcal{E}^{ik})} \leq C\left(1+\log(H/h)\right)|w^{(i)}|^2_{H^1(\Omega_i)},$$

where the last inequality is a result of Lemma 4.

Face terms: Denote one edge of the face $\mathcal{F}^{ij}$ by $\mathcal{E}^{ik}$ and the common edge average value of $w^{(i)}$ and $w^{(j)}$ on $\mathcal{E}^{ik}$ by $\overline{w}_{\mathcal{E}^{ik}}$. We have

$$|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-w^{(j)})|^2_{H^1(\Omega_i)} \leq 2|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} + 2|\vartheta_{\mathcal{F}^{ij}}(w^{(j)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)}.$$

For the first term, we note that $\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}})$ is supported only in the elements next to the face $\mathcal{F}^{ij}$ in the subdomain Ω_i. Let us denote such an element by K and denote its nodes on the face $\mathcal{F}^{ij}$ by x_s, $s=1,...,n_K$, where n_K is the number of such nodes. We then have

$$\int_K \left|\nabla(\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}}))\right|^2 \leq Ch^3 \sum_{s=1}^{n_K} \frac{(w^{(i)}(x_s)-\overline{w}_{\mathcal{E}^{ik}})^2}{h^2}.$$

Summing over the elements which are next to $\mathcal{F}^{ij}$, we have

$$|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\overline{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} \leq \frac{1}{h}\|w^{(i)}-\overline{w}_{\mathcal{E}^{ik}}\|^2_{L^2(\mathcal{F}^{ij})} \leq C\frac{H}{h}(1+\log\frac{H}{h})|w^{(i)}|^2_{H^1(\Omega_i)},$$

where the last step is a result of Lemma 4.

Combining the bounds on the edge and face terms in the right hand side of equation (19), we have $\Phi_1(H,h) = C\frac{H}{h}(1+\log(H/h))$, in Assumption 1, for the case where the set of coarse level primal degrees of freedom consist of both the subdomain vertex and edge average degrees of freedom.

Better stability results can be obtained for the average operator $E_{D,2}$ which employs a discrete harmonic extension of the interface jump to the interior of subdomains. For two-dimensional problems, the components of $E_{D,2}w$ in the subdomain Ω_i can be written, as in equation (7),

$$(E_{D,2}w)^{(i)} = w^{(i)} - \sum_{\mathcal{E}^{ij}\subset\partial\Omega_i} \delta_j^\dagger(\mathcal{E}^{ij})(I^h \oplus \mathcal{H}^{(i)})(\vartheta_{\mathcal{E}^{ij}}(w^{(i)} - w^{(j)})),$$

for any given $w \in \widetilde{W}$. A bound on the edge terms can be found in [30, Lemma 3.3]. For three dimensional problems, where both subdomain vertex and edge average degrees of freedom are chosen as primal variables, the results can be found in [27, Lemma 6.34]. In both cases, $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$, in Assumption 1.

To summarize, for two-dimensional problems, when only subdomain corner variables are chosen as coarse level primal variables, $\Phi_1(H,h) = C\frac{H}{h}(1+\log\frac{H}{h})$, for the average operator $E_{D,1}$, and $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$, for $E_{D,2}$, in Assumption 1. When the edge average degrees of freedom are also chosen as primal degrees of freedom for two-dimensional problems, $\Phi_1(H,h) = CH/h$, for $E_{D,1}$. For three-dimensional problems, we have $\Phi_1(H,h) = C\frac{H}{h}(1+\log\frac{H}{h})$, for $E_{D,1}$, and $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$ for $E_{D,2}$, when both the subdomain vertex and edge average degrees of freedom are chosen as primal variables.

7 Inexact solvers

A BDDC preconditioner which uses inexact solvers is defined by

$$M_3^{-1} = (\widetilde{R}_D^T - \mathcal{H}_V J_D)\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}_V^T),$$

where $\mathcal{H}_V$ is the direct sum of subdomain operators $-A_{II_V}^{(i)^{-1}}A_{\Gamma I}^{(i)^T}$. Here $A_{II_V}^{(i)^{-1}}$ and $\widetilde{A}_V^{-1}$ can represent the action of one or several multigrid V-cycles or W-cycles for solving the subdomain Dirichlet problems and the global partially subassembled problem (4), with a zero initial guess, respectively. They can be written in terms of the exact solution operators as, cf. [3], [13],

$$A_{II_V}^{(i)^{-1}} = (I - M_i^\mu)A_{II}^{(i)^{-1}},\ i = 1,2,...,N; \quad \widetilde{A}_V^{-1} = (I - \widetilde{M}^\nu)\widetilde{A}^{-1},$$

where M_i and $\widetilde{M}$ are error propagation matrices corresponding to one multigrid V-cycle or W-cycle iteration. μ and ν are the number of iterations used.

A few nested levels of meshes for the multigrid implementation, for solving the partially assembled problem (4), are shown in Figure 1, with nine subdomains in two dimensions and for the case where the coarse level primal variables are those at the subdomain corners. From the structure of the matrix in (4), we can see that at each level of the V-cycle, a Gauss-Seidel smoothing step can be implemented subdomain by subdomain independently at first, and then followed by a Gauss-Seidel step on the coarse level primal variables, which has to be done sequentially. In our implementation, the coarsest level mesh in the multigrid methods is always given in terms of the nodes related to the coarse level primal variables, e.g., as in Figure 1. At the coarsest level, the problem is solved exactly and the coarse level problem matrix is the stiffness matrix of the continuous bilinear finite element space on the coarsest level mesh. When the size of the coarse level problem is relatively small, the main effort of a multigrid V-cycle for solving the partially subassembled problem (4) will be related to subdomain computations that can be carried out in parallel.

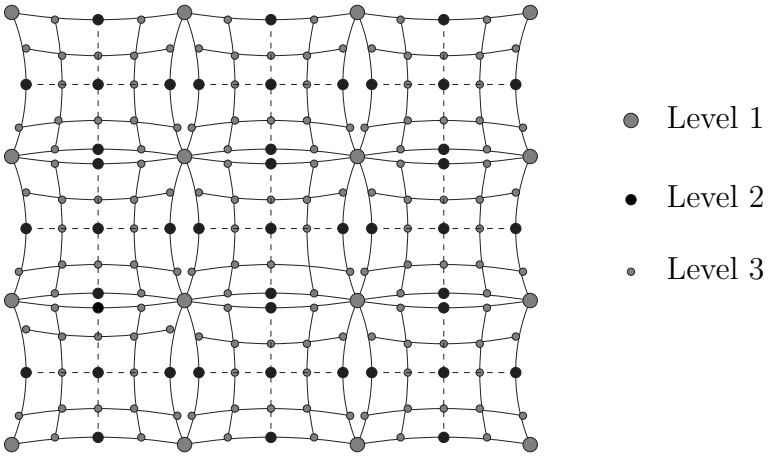

Fig. 1. Three nested meshes for the partially subassembled problem.

In order to show that the condition number bound of the preconditioned operator $M_3^{-1}A$ is as strong as that of $M_2^{-1}A$, we need assume that the convergence rates of the multigrid solvers is independent of the problem size, i.e., we assume

Assumption 2 *There is a positive constant $\eta_* < 1$, which is independent of H, h, and the number of subdomains, such that, $\|M_i\|_{A_{II}^{(i)}} \leq \eta_*$ and $\|\widetilde{M}\|_{\widetilde{A}} \leq \eta_*$.*

The bound $\|M_i\|_{A_{II}^{(i)}} \leq \eta_*$ is known for the subdomain Dirichlet problems, cf. [2] and [3]. A result of Assumption 2 is that the bilinear forms corresponding to the multigrid approximation are spectrally equivalent to the original bilinear forms, cf. [13], i.e., there are positive constants c and C, which are independent of H, h, and the number of subdomains, such that,

$$cu_I^{(i)^T} A_{II_V}^{(i)} u_I^{(i)} \leq u_I^{(i)^T} A_{II}^{(i)} u_I^{(i)} \leq C u_I^{(i)^T} A_{II_V}^{(i)} u_I^{(i)}, \quad i = 1, 2, ..., N, \tag{20}$$

$$cu^T\widetilde{A}_V u \leq u^T\widetilde{A}u \leq Cu^T\widetilde{A}_V u. \tag{21}$$

Using equations (20) and (21), Assumption 1 is also valid for the average operator defined by $\widetilde{R}(\widetilde{R}_D^T - \mathcal{H}_V J_D)$ with the same function $\Phi_2(H,h)$ as for $E_{D,2}$ with exact solvers. With a small modification in the proof of Theorem 2, we can also prove a similar condition number bound for the preconditioned operator $M_3^{-1}A$.

Theorem 3 *Let Assumptions 1 and 2 hold. The preconditioned operator* $M_3^{-1}A = (\widetilde{R}_D^T - \mathcal{H}_V J_D)\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}_V^T)A$*, is symmetric, positive definite with respect to the bilinear from* $\langle\cdot,\cdot\rangle_A$ *and*

$$c\langle u,u\rangle_A \leq \left\langle M_3^{-1}Au,u\right\rangle_A \leq C\Phi_2(H,h)\langle u,u\rangle_A\,,\ \forall u\in\widehat{W},$$

where c *and* C *are positive constants independent of* H*,* h*, and the number of subdomains.*

Another approach to solving the partially subassembled problem (4) inexactly, would be to construct submatrices $\widehat{A}^{(i)}$ which are spectrally equivalent to $A^{(i)}$. Such ideas have proven very successful in the design of many primal iterative substructuring algorithms; see Chapter 5 and in particular Section 5.2 of [27]. We note that all the matrices $A^{(i)}$ corresponding to interior subdomains will be singular and that therefore any such a pair of $A^{(i)}$ and $\widehat{A}^{(i)}$ must have a common null space. Another requirement is that there exists a fast algorithm for solving the linear systems with the submatrices $\widehat{A}_{rr}^{(i)}$. If the pairs of subdomain matrices are spectrally equivalent, it can be easily shown that the matrix obtained by partially subassembling the $\widehat{A}^{(i)}$ will be spectrally equivalent to $\widetilde{A}$. This then straightforwardly leads to a strong overall result if the inexact Dirichlet solvers also are of good quality. Algorithmically, an algorithm based on such a preconditioner would closely resemble the algorithm with exact subdomain solvers; an approximation of $\widetilde{S}_\Pi$ in (5) would be computed and then factored replacing, in the algorithm, the $A^{(i)}$ by the $\widehat{A}^{(i)}$.

8 Numerical experiments

In our numerical experiments, a Poisson equation on a square domain with Dirichlet boundary conditions is discretized using bilinear elements on a uniform square mesh. The FETI–DP algorithm with a lumped preconditioner, the standard BDDC algorithm, and the preconditioners M_1^{-1}, M_2^{-1}, and M_3^{-1} are used to solve the discrete problem. We implement the standard BDDC algorithm in terms of $\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}S_\Gamma$ for solving the interface problem. The preconditioned conjugate gradient method is always used in the iterations, and

it is stopped when the L_2-norm of the residual has been reduced by a factor 10^{-6}. In all the tables below, the condition number bounds are estimated by using the smallest and largest eigenvalues obtained from the tridiagonal Lanczos matrix generated by the preconditioned conjugate gradient iterations.

In Tables 1 and 2, we compare the condition number bounds and iteration counts of $M_1^{-1}A$ and $M_2^{-1}A$ with those of the FETI–DP algorithm with a lumped preconditioner and those of the BDDC algorithm, respectively. Two different sets of coarse level primal variables are used, either only the subdomain corner variables or both the subdomain corner and edge average variables. We see that both condition number bounds and iteration counts match very well for $M_1^{-1}A$ and the FETI–DP algorithm with a lumped preconditioner, and for $M_2^{-1}A$ and the BDDC algorithm. The convergence rates are always independent of the number of subdomains. With an increase of the subdomain problem size, the condition number bounds of $M_1^{-1}A$ and of the lumped FETI–DP algorithm grow quickly, especially for the case when only subdomain corner variables are used as coarse level primal variables. Better condition number bounds are obtained for $M_2^{-1}A$ and the BDDC algorithm, where the subdomain Dirichlet problems are solved directly; we see from Table 2 that the growth of the condition numbers is then slow with an increase of the subdomain problem size.

Tables 3 and 4 show the condition number bounds and iteration counts of $M_3^{-1}A$, where inexact solvers are used, and they are also compared with those of $M_2^{-1}A$. We have chosen three different sets of coarse level primal variables:

- the case with subdomain corner variables only;
- the case with corner variables, variables at the middle points of the edges, and variables at the subdomain centers;
- the case with edge average variables and the subdomain corner and center variables.

On each level of the V-cycle and W-cycle, two Gauss-Seidel iterations were used for both the pre- and the post-smoothing steps, as represented by V_{22} and W_{22} in Tables 3 and 4. One V-cycle is always used for solving the subdomain Dirichlet problems. The $3W_{22}$, $2V_{22}$, and $1V_{22}$ in the tables, represent the use of three W-cycles, two V-cycles, and one V-cycle, respectively, for solving the partially subassembled problem in the algorithm. Table 3 shows that when $3W_{22}$ is used to solve the partially subassembled problem, the condition number bounds and iteration counts are the same as for the algorithm using exact solutions. If only one or two V-cycles are used, we can see from Table 4 that the changes of the condition number bounds and iteration counts are very small, for the cases where the coarse level degrees of freedom does not involve any edge averages. In fact better convergence rates are obtained when $1V_{22}$ was used to solve the partially subassembled problem than when it is solved

Table 1
Condition number bounds (κ) and iteration counts ($it.$) of FETI–DP (lumped) and $M_1^{-1}A$

		With Edge Aver.				Corners only			
		FETIDP		$M_1^{-1}A$		FETIDP		$M_1^{-1}A$	
H/h	# of Sub.	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$
	4×4	1.9	8	1.9	7	8.3	12	8.8	12
	8×8	2.0	8	2.0	7	10.8	19	11.3	17
8	12×12	2.0	8	2.0	7	11.2	19	11.8	17
	16×16	2.0	8	2.0	7	11.3	19	11.6	17
	20×20	2.0	8	2.0	7	11.3	19	11.7	17
4		1.1	5	1.1	4	3.3	9	3.5	9
8	4×4	1.9	8	1.9	7	8.3	12	8.8	12
16		3.8	12	3.9	9	19.6	16	21.4	16
32		8.0	17	8.2	13	56.7	22	63.5	20

Table 2
Condition number bounds and iteration counts of BDDC and $M_2^{-1}A$

		With Edge Aver.				Corners only			
		BDDC		$M_2^{-1}A$		BDDC		$M_2^{-1}A$	
H/h	# of Sub.	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$
	4×4	1.2	5	1.2	5	2.7	8	2.7	8
	8×8	1.2	5	1.2	5	3.0	10	3.0	9
8	12×12	1.2	5	1.2	5	3.1	10	3.1	10
	16×16	1.2	5	1.2	5	3.1	10	3.1	10
	20×20	1.2	5	1.2	5	3.1	10	3.1	10
4		1.1	4	1.1	4	2.0	7	2.0	7
8	4×4	1.2	5	1.2	5	2.7	8	2.7	8
16		1.4	5	1.4	5	3.6	9	3.6	9
32		1.7	6	1.7	6	4.6	10	4.6	9

exactly. For the case where the edge average degrees of freedom were part of the primal variable set, the use of $2V_{22}$ is sufficient to guarantee small changes in the convergence rates in all the experiments.

The CPU time used for running the algorithms with multigrid has not been

Table 3
Condition number bounds and iteration counts of $M_2^{-1}A$ and $M_3^{-1}A$ with three W-cycles

		With Edge Aver.				With Edge Mid.				Corners only			
		$M_2^{-1}A$		$3W_{22}$		$M_2^{-1}A$		$3W_{22}$		$M_2^{-1}A$		$3W_{22}$	
H/h	# of Sub.	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$
	4×4	1.2	5	1.2	5	1.7	6	1.7	6	2.7	8	2.7	8
	8×8	1.2	5	1.2	5	1.8	7	1.8	7	3.0	9	3.0	9
8	12×12	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
	16×16	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
	20×20	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
4		1.1	4	1.1	4	1.3	5	1.3	5	2.0	7	2.0	7
8	4×4	1.2	5	1.2	5	1.7	6	1.7	6	2.7	8	2.7	8
16		1.4	5	1.4	5	2.2	7	2.2	7	3.6	9	3.5	9
32		1.7	6	1.6	6	2.8	8	2.7	8	4.6	9	4.5	10

compared with those of the algorithms using direct solvers. Here we only consider the floating point operation counts (flops) required by the algorithms and show that, for problems of large size, using multigrid V-cycles in the algorithms will be less expensive than using exact solvers. Let us denote the size of the subdomain problems by N. When using a direct solver for solving the subdomain level problems, the factorization step is only implemented once in a preprocessing step of the algorithm reducing the rest of the work of an iteration to forward eliminations and backward substitutions. The best possible bounds for a two dimensional discrete Laplacian is given in [12, Section 8.1] and best possible bounds are also known for three dimensions, see [7]. (In our discussion, we will assume that an optimal ordering is used for the exact solvers.) In each iteration step, the forward and backward substitutions asymptotically require more flops, $O(N \log N)$ in two dimensions and $O(N^{4/3})$ in three dimensions, than $O(N)$ required by one multigrid V-cycle. In addition, the factorization step of a direct solver also requires $O(N^{3/2})$ flops in 2D and $O(N^2)$ in 3D. This shows that replacing the exact solvers in the BDDC algorithms by multigrid V-cycles, where essentially the same iteration count can be retained, will be more effective.

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Table 4
Condition number bounds and iteration counts of $M_3^{-1}A$ with one and two V-cycles

		With Edge Aver.				With Edge Mid.				Corners only			
		$2V_{22}$		$1V_{22}$		$2V_{22}$		$1V_{22}$		$2V_{22}$		$1V_{22}$	
H/h	# of Sub.	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$	κ	$it.$
	4×4	1.3	5	2.0	8	1.6	6	1.6	7	2.4	8	1.9	8
	8×8	1.3	6	2.0	8	1.6	6	1.6	7	2.4	8	2.0	8
8	12×12	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
	16×16	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
	20×20	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
4		1.1	4	1.3	6	1.3	5	1.3	6	1.9	7	1.7	7
8	4×4	1.3	5	2.0	8	1.6	6	1.6	7	2.4	8	1.9	8
16		1.5	6	2.1	8	1.9	7	1.8	7	2.8	9	2.1	8
32		1.8	7	7.9	13	2.2	7	1.9	7	3.1	9	2.3	8

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