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TWO-LEVEL SCHWARZ ALGORITHMS, USING OVERLAPPING SUBREGIONS, FOR MORTAR FINITE ELEMENT METHODS *

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Abstract. Preconditioned conjugate gradient methods based on two-level overlapping Schwarz methods often perform quite well. Such a preconditioner combines a coarse space solver with local components which are defined in terms of subregions which form an overlapping covering of the region on which the elliptic problem is defined. Precise bounds on the rate of convergence of such iterative methods have previously been obtained in the case of conforming lower order and spectral finite elements as well as in a number of other cases. In this paper, this domain decomposition algorithm and analysis are extended to mortar finite elements. It is established that the condition number of the relevant iteration operator is independent of the number of subregions and varies with the relative overlap between neighboring subregions linearly as in the conforming cases previously considered.

Key words. domain decomposition, elliptic finite element problems, preconditioned conjugate gradients, mortar finite elements, overlapping Schwarz algorithms

AMS(MOS) subject classifications. 65F10, 65N30, 65N55

1. Introduction. In this paper, the well-known two level Schwarz method, see, e.g., [15, Chapter 3], is extended to mortar finite element methods. Mortar finite element methods were first introduced in [7]. They are nonconforming finite element methods based on a partitioning, not necessarily geometrically conforming, of the region Ω into substructures Ω_i. Thus, in three dimensions, vertices and edges of one substructure can fall in the interior of edges and/or faces of its neighbors and in two dimensions vertices can divide edges of neighboring substructures. In each of the substructures, we choose a conforming standard finite element or a spectral element method without much regard for its neighbors. Even if the substructures geometrically conform, e.g., when the set of substructures forms a regular finite element triangulation, the local finite element meshes need not. We can also use spectral finite element spaces of different order in different substructures and we can also mix finite elements and spectral elements. In this paper, we will work out a theory only for the case of piece-wise linear mortar finite elements; we treat both the more conventional mortar finite elements and those introduced by Wohlmuth [16, 17].

We note that Achdou and Maday have considered a related problem in [1]. However, in their paper, the principal issue is to establish the convergence and best possible error bounds for finite element methods based on overlapping subdomains. Typically, the meshes in the regions common to two or more overlapping subdomains do not match and mortar conditions are used to introduce a weak continuity between the boundary values of one component of the finite element solution and the interior values of different components along the boundary of the first subdomain. In the final subsection of their paper a convergence result similar to ours, and that for standard conforming elements is formulated and established. We note that in our paper, we instead consider overlapping Schwarz methods for the standard mortar methods.

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For references to earlier work by Cai, Dryja, and Sarkis, which is related to Achdou's and Maday's work, see the reference section of [1].

Finally, a word about the history of this project. The second author worked on algorithms of this kind almost ten years ago; the work was then not completed but some results were presented in a talk at the 1996 ECCOMAS conference in Paris. The basic idea of using three independent decompositions of the region, including one for a conforming finite element space on a regular coarse grid was inspired by a paper by Chan, Smith, and Zou [9]. Around the same time, Dan Stefanica conducted numerical experiments which demonstrated that there is very little difference in the performance of the two-level overlapping Schwarz method for a mortar case and a regular conforming finite element case if the subdomains and the overlap are chosen similarly. The work then lay dormant until it recently was reexamined by the present authors; many details have now been added and a more complete theory has now been developed.

2. The Elliptic Problem and Mortar Finite Element Methods. To simplify the notation, we consider only Poisson's equation. As usual, we formulate our elliptic problem as: find $u \in V$, such that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = f(v) \quad \forall \, v \in \; V. \tag{2.1}$$

The definition of $V \subset H^1(\Omega)$ incorporates the boundary conditions and the region Ω is assumed to be bounded and polyhedral; a homogeneous Dirichlet condition is imposed on a nonempty subset $\partial\Omega_D$ of the boundary $\partial\Omega$ of Ω and a natural boundary condition is given on $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. (Inhomogeneous Neumann boundary data can be incorporated into the right hand side of (2.1).) It is well known that the bilinear form $a(\cdot,\cdot)$ is self-adjoint, elliptic, and bounded in $V \times V$. Our analysis is equally valid for two and three dimensions. The bilinear form $a(u,v)$ is directly related to the Sobolev space $H^1(\Omega)$ that is defined by the semi-norm and norm

$$|u|^2_{H^1(\Omega)} = a(u,u) \text{ and } \|u\|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

respectively.

The discretization of an elliptic, second order problem starts by partitioning the computational domain Ω into a union of nonoverlapping substructures, $\{\Omega_i\}_{i=1}^I$, and an interface Γ, defined by $(\cup_{i\neq j}\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega_D$, which is a set of points that belong to the boundaries of at least two substructures. The restriction to an individual substructure Ω_i, of the mortar finite element space considered in detail in this paper, will just be a standard piecewise linear finite element space defined on a quasi uniform mesh. The meshes of two neighboring substructures do not necessarily match on their common interface and the elements of the discrete space V^h are typically discontinuous across the interface Γ. Instead of pointwise continuity, the interface jumps are made orthogonal to a carefully chosen space of trial functions. In our work, we primarily consider the second generation mortar element methods for which continuity is not even imposed at the vertices or wire baskets (the union of the edges and vertices) of the substructures. Even if the meshes match across the interface between adjacent substructures, the mortar finite element functions will not, generally, be pointwise continuous.

This weak continuity is introduced in terms of a set of *mortars* $\{\gamma_m\}_{m=1}^M$ obtained by selecting open edges/faces of the substructures such that

$$\Gamma = \cup_{m=1}^M \overline{\gamma}_m, \;\; \gamma_m \cap \gamma_n = \emptyset \text{ if } m \neq n.$$

Each edge/face, and mortar γ_m, is viewed as belonging to just one substructure. The remaining edges/faces are the *nonmortars* and are denoted by δ_n. The restrictions of the triangulations of the different substructures to the mortars and nonmortars typically will not match and are denoted by γ_m^h and δ_n^h, respectively; discontinuous mortar finite element functions have two different traces on the interface Γ given by one-sided limits of finite element functions defined on the individual substructures. The continuity across the interface of a conforming finite element method is replaced by weak continuity across the individual nonmortars: for each n, we define a space of test functions $M(\delta_n)$ given by the restriction, to the nonmortar δ_n, of the finite element space defined on the substructure of which δ_n is an edge/face. In two dimensions, the elements of $M(\delta_n)$ are subject to the constraints that they are constant in the first and last mesh intervals of δ_n^h. In three dimensions, the value of a test function of $M(\delta_n)$ at a node on $\partial\delta_n$ is given by a fixed convex combination of nodal values at its next neighbors in δ_n; cf. Ben Belgacem and Maday [5]. We will call this the standard Lagrange multiplier space. In the spectral case, we would use polynomials of a degree two less as test functions.

Lagrange multiplier spaces with dual bases have been developed by Wohlmuth [16, 17]. Each basis function associated with these Lagrange multiplier spaces is supported on a few mesh intervals just as for the standard Lagrange multiplier spaces. They are discontinuous and lead to a diagonal matrix instead of the mass matrix appearing in the standard mortar matching condition. Our algorithm and our proofs can be applied both to the standard and dual Lagrange multiplier spaces and $M(\delta_n)$ can therefore represent either the standard or the dual Lagrange multiplier space.

In this paper, we consider partitions $\{\Omega_i\}_{i=1}^I$, where the Ω_i are geometrically nonconforming. We assume that $\{\Omega_i\}_{i=1}^I$ form a regular partition of Ω, i.e., the size of Ω_i is comparable to that of its neighboring substructures. We will impose some assumptions on the meshes and the Lagrange multiplier space $M(\delta_n)$. A nonmortar $\delta_n \subset \partial\Omega_i$ can be partitioned into several edges/faces $\{\delta_{n,j}\}_j$ by mortar neighbors $\Omega_{m(n,j)}$ with boundaries which intersect $\partial\Omega_i$ along $\delta_{n,j}$, i.e., $\delta_{n,j} = \partial\Omega_{m(n,j)} \cap \partial\Omega_i$. We will use the following assumptions on the meshes and the Lagrange multiplier space in some of our work.

ASSUMPTION 1. *Each subpartition $\delta_{n,j}$ of a nonmortar is the union of entire elements.*

ASSUMPTION 2. *The Lagrange multiplier space $M(\delta_{n,j})$ are defined on each edge/face of the partition $\delta_{n,j}$ individually. Standard or dual Lagrange multiplier spaces are thus given on each $\delta_{n,j}$ which inherits the triangulation from δ_n^h. The Lagrange multiplier space $M(\delta_n)$ on δ_n is then defined by*

$$M(\delta_n) = \prod_{\delta_{n,j}} M(\delta_{n,j}).$$

With these assumptions, mortar methods provide a best approximation even for geometrically nonconforming partitions. Without them, an additional factor $|\log(h)|$ will appear in the error bound; see [2]. See also [3, 4, 5, 7] where error bounds of the same type as for standard conforming methods are derived. We will first analyze two-level overlapping Schwartz algorithms for mortar methods under Assumptions 1 and 2 and we will later derive slightly weaker result after removing these assumptions.

The *mortar projection* π_n maps all of $L_2(\delta_n)$ onto the finite element space defined on the nonmortar mesh δ_n^h. For two dimensions and for a given $w \in L_2(\delta_n)$ with given values at v_{n_1} and v_{n_2}, the endpoints of δ_n, we define $\pi_n(w, w^{(n)}(v_{n_1}), w^{(n)}(v_{n_2}))$ on δ_n^h by

$$\int_{\delta_n} (w - \pi_n(w, w^{(n)}(v_{n_1}), w^{(n)}(v_{n_2}))\psi ds = 0 \ \ \forall \psi \in M(\delta_n). \tag{2.2}$$

We note that only the values at the interior nodes of δ_n are determined by this condition; the values $w^{(n)}(v_{n_1})$ and $w^{(n)}(v_{n_2})$ are genuine degrees of freedom. Similarly, for three dimensions, the values in the interior of δ_n are determined not only by the values on the part of Γ opposite the nonmortar, but also by the nodal values on $\partial\delta_n$.

As when working with other nonconforming methods, the original bilinear form $a(\cdot,\cdot)$ is replaced by $a^\Gamma(\cdot,\cdot)$ defined as the sum of the contributions from the individual substructures to $a(\cdot,\cdot)$:

$$a^\Gamma(u_h, v_h) = \sum_{i=1}^{I} a_{\Omega_i}(u_h, v_h). \tag{2.3}$$

For $u_h = v_h$, we obtain the square of what is often called *a broken norm*. The norm has been broken along Γ and it is finite for any element of the mortar space even if it is discontinuous across Γ. The resulting discrete variational problem gives rise to a linear system with a symmetric, positive definite matrix.

After these preparations, the mortar finite element space V^h, and the problem as a whole, can be fully defined. The discrete problem is then: find $u \in V^h$ such that

$$a^\Gamma(u, v) = f^\Gamma(v) \ \ \forall v \in V^h, \tag{2.4}$$

where $a^\Gamma(u, v)$ is defined in formula (2.3) and, similarly, $f^\Gamma(v)$ is the sum of contributions from the different substructures.

3. The Dryja-Widlund Algorithm. We now describe the additive Schwarz method introduced in Dryja and Widlund [10]; cf. also Smith, Bjørstad, and Gropp [14, Chap.5] and, for many details, Toselli and Widlund [15, Chap.3]. This additive Schwarz method for an overlapping subdomain partition performs quite well even for partitions with small overlap as first established in Dryja and Widlund [11]. The condition number bound given in [11] has also been proven to be optimal by Brenner [8]. We now use two additional decompositions of the region Ω, in addition to the set of substructures $\{\Omega_i\}$, used to define the mortar finite element problem, namely a set of overlapping subregions $\{\widetilde{\Omega}_j\}$ and an independent coarse mesh $\{\tau_l^H\}$. Let X_i^h be the finite element space on the substructure Ω_i equipped with a quasi-uniform triangulation $\mathcal{T}^h(\Omega_i)$. Throughout this paper, we will impose the following assumptions on these partitions:

ASSUMPTION 3. *The diameter H_i of a substructure Ω_i is comparable to the diameter H of any triangle τ_l^H that intersects it.*

ASSUMPTION 4. *The diameter H_i of a substructure Ω_i satisfies*

$$H_i \le C\widetilde{H}_j,$$

where $\widetilde{H}_j$ is the diameter of any subregion $\widetilde{\Omega}_j$ that intersects it.

ASSUMPTION 5. *The mesh sizes of the substructures that intersect along a common edge/face are comparable.*

The $\widetilde{\Omega}_j$ can be quite arbitrary; a local subspace V_j will be associated with each of them, essentially by making all genuine degrees of freedom associated with nodes outside $\widetilde{\Omega}_j$ equal to zero. More precisely, the space V_j is given by

$$V_j = \left\{ v \in \prod_{i=1}^{I} X_i^h \ : \ v(x) = 0 \text{ for } x \in \Omega \setminus \widetilde{\Omega}_j, \text{ or } x \in \delta_n \right\}, \quad j = 1, \cdots, N,$$

where δ_n denotes any nonmortar edges/faces. The space V_0 is V^H, the space of continuous, piecewise linear functions on an independent coarse mesh given by its elements τ_l^H. We further impose zero Dirichlet conditions, on the elements of V_j, on $\partial\widetilde{\Omega}_j \cap \partial\Omega_D$ and on the elements of V_0, on $\partial\Omega_D$.

We remark that the overlap can be quite small. Thus, if no degrees of freedom are shared between neighboring subregions, the overlap is on the order of h, the diameter of the elements of the fine discretization. Our analysis applies in this case as well in which case our Schwarz method corresponds to a block Jacobi preconditioner augmented by a coarse solver.

It is now appropriate essentially to follow the description and analysis of Schwarz methods given in Smith, Bjørstad, and Gropp [14] and Toselli and Widlund [15]. Our iterative method is given in terms of N+l finite element spaces $V_j^h, j = 0, \ldots, N$, which are subspaces of V^h and are associated with the space V_j:

$$V_j^h = I^m(V_j).$$

The interpolation operator $I^m : \prod_{i=1}^I C(\Omega_i) \to V^h$ is defined by

$$I^m(u) = \sum_{i=1}^{I} \left(I_i^h(u) + \sum_{\delta_n \subset \partial\Omega_i} \widetilde{\pi}_n \left(I^h_{m(\delta_n)}(u) - I_i^h(u) \right) \right), \tag{3.1}$$

where $I_i^h(u)$ is the nodal value interpolant in the space X_i^h and $\widetilde{\pi}_n(w)$ is the zero extension of $\pi_n(w)$ to $\overline{\Omega}_i$. Here $\pi_n(w)$ denotes $\pi_n(w,0,0)$; see (2.2). (In the following, we will use this simple notation $\pi_n(w)$ in stead of $\pi_n(w,0,0)$.) It has been shown that $\pi_n(w)$ is L^2-stable but not H^1-stable; see [17, Chap.1]. We recall that δ_n denotes a nonmortar edge/face of $\partial\Omega_i$ and that $\{\delta_{n,j}\}_j$ is the partition of δ_n described in Section 2, i.e., $\delta_{n,j} = \partial\Omega_{m(n,j)} \cap \partial\Omega_i$. The interpolant $I^h_{m(\delta_n)}(u)$ is defined by

$$I^h_{m(\delta_n)}(u) = I^h_{m(n,j)}(u) \ \text{ on } \delta_{n,j},$$

and it can thus be discontinuous across the boundaries of $\delta_{n,j}$. The mortar finite element space V^h can then be represented as the sum

$$V^h = V_0^h + V_1^h + \cdots + V_N^h \,. \tag{3.2}$$

REMARK 1. *The local spaces $\{V_j^h\}_{j=1}^N$, in which our Schwarz algorithm will be considered, consist of functions defined on the whole domain Ω not just on the subregion $\widetilde{\Omega}_j$ as in the standard Schwarz algorithms described in [15, Chap.3]. Therefore the trivial extension operator from V_j^h to V^h will not appear in our algorithm. We note that the support of each function in V_j^h is contained in the union of the substructures Ω_i that intersect the subregion $\widetilde{\Omega}_j$.*

It is often more economical to use approximate rather than exact solvers for the subspace problems. The approximate solvers can be described in terms of inner products $\tilde{a}_j(\cdot,\cdot)$ defined on $V_j^h \times V_j^h$. One assumption that needs to be checked for each of them is the existence of a constant ω such that

$$a^\Gamma(u,u) \le \omega \tilde{a}_j(u,u) \ \ \forall u \in V_j^h \,. \tag{3.3}$$

In terms of matrices, this inequality becomes a one-sided bound of a submatrix of the stiffness matrix, given by $a^\Gamma(\cdot,\cdot)$ and V_j^h, in terms of the matrix given by $\tilde{a}_j(\cdot,\cdot)$.

A projection-like operator $T_j : V^h \to V_j^h$, is now defined for each j by

$$\tilde{a}_j(T_j u, \phi_h) = a^\Gamma(u,\phi_h) \quad \forall \phi_h \in V_j^h \,. \tag{3.4}$$

It is easy to show that the operator T_j is positive semidefinite and symmetric with respect to $a^\Gamma(\cdot,\cdot)$ and that the minimal constant ω in equation (3.3) is $\|T_j\|_a$, i.e.,

$$\|T_j\|_a \leq \omega; \tag{3.5}$$

see [15, Chap2]. Additive and multiplicative Schwarz methods can now be defined straightforwardly in terms of polynomials of the operators T_j. We note that if exact solvers, and thus genuine projections P_j, are used, then $\omega = 1$. The operator relevant to an additive Schwarz operator is $T = \sum_{j=0}^{N} T_j$. In the case of no coarse space and the local spaces forming a direct sum, this operator is a block-Jacobi operator, with one block for each subspace.

In order to estimate the rate of convergence of our special, or any other, additive Schwarz methods, we need upper and lower bounds for the spectrum of the operator relevant in the conjugate gradient iteration. A lower bound can be obtained by using the following lemma; see, e.g., Zhang [18], Smith, Bjørstad, and Gropp [14], or Toselli and Widlund [15, Chap 2].

LEMMA 1. *Let T_j be the operators defined in equation (3.4) and let $T = T_0 + T_1 + \cdots + T_N$. Then,*

$$a(T^{-1}u,u) = \min_{u=\sum u_j} \sum \tilde{a}_j(u_j,u_j),\ u_j \in V_j^h.$$

Therefore, if a representation, $u = \sum u_j$, can be found, such that

$$\sum \tilde{a}_j(u_j,u_j) \leq C_0^2 a(u,u)\ \ \forall u \in V^h,$$

then,

$$\lambda_{min}(T) \geq C_0^{-2}.$$

For the algorithms considered in this paper, and many other domain decomposition algorithms, it is easy to show that there is an upper bound for T which is proportional to ω.

In this paper, our results are only formulated for additive algorithms and with exact solvers for the subdomain problems. The corresponding bounds for the multiplicative variants, etc., can easily be worked out using the general Schwarz theory; see, e.g., Smith, Bjørstad, and Gropp [14], or Toselli and Widlund [15, Chap.2].

4. The lower bound. We will find a lower bound of the two-level Schwarz algorithm. A stable decomposition of $u = \sum_{j=0}^{N} u_j$ will be provided with

$$C_0^2 = C \max_{j=1,\cdots,N} \left\{ (1 + H_j/\delta_j) \right\}.$$

Here H_j is the diameter of the subregion $\widetilde{\Omega}_j$ and δ_j is the overlapping width of $\widetilde{\Omega}_j$, i.e., the minimal width of the subset of $\widetilde{\Omega}_j$ which is common to some neighbors, and C is a constant independent of the mesh sizes, the subregion diameters and the number of subregions; see Figure 1. We first assume that the five assumptions hold and later derive a bound for C_0^2 with an additional $\log(H/h)$ factor for the general case for which Assumptions 1 and 2 are removed.

4.1. Technical Tools. In this section, we will collect a number of technical tools that are used in proving our main results. Some of the tools can be borrowed directly from Toselli and Widlund [15, Chap.3], but some work also needs to be done that is directly related to the mortar finite element method.

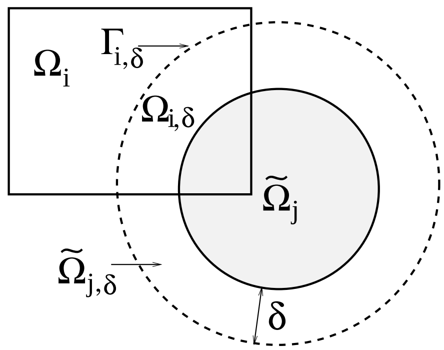

FIG. 1. *The substructure Ω_i intersects the subregion $\widetilde{\Omega}_j$ (interior of the dashed circle); $\widetilde{\Omega}_{j,\delta}$ (the part between the dashed and the solid circles) is the support of $\nabla\theta_j$, $\Omega_{i,\delta}$ is the part of $\widetilde{\Omega}_{j,\delta}$ which belongs to Ω_i, and $\Gamma_{i,\delta}$ is a part of the boundary of $\Omega_{i,\delta}$ that divides Ω_i into two parts.*

As before, $\Omega \subset R^d, d =$2 or 3, is a bounded, polygonal region, $\{\Omega_i\}_{i=1}^I$ is a nonoverlapping decomposition of Ω into substructures, and $\{\widetilde{\Omega}_j\}_{j=1}^N$ that of a set of overlapping subregions. Let $\{\widetilde{\theta}_j\}_{j=1}^N$ be a partition of unity for the overlapping partition $\{\widetilde{\Omega}_j\}_{j=1}^N$ of Ω, with the following properties (see, e.g., [15, Section 3.2]):

$$
\begin{aligned}
&0 \leq \tilde{\theta}_j(x) \leq 1, \quad x \in \widetilde{\Omega}_j,\\
&\text{supp}\,(\widetilde{\theta}_j) \subset \widetilde{\Omega}_j,\\
&\sum_{j=1}^N \widetilde{\theta}_j = 1,\\
&|\nabla\widetilde{\theta}_j| \leq \frac{C}{\delta_j}.
\end{aligned}
$$

We will employ a modified partition of unity θ_j obtained by interpolating $\widetilde{\theta}_j$ on the triangulations $\{\mathcal{T}^h(\Omega_i)\}_{i=1}^I$. The θ_j will be discontinuous across substructure interfaces. However, we can easily check that the modified partition of unity $\{\theta_j\}_{j=1}^N$ has the same properties as $\{\widetilde{\theta}_j\}_{j=1}^N$ when restricted to any substructure Ω_i because these properties hold for each elements of $\{\mathcal{T}^h(\Omega_i)\}_{i=1}^I$.

We now consider the case in Figure 1. The substructure Ω_i intersects the subregion $\widetilde{\Omega}_j$. We denote the support of $\nabla\theta_j$ by $\widetilde{\Omega}_{j,\delta}$ and the intersection of Ω_i and $\widetilde{\Omega}_{j,\delta}$ by $\Omega_{i,\delta}$. As in the Figure 1, we select $\Gamma_{i,\delta}$, as a part of the boundary of $\Omega_{i,\delta}$ that divides the domain Ω_i into two parts. We will prove the following lemma that is similar to the one provided in [15, Lemma 3.10]:

LEMMA 2. *Let u be an arbitrary element of $H^1(\Omega_i)$. Then,*

$$\|u\|^2_{L^2(\Omega_{i,\delta})} \leq C\,\delta^2\big((1+H_i/\delta)|u|^2_{H^1(\Omega_i)} + 1/(H_i\delta)\|u\|^2_{L^2(\Omega_i)}\big)\ ,$$

where H_i denotes the diameter of Ω_i and δ is the overlapping width of $\widetilde{\Omega}_j$, a subregion that intersects Ω_i.

Proof. Let us cover $\Omega_{i,\delta}$ by shape-regular patches $\{P_l\}_l$ with $O(\delta)$ diameters. We may assume that the $P_{l,\Gamma}(:= \partial P_l \cap \Gamma_{i,\delta})$ have positive measure. By using a Friedrichs inequality (see Toselli and Widlund [15, Lemma A.17]) for each patch P_l and summing over all patches, we obtain

$$\|u\|^2_{L^2(\Omega_{i,\delta})} \le C\left(\delta^2 |u|^2_{H^1(\Omega_{i,\delta})} + \delta \|u\|^2_{L^2(\Gamma_{i,\delta})}\right). \tag{4.1}$$

From the embedding $H^{1/2}(\Gamma_{i,\delta}) \subset L^2(\Gamma_{i,\delta})$, a trace theorem, and a scaling argument, we obtain

$$\begin{aligned}
\|u\|^2_{L^2(\Gamma_{i,\delta})} &= H_i^{d-1} \|\widehat{u}\|^2_{L^2(\widehat{\Gamma}_{i,\delta})} \\
&\le C H_i^{d-1} \|\widehat{u}\|^2_{H^{1/2}(\widehat{\Gamma}_{i,\delta})} \\
&\le C H_i^{d-1} \|\widehat{u}\|^2_{H^1(\widehat{\Omega}_{i,1})} \\
&\le C H_i^{d-1} \left(|\widehat{u}|^2_{H^1(\widehat{\Omega}_i)} + \|\widehat{u}\|^2_{L^2(\widehat{\Omega}_i)} \right) \\
&\le C H_i^{d-1} \left(H_i^{2-d} |u|^2_{H^1(\Omega_i)} + H_i^{-d} \|u\|^2_{L^2(\Omega_i)} \right).
\end{aligned}$$

Here the hat designates a dilated domain with diameter 1 or a function defined on the scaled domain, and $\Omega_{i,1}$ is a part of Ω_i divided by $\Gamma_{i,\delta}$. By combining the above estimate with (4.1), the desired bound follows. ◻

We also have the following generalized Poincaré-Friedrichs inequality (see Nečas [13]).

LEMMA 3. *Let Φ be a seminorm on $H^1(\Omega)$ with the following properties*

(1) $\Phi(\phi) \le C_1 \|\phi\|_{1,\Omega}, \quad \forall \phi \in H^1(\Omega),$
(2) For a constant function $c, \quad \Phi(c) = 0$ *iff* $c = 0.$

Then we have a generalized Poincaré-Friedrichs inequality for $H^1(\Omega)$

$$\|\phi\|_{0,\Omega} \le C H^{d/2} \left(H^{(2-d)/2} |\phi|_{1,\Omega} + H^{k(\Phi)} \Phi(\phi) \right) \quad \forall \phi \in H^1(\Omega),$$

where d is the dimension of the domain Ω, H is the diameter of Ω, and the constant C is independent of H; $\Phi(\phi)$ is homogeneous of degree $k(\Phi)$, i.e., $k(\Phi)$ is the real number which makes $H^{k(\Phi)}\Phi(\phi)$ invariant to scaling.

A prime example is provided by

$$\Phi(\phi) = \left| \int_\gamma \phi \, ds \right|, \quad \forall \phi \in H^1(\Omega).$$

Then, the two assumptions of Lemma 3 hold for $\Phi(\phi)$ and the application of the Poincaré-Friedrichs inequality for ϕ with a zero average on γ, i.e., $\Phi(\phi) = 0$, gives

$$\|\phi\|_{0,\Omega} \le C H |\phi|_{1,\Omega}. \tag{4.2}$$

We will now consider two cases. In the first, the meshes and Lagrange multipliers satisfy Assumptions 1 and 2 on the nonconformity of the subdomain partition $\{\Omega_i\}_{i=1}^I$. In the second, we will drop these assumptions. In the later case, the Lagrange multiplier space $M(\delta_n)$ is then a standard or dual Lagrange multiplier space defined on the triangulation δ_n^h, without partitioning it into $\{\delta_{n,j}\}_j$. The following approximation properties hold for both the standard and the dual Lagrange multiplier spaces; see [7, 12, 17].

LEMMA 4. *Let $0 < \alpha \le 1/2$. For $v \in H^\alpha(\delta_{n,j})$, there exists a $\psi \in M(\delta_{n,j})$ such that*

$$\|v - \psi\|_{0,\delta_{n,j}} \le C h^\alpha |v|_{H^\alpha(\delta_{n,j})},$$

where h denotes the diameter of the elements of the nonmortar δ_n.

LEMMA 5. *Let $0 < \alpha \leq 1/2$. For $v \in H^\alpha(\delta_n)$, there exists $\psi \in M(\delta_n)$ such that*

$$\|v - \psi\|_{(H^\alpha(\delta_n))'} \leq Ch^{2\alpha}|v|_{H^\alpha(\delta_n)},$$

where h denotes the diameter of the nonmortar elements and $(H^\alpha(\delta_n))'$ is the dual space of $H^\alpha(\delta_n)$.

LEMMA 6. *Let the meshes and Lagrange multiplier spaces satisfy Assumptions 1 and 2. Then, for $v = (v_1, \cdots, v_I) \in V^h$, we have*

$$\|v_i - v_j\|_{0,\delta_{n,j}} \leq Ch_i^{1/2}\left(|v_i|_{1,\Omega_i} + |v_j|_{1,\Omega_j}\right),$$

where Ω_i and Ω_j are the nonmortar and mortar substructures of the interface $\delta_{n,j} = \partial\Omega_i \cap \partial\Omega_j$.

Proof. We have

$$\begin{aligned}\|v_i - v_j\|^2_{0,\delta_{n,j}} &= \int_{\delta_{n,j}} (v_i - v_j)(v_i - v_j - \psi)\, ds \\ &\leq \|v_i - v_j\|_{0,\delta_{n,j}}\|v_i - v_j - \psi\|_{0,\delta_{n,j}}.\end{aligned}$$

This inequality holds for an arbitrary $\psi \in M(\delta_{n,j})$. Applying Lemma 4 with $\alpha = 1/2$ and a trace theorem, we obtain

$$\min_{\psi \in M_{\delta_{n,j}}} \|v_i - v_j - \psi\|_{0,\delta_{n,j}} \leq Ch_i^{1/2}\left(|v_i|_{1,\Omega_i} + |v_j|_{1,\Omega_j}\right).$$

□

We now consider a general case without the extra Assumptions 1 and 2 on the meshes and Lagrange multiplier spaces. The set of nonmortars $\{\delta_n\}_n$ is selected from the edges/faces of the subdomain partition and the Lagrange multiplier spaces $M(\delta_n)$ are defined on the finite elements associated with the nonmortar interfaces δ_n. We recall that any nonmortar edge/face $\delta_n \subset \partial\Omega_i$ is partitioned into

$$\overline{\delta}_n = \cup_j \overline{\delta}_{n,j}, \quad \delta_{n,j} = \delta_n \cap \partial\Omega_{n_j}.$$

The mortar matching condition is then

$$\int_{\delta_n} (v_{i(n)} - \phi)\psi\, ds = 0, \quad \forall \psi \in M(\delta_n), \tag{4.3}$$

where ϕ is given by $\phi = v_{n_j}$ on $\delta_{n,j}$. We see that $\phi \in H^{1/2-\epsilon}(\delta_n)$ for any $0 < \epsilon \leq 1/2$. Moreover the following estimate holds for ϕ; see [6]:

LEMMA 7. *Let each subdomain Ω_{n_j} be scaled by H_i, the diameter of the subdomain Ω_i. Then, for any $0 < \epsilon \leq 1/2$, we have,*

$$\sqrt{\epsilon}\|\phi\|_{H^{1/2-\epsilon}(\delta_n)} \leq C\sum_j \|v_{n_j}\|_{1,\Omega_{n_j}},$$

where ϕ is given by $\phi = v_{n_j}$ on $\delta_n \cap \partial\Omega_{n_j}$.

In the general case, without Assumptions 1 and 2, the space V^h consists of functions $v = (v_1, \cdots, v_I)$ satisfying the mortar matching condition (4.3) on each nonmortar edge/face

δ_n. Let us denote by $\{\psi_l\}_l$ a basis for the Lagrange multiplier space $M^h(\delta_n)$. We also select $\{\psi_{j_k}\}_k$ from $\{\psi_l\}_l$ such that supp $(\psi_{j_k}) \subset \delta_{n,j}\,(= \delta_n \cap \partial\Omega_{n_j})$, and set $\psi_{n,j} = \sum_k \psi_{j_k}$; we assume that at least one such ψ_{j_k} exists for every $\delta_{n,j}$. We will then show that the L^2-norm of the jump across δ_n is bounded by the sum of H^1-seminorms of the functions on the subdomains Ω_k for which $\partial\Omega_k$ intersects δ_n with a positive measure.

LEMMA 8. *Let $\delta_n \subset \partial\Omega_i$ be a nonmortar edge/face. For the general case, without Assumptions 1 and 2, we have*

$$\|v_i - \phi\|_{0,\delta_n} \le C h_i^{1/2} \left(\mathit{log}\, \frac{H_i}{h_i} \right)^{1/2} \left(|v_i|_{1,\Omega_i} + \sum_j |v_{n_j}|_{1,\Omega_{n_j}} \right),$$

for $v = (v_1, \cdots, v_I) \in V^h$ where ϕ is given by $\phi = v_{n_j}$ on $\delta_{n,j}$.

Proof. We first dilate Ω_i and Ω_{n_j} so that the diameter of Ω_i is 1. The triangles/tetrahedra of each subdomain are then also scaled by the diameter H_i. We obtain,

$$\begin{aligned}
\|v_i - \phi\|^2_{0,\delta_n} &= \int_{\delta_n} (v_i - \phi)(v_i - \phi - \psi)\, ds \\
&\le \|v_i - \phi\|_{H^{1/2-\epsilon}(\delta_n)} \|v_i - \phi - \psi\|_{(H^{1/2-\epsilon}(\delta_n))'} \\
&\le C \|v_i - \phi\|_{H^{1/2-\epsilon}(\delta_n)} h_i^{2(1/2-\epsilon)} |v_i - \phi|_{H^{1/2-\epsilon}(\delta_n)} \\
&\le C h_i^{1-2\epsilon} \|v_i - \phi\|^2_{H^{1/2-\epsilon}(\delta_n)}. \qquad (4.4)
\end{aligned}$$

Here $\psi \in M(\delta_n)$ is the best approximation and h_i is the scaled mesh size. We have also used the mortar matching condition and Lemma 5 for the function $v_i - \phi \in H^{1/2-\epsilon}(\delta_n)$.

We now define

$$\tilde{v}_i = v_i - c_{ij}, \quad \tilde{\phi} = v_{n_j} - c_{ij} \quad \text{ on } \delta_{n,j},$$

where

$$c_{ij} = \frac{\int_{\delta_{n,j}} v_i \psi_{n,j}\, ds}{\int_{\delta_{n,j}} \psi_{n,j}\, ds} = \frac{\int_{\delta_{n,j}} v_{n_j} \psi_{n,j}\, ds}{\int_{\delta_{n,j}} \psi_{n,j}\, ds}.$$

The equality above holds because of the mortar matching condition for $v = (v_1, \cdots, v_I) \in V^h$ and the fact that the function $\psi_{n,j} \in M^h(\delta_n)$ is supported in $\delta_{n,j}$. We also have

$$\tilde{v}_i - \tilde{\phi} = v_i - \phi \quad \text{ in } L^2(\delta_n), \quad \tilde{v}_i - \tilde{\phi} \in H^{1/2-\epsilon}(\delta_n).$$

From these properties and applying (4.4) to $\widetilde{v}_i - \widetilde{\phi}_i$, we obtain

$$\|v_i - \phi\|^2_{0,\delta_n} \le C h_i^{1-2\epsilon} \|\tilde{v}_i - \tilde{\phi}\|^2_{H^{1/2-\epsilon}(\delta_n)}.$$

Applying Lemma 7 to $\tilde{v}_i$ and $\tilde{\phi}$ gives

$$\|v_i - \phi\|_{0,\delta_n} \le C h_i^{1/2-\epsilon} \epsilon^{-1/2} \sum_j \left(\|v_i - c_{ij}\|_{1,\Omega_i} + \|v_{n_j} - c_{ij}\|_{1,\Omega_{n_j}} \right).$$

Let

$$\Phi_{ij}(w) = \left| \int_{\delta_{n,j}} w\, \psi_{n,j}\, ds \right|.$$

Since $\psi_{n,j}$ is bounded from above by a constant, independent of the mesh parameters, and $\int_{\delta_{n,j}} \psi_{n,j}\, ds > 0$, $\Phi_{ij}(w)$ satisfies the two properties of Lemma 3; positivity of the integral also holds for the dual Lagrange multiplier case. By applying Lemma 3 to $v_i - c_{ij}$ and $v_{n_j} - c_{ij}$, with the seminorm Φ_{ij}, we obtain

$$\|v_i - c_{ij}\|_{1,\Omega_i} \le C|v_i|_{1,\Omega_i}, \quad \|v_{n_j} - c_{ij}\|_{1,\Omega_{n_j}} \le C|v_{n_j}|_{1,\Omega_{n_j}}.$$

Therefore,

$$\|v_i - \phi\|_{0,\delta_n} \le C h_i^{1/2-\epsilon} \epsilon^{-1/2} \left(|v_i|_{1,\Omega_i} + \sum_j |v_{n_j}|_{1,\Omega_{n_j}} \right).$$

Letting $\epsilon = 1/|\log h_i|$ gives $\log(h_i^{-\epsilon}) = 1$ and results in the bound

$$\|v_i - \phi\|_{0,\delta_n} \le C h_i^{1/2} |\log h_i|^{1/2} \left(|v_i|_{1,\Omega_i} + \sum_j |v_{n_j}|_{1,\Omega_{n_j}} \right). \tag{4.5}$$

By considering the scaling, we find

$$\|v\|_{0,\delta_n} = H_i^{(d-1)/2} \|\hat{v}\|_{0,\hat{\delta}_n}, \quad |v|_{1,\Omega_i} = H_i^{(d-2)/2} |\hat{v}|_{1,\hat{\Omega}_i}. \tag{4.6}$$

Here $\hat{\delta}_n$ and $\hat{\Omega}_i$ denote the scaled domains and $\hat{v}$ denotes the function defined on the scaled set $\hat{\delta}_n$ or $\hat{\Omega}_i$. We then obtain

$$\begin{aligned}
\|v_i - \phi\|_{0,\delta_n} &= H_i^{(d-1)/2} \|\hat{v}_i - \hat{\phi}\|_{0,\hat{\delta}_n} \\
&\le C H_i^{(d-1)/2} \hat{h}_i^{1/2} |\log \hat{h}_i|^{1/2} \left(|\hat{v}_i|_{1,\hat{\Omega}_i} + \sum_j |\hat{v}_{n_j}|_{1,\hat{\Omega}_{n_j}} \right) \\
&\le C H_i^{(d-1)/2} H_i^{-(d-2)/2} \hat{h}_i^{1/2} |\log \hat{h}_i|^{1/2} \left(|v_i|_{1,\Omega_i} + \sum_j |v_{n_j}|_{1,\Omega_{n_j}} \right) \\
&\le C H_i^{1/2} \left(\frac{h_i}{H_i} \right)^{1/2} \left(\log \frac{H_i}{h_i} \right)^{1/2} \left(|v_i|_{1,\Omega_i} + \sum_j |v_{n_j}|_{1,\Omega_{n_j}} \right).
\end{aligned}$$

Here we have used (4.5), (4.6) and that $\hat{h}_i = h_i/H_i$. □

4.2. The Stability of a Certain Interpolation Operator. Let $I^H : V^h \to V^H$ be a stable quasi interpolant in both the H^1- and L^2-norms in the following sense:

$$\sum_{i=1}^{I} |I^H u|_{1,\Omega_i}^2 \le C \sum_{i=1}^{I} |u|_{1,\Omega_i}^2,$$
$$\sum_{i=1}^{I} \frac{1}{H_i^2} \|u - I^H u\|_{0,\Omega_i}^2 \le C \sum_{i=1}^{I} |u|_{1,\Omega_i}^2,$$

where H_i denotes the diameter of Ω_i. We then obtain the same bound for $u_0 = I^m(I^H u)$.

LEMMA 9. *Let $u_0 = I^m(I^H u)$ for $u \in V^h$. Then u_0 satisfies*

$$\sum_{i=1}^{I} |u_0|^2_{1,\Omega_i} \le C \sum_{i=1}^{I} |u|_{1,\Omega_i},$$

$$\sum_{i=1}^{I} \frac{1}{H_i^2} \|u - u_0\|^2_{0,\Omega_i} \le C \sum_{i=1}^{I} |u|^2_{1,\Omega_i}.$$

Proof. We find, using (3.1), that

$$|u_0|^2_{1,\Omega_i} \le C \left\{ |I_i^h(I^H u)|^2_{1,\Omega_i} + \sum_{\delta_n \subset \partial\Omega_i} \left| \widetilde{\pi}_n \left(I^h_{m(\delta_n)}(I^H u) - I_i^h(I^H u) \right) \right|^2_{1,\Omega_i} \right\}.$$

From the H^1-stability of the nodal value interpolant I_i^h for functions in V^H (see [15, Lemma 3.8]), the first term above is bounded by

$$|I_i^h(I^H u)|^2_{1,\Omega_i} \le C |I^H u|^2_{1,\Omega_i}. \tag{4.7}$$

We estimate the second term by

$$\begin{aligned}
&\left| \widetilde{\pi}_n \left(I^h_{m(\delta_n)}(I^H u) - I_i^h(I^H u) \right) \right|^2_{1,\Omega_i} \\
&\le C h_i^{-1} \left\| \pi_n \left(I^h_{m(\delta_n)}(I^H u) - I_i^h(I^H u) \right) \right\|^2_{0,\delta_n} \qquad (4.8)\\
&\le C h_i^{-1} \left\{ \|I^h_{m(\delta_n)}(I^H u) - I^H u\|^2_{0,\delta_n} + \|I_i^h(I^H u) - I^H u\|^2_{0,\delta_n} \right\} \\
&\le C h_i^{-1} \left\{ \sum_{\delta_{n,j} \subset \delta_n} h_{m(n,j)} |I^H u|^2_{1,\Omega_{m(n,j)}} + h_i |I^H u|^2_{1,\Omega_i} \right\}, \qquad (4.9)
\end{aligned}$$

where $\delta_{n,j} = \partial\Omega_{m(n,j)} \cap \partial\Omega_i$. We have used an inverse inequality, the stability of π_n in $L^2(\delta_n)$, and the approximation property of the nodal value interpolation operator for $I^H u \in V^H$ provided by [15, Lemma 3.8]. Adding (4.7) and (4.9) over all nonmortar sides and subdomains and using Assumption 5 and the H^1-stability of the coarse interpolation operator I^H, we obtain

$$\sum_{i=1}^{I} |u_0|^2_{1,\Omega_i} \le C \sum_{i=1}^{I} |u|^2_{1,\Omega_i}.$$

We now estimate

$$\begin{aligned}
&\|u - u_0\|^2_{0,\Omega_i} \\
&\le C \left\{ \left\| u - I_i^h(I^H u) \right\|^2_{0,\Omega_i} + \sum_{\delta_n \subset \partial\Omega_i} \left\| \widetilde{\pi}_n \left(I^h_{m(\delta_n)}(I^H u) - I_i^h(I^H u) \right) \right\|^2_{0,\Omega_i} \right\}. \qquad (4.10)
\end{aligned}$$

The first term is bounded by

$$\begin{aligned}
\|u - I_i^h(I^H u)\|^2_{0,\Omega_i} &\le 2\|u - I^H u\|^2_{0,\Omega_i} + 2\|I_i^h(I^H u) - I^H u\|^2_{0,\Omega_i} \\
&\le C \left\{ \|u - I^H u\|^2_{0,\Omega_i} + h_i^2 |I^H u|^2_{1,\Omega_i} \right\}.
\end{aligned}$$

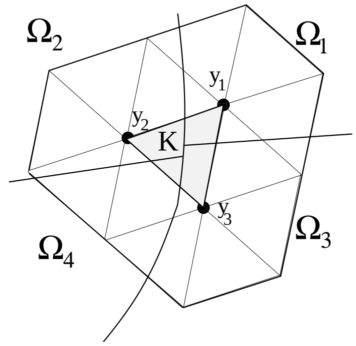

FIG. 2. *The region w_K divided by a geometrically nonconforming subdomain partition.*

By using (4.8) and (4.9), we bound the second term of (4.10) as follows:

$$\begin{aligned}
&\left\|\widetilde{\pi}_n \left(I^h_{m(\delta_n)}(I^H u) - I^h_i(I^H u)\right)\right\|^2_{0,\Omega_i} \\
&\le C h_i \left\|\pi_n \left(I^h_{m(\delta_n)}(I^H u) - I^h_i(I^H u)\right)\right\|^2_{0,\delta_n} \\
&\le C \left(h_i^2 |I^H u|^2_{1,\Omega_i} + \sum_{\delta_{n,j}\subset\delta_n} h_i h_{m(n,j)} |I^H u|^2_{1,\Omega_{m(n,j)}} \right) \\
&\le C \left(H_i^2 |I^H u|^2_{1,\Omega_i} + \sum_{\delta_{n,j}\subset\delta_n} H^2_{m(n,j)} |I^H u|^2_{1,\Omega_{m(n,j)}} \right).
\end{aligned}$$

In (4.10), summing the second term over the nonmortar sides gives

$$\|u-u_0\|^2_{0,\Omega_i} \le C \left(\|u - I^H u\|^2_{0,\Omega_i} + \sum_{\delta_n\subset\partial\Omega_i} \sum_{|\partial\Omega_l\cap\delta_n|>0} H_l^2 |I^H u|^2_{1,\Omega_l} \right).$$

From the assumption that the diameter of Ω_i is comparable to those of its neighbors Ω_l, a coloring argument, and the L^2- and H^1-stability of the interpolation $I^H u$, we obtain the second bound of the lemma. □

We now introduce our coarse interpolation operator $I^H : V^h \to V^H$. Let K be a triangle/tetrahedron in the coarse triangulation of Ω. Each vertex y_l of the triangle belongs to at least one substructure Ω_k (or to $\partial\Omega_k$) of the nonoverlapping partition. We denote the subdomain containing the vertex y_l by Ω_l. The set w_K is the union of the elements in T^H of which boundary intersects the boundary of the given element K. We consider a case as in Figure 2. The interpolation is defined by the values

$$(I^H u)(y_l) = \frac{1}{|w_{y_l}|}\int_{w_{y_l}} u\,dx,$$

where $w_{y_l} = w_K \cap \Omega_l$ and $|w_{y_l}|$ denotes the volume of w_{y_l}. In the following, we show that this coarse interpolation operator is stable in both the H^1- and L^2-norms.

LEMMA 10. *The coarse interpolant $I^H : V^h \to V^H$ satisfies*

$$\sum_{i=1}^{I} |I^H u|^2_{1,\Omega_i} \le C \sum_{i=1}^{I} |u|^2_{1,\Omega_i},$$
$$\sum_{i=1}^{I} \frac{1}{H_i^2} \|u - I^H u\|^2_{0,\Omega_i} \le C \sum_{i=1}^{I} |u|^2_{1,\Omega_i}.$$

Proof. We first estimate

$$\|I^H u\|^2_{0,K} \le C \sum_{l=1}^{3} |(I^H u)(y_l)|^2 \|\phi_l\|^2_{0,K} \le C \sum_{l=1}^{3} \|u(x)\|^2_{0,w_{y_l}} \frac{|K|}{|w_{y_l}|}, \tag{4.11}$$

where ϕ_l is the nodal basis function of the vertex y_l of the coarse triangle K. In general, we can have more than one subdomain Ω_k which intersects K and does not contain any vertices of K. For simplicity, we assume that we have only one such subdomain and denote it by Ω_4 (see Figure 2).

Let us denote by c_l the average of u over the subdomain Ω_l, and by K_l the common part of K and Ω_l, and let

$$c_l = \frac{1}{|\Omega_l|} \int_{\Omega_l} u\, dx, \quad K_l = K \cap \Omega_l, \quad \forall\, l = 1, \cdots, 4. \tag{4.12}$$

We then obtain

$$\begin{aligned}
\|u - I^H u\|^2_{0,K} &= \|u - c_1 - I^H(u - c_1)\|^2_{0,K} \\
&\le 2\|u - c_1\|^2_{0,K} + 2\|I^H(u - c_1)\|^2_{0,K} \\
&\le C \left\{ \|u - c_1\|^2_{0,K} + \sum_{l=1}^{3} \|u - c_1\|^2_{0,w_{y_l}} \frac{|K|}{|w_{y_l}|} \right\} \qquad (4.13) \\
&\le C \left\{ \sum_{l=1}^{3} \|u - c_1\|^2_{0,w_{y_l}} + \|u - c_1\|^2_{0,K_4} \right\}. \qquad (4.14)
\end{aligned}$$

Here we have used the identity $I^H(c_1) = c_1$, the estimate (4.11) and the fact that the factor $|K|/|w_{y_l}|$ is bounded from above independently of any mesh parameters.

From the Poincaré inequality and Assumption 3, we have

$$\|u - c_l\|^2_{0,w_{y_l}} \le C H_K^2\, |u|^2_{1,\Omega_l}, \quad l = 1, 2, 3.$$

We now consider

$$\|u - c_1\|^2_{0,w_{y_2}} \le 2\|u - c_2\|^2_{0,w_{y_2}} + 2\|c_2 - c_1\|^2_{0,w_{y_2}}.$$

Let

$$c_{12} = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u|_{\Omega_1}\, ds = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u|_{\Omega_2}\, ds,$$

where Γ_{12} is the common edge/face of Ω_1 and Ω_2. The identity follows from the mortar matching condition for the function u. We then have

$$\|c_2 - c_1\|^2_{0,w_{y_2}} \le C \left\{ |c_2 - c_{12}|^2 + |c_1 - c_{12}|^2 \right\} |w_{y_2}|.$$

The first term in the above equation is written as

$$\begin{aligned} c_2 - c_{12} &= \frac{1}{|\Omega_2|}\int_{\Omega_2} u_2\,dx - \frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_2\,ds \\ &= \frac{1}{|\Omega_2|}\int_{\Omega_2}\left(u_2 - \frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_2\,ds\right)\,dx, \end{aligned}$$

where $u_2 = u|_{\Omega_2}$. Let

$$\widetilde{u}_2 = u_2 - \frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_2.$$

Applying the Poincaré inequality to $\widetilde{u}_2$ and using the Hölder inequality, we obtain

$$|c_2 - c_{12}|^2 \le C H_2^{2-d} |u|_{1,\Omega_2}^2.$$

Similarly, we obtain

$$|c_1 - c_{12}|^2 \le C H_1^{2-d} |u|_{1,\Omega_1}^2.$$

We then have

$$\|c_2 - c_1\|_{0,w_{y_2}}^2 \le C H_K^2 (|u|_{1,\Omega_1}^2 + |u|_{1,\Omega_2}^2).$$

Here we have used that $|w_{y_2}| \le H_K^d$, for $d = 2, 3$ and Assumption 3. The estimate of the remaining terms in (4.14) can be done similarly and it gives

$$\|u - I^H u\|_{0,K}^2 \le C H_K^2 \sum_{l=1}^{4} |u|_{1,\Omega_l}^2. \tag{4.15}$$

By summing the above inequality over all K which intersect Ω_i, we obtain

$$\begin{aligned} \frac{1}{H_i^2}\|u - I^H u\|_{0,\Omega_i}^2 &\le \frac{1}{H_i^2}\sum_{K\cap\Omega_i\neq\emptyset} \|u - I^H u\|_{0,K}^2 \\ &\le C\frac{1}{H_i^2}\sum_{K\cap\Omega_i\neq\emptyset} H_K^2 \left(\sum_{\Omega_l\cap K\neq\emptyset} |u|_{1,\Omega_l}^2\right). \end{aligned}$$

The fact that the H_i is comparable to H_K and a coloring argument give the first estimate of the lemma. We note that we also have the following estimate from (4.13) and (4.15)

$$\|u - c_1\|_{0,K}^2 \le C H_K^2 \sum_{l=1}^{4} |u|_{1,\Omega_l}. \tag{4.16}$$

We now estimate

$$\begin{aligned} |I^H u|_{1,K}^2 &= |I^H u - c_1|_{1,K}^2 \\ &\le C H_K^{-2} \|I^H u - c_1\|_{0,K}^2 \\ &\le C H_K^{-2}\left(\|I^H u - u\|_{0,K}^2 + \|u - c_1\|_{0,K}^2\right), \end{aligned}$$

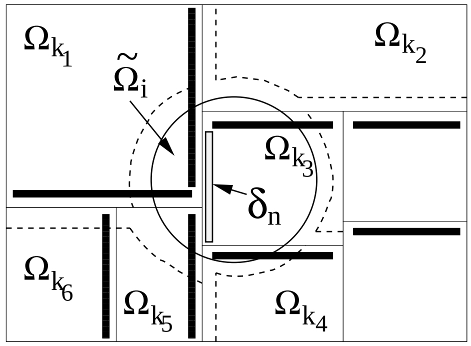

FIG. 3. *Nonconforming subdomain partition: mortar sides of interfaces (black bars), support of the functions $u_i \in V_i^h\,(= I^m(V_i))$ corresponding to the overlapping subdomain $\widetilde{\Omega}_i$ (interior of the dotted line); the subdomain Ω_{k_3} meets Ω_{k_1} and Ω_{k_5} along the nonmortar interface δ_n.*

where c_1 is the constant defined in (4.12). We have used an inverse inequality. By using (4.15) and (4.16), we obtain

$$|I^H u|_{1,K}^2 \leq C \sum_{l=1}^{4} |u|_{1,\Omega_l}^2.$$

The second estimate of the lemma follows by summing the above term over all triangles K and a coloring argument. □

REMARK 2. *For the general case, without Assumptions 1 and 2, we choose*

$$c_{12} = \frac{\int_{\Gamma_{12}} u|_{\Omega_1} \psi_{12}\, ds}{\int_{\Gamma_{12}} \psi_{12}\, ds} = \frac{\int_{\Gamma_{12}} u|_{\Omega_2} \psi_{12}\, ds}{\int_{\Gamma_{12}} \psi_{12}\, ds},$$

where ψ_{12} is the sum of the basis functions for $M^h(\delta_n)$ that are supported in Γ_{12}. The identity holds for $u \in V^h$. The arguments in the proof of Lemma 10 can also be applied to this general case and give the same bounds.

LEMMA 11. *Under Assumptions 1 and 2, and for $u \in V^h$, there exists a stable decomposition*

$$u = u_0 + u_1 + \cdots + u_N$$

such that

$$\sum_{i=0}^{N} a^\Gamma(u_i, u_i) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i}\right) \right\} a^\Gamma(u, u),$$

where H_i and δ_i denote the diameter of the subregion $\widetilde{\Omega}_i$ and the overlapping width of $\widetilde{\Omega}_i$.

Proof. We take $u_0 = I^m(I^H(u))$ using the interpolants I^m and I^H provided in Lemmas 9 and 10. We then define

$$u_i = I^m(\widetilde{u}_i), \quad \widetilde{u}_i = \theta_i(u - u_0) \quad \text{for } i = 1, \cdots, N.$$

From $u - u_0 \in V^h$ and $\sum_{i=1}^{N} \theta_i = 1$, we see that

$$u - u_0 = I^m(u - u_0) = \sum_{i=1}^{N} u_i.$$

The function u_i is supported as in Figure 3 and it can be written as

$$u_i = I^m(\widetilde{u}_i) = \sum_{l=1}^{6} \left(I^h_{k_l}(\widetilde{u}_i) + \sum_{\delta_n \subset \partial\Omega_{k_l}} \widetilde{\pi}_{\delta_n} \left(I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i) \right) \right).$$

Here $I^h_{m(\delta_n)}(\widetilde{u}_i)$, on δ_n in Figure 3, is given by

$$I^h_{m(\delta_n)}(\widetilde{u}_i) = \begin{cases} I^h_{k_1}(\widetilde{u}_i) \text{ on } \delta_{n,1} = \partial\Omega_{k_1} \cap \delta_n, \\ I^h_{k_5}(\widetilde{u}_i) \text{ on } \delta_{n,5} = \partial\Omega_{k_5} \cap \delta_n. \end{cases}$$

We will now prove that

$$\sum_{i=1}^{N} a^\Gamma(u_i, u_i) \le C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i} \right) \right\} a^\Gamma(u, u).$$

The required bound then follows by combining with Lemma 9. We consider

$$\begin{aligned} a^\Gamma(u_i, u_i) &= \sum_{l=1}^{6} |u_i|^2_{1,\Omega_{k_l}} \\ &= \sum_{l=1}^{6} \left| I^h_{k_l}(\widetilde{u}_i) + \sum_{\delta_n \subset \partial\Omega_{k_l}} \widetilde{\pi}_{\delta_n} \left(I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i) \right) \right|^2_{1,\Omega_{k_l}}. \end{aligned} \tag{4.17}$$

We note that $\widetilde{u}_i|_{\Omega_{k_l}}$ is a continuous and piecewise quadratic function defined on $T^h(\Omega_{k_l})$. From [15, Lemma 3.9], we have

$$|I^h_{k_l}(\widetilde{u}_i)|^2_{1,\Omega_{k_l}} \le C|\widetilde{u}_i|^2_{1,\Omega_{k_l}}. \tag{4.18}$$

For the second term of (4.17), we obtain

$$\begin{aligned} \left| \widetilde{\pi}_{\delta_n} \left(I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i) \right) \right|^2_{1,\Omega_{k_l}} &\le C h_{k_l}^{-2} h_{k_l} \left\| \pi_{\delta_n} \left(I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i) \right) \right\|^2_{0,\delta_n} \\ &\le C h_{k_l}^{-1} \left\| I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i) \right\|^2_{0,\delta_n}. \end{aligned} \tag{4.19}$$

Here we have used an inverse inequality, the quasi-uniformity of the triangulation in the subdomain Ω_{k_l}, and the L^2-continuity of the mortar projection π_{δ_n}. We now consider the term $\|I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_l}(\widetilde{u}_i)\|^2_{0,\delta_n}$, for δ_n and $l = 3$ in the Figure 3:

$$\begin{aligned} &\left\| I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_3}(\widetilde{u}_i) \right\|^2_{0,\delta_n} \\ &= \left\| I^h_{k_1}(\widetilde{u}_i) - I^h_{k_3}(\widetilde{u}_i) \right\|^2_{0,\delta_{n,1}} + \left\| I^h_{k_5}(\widetilde{u}_i) - I^h_{k_3}(\widetilde{u}_i) \right\|^2_{0,\delta_{n,5}} \\ &\le C \left(\left\| I^h_{k_1}(\widetilde{u}_i) - \widetilde{u}_i|_{\Omega_{k_1}} \right\|^2_{0,\delta_{n,1}} + \left\| I^h_{k_3}(\widetilde{u}_i) - \widetilde{u}_i|_{\Omega_{k_3}} \right\|^2_{0,\delta_{n,1}} + \left\| \widetilde{u}_i|_{\Omega_{k_1}} - \widetilde{u}_i|_{\Omega_{k_3}} \right\|^2_{0,\delta_{n,1}} \right. \\ &\quad \left. + \left\| I^h_{k_5}(\widetilde{u}_i) - \widetilde{u}_i|_{\Omega_{k_5}} \right\|^2_{0,\delta_{n,5}} + \left\| I^h_{k_3}(\widetilde{u}_i) - \widetilde{u}_i|_{\Omega_{k_3}} \right\|^2_{0,\delta_{n,5}} + \left\| \widetilde{u}_i|_{\Omega_{k_5}} - \widetilde{u}_i|_{\Omega_{k_3}} \right\|^2_{0,\delta_{n,5}} \right), \end{aligned}$$

where $\delta_{n,j} = \partial\Omega_{k_j} \cap \partial\Omega_{k_3}$ for $j = 1, 5$.

Let $w = u - u_0$. We now consider

$$
\begin{aligned}
&\|\widetilde{u}_i|_{\Omega_{k_1}} - \widetilde{u}_i|_{\Omega_{k_3}}\|^2_{0,\delta_{n,1}} \\
&= \|I^h_{k_1}(\widetilde{\theta}_i) w|_{\Omega_{k_1}} - I^h_{k_3}(\widetilde{\theta}_i) w|_{\Omega_{k_3}}\|^2_{0,\delta_{n,1}} \\
&\leq C \left(\sum_{l=1,3} \|(I^h_{k_l}(\widetilde{\theta}_i) - \widetilde{\theta}_i) w|_{\Omega_{k_l}}\|^2_{0,\delta_{n,1}} + \|\widetilde{\theta}_i (w|_{\Omega_{k_1}} - w|_{\Omega_{k_3}})\|^2_{0,\delta_{n,1}} \right). \qquad (4.20)
\end{aligned}
$$

Using the approximation property of the nodal value interpolant, $\|\nabla \widetilde{\theta}_i\|_\infty \leq C/\delta_i$, and a trace theorem, the first term above can be estimated

$$
\begin{aligned}
\|(I^h_{k_l}(\widetilde{\theta}_i) - \widetilde{\theta}_i) w|_{\Omega_{k_l}}\|^2_{0,\delta_{n,1}} &\leq \|I^h_{k_l}(\widetilde{\theta}_i) - \widetilde{\theta}_i\|^2_{0,\delta_{n,1}} \|w|_{\Omega_{k_l}}\|^2_{0,\delta_{n,1}} \\
&\leq C h_{k_l} |\widetilde{\theta}_i|^2_{1,\Omega_{k_l}} \|w\|^2_{1,\Omega_{k_l}} \\
&\leq C h_{k_l} \frac{1}{\delta_i^2} |\Omega_{k_l,\delta_i}| \|w\|^2_{1,\Omega_{k_l}},
\end{aligned}
$$

where $|\Omega_{k_l,\delta_i}|$ denotes the volume of the set Ω_{k_l,δ_i}, that is the support of $\nabla \widetilde{\theta}_i$ contained in Ω_{k_l}. In general, we have $|\Omega_{k_1,\delta_i}| \leq C \delta_i^{d-1} H_{k_l}$ with $d = 2$ or 3. Using this, we obtain

$$
\|(I^h_{k_l}(\widetilde{\theta}_i) - \widetilde{\theta}_i) w|_{\Omega_{k_l}}\|^2_{0,\delta_{n,1}} \leq C h_{k_l} \left(1 + \frac{H_{k_l}}{\delta_i}\right) \left(|w|^2_{1,\Omega_{k_l}} + \frac{1}{H^2_{k_l}} \|w\|^2_{0,\Omega_{k_l}} \right). \qquad (4.21)
$$

Using Lemma 6, the second term in (4.20) is bounded by

$$
\begin{aligned}
&\left\| \widetilde{\theta}_i (w|_{\Omega_{k_j}} - w|_{\Omega_{k_3}}) \right\|^2_{0,\delta_{n,j}} \qquad (4.22)\\
&\leq C \left\| \widetilde{\theta}_i \right\|^2_{\infty,\delta_{n,j}} \left\| w|_{\Omega_{k_j}} - w|_{\Omega_{k_3}} \right\|^2_{0,\delta_{n,j}} \\
&\leq C h_{k_3} \left(|w|^2_{1,\Omega_{k_j}} + |w|^2_{1,\Omega_{k_3}} \right), \quad j = 1, 5.
\end{aligned}
$$

Combining (4.20) with (4.21) and (4.22), and the approximation property of the nodal interpolation operators $I^h_{k_j},\ j = 1,3,5$, for the functions $\widetilde{u}_i$, that are continuous and piecewise quadratic on $T^h(\Omega_{k_j})$, lead to the following estimate:

(4.23)

$$
\begin{aligned}
\left\| I^h_{m(\delta_n)}(\widetilde{u}_i) - I^h_{k_3}(\widetilde{u}_i) \right\|^2_{0,\delta_n} \leq C h_{k_3} \Bigg(&\sum_{j=1,3,5} |\widetilde{u}_i|^2_{1,\Omega_{k_j}} \\
&+ \left(1 + \frac{H_i}{\delta_i}\right) \sum_{j=1,3,5} \left(|w|^2_{1,\Omega_{k_j}} + \frac{1}{H^2_{k_j}} \|w\|^2_{0,\Omega_{k_j}} \right) \Bigg),
\end{aligned}
$$

where H_i is the diameter of the subregion $\widetilde{\Omega}_i$. Here we have used Assumptions 4 and 5.

Combining the estimates in (4.23), (4.19), and (4.18) with (4.17), we obtain

$$
a^\Gamma(u_i, u_i) \leq C \left(\sum_{l \in \mathcal{S}_i} |\widetilde{u}_i|^2_{1,\Omega_{k_l}} + \left(1 + \frac{H_i}{\delta_i}\right) \sum_{l \in \mathcal{S}_i} \left(|u - u_0|^2_{1,\Omega_{k_l}} + \frac{1}{H_{k_l}} \|u - u_0\|^2_{0,\Omega_{k_l}} \right) \right),
$$

where $\mathcal{S}_i = \{l \,:\, \Omega_{k_l} \cap \widetilde{\Omega}_i \neq \emptyset\}$, the set of indices k_l of the substructures which intersect the subregion $\widetilde{\Omega}_i$. The first term of the above equation is estimated as follows:

$$\begin{aligned}
|\widetilde{u}_i|^2_{1,\Omega_{k_l}} &= \left\| \nabla \left(\widetilde{\theta}_i (u-u_0) \right) \right\|^2_{0,\Omega_{k_l}} \\
&\leq C \left\{ \int_{\Omega_{k_l}} \left| (u-u_0) \nabla \widetilde{\theta}_i \right|^2 dx + \int_{\Omega_{k_l}} \left| \widetilde{\theta}_i \nabla (u-u_0) \right|^2 dx \right\} \\
&\leq C \left\{ \frac{1}{\delta_i^2} \int_{\Omega_{k_l,\delta_i}} (u-u_0)^2 \, dx + |u-u_0|^2_{1,\Omega_{k_l}} \right\},
\end{aligned}$$

where Ω_{k_l,δ_i} is the support of $\nabla \widetilde{\theta}_i$ contained in Ω_{k_l}. We then obtain by applying Lemma 2 to $\int_{\Omega_{k_l,\delta_i}} (u-u_0)^2 \, dx$

$$\frac{1}{\delta_i^2} \int_{\Omega_{k_l,\delta_i}} (u-u_0)^2 \, dx \leq C \left(\left(1 + \frac{H_{k_l}}{\delta_i} \right) |u-u_0|^2_{1,\Omega_{k_l}} + \frac{1}{H_{k_l} \delta_i} \|u-u_0\|^2_{0,\Omega_{k_l}} \right).$$

Using Assumption 4, we have

$$a^{\Gamma}(u_i,u_i) \leq C \left(1 + \frac{H_i}{\delta_i} \right) \left(\sum_{l \in \mathcal{S}_i} |u-u_0|^2_{1,\Omega_{k_l}} + \sum_{l \in \mathcal{S}_i} \frac{1}{H^2_{k_l}} \|u-u_0\|^2_{0,\Omega_{k_l}} \right).$$

By summing the above estimate over all the subregions $\widetilde{\Omega}_i$, using a coloring argument and the estimates in Lemma 9, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} a^{\Gamma}(u_i,u_i) &\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i} \right) \right\} \left(\sum_{l=1}^{N} |u-u_0|^2_{1,\Omega_l} + \sum_{l=1}^{N} \frac{1}{H_l^2} \|u-u_0\|^2_{0,\Omega_l} \right) \\
&\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i} \right) \right\} a^{\Gamma}(u,u).
\end{aligned}$$

□

REMARK 3. *In the above Lemma, we use Assumption 5 that the mesh sizes are comparable between neighboring subdomains. On any interface of two subdomains, denote by h_m and h_{nm} the mesh sizes of the mortar subdomain and the nonmortar subdomain, respectively. If they satisfy*

$$h_m \leq C h_{nm} \tag{4.24}$$

then the result of Lemma 11 holds without the assumption of comparable meshes between neighboring subdomains. However condition (4.24) *is the opposite from the one considered in previous work on the mortar methods.*

By combining the bound in Lemma 11 with Lemma 1 and the upper bound (3.5), we obtain the following condition number bound:

THEOREM 4.1. *With Assumptions 1 and 2, the two-level additive algorithm satisfies*

$$\kappa \left(\sum_{i=0}^{N} T_i \right) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i} \right) \right\},$$

where C depends on the constant ω in (3.5).

For the general case, we bound the term in (4.22) by using Lemma 8

$$\sum_{j=1,5} \|\widetilde{\theta}_i(w|_{\Omega_{k_j}} - w|_{\Omega_{k_3}})\|^2_{0,\delta_n,j} \le C h_{k_3} \log \left(\frac{H_{k_3}}{h_{k_3}}\right) \sum_{j=1,3,5} |w|^2_{1,\Omega_{k_j}}.$$

This gives the bound in the general case.

$$\sum_{i=0}^{N} a^{\Gamma}(u_i, u_i) \le C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i}\right) \max_{\Omega_{k_l} \cap \operatorname{supp}(V_i^h) \neq \emptyset} \left\{ \log \left(\frac{H_{k_l}}{h_{k_l}}\right) \right\} \right\} a^{\Gamma}(u, u),$$

where $\operatorname{supp}(V_i^h)$ denotes the support of the functions in the space V_i^h. By combining this bound with Lemma 1 and the upper bound (3.5), we obtain the following condition number bound:

THEOREM 4.2. *Without Assumptions 1 and 2, the two-level additive algorithm satisfies*

$$\kappa \left(\sum_{i=0}^{N} T_i\right) \le C \max_{i=1,\cdots,N} \left\{ \left(1 + \frac{H_i}{\delta_i}\right) \max_{\Omega_{k_l} \cap \mathit{supp}(V_i^h) \neq \emptyset} \left\{ \mathit{log} \left(\frac{H_{k_l}}{h_{k_l}}\right) \right\} \right\},$$

where C depends on the constant ω in (3.5).

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