A BDDC ALGORITHM FOR FLOW IN POROUS MEDIA WITH A HYBRID FINITE ELEMENT DISCRETIZATION

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Abstract. The BDDC (balancing domain decomposition by constraints) methods have been applied successfully to solve the large sparse linear algebraic systems arising from conforming finite element discretizations of elliptic boundary value problems. In this paper, the scalar elliptic problems for flow in porous media are discretized by a hybrid finite element method which is equivalent to a nonconforming finite element method. The BDDC algorithm is extended to these problems which originate as saddle point problems. Edge/face average constraints are enforced across the interface and the same rate of convergence is obtained as in conforming cases. The condition number of the preconditioned system is estimated and numerical experiments are discussed.

Key words. BDDC, domain decomposition, saddle point problem, condition number, hybrid finite element method

AMS subject classifications 65N30, 65N55 65F10

1. Introduction.

Mixed formulations of elliptic problems, see [3], have many applications, e.g., for flow in porous media, for which a good approximation to the velocity, which involves derivatives of the solution of the differential equations, is required. These discretizations lead to large, sparse, symmetric, indefinite linear systems.

In our recent paper [24], we extended the BDDC algorithm to this mixed formulation of elliptic problems. The BDDC algorithms are nonoverlapping domain decomposition methods, introduced by Dohrmann [6] and further analyzed in [15, 16], are similar to the balancing Neumann-Neumann algorithms, see [14, 7]. However, the BDDC methods have different coarse components which are formed by a small number of continuity constraints enforced across the interface throughout the iterations. An important advantage of using such coarse problems is that the Schur complements and all other matrices that arise in the computation will be invertible.

In [24], the original saddle point problem is reduced to finding a correction pair which stays in the divergence free, benign subspace, as in [8, 17, 18, 19]. Then the BDDC method, with edge/face constraints, is applied to the reduced system. It is similar to the BDDC algorithm proposed for the Stokes case in [13]. The analysis of this approach is focused on estimating the norm of the average operator. Several useful technical tools for the Raviart Thomas finite elements, originally given in [26, 22, 25], are used and the algorithm converges at a rate similar to that of simple elliptic cases.

The hybrid finite element discretization is equivalent to a nonconforming finite element method. Two-level domain decomposition methods have been developed for a nonconforming approximation in [21, 20]. The condition number bounds are independent of the jumps in the coefficients of the original equations and grow only logarithmically with the number of degrees of freedom in each subdomain, a result which is the same as for a conforming case.

A non-overlapping domain decomposition algorithm for the hybrid formulation, called Method II, was proposed already in [10]. It is an unpreconditioned conjugate gradient method for certain interface variables. The rate of convergence is independent of the coefficients, but depends mildly on the number of degrees of freedom in the

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subdomains. Problems related to singular local Neumann problems arising in the preconditioners were also addressed in [10]. In addition, other non-overlapping domain decomposition methods were proposed with improved rates of convergence in [9] and [5].

A Balancing Neumann-Neumann (BNN) method was extended and analyzed in [4] for Method II of [10], see also [21] for a nonconforming case. The same rate of convergence was obtained as for the conforming case. We will extend the BDDC algorithm to Method II of [10] in this paper. In contrast to [4], we need not solve any singular systems with BDDC.

The method proposed here differs from the one in [24]. We reduce the original saddle point problem to a positive definite system for the pressure by introducing the Lagrange multipliers on the interface of the subdomains and eliminating the velocity in each subdomain. Thus, we need not find a velocity that satisfies the divergence constraint at the beginning of the computation and then restrict the iterates to the divergence free, benign subspace. Our approach is quite similar to the work on the FETI-DP methods as described in [23, Chapter 6]. We use the BDDC preconditioner to solve the interface problem for the Lagrange multipliers, which can be interpreted as an approximation to the trace of the pressure. By enforcing a suitable set of constraints, we obtain the same convergence rate as for a conforming finite element case. As in other studies of BDDC, our analysis will focus on the estimate of the norm of the average operator. However, we cannot use properties of the Raviart-Thomas finite elements directly since we work with the Lagrange multipliers. The technical tools, originally given in [21, 20, 4], are needed to make a connection between the hybrid finite element method and a conforming finite element method.

The rest of the paper is organized as follows. The mixed formulation for the elliptic problem and its hybrid finite element discretization are described in Section 2. In Section 3, we reduce our problem to a symmetric positive definite interface problem. We introduce the BDDC preconditioner for the interface system in Section 4 and give some auxiliary results in Section 5. In Section 6, we provide an estimate of the condition number for the system with the BDDC preconditioner which is of the form $C\left(1+\log\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively. Finally, some computational results are presented in Section 7.

2. An elliptic problem and its discretization by hybrid finite elements.

We consider the following elliptic problem on a bounded polygonal domain Ω, in two or three dimensions, with a Dirichlet boundary condition:

$$\begin{cases} -\nabla\cdot(a\nabla p) = f & \text{in} \quad \Omega, \\ p = g & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where a is a positive definite matrix function with the entries in $L^\infty(\Omega)$ satisfying

$$\xi^T a(\mathbf{x})\xi \geq \alpha\|\xi\|^2, \quad \text{for a.e.} \quad \mathbf{x}\in\Omega, \tag{2.2}$$

for some positive constant α, and with $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Omega)$.

The equation (2.1) has a unique solution p. Without loss of generality, we assume that $g=0$. We use the Dirichlet boundary condition for convenience. The algorithm can also be extended to other boundary conditions.

We assume that we are interested in computing $-a\nabla p$ directly as often required in flow in porous media. We then introduce the velocity $\mathbf{u}$:

$$\mathbf{u} = -a\nabla p. \tag{2.3}$$

We obtain the following system for the velocity $\mathbf{u}$ and the pressure p:

$$\begin{cases} \mathbf{u} = -a\nabla p & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = f & \text{in} \quad \Omega, \\ p = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{2.4}$$

Let $c(\mathbf{x}) = a(\mathbf{x})^{-1}$ and define a Hilbert space by

$$H(div, \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$

with the norm

$$\|\mathbf{v}\|^2_{H(div,\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{v}\|^2_{L^2(\Omega)}.$$

The weak form of (2.4) is as follows: find $\mathbf{u} \in H(div, \Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & 0 & \forall \mathbf{v} \in H(div, \Omega), \\ b(\mathbf{u}, q) & = & -\int_\Omega fq d\mathbf{x} & \forall q \in L^2(\Omega), \end{cases}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}^T c(\mathbf{x}) \mathbf{v} d\mathbf{x}$ and $b(\mathbf{u}, q) = -\int_\Omega (\nabla \cdot \mathbf{u}) q d\mathbf{x}$.

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse rectangles/hexahedra and that the number of such rectangles/hexahedra forming an individual subdomain is uniformly bounded. We also assume $a(\mathbf{x})$, the coefficient of (2.1), is constant in each subdomain. Let $\mathcal{T}$ be a triangulation of Ω, let $\widehat{\mathbf{W}}$ be the lowest order Raviart-Thomas finite element space, see [3, Chapter III, 3], and let Q be the space of piecewise constants, which are finite dimensional subspaces of $H(div, \Omega)$ and $L^2(\Omega)$, respectively. The pair $\widehat{\mathbf{W}}$ and Q satisfies a uniform inf-sup condition, see [3, Chapter IV. 1.2].

We have

$$\widehat{\mathbf{W}} = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$, $c_T \in \mathbb{R}$, and the normal component of $\mathbf{v}$ is continuous across the inter-element boundary.

The finite element discrete problem is: find $\mathbf{u}_h \in \widehat{\mathbf{W}}$ and $p_h \in Q$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & 0 & \forall \mathbf{v}_h \in \widehat{\mathbf{W}}, \\ b(\mathbf{u}_h, q_h) & = & -\int_\Omega f q_h d\mathbf{x} & \forall q_h \in Q. \end{cases} \tag{2.5}$$

Let $\widehat{\mathbf{W}}^{(i)}$ be the subdomain subspace of $\widehat{\mathbf{W}}$, i.e.,

$$\widehat{\mathbf{W}}^{(i)} = \{\mathbf{v} \in L^2(\Omega_i)^2 \text{ or } L^2(\Omega_i)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$, $c_T \in \mathbb{R}$, and the normal component of $\mathbf{v}$ is continuous across the inter-element boundaries.

We also define $\mathbf{W}$ and $\mathbf{W}^{(\mathbf{i})}$ which are similar to $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{W}}^{(i)}$, respectively. However, they have no continuity constraints on the normal components of the functions, i.e.,

$$\mathbf{W} = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}\},$$

and

$$\mathbf{W}^{(i)} = \{\mathbf{v} \in L^2(\Omega_i)^2 \text{ or } L^2(\Omega_i)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \ \ \forall T \in \mathcal{T}\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$ and $c_T \in \mathbb{R}$.

We thus relax the continuity of the normal components on the element interface in $\mathbf{W}$ and $\mathbf{W}^{(i)}$. Instead, we will introduce Lagrange multipliers to enforce the continuity of the Raviart-Thomas space. As in [10, 4], in an implementation, we only need to use inter-element Lagrange multiplier on the subdomain interfaces.

Let $\mathcal{F}$ denote the set of edges/faces in $\mathcal{T}$ and $\mathcal{F}^\partial$ be a subset of $\mathcal{F}$ which contains the edges/faces on $\partial\Omega$. Then the Lagrange multiplier space $\widehat{\Lambda}$ is the set of functions on $\mathcal{F} \setminus \mathcal{F}^\partial$ which take constant values on individual edges/faces of $\mathcal{F}$ and vanish on $\mathcal{F}^\partial$; see [3, Section V1.2].

We can then reformulate the mixed problem (2.5) as follows: find $(\mathbf{u}, p, \lambda) \in \mathbf{W} \times Q \times \widehat{\Lambda}$ such that for all $(\mathbf{v}, q, \mu) \in \mathbf{W} \times Q \times \widehat{\Lambda}$

$$(2.6) \qquad \begin{cases} \sum_{T\in\mathcal{T}} \left(\int_T \mathbf{u}^T c\mathbf{v} - \int_T \nabla\cdot\mathbf{v} p d\mathbf{x} + \int_{\partial T} \lambda \mathbf{v}\cdot\mathbf{n}_T d\mathbf{s}\right) & = & 0, \\ -\sum_{T\in\mathcal{T}} \int_T q\nabla\cdot\mathbf{u} & = & -\int_\Omega f q d\mathbf{x}, \\ \sum_{T\in\mathcal{T}} \int_{\partial T} \mu\mathbf{u}\cdot\mathbf{n}_T d\mathbf{s} & = & 0. \end{cases}$$

The additional function λ is naturally interpreted as an approximation to the trace of p on the boundary of the elements. A proof of the equivalence of (2.5) and (2.6) can be found in [1, 2].

Correspondingly, the matrix form of (2.6) is

$$(2.7) \qquad \begin{bmatrix} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}.$$

3. The problem reduced to the subdomain interface. We denote the discrete space of nodal values of $Q \times \widehat{\Lambda}$ by $\widehat{\mathcal{P}}$. We note that $\widehat{\mathcal{P}}$ has the natural interpretation as the space of values of the pressure p in the interior and on the edges/faces of the elements. By this definition, $\widehat{\mathcal{P}}$ is isomorphic to $Q \times \widehat{\Lambda}$; we can then write an element of $\widehat{\mathcal{P}}$ as $\hat{p} = [p, \lambda]$.

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i\neq j}\partial\Omega_{i,h} \cap \partial\Omega_{j,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$.

We can write the discrete pressure spaces $\widehat{\mathcal{P}}$ as

$$(3.1) \qquad \widehat{\mathcal{P}} = Q \bigoplus \widehat{\Lambda}.$$

The space Q is a direct sum of subdomain interior pressure spaces $Q^{(i)}$, i.e.,

$$Q = \bigoplus_{i=1}^{N} Q^{(i)}.$$

The elements of $Q^{(i)}$ are restrictions of elements in Q to Ω_i.

We can further decompose $\widehat{\Lambda}$ into

$$\widehat{\Lambda} = \Lambda_I \bigoplus \widehat{\Lambda}_\Gamma,$$

where $\widehat{\Lambda}_\Gamma$ denotes the set of degrees of freedom associated with Γ and Λ_I is a direct sum of subdomain interior degrees of freedom, i.e,

$$\Lambda_I = \bigoplus_{i=1}^{N} \Lambda_I^{(i)}.$$

We denote the subdomain interface pressure space by $\widehat{\Lambda}_\Gamma^{(i)}$ and the associated product space by $\widehat{\Lambda}_\Gamma = \prod_{i=1}^N \widehat{\Lambda}_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface pressure space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\widehat{\Lambda}_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ.

The global saddle point problem (2.7) is assembled from subdomain problems

$$\begin{bmatrix} A^{(i)} & B_1^{(i)^T} & B_{2,I}^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\ B_1^{(i)} & 0 & 0 & 0 \\ B_{2,I}^{(i)} & 0 & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \\ 0 \\ 0 \end{bmatrix}, \tag{3.2}$$

where $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_I^{(i)}, \lambda_\Gamma^{(i)}) \in (\mathbf{W}^{(i)}, Q^{(i)}, \Lambda_I^{(i)}, \widehat{\Lambda}_\Gamma^{(i)})$.

We define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \widehat{\Lambda}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)}\lambda_\Gamma^{(i)}$ such that

$$\begin{bmatrix} A^{(i)} & B_1^{(i)^T} & B_{2,I}^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\ B_1^{(i)} & 0 & 0 & 0 \\ B_{2,I}^{(i)} & 0 & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ -S_\Gamma^{(i)}\lambda_\Gamma^{(i)} \end{bmatrix}. \tag{3.3}$$

We note that $A^{(i)}$ is block diagonal, with each block corresponding to an element $T \subset \mathcal{T}(\Omega_i)$. We first eliminate the velocity $\mathbf{u}^{(i)}$ and we obtain a system for the $p^{(i)}$, $\lambda_I^{(i)}$, and $\lambda_\Gamma^{(i)}$. We then eliminate the degrees of freedom interior to the subdomain, i.e., the $p^{(i)}$ and $\lambda_I^{(i)}$.

As we mentioned before, in practice, for each subdomain Ω_i, we only need to use the inter-element multipliers on the interface of the subdomains. Let $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_\Gamma^{(i)}) \in (\widehat{\mathbf{W}}^{(i)}, Q^{(i)}, \widehat{\Lambda}_\Gamma^{(i)})$ and we obtain the following subdomain problems

$$\begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \\ 0 \end{bmatrix}. \tag{3.4}$$

We note that $\hat{A}^{(i)}$ is no longer block diagonal by element. We eliminate the velocity $\mathbf{u}^{(i)}$ and the pressure $p^{(i)}$ and obtain the following Schur complement for $\lambda_\Gamma^{(i)}$

$$\begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} & B_{2,\Gamma}^{(i)^T} \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ -S_\Gamma^{(i)}\lambda_\Gamma^{(i)} \end{bmatrix}. \tag{3.5}$$

Here we use the same notation $S_\Gamma^{(i)}$ since this matrix, in fact, is the same as in (3.3). This follows from the equivalence of (2.5) and (2.6). The action of $S_\Gamma^{(i)}$ can then be evaluated by solving a Dirichlet problem in the variational form: find $\{\mathbf{u}_i, p_i\} \in \widehat{\mathbf{W}}^{(i)} \times Q^{(i)}$ such that

$$
\begin{aligned}
&\int_{\Omega_i} \mathbf{u}_i^T c \mathbf{v}_i d\mathbf{x} - \int_{\Omega_i} \nabla \cdot \mathbf{v}_i d\mathbf{x} = -\int_{\partial\Omega_i \partial\Omega} \lambda_\Gamma^{(i)} \mathbf{v}_i \cdot \mathbf{n} d\mathbf{s} \quad \forall\ \mathbf{v}_i \in \widehat{\mathbf{W}}^{(i)}, \\
&\int_{\Omega_i} \nabla \cdot \mathbf{u}_i q_i = 0 \quad \forall\ q_i \in Q^{(i)},
\end{aligned} \tag{3.6}
$$

then set $S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = -B_{2,\Gamma}^{(i)} \mathbf{u}_i$. We note that these Dirichlet problems are always well posed and that $S_\Gamma^{(i)}$ is symmetric and positive definite. We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ.

Given the definition of $S_\Gamma^{(i)}$, the subdomain problem (3.4) corresponds to the subdomain interface problem

$$
S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = g_\Gamma^{(i)}, \quad i = 1, 2, ..., N,
$$

where

$$
g_\Gamma^{(i)} = -\begin{bmatrix} B_{2,\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} \\ B_1^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \end{bmatrix}.
$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$
\widehat{S}_\Gamma \lambda_\Gamma = g_\Gamma, \tag{3.7}
$$

where $g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} g_\Gamma^{(i)}$, and

$$
\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}. \tag{3.8}
$$

Thus, $\widehat{S}_\Gamma$ is a symmetric, positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (3.7) with a preconditioned conjugate gradient method.

4. The BDDC preconditioner. We introduce a partially assembled interface pressure space $\widetilde{\mathbf{\Lambda}}_\Gamma$ by

$$
\widetilde{\mathbf{\Lambda}}_\Gamma = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \mathbf{\Lambda}_\Delta = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \left(\prod_{i=1}^N \mathbf{\Lambda}_\Delta^{(i)} \right).
$$

Here, $\widehat{\mathbf{\Lambda}}_\Pi$ is the coarse level, primal interface pressure space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see [12] and [11]. The space $\mathbf{\Lambda}_\Delta$ is the direct sum of the $\mathbf{\Lambda}_\Delta^{(i)}$, which are spanned by the remaining interface pressure degrees of freedom with a zero average over each edge/face. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints on the pressure across the interface but retain all primal continuity constraints, which makes all the linear systems nonsingular. This is the main difference from the BNN method in [4], where we encounter singular local problems.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{\Lambda}}_\Gamma$ to the components $\mathbf{\Lambda}_\Gamma^{(i)}$ related to the subdomain Ω_i. $R_\Delta^{(i)}$ maps functions from $\widehat{\mathbf{\Lambda}}_\Gamma$ to $\mathbf{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\mathbf{\Lambda}}_\Gamma$ to its subspace $\widehat{\mathbf{\Lambda}}_\Pi$ and $R_\Pi^{(i)}$ is the operator which maps vectors in $\widehat{\mathbf{\Lambda}}_\Pi$ into their components in $\Lambda_\Pi^{(i)}$. $\overline{R}_\Gamma : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma$ is the direct sum of the $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\mathbf{\Lambda}}_\Gamma \to \widetilde{\mathbf{\Lambda}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{a_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} a_j^\gamma(x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We note that $\delta_i^\dagger(x)$ is constant on each edge/face, since we assume that the $a_i(x)$ is constant in each subdomain, and the nodes on each edge/face are shared by the same subdomains. Multiplying each row of $R_\Delta^{(i)}$, with the scaling factor $\delta_i^\dagger(x)$, gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. Furthermore, $\widetilde{R}_\Delta^{(i)}$ maps functions from $\widetilde{\mathbf{\Lambda}}_\Gamma$ to $\mathbf{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $\widetilde{R}_{\Gamma\Pi}$ is a restriction operator from $\widetilde{\mathbf{\Lambda}}_\Gamma$ to its subspace $\widehat{\mathbf{\Lambda}}_\Pi$.

We also denote by $\widetilde{\mathbf{F}}_\Gamma$, the right hand side space corresponding to $\widetilde{\mathbf{\Lambda}}_\Gamma$. We will use the same restriction, extension, and scaled restriction operators for the space $\widetilde{\mathbf{F}}_\Gamma$ as for $\widetilde{\mathbf{\Lambda}}_\Gamma$.

The interface pressure Schur complement $\widetilde{S}_\Gamma$, on the partially assembled interface pressure space $\widetilde{\Lambda}_\Gamma$, is partially assembled from subdomain Schur complements $S_\Gamma^{(i)}$, i.e.,

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{4.1}$$

$\widetilde{S}_\Gamma$ can also be defined by: for any given $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, $\widetilde{S}_\Gamma\lambda_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ satisfies

$$H \begin{bmatrix} \mathbf{u}^{(1)} \\ p_I^{(1)} \\ \lambda_\Delta^{(1)} \\ \vdots \\ \mathbf{u}^{(N)} \\ p_I^{(N)} \\ \lambda_\Delta^{(N)} \\ \lambda_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ -(\widetilde{S}_\Gamma\lambda_\Gamma)_\Delta^{(1)} \\ \vdots \\ \mathbf{0} \\ 0 \\ -(\widetilde{S}_\Gamma\lambda_\Gamma)_\Delta^{(N)} \\ -(\widetilde{S}_\Gamma\mathbf{w}_\Gamma)_\Pi \end{bmatrix}, \tag{4.2}$$

where

$$
(4.3) \qquad H = \begin{bmatrix}
\hat{A}^{(1)} & {B_1^{(1)}}^T & {B_{2,\Delta}^{(1)}}^T & & & & & {\widetilde{B}_{2,\Pi}^{(1)}}^T \\
B_1^{(1)} & 0 & 0 & & & & & 0 \\
B_{2,\Delta}^{(1)} & 0 & 0 & & & & & 0 \\
 & & & \ddots & \vdots & & & \\
 & & & & \hat{A}^{(N)} & {B_1^{(N)}}^T & {B_{2,\Delta}^{(N)}}^T & {\widetilde{B}_{2,\Pi}^{(N)}}^T \\
 & & & & B_1^{(N)} & 0 & 0 & 0 \\
 & & & & B_{2,\Delta}^{(N)} & 0 & 0 & 0 \\
\widetilde{B}_{2,\Pi}^{(1)} & 0 & 0 & \dots & \widetilde{B}_{2,\Pi}^{(N)} & 0 & 0 & 0
\end{bmatrix},
$$

and

$$
\widetilde{B}_{2,\Pi}^{(i)} = {R_\Pi^{(i)}}^T B_{2,\Pi}^{(i)}.
$$

Given the definition of $\widetilde{S}_\Gamma$ on the partially assembled interface pressure space $\widetilde{\Lambda}_\Gamma$, we can also obtain $\widehat{S}_\Gamma$, introduced in (3.7), from $\widetilde{S}_\Gamma$ by assembling the dual interface pressure part on the subdomain interface, i.e.,

$$
(4.4) \qquad \widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma.
$$

The BDDC preconditioner for solving the global interface problem (3.7) is

$$
(4.5) \qquad M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}.
$$

Here, from a block Cholesky factorization, see [12, 13], we have

$$
(4.6) \qquad \widetilde{S}_\Gamma^{-1} = -\sum_{i=1}^N \left[0 \; 0 \; {\widetilde{R}_\Delta^{(i)}}^T\right] \begin{bmatrix} \hat{A}^{(i)} & {B_1^{(i)}}^T & {B_{2,\Delta}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Delta}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \widetilde{R}_\Delta^{(i)} \end{bmatrix} + \Phi S_{CC}^{-1} \Phi^T,
$$

$$
S_{CC} = \sum_{i=1}^N {R_\Pi^{(i)}}^T \left\{ \begin{bmatrix} B_{2,\Pi}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)} & {B_1^{(i)}}^T & {B_{2,\Delta}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Delta}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} {B_{2,\Pi}^{(i)}}^T \\ 0 \\ 0 \end{bmatrix} \right\} R_\Pi^{(i)},
$$

and the matrix Φ is defined by

$$
\Phi = \widetilde{R}_{\Gamma\Pi}^T - \sum_{i=1}^N \left[0 \; 0 \; {\widetilde{R}_\Delta^{(i)}}^T\right] \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)} & {B_{2,\Delta}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Pi}^{(i)} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} {B_{2,\Delta}^{(i)}}^T \\ 0 \\ 0 \end{bmatrix} R_\Pi^{(i)}.
$$

The preconditioned BDDC algorithm is then of the form: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$
(4.7) \qquad \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma.
$$

This preconditioned problem is symmetric positive definite and we can use the preconditioned conjugate gradient method to solve it.

5. Some auxiliary results. In this section, we will collect a number of results which are needed in our theory.

In order to connect our hybrid finite element discretization to a conforming finite element method, we need to introduce a new mesh on each subdomain. The idea follows [20, 21, 4]. In order to be complete and for the readers unfamiliar with these technical tools, we give the construction of the new mesh, the definitions of two important maps, and some useful lemmas, which were originally given in [4, 20, 21].

Given an element $\tau \in \mathcal{T}$, let $\hat{\tau}$ be a subtriangulation of τ which includes the vertices of τ and the nodal points in τ for the degrees of the freedom of $Q \times \Lambda$. We then obtain a quasi-uniform sub-triangulation $\widehat{\mathcal{T}}$. We partition the vertices in the new mesh $\widehat{\mathcal{T}}$ into two sets. The nodes in $\mathcal{T}$ are called primary and the rest are called secondary. We say that two vertices in the triangulation $\widehat{\mathcal{T}}$ are adjacent if there is an edge of $\widehat{\mathcal{T}}$ between them.

Let $U_h(\Omega)$ be the continuous piecewise linear finite element function space with respect to the new triangulation $\widehat{\mathcal{T}}$. For a subdomain Ω_i, $U_h(\Omega_i)$ and $U_h(\partial\Omega_i)$ are defined by restrictions:

$$U_h(\Omega_i) = \{u|_{\Omega_i} : \quad u \in U_h(\Omega)\}, \qquad U_h(\partial\Omega_i) = \{u|_{\partial\Omega_i} : \quad u \in U_h(\Omega)\}.$$

Define a mapping $I_h^{\Omega_i}$ from any function ϕ defined at the primary vertices in Ω_i to $U_h(\Omega_i)$ by

$$I_h^{\Omega_i}\phi(x) = \begin{cases} \phi(x), \text{ if } x \text{ is a primary vertex;} \\ \\ \text{the average of all adjacent primary vertices on } \partial\Omega_i, \\ \ \text{if } x \text{ is a secondary vertex on } \partial\Omega_i; \\ \\ \text{the average of all adjacent primary vertices,} \\ \ \text{if } x \text{ is a secondary vertex in the interior of } \Omega_i; \\ \\ \text{the linear interpolation of the vertex values,} \\ \ \text{if } x \text{ is not a vertex of } \widehat{\mathcal{T}}. \end{cases} \tag{5.1}$$

We note that $I_h^{\Omega_i}$ defines a map from $Q(\Omega_i) \times \Lambda(\Omega_i)$ to $U_h(\Omega_i)$ and also a map from $U_h(\Omega_i)$ to $U_h(\Omega_i)$.

Let $I_h^{\partial\Omega_i}$ be the mapping from a function ϕ, defined at the primary vertices on $\partial\Omega_i$, to $U_h(\partial\Omega_i)$ and defined by $I_h^{\partial\Omega_i}\phi = (I_h^{\Omega_i}\hat{p})|_{\partial\Omega_i}$, where $\hat{p}$ is any functions in $Q(\Omega_i) \times \Lambda(\Omega_i)$ such that $\hat{p}|_{\partial\Omega_i} = \phi$. The map is well defined since the boundary values of $I_h^{\Omega_i}\hat{p}$ only depend on the boundary values of $\hat{p}$.

Let

$$\widetilde{U}_h(\Omega_i) = \{\psi = I_h^{\Omega_i}\phi, \quad \phi \in U_h(\Omega_i)\} \quad \text{and} \quad \widetilde{U}_h(\partial\Omega_i) = \{\psi|_{\partial\Omega}, \quad \psi \in \widetilde{U}_h(\Omega_i)\}.$$

We list some useful lemmas from [4].

Lemma 1. *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$|I_h^{\Omega_i}\phi|_{H^1(\Omega_i)} \le C|\phi|_{H^1(\Omega_i)}, \quad \forall \phi \in U_h(\Omega_i), \tag{5.2}$$

$$\|I_h^{\Omega_i}\phi\|_{L^2(\Omega_i)} \le C\|\phi\|_{L^2(\Omega_i)}, \quad \forall \phi \in U_h(\Omega_i). \tag{5.3}$$

Proof: See [4, Lemms 6.1].

□

Lemma 2. *For $\hat{\phi} \in \widetilde{U}_h(\partial\Omega_i)$, there exist two positive constants C_1 and C_2, independent of h and $|\Omega_i|$, such that*

$$C_1\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \le \inf_{\phi\in\widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i}=\hat{\phi}} \|\phi\|_{H^1(\Omega_i)} \le C_2\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)}, \tag{5.4}$$

$$C_1|\hat{\phi}|_{H^{1/2}(\partial\Omega_i)} \le \inf_{\phi\in\widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i}=\hat{\phi}} |\phi|_{H^1(\Omega_i)} \le C_2|\hat{\phi}|_{H^{1/2}(\partial\Omega_i)}. \tag{5.5}$$

Proof: See [4, Lemms 6.2].

□

Lemma 3. *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$\|I_h^{\partial\Omega_i}\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \le C\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \qquad \forall\ \hat{\phi} \in U_h(\partial\Omega_i). \tag{5.6}$$

Proof: See [4, Lemms 6.3].

□

Lemma 4. *There exist positive constants C_1 and C_2 independent of H, h, and the coefficient of (2.1), such that for all $\lambda_i \in \Lambda_\Gamma^{(i)}$,*

$$a_iC_1|I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)} \le |\lambda_i|^2_{S_\Gamma^{(i)}} \le a_iC_2|I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)}. \tag{5.7}$$

Proof: See [4, Theorem 6.5].

□

We define the interface averages operator E_D, by

$$E_D = \widetilde{R}_\Gamma\widetilde{R}_{D,\Gamma}^T, \tag{5.8}$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the boundary points of the subdomain.

The interface average operator E_D has the following property:

Lemma 5.

$$|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \le C\left(1+\log\frac{H}{h}\right)^2|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma},$$

for any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, where C is a positive constant independent of H, h, and the coefficient of (2.1),

Proof: Given any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, we have

$$\begin{aligned}
&|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}\\
&\le 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + |\lambda_\Gamma - E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}\right)\\
&\le 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + |\overline{R}_\Gamma\left(\lambda_\Gamma - E_D\lambda_\Gamma\right)|^2_{S_\Gamma}\right)\\
&= 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + \sum_{i=1}^N|\overline{R}_\Gamma^{(i)}\left(\lambda_\Gamma - E_D\lambda_\Gamma\right)|^2_{S_\Gamma^{(i)}}\right).
\end{aligned} \tag{5.9}$$

Let $\lambda_i = \overline{R}^{(i)}\lambda_\Gamma$ and set

$$v_i(x) := \overline{R}_\Gamma^{(i)}(\lambda_\Gamma - E_D\lambda_\Gamma)(x) = \sum_{j\in\mathcal{N}_x}\delta_j^\dagger(\lambda_i(x) - \lambda_j(x)), \quad x \in \partial\Omega^i\cap\Gamma. \tag{5.10}$$

Here $\mathcal{N}_x$ is the set of indices of the subdomains that have x on their boundaries. Since a fine edge/face only belongs to exactly two subdomains, we have, for an edge/face $\mathcal{F}^{ij} \subset \partial\Omega_i$ that is also shared by Ω_j,

$$v_i = \delta_j^\dagger \lambda_i - \delta_j^\dagger \lambda_j, \text{ on } \mathcal{F}^{ij}. \tag{5.11}$$

We note that the simple inequality

$$a_i {\delta_j^\dagger}^2 \leq \min(a_i, a_j), \tag{5.12}$$

holds for $\gamma \in [1/2, \infty)$.

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/faces. Let $\zeta_\mathcal{F}$ be the characteristic function of $\mathcal{F}$, i.e., the function that is identically one on $\mathcal{F}$ and zero on $\partial\Omega_i \backslash \mathcal{F}$. We clearly have

$$\sum_{\mathcal{F} \subset \partial\Omega_i} \zeta_\mathcal{F}(x) = 1, \qquad \text{almost everywhere on } \ \partial\Omega_i \backslash \partial\Omega.$$

Let $\vartheta_\mathcal{F}$ be the partition of unity functions associated with the edges/faces for a function in the space $U_h(\Omega_i)$, which is defined in [23, Lemma 4.23].

We have

$$|v_i|^2_{S_\Gamma^{(i)}} \leq C \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} |\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}}. \tag{5.13}$$

By Lemma 4, with $\overline{\lambda}_{i,\mathcal{F}^{ij}}$ the average over $\mathcal{F}^{ij}$,

$$\begin{aligned}
& |\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}} \\
\leq\ & C_2 a_i |I_h^{\partial\Omega_i}(\zeta_{\mathcal{F}^{ij}} v_i)|^2_{H^{1/2}(\partial\Omega_i)} \\
=\ & C_2 a_i |I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}} \delta_j^\dagger (\lambda_i - \lambda_j)\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
=\ & C_2 a_i {\delta_j^\dagger}^2 |I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}} (\lambda_i - \lambda_j)\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
\leq\ & 2C_2 a_i {\delta_j^\dagger}^2 \Big(|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}} (\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
+\ & |I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}} (\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \Big).
\end{aligned} \tag{5.14}$$

We estimate these two terms in (5.14) separately.

The first term is estimated as follows:

$$\begin{aligned}
& a_i {\delta_j^\dagger}^2 |I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}} (\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
\leq\ & a_i |I_h^{\partial\Omega_i}\left(\vartheta_{\mathcal{F}^{ij}} I_h^{\partial\Omega_i} (\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
\leq\ & a_i \|\vartheta_{\mathcal{F}^{ij}} I_h^{\partial\Omega_i} (\lambda_i - \overline{\lambda}_{i,\mathcal{F}^{ij}})\|^2_{H^{1/2}(\partial\Omega_i)} \\
\leq\ & a_i \|\vartheta_{\mathcal{F}^{ij}} (I_h^{\partial\Omega_i} \lambda_i - (\overline{I_h^{\partial\Omega_i} \lambda_i})_{\mathcal{F}^{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \\
\leq\ & C a_i \left(1 + \log \frac{H}{h}\right)^2 |I_h^{\partial\Omega_i} \lambda_i|^2_{H^{1/2}(\partial\Omega_i)},
\end{aligned} \tag{5.15}$$

where we use (5.11) and the definition of $I_h^{\partial\Omega_i}$ for the first inequality. Using Lemma 3, we obtain the second inequality. We use $I_h^{\partial\Omega_i}(\overline{\lambda}_{i,\mathcal{F}^{ij}}) = (\overline{I_h^{\partial\Omega_i} \lambda_i})_{\mathcal{F}^{ij}}$ and [23, Lemma 4.26] for the penultimate and final inequalities.

For the second term in (5.14), similarly as for the first term, we have,

$$
\begin{aligned}
& a_i \delta_j^{\dagger^2} |I_h^{\partial\Omega_i} \left(\zeta_{\mathcal{F}^{ij}}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
& \leq a_j |I_h^{\partial\Omega_i} \left(\vartheta_{\mathcal{F}^{ij}} I_h^{\partial\Omega_j}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} \\
& \leq a_j \|\vartheta_{\mathcal{F}^{ij}} I_h^{\partial\Omega_j}(\lambda_j - \overline{\lambda}_{j,\mathcal{F}^{ij}})\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\
& \leq a_j \|\vartheta_{\mathcal{F}^{ij}} (I_h^{\partial\Omega_j}\lambda_j - (\overline{I_h^{\partial\Omega_j}\lambda_j})_{\mathcal{F}^{ij}})\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\
& \leq C a_j \left(1 + \log \frac{H}{h}\right)^2 |I_h^{\partial\Omega_j}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)},
\end{aligned} \tag{5.16}
$$

where we use (5.11) and the definition of $I_h^{\partial\Omega_i}$ and $I_h^{\partial\Omega_j}$ for the first inequality. Using Lemma 3, we obtain the second inequality. We use $I_h^{\partial\Omega_j}(\overline{\lambda}_{j,\mathcal{F}^{ij}}) = (\overline{I_h^{\partial\Omega_j}\lambda_j})_{\mathcal{F}^{ij}}$ and [23, Lemma 4.26] for the penultimate and final inequalities.

Combining (5.15), (5.16), (5.14), and (5.13), we have

$$
\begin{aligned}
& |v_i|^2_{S_\Gamma^{(i)}} \\
& \leq C C_2 \left(1 + \log \frac{H}{h}\right)^2 \left(a_i |I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)} + a_j |I_h^{\partial\Omega_j}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)}\right) \\
& \leq C \frac{C_2}{C_1} \left(1 + \log \frac{H}{h}\right)^2 \left(|\lambda_i|^2_{S_\Gamma^{(i)}} + |\lambda_j|^2_{S_\Gamma^{(j)}}\right),
\end{aligned} \tag{5.17}
$$

where we use Lemma 4 again for the final inequality.

Using (5.9), (5.10), and (5.17), we obtain

$$
|E_D \lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C \left(1 + \log \frac{H}{h}\right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}.
$$

□

6. Condition number estimate for BDDC preconditioner. We are now ready to formulate and prove our main result; it follows exactly in the same way as the proof of [13, Theorem 1] by using Lemma 5.

THEOREM 1. *The preconditioned operator $M^{-1}\widehat{S}_\Gamma$ is symmetric, positive definite with respect to the bilinear from $\langle \cdot, \cdot \rangle_{\widehat{S}_\Gamma}$ on the space $\widehat{\Lambda}$ and*

$$
\langle \lambda, \lambda \rangle_{\widehat{S}_\Gamma} \leq \left\langle M^{-1}\widehat{S}_\Gamma \lambda, \lambda \right\rangle_{\widehat{S}_\Gamma} \leq C \left(1 + \log \frac{H}{h}\right)^2 \langle \lambda, \lambda \rangle_{\widehat{S}_\Gamma}, \ \forall \lambda \in \widehat{\Lambda}_\Gamma. \tag{6.1}
$$

Here, C is a constant which is independent of h and H.

7. Numerical experiments. We have applied our BDDC algorithms to the model problem (2.1), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (2.1) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure, respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual has been reduced by a factor of 10^{-6}.

Table 1

Eigenvalue bounds and iteration counts for BDDC preconditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ *and* $a \equiv 1$.

Number of Subdomains	Iterations	Condition number
4×4	7	2.53
8×8	10	3.01
12×12	10	3.06
16×16	10	3.06
20×20	10	3.06

Table 2

Eigenvalue bounds and iteration counts for the BDDC preconditioner with a change of the size of the subdomain problems. 8×8 *subdomains and* $a \equiv 1$.

$\frac{H}{h}$	Iterations	Condition number
4	8	2.23
8	10	3.01
12	11	3.54
16	11	3.95
20	11	4.29

Table 3

Eigenvalue bounds and iteration counts for BDDC preconditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ *and* a *is in a checkerboard pattern.*

Number of Subdomains	Iterations	Condition number
4×4	8	2.98
8×8	10	2.97
12×12	11	2.98
16×16	11	2.98
20×20	10	2.98

Table 4

Eigenvalue bounds and iteration counts for the BDDC preconditioner with a change of the size of the subdomain problems. 8×8 *subdomains and* a *is in a checkerboard pattern.*

$\frac{\hat{H}}{h}$	Iterations	Condition number
4	9	2.19
8	10	2.97
12	11	3.51
16	12	3.92
20	13	4.26

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we take the coefficient $a \equiv 1$. Table 1 gives the iteration counts and the estimate of the condition numbers, with a change of the number of subdomains. We find that the condition number is independent of the number of subdomains. Table 2 gives results with a change of the size of the subdomain problems.

In the second set of experiments, we take the coefficient $a = 1$ in half the subdomains and $a = 100$ in the neighboring subdomains, in a checkerboard pattern. Table 3 gives the iteration counts, and condition number estimates with a change of the number of subdomains. We find that the condition numbers are independent of the number of subdomains. Table 4 gives results with a change of the size of the subdomain problems.

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