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A FETI-DP FORMULATION OF THREE DIMENSIONAL ELASTICITY PROBLEMS WITH MORTAR DISCRETIZATION *

HYEA HYUN KIM †

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Abstract. In this paper, a FETI-DP formulation for the three dimensional elasticity problem on non-matching grids over a geometrically conforming subdomain partition is considered. To resolve the nonconformity of the finite elements, a mortar matching condition on the subdomain interfaces (faces) is imposed. By introducing Lagrange multipliers for the mortar matching constraints, the resulting linear system becomes similar to that of a FETI-DP method. In order to make the FETI-DP method efficient for solving this linear system, a relatively large set of primal constraints, which include average and momentum constraints over interfaces (faces) as well as vertex constraints, is introduced. A condition number bound $C(1+\log(H/h))^2$ for the FETI-DP formulation with a Neumann-Dirichlet preconditioner is then proved for the elasticity problems with discontinuous material parameters when only some faces are chosen as primal faces on which the average and momentum constraints will be imposed. An algorithm which selects a quite small number of primal faces is also discussed.

Key words. FETI-DP, mortar methods, preconditioner, Stokes problem

AMS subject classifications. 65N30, 65N55

1. Introduction.

We will develop an efficient FETI-DP algorithm for solving linear systems arising from non-conforming discretization of compressible elasticity problems in three dimensions. We consider a non-conforming discretization given by finite elements on triangulations which are nonmatching across subdomain interfaces. We note that nonmatching triangulations are important for generation of meshes, especially in three spatial dimensions, for problems with singular points or joints, and for problems with jumps in diffusion coefficients or material parameters.

Mortar methods have been developed as non-conforming approximation in order to obtain as accurate an approximate solution as for conforming approximations; see [7, 4, 5, 16, 31]. For this purpose, a mortar matching condition is imposed on the subdomain solutions across the interfaces. The jumps of the solutions on the common interfaces are orthogonal to a certain Lagrange multiplier space. This condition can be enforced directly by using non-conforming finite element functions or weakly by introducing Lagrange multipliers. The second approach leads to a saddle-point

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† Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA. (hhk2@cims.nyu.edu)

problem similar to that considered in FETI-type algorithms.

FETI-type algorithms were originally developed for second order elliptic problems with conforming discretizations. These algorithms belong to the iterative substructuring domain decomposition methods with dual variables. A separate set of interface unknowns is assigned to each subdomain and point-wise continuity of solutions across interfaces is enforced using dual Lagrange multipliers, leading to a saddle point problem. The local unknowns are then eliminated and the resulting linear system for the dual variables is solved iteratively with a preconditioner. These algorithms have evolved from one-level FETI into two-level FETI, and FETI-DP methods; see [15, 14, 12]. FETI-DP methods were introduced in [12] for plane linear elasticity problems and further extended to three dimensional problems in [13] by introducing an additional set of primal constraints. In FETI-DP methods, a certain set of primal constraints are enforced throughout iterations while the remaining constraints are imposed weakly by dual Lagrange multipliers. FETI-DP algorithms have been further developed for three dimensional elliptic problems with discontinuous coefficients by Klawonn, Widlund and Dryja [24]. They introduced a Dirichlet preconditioner scaled with a weight matrix depending on the coefficients and showed that this preconditioner gives a condition number bound $C(1 + \log(H/h))^2$ with the constant C independent of the coefficients and mesh parameters.

FETI-type algorithms have also been applied to solving saddle-point problems resulting from mortar discretization. A numerical study in [29] showed that FETI methods applied to these saddle-point problems are as efficient as the original FETI methods for conforming discretizations. Further FETI-DP algorithms for two-dimensional elliptic problems were developed and the condition number bound of these algorithms were analyzed in [10, 11] but these results depend on the ratio of the mesh sizes between neighboring subdomains. The author with Lee [19] developed a different FETI-DP algorithm for two-dimensional elliptic problems with discontinuous coefficients and showed that a condition number bound $C(1+\log(H/h))^2$ with the constant C independent of the coefficients and mesh parameters. This preconditioner is similar to previously developed FETI-DP preconditioners [24, 10, 11] except that its scaling matrix has zero value for the unknowns except on nonmortar interfaces. We call this preconditioner a Neumann-Dirichlet preconditioner. This algorithm has later been extended to the Stokes problem and three-dimensional elliptic problems with heterogeneous coefficients; see [18, 17].

The aim of our present study is to extend the FETI-DP algorithm of [19] to three-dimensional compressible elasticity problems with mortar discretization. FETI-DP methods for three dimensional elasticity problems with conforming discretization have been studied extensively both theoretically and numerically; see [13, 23, 28, 20].

In [13], Farhat *et al.* introduced face averages and vertex constraints as the set of primal constraints and observed that these additional constraints give a scalable method. Later Klawonn and Widlund [23] considered various primal constraints for elasticity problems with discontinuous Lamé parameters. In their work, some faces and edges are selected as fully primal faces and fully primal edges. They work with edge average constraints on a fully primal face, and edge average and edge moment constraints on a fully primal edge in order to get a scalable algorithm and to make the subdomain problems invertible. However, edge constraints are not compatible with mortar matching constraints. In our FETI-DP formulation, we therefore introduce face average and face moment constraints on the faces. Further, we reduce the number of primal constraints by selecting only some of the faces as primal faces for which the face average and face moment constraints are applied.

This paper is organized as follows. In Section 2, we introduce the compressible elasticity problems and Korn inequalities which will be used in our analysis. In Section 3, a non-conforming approximation space is introduced for the model elasticity problems and mortar matching constraints are considered as weak continuity constraints in our FETI-DP formulation. We then construct primal constraints for the FETI-DP formulation. Section 4 is devoted to condition number analysis of our FETI-DP algorithm with the primal constraints introduced in Section 3. In the final section, we propose an algorithm which selects a quite small number of primal faces and show the performance of this algorithm both for cases with continuous and discontinuous material parameters.

Throughout this paper, C denotes a generic constant independent of mesh parameters, the number of subdomains, and coefficients of the elasticity problems. We will use h_i and H_i to denote the mesh size and the subdomain size of Ω_i, respectively.

2. A model problem and Korn's inequality. Let Ω be a polyhedral domain in $\mathbf{R}^3$. The Sobolev space $H^1(\Omega)$ is the set of functions in $L^2(\Omega)$ which are square integrable up to first weak derivatives and equipped with the standard Sobolev norm;

$$\|v\|_{1,\Omega}^2 := |v|_{1,\Omega}^2 + \|v\|_{0,\Omega}^2,$$

where $|v|_{1,\Omega}^2 = \int_\Omega \nabla v \cdot \nabla v \, dx$ and $\|v\|_{0,\Omega} = \int_\Omega v^2 \, dx$.

We assume that $\partial\Omega$ is divided into two parts $\partial\Omega_D$ and $\partial\Omega_N$ on which a Dirichlet boundary condition and a natural boundary condition are specified, respectively. The subspace $H_D^1(\Omega) \subset H^1(\Omega)$ is a set of functions having zero trace on $\partial\Omega_D$. We introduce the vector valued Sobolev space

$$\mathbf{H}_D^1(\Omega) = \prod_{i=1}^3 H_D^1(\Omega), \quad \mathbf{H}^1(\Omega) = \prod_{i=1}^3 H^1(\Omega)$$

equipped with the norm

$$\|\mathbf{v}\|_{1,\Omega}^2 := |\mathbf{v}|_{1,\Omega}^2 + \|\mathbf{v}\|_{0,\Omega}^2,$$

where $|\mathbf{v}|_{1,\Omega}^2 = \sum_{i=1}^3 |v_i|_{1,\Omega}^2$ and $\|\mathbf{v}\|_{0,\Omega}^2 = \sum_{i=1}^3 \|v_i\|_{0,\Omega}^2$ for $\mathbf{v} = (v_1, v_2, v_3)$.

We then consider the elasticity problem:

find $\mathbf{u} \in \mathbf{H}_D^1(\Omega)$ such that

$$\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\, d\mathbf{x} + \int_\Omega G(\mathbf{x})\beta(\mathbf{x})\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{v}\, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{H}_D^1(\Omega), \tag{2.1}$$

where $G = E/(1+\nu)$ and $\beta = \nu/(1-2\nu)$ are material parameters depending on the Young's modulus $E > 0$ and the Poisson ratio $\nu \in (0, 1/2]$. We assume that ν is bounded away from $1/2$ so that we exclude the case of incompressible elasticity problems. The linearized strain tensor is defined by

$$\varepsilon(\mathbf{u})_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2,3,$$

and the tensor product and the force term are given by

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^3 \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{g}\cdot\mathbf{v} d\sigma.$$

Here $\mathbf{f}$ is the body force and $\mathbf{g}$ is the surface force on the natural boundary part $\partial\Omega_N$.

The space $\mathbf{ker}(\varepsilon)$ has the following six rigid body motions as its basis, which are three translations

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

and three rotations

$$\mathbf{r}_4 = \frac{1}{H}\begin{pmatrix} x_2 - \widehat{x}_2 \\ -x_1 + \widehat{x}_1 \\ 0 \end{pmatrix}, \ \mathbf{r}_5 = \frac{1}{H}\begin{pmatrix} -x_3 + \widehat{x}_3 \\ 0 \\ x_1 - \widehat{x}_1 \end{pmatrix}, \ \mathbf{r}_6 = \frac{1}{H}\begin{pmatrix} 0 \\ x_3 - \widehat{x}_3 \\ -x_2 + \widehat{x}_2 \end{pmatrix}. \tag{2.3}$$

Here $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \Omega$ and H is the diameter of Ω. This shift and the scaling make the L_2-norm of the six vectors scale in the same way with H. When Ω is partitioned into a set of subdomains, the elasticity problem given on a floating subdomain has purely natural boundary condition. The Korn inequalities provided in Section 2 of [23] concern this case. Let $\Sigma \subset \partial\Omega$ with positive measure. We define an L_2-inner product $(\mathbf{u}, \mathbf{r})_\Sigma$ by integrating $\mathbf{u}\cdot\mathbf{r}$ over Σ

$$(\mathbf{u}, \mathbf{r})_\Sigma = \int_\Sigma \mathbf{u}\cdot\mathbf{r}\, ds.$$

The following Korn inequality is provided in [23, Lemma 5]:

LEMMA 2.1. *Let Ω be a Lipschitz domain and Σ be a subset of $\partial\Omega$ with positive measure. Then there exist a constant $c > 0$, invariant under dilation, such that*

$$c|\mathbf{u}|_{1,\Omega} \leq \|\varepsilon(\mathbf{u})\|_{0,\Omega} \leq |\mathbf{u}|_{1,\Omega},$$

where $\mathbf{u} \in \mathbf{H}^1(\Omega)$ satisfies $(\mathbf{u}, \mathbf{r})_\Sigma = 0$ for all $\mathbf{r} \in \mathbf{ker}(\varepsilon)$.

Furthermore, we have similar inequalities for semi-norms defined in the space $\mathbf{H}^{1/2}(\Sigma)$ which is the trace space of $\mathbf{H}^1(\Omega)$ for $\Sigma \subset \partial\Omega$. For $\mathbf{u} \in \mathbf{H}^{1/2}(\Sigma)$, we define two semi-norms by

$$|\mathbf{u}|_{1/2,\Sigma} := \inf_{\substack{\mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{v}|_\Sigma = \mathbf{u}}} |\mathbf{v}|_{1,\Omega}, \quad |\mathbf{u}|_{E(\Sigma)} := \inf_{\substack{\mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{v}|_\Sigma = \mathbf{u}}} \|\varepsilon(\mathbf{v})\|_{0,\Omega}.$$

LEMMA 2.2. *Let Ω be a Lipschitz domain and Σ be a subset of $\partial\Omega$ with positive measure. Then there exists a constant $C > 0$, invariant under dilation, such that*

$$C|\mathbf{u}|_{1/2,\Sigma} \leq |\mathbf{u}|_{E(\Sigma)} \leq |\mathbf{u}|_{1/2,\Sigma},$$

for $\mathbf{u} \in \mathbf{H}^{1/2}(\Sigma)$ satisfying $(\mathbf{u}, \mathbf{r})_\Sigma = 0 \quad \forall \mathbf{r} \in \mathbf{ker}(\varepsilon)$.

The lemma can be found in [23, Lemma 6]. Another important inequality, which follows from this inequality and is useful for our analysis of the condition number bound, is given in (see [23, Lemma 7]):

LEMMA 2.3. *Let Ω be a Lipschitz domain of diameter H and $\Sigma \subset \partial\Omega$ be an open subset with positive measure. Then there exists a constant $C > 0$, not depending on H, such that*

$$\inf_{\mathbf{r} \in \mathbf{ker}(\varepsilon)} \|\mathbf{u} - \mathbf{r}\|_{0,\Sigma}^2 \leq CH|\mathbf{u}|_{E(\Sigma)}^2 \quad \forall \mathbf{u} \in \mathbf{H}^{1/2}(\Sigma).$$

3. FETI-DP formulation.

3.1. Domain decomposition with mortar discretization. We divide the domain Ω into a geometrically conforming partition $\{\Omega_i\}_{i=1}^N$, that is shape regular. We consider a compressible elasticity problem with coefficients $G(\mathbf{x})$ and $\beta(\mathbf{x})$ positive constants in each subdomain

$$G(\mathbf{x})|_{\Omega_i} = G_i, \quad \beta(\mathbf{x})|_{\Omega_i} = \beta_i.$$

The conforming P_1-finite element space $\mathbf{X}_i$ is associated to a quasi-uniform triangulation T_i of each subdomain Ω_i. In addition, functions in the space $\mathbf{X}_i$ satisfy the Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega_D$. The triangulations $\{T_i\}_{i=1}^N$ may not match across the subdomain interfaces. We associate the finite element space $\mathbf{W}_i$ to the boundary of subdomain Ω_i; it is the trace space of $\mathbf{X}_i$ on $\partial\Omega_i$.

In the three dimensional case, a pair of subdomains can have a face, an edge, or a vertex in common. We will primarily consider only the common faces as the interfaces of subdomains. On each face $F^{ij} = \partial\Omega_i \cap \partial\Omega_j$, we will choose one of the two subdomains as the mortar side and the other as the nonmortar side depending on the coefficients $G(\mathbf{x})$, i.e., we will choose the subdomain with smaller $G(\mathbf{x})$ as the nonmortar side. We then introduce the finite element space

$$\mathbf{W}_{ij} = \left\{ \mathbf{w} \in \mathbf{H}_0^1(F^{ij}) \;:\; \mathbf{w} = \mathbf{v}|_{F^{ij}} \text{ for } \mathbf{v} \in \mathbf{X}_{n(ij)} \right\},$$

where $n(ij)$ denotes the nonmortar side of F^{ij}. This space is spanned by a nodal basis $\{\boldsymbol{\phi}_k\}_{k=1}^{n_{ij}}$ related to the interior nodes of F^{ij} with respect to the triangulation $T^{n(ij)}$ of the nonmortar side. Based on this space, we construct a dual Lagrange multiplier space $\mathbf{M}_{ij}$ with a basis $\{\boldsymbol{\psi}_k\}_{k=1}^{n_{ij}}$ satisfying

$$\int_{F^{ij}} \boldsymbol{\phi}_l \cdot \boldsymbol{\psi}_k \, ds = \delta_{lk} \int_{F^{ij}} \boldsymbol{\phi}_l \, ds \quad \forall\, l, k = 1, \cdots, n_{ij}.$$

We refer to [16] for a detailed construction of the dual Lagrange multiplier space. The standard Lagrange multiplier space was introduced in [6] for three spatial dimensions. However the dual Lagrange multiplier space is more computationally efficient as well as easier to implement compared to the standard one. The mortar matching condition is then written as

$$\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda} \, ds = 0 \quad \forall \boldsymbol{\lambda} \in \mathbf{M}_{ij},\ \forall F_{ij}. \tag{3.1}$$

For each subdomain Ω_i, we define the set m_i containing the subdomain indices j which are the mortar sides of the faces $F \subset \partial\Omega_i$:

$$m_i := \{ j \;:\; \Omega_j \text{ is the mortar side of } F \, (:= \partial\Omega_i \cap \partial\Omega_j) \; \forall F \subset \partial\Omega_i \}.$$

We then introduce the finite element spaces on the interfaces

$$\mathbf{W} = \prod_{i=1}^{N} \mathbf{W}_i,$$
$$\mathbf{W}_n = \prod_{i=1}^{N} \prod_{j \in m_i} \mathbf{W}_{ij},$$
$$\mathbf{M} = \prod_{i=1}^{N} \prod_{j \in m_i} \mathbf{M}_{ij}.$$

Here the space $\mathbf{W}_n$ consists of functions defined on the nonmortar faces, while the space $\mathbf{W}$ consists of functions defined on the whole interfaces, i.e. both on nonmortar and mortar faces.

3.2. Primal constraints in the FETI-DP formulation. Solving linear systems arising from the mortar discretization is a difficult task [3, 25, 2]. Construction of coarse finite element space in Schwarz-type algorithms or iterative substructuring algorithms, that provides scalability of the algorithms, is challenging in particular for three-dimensional problems with a geometrically non-conforming subdomain partition [1, 8]. On the other hand, the coarse problem in FETI-DP type algorithms follows from algebraic elimination of primal unknowns associated to the primal constraints. The selection of the primal constraints is important in achieving a scalable FETI-DP algorithm as well as in obtaining invertible subdomain problems.

For the case of point-wise matching constraints in conforming discretizations, there have been studies for three dimensional elliptic problems [24, 21], three dimensional elasticity problems [22], and the Stokes problem [26, 27]. Face average or edge average constraints were introduced for three dimensional elliptic problems and condition number bounds in terms of polylogarithmic functions of the subdomain problem size were shown for problems with discontinuous coefficients [24]. Klawonn and Widlund [22] considered edge average and edge moment constraints, and vertex constraints for elasticity problems to control the rigid body motions of the subdomains as well as to obtain a scalable method. Furthermore they introduced the concepts of an acceptable face path and an acceptable vertex path in an attempt to reduce the number of primal constraints. Using constraints depending on edges is more promising than relying on faces when there are general distributions of jumps in the coefficients. Moreover the exchange of information between subdomains is related to a smaller set of unknowns. Numerical results support that edge constraints are more effective than face constraints [21].

For the case of mortar constraints, we are able to construct primal constraints based on faces. In [17], we introduced face average constraints for three dimensional elliptic problems with mortar discretizations and showed that the condition number is bounded by a polylogarithmic function of the subdomain problem size and is independent of the coefficients of elliptic problems.

Our purpose is to select primal constraints for the elasticity problem with mortar constraints. We will now introduce six primal constraints on each face based on the idea in a recent study [23] by Klawonn and Widlund. On a face F^{ij}, we consider the rigid body motions $\{\mathbf{r}_i\}_{i=1}^6$ as in (2.2) and (2.3), where H is the diameter of the face F^{ij} and $\widehat{\mathbf{x}}$ is a point in F^{ij}. We define a projection $\mathbf{Q} : \mathbf{H}^{1/2}(F^{ij}) \to \mathbf{M}_{ij}$ by

$$\int_{F^{ij}} (\mathbf{Q}(\mathbf{w}) - \mathbf{w}) \cdot \boldsymbol{\phi} \, ds = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{W}_{ij}.$$

We then consider the projected rigid body motions $\{\mathbf{Q}(\mathbf{r}_i)\}_{i=1}^6$. Since the translational rigid body motions $\{\mathbf{r}_i\}_{i=1}^3$ are contained in $\mathbf{M}_{ij}$, $\mathbf{Q}(\mathbf{r}_i) = \mathbf{r}_i$ for $i = 1, 2, 3$. We now introduce the following constraints on the face F^{ij}

$$\int_{F^{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{Q}(\mathbf{r}_l)\, ds = 0 \quad \forall l = 1, \cdots, 6.$$

For $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=1}^3$, these constraints are nothing but the average matching conditions across the interface. The remaining constraints with $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^6$, are similar to the moment matching constraints which were introduced for fully primal edges in [23] except that our constraints use the projected rotational rigid body motions and are imposed on faces. In the following, we call these constraints of $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^6$ the moment constraints.

Even though we have introduced the set of primal constraints in order to make the FETI-DP method more efficient, the enlarged coarse problem can be a bottle neck of the computation. To compromise between the number of iterations of the FETI-DP method and the size of coarse problem, we will not impose the primal constraints over all faces. Among the faces, we select some as primal faces and we impose the six constraints only over them. For the remaining (non-primal faces), we assume that they satisfy an acceptable face path condition. This assumption makes it possible for the FETI-DP method with primal faces to have a condition number bound comparable to when all faces are chosen to be primal. We now define an acceptable face path.

Definition 3.1. **(Acceptable face path)** *For a pair of subdomains* (Ω_i, Ω_j) *having the common face* F^{ij} *with* $G_i \leq G_j$, *an acceptable face path is a path*

$$\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$$

from Ω_i *to* Ω_j *such that the coefficient* G_{k_l} *of* Ω_{k_l} *satisfies the condition*

$$TOL * (1 + log(H_i/h_i))^{-1} \left(1 + log(H_{k_l}/h_{k_l})\right)^2 * G_{k_l} \geq G_i$$

and the path from one subdomain to another is always through a primal face.

Some of faces are chosen as primal faces and the remaining are non-primal faces. In Section 5, we will introduce an algorithm which selects relatively few primal faces as well as keeps the condition number bound of the resulting FETI-DP operator within $C \max_{i=1,\cdots,N} \left\{(1 + \log(H_i/h_i))^2\right\}$. Here the constant C depends on the parameters TOL and L, the maximum number of subdomains on the acceptable face path. Furthermore, we choose some of vertices as primal vertices at which we will impose a point-wise matching condition. We assume that enough primal vertices are taken so as to make each local problem invertible. Based on these primal constraints, we introduce the following subspaces

$$\begin{aligned}\widetilde{\mathbf{W}} &:= \{\mathbf{w} \in \mathbf{W} \ : \ \mathbf{w} \text{ satisfies the vertex constraints at the primal vertices} \\ &\qquad \text{and the six face constraints across each primal faces}\}\,, \\ \widetilde{\mathbf{W}}_n &:= \{\mathbf{w}_n \in \mathbf{W}_n \ : \ \mathbf{w}_n \text{ has zero average and zero moment} \\ &\qquad \text{on each primal faces}\}\,.\end{aligned}$$

For $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, let $E(\mathbf{w}_n) \in \mathbf{W}$ be the zero extension of $\mathbf{w}_n$ to the whole interface, i.e., mortar and nonmortar faces. We can easily see that $E(\mathbf{w}_n) \in \widetilde{\mathbf{W}}$.

3.3. The FETI-DP equation. Let A_i denote the stiffness matrix of the bilinear form

$$a_i(\mathbf{u}_i, \mathbf{v}_i) := G_i \int_{\Omega_i} \varepsilon(\mathbf{u}_i) : \varepsilon(\mathbf{v}_i)\, dx + G_i\beta_i \int_{\Omega_i} \nabla \cdot \mathbf{u}_i \nabla \cdot \mathbf{v}_i\, dx,$$

and let S_i be the Schur complement of the matrix A_i. Moreover the matrix B_i is the mortar matching matrix corresponding to the unknowns of $\partial\Omega_i$. The mortar matching condition for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in W$ can be written as

$$\sum_{i=1}^{N} B_i \mathbf{w}_i = 0.$$

We note that we choose some of the vertices as primal vertices at which we will impose the point-wise matching condition. Let V_c be the set of unknowns at the global primal vertices and let $V_c^{(i)}$ be the set of unknowns at the primal vertices in the subdomain Ω_i. The mapping $R_c^{(i)} : V_c \to V_c^{(i)}$ is the restriction from the unknowns at the global primal vertices to the unknowns at the local primal vertices. The matrix B_i and the vector $\mathbf{w}_i \in \mathbf{W}_i$ are ordered as

$$B_i = \begin{pmatrix} B_r^{(i)} & B_c^{(i)} \end{pmatrix}, \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^{(i)} \\ \mathbf{w}_c^{(i)} \end{pmatrix},$$

where c stands for the unknowns at the primal vertices and r stands for the remaining unknowns. We then assemble vectors and matrices from the subdomains

$$\mathbf{w}_r = \begin{pmatrix} \mathbf{w}_r^{(1)} \\ \vdots \\ \mathbf{w}_r^{(N)} \end{pmatrix}, \quad B_r = \begin{pmatrix} B_r^{(1)} & \dots & B_r^{(N)} \end{pmatrix}, \quad B_c = \sum_{i=1}^{N} B_c^{(i)} R_c^{(i)}.$$

The face constraints are selected from the mortar matching constraints and they can be written as

$$R^t (B_r \mathbf{w}_r + B_c \mathbf{w}_c) = 0, \tag{3.2}$$

where the matrix R gives the face constraints as linear combinations of rows of the matrix $\begin{pmatrix} B_r & B_c \end{pmatrix}$.

By introducing the Lagrange multipliers $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ for the primal face constraints and for the mortar matching constraints, respectively, we get the following mixed formulation of (2.1)

$$\begin{pmatrix} S_{rr} & S_{rc} & B_r^t R & B_r^t \\ S_{cr} & S_{cc} & B_c^t R & B_c^t \\ R^t B_r & R^t B_c & 0 & 0 \\ B_r & B_c & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_r \\ \mathbf{w}_c \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_r \\ \mathbf{g}_c \\ 0 \\ 0 \end{pmatrix}.$$

We now eliminate the unknowns other than $\boldsymbol{\lambda}$ and obtain

$$F_{DP} \boldsymbol{\lambda} = \boldsymbol{d}.$$

This matrix F_{DP} satisfies the well-known relation

$$\langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle},$$

where

$$S = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_N \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \dots & B_N \end{pmatrix}.$$

We now introduce the Neumann-Dirichlet preconditioner M^{-1} given by

$$\langle M \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{\mathbf{W}}_n} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle}, \tag{3.3}$$

where $E(\mathbf{w}_n)$ is the zero extension of $\mathbf{w}_n$ into the space $\mathbf{W}$. From the observation that the extension $E(\mathbf{w}_n)$ belongs to $\widetilde{\mathbf{W}}$ for $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, we get

$$\langle M \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{\mathbf{W}}_n} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle} \leq \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle} = \langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \tag{3.4}$$

Therefore the lower bound of the FETI-DP operator is bounded from below by the value 1.

The explicit form of the preconditioner

$$M^{-1} = P^t \sum_{i=1}^{N} (B_i)(D_i)^t S_i D_i B_i^t P \tag{3.5}$$

is similar to other FETI-DP preconditioners except that the scaling matrix D_i is given differently and a certain projection P appears. We now derive the explicit form in detail. Let us define the space

$$M_R = \left\{ \boldsymbol{\lambda} \in M \ : \ R^t \boldsymbol{\lambda} = 0 \right\},$$

where R is the matrix given in (3.2) related to the primal constants. We then introduce the l^2-orthogonal projections P and P_n

$$P : M \to M_R, \quad P_n : W_n \to \widetilde{W}_n.$$

Since the constraints on the spaces M_R and $\widetilde{W}_n$ are given based on the nonmortar faces which are primal, these projections are composed of diagonal blocks of projections defined on each nonmortar faces

$$P = \text{diag}_{i=1,\cdots,N}\text{diag}_{j\in m_i}\left(P^{ij}\right), \quad P_n = \text{diag}_{i=1,\cdots,N}\text{diag}_{j\in m_i}\left(P_n^{ij}\right). \tag{3.6}$$

Here P^{ij} and P_n^{ij} are l^2-orthogonal projections given on the nonmortar face F^{ij}

$$P^{ij} : M|_{F^{ij}} \to M_R|_{F^{ij}}, \quad P_n^{ij} : W_n|_{F^{ij}} \to \widetilde{W}_n|_{F^{ij}}.$$

Let us define the restriction

$$R_{ij} : \mathbf{W}_n \to \mathbf{W}_{ij}$$

and the extension

$$E_{ij}^i : \mathbf{W}_{ij} \to \mathbf{W}_i.$$

We then express the zero extension $E(\mathbf{w}_n) = (\mathbf{w}_1, \cdots, \mathbf{w}_N)$ by

$$\mathbf{w}_i = E_i \mathbf{w}_n \text{ with } E_i = \sum_{j\in m_i} E_{ij}^i R_{ij}. \tag{3.7}$$

By using this notation, we rewrite formula (3.3) as

$$\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle = \max_{\mathbf{w}_n \in \tilde{\mathbf{W}}_n} \frac{\langle \widehat{B}\mathbf{w}_n, \boldsymbol{\lambda}\rangle^2}{\langle \widehat{S}\mathbf{w}_n, \mathbf{w}_n\rangle},$$

where

$$\widehat{S} = \sum_{i=1}^{N} E_i^t S_i E_i, \quad \widehat{B} = \text{diag}_{i=1}^N \text{diag}_{j\in m_i}\left(B_{ij}\right). \tag{3.8}$$

Here the matrix B_{ij} is a block of B_i corresponding to the unknowns of the nonmortar face F^{ij}. It is easy to check that

$$\widehat{B} : \widetilde{\mathbf{W}}_n \to \mathbf{M}_R$$

is one-to-one for $\dim(\widetilde{\mathbf{W}}_n) = \dim(\mathbf{M}_R)$ and $\widehat{B}(\widetilde{\mathbf{W}}_n) \subset \mathbf{M}_R$, and that $\widehat{S}$ is symmetric and positive definite on $\widetilde{\mathbf{W}}_n$. Therefore the maximum occurs for $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, such that

$$P_n^t \widehat{S} P_n \mathbf{w}_n = P_n^t \widehat{B}^t P \boldsymbol{\lambda}.$$

Let

$$B_p = P^t \widehat{B} P_n, \quad S_p = P_n^t \widehat{S} P_n. \tag{3.9}$$

As mentioned before, these operators are invertible and their inverses are denoted by B_p^{-1} and S_p^{-1}, respectively. Since B_p is block diagonal, B_p^{-1} can be written as

$$B_p^{-1} = \text{diag}_{i=1}^N \text{diag}_{j\in m_i} \left({B_p^{ij}}^{-1} \right), \quad B_p^{ij} = {P^{ij}}^t B_{ij} P_n^{ij}. \tag{3.10}$$

By using the expressions in (3.6)-(3.10), we obtain

$$\begin{aligned} \widehat{F}_{DP}^{-1} &= (B_p^t)^{-1} \widehat{S}_p B_p^{-1}, \\ &= \sum_{i=1}^N B_{i,n}^t S_i B_{i,n}. \end{aligned} \tag{3.11}$$

Here $B_{i,n}$ is given by

$$B_{i,n} = \begin{pmatrix} \text{diag}_{j\in m_i} \left(P_n^{ij} {B_p^{ij}}^{-1} \right) \\ 0 \end{pmatrix} R_i,$$

where $R_i : \mathbf{M} \to \Pi_{j\in m_i} \mathbf{M}_{ij}$ is the restriction and the zero submatrix corresponds to the unknowns of the other than nonmortar faces, i.e. mortar faces and boundaries of faces that belong to Ω_i.

We now derive the factor D_i in the right hand side of (3.5). (However, we will use the formula (3.11) in implementation.) The matrix $B_{i,n}$ can be written as

$$\begin{aligned} B_{i,n} &= \begin{pmatrix} \text{diag}_{j\in m_i} \left(P_n^{ij} {B_p^{ij}}^{-1} \right) \\ 0 \end{pmatrix} R_i, \\ &= \begin{pmatrix} \text{diag}_{j\in m_i} \left(P_n^{ij} ({B_p^{ij}}^t B_p^{ij})^{-1} {B_p^{ij}}^t \right) \\ 0 \end{pmatrix} R_i, \\ &= \begin{pmatrix} \text{diag}_{j\in m_i} \left(P_n^{ij} ({B_p^{ij}}^t B_p^{ij})^{-1} {P_n^{ij}}^t B_{ij}^t P^{ij} \right) \\ 0 \end{pmatrix} R_i, \\ &= \begin{pmatrix} D_{i,n} & 0 \\ 0 & 0 \end{pmatrix} B_i^t P, \end{aligned}$$

where

$$D_{i,n} = \text{diag}_{j\in m_i} \left(P_n^{ij} ({B_p^{ij}}^t B_p^{ij})^{-1} {P_n^{ij}}^t \right).$$

Therefore, the scaling matrix D_i in (3.5) is given by

$$D_i = \begin{pmatrix} D_{i,n} & 0 \\ 0 & 0 \end{pmatrix}.$$

The scaling matrix provides each subdomain problem with a zero Neumann boundary condition on the mortar faces and Dirichlet boundary conditions on the remaining part of the subdomain boundary. Hence we call it a Neumann-Dirichlet preconditioner.

4. Condition number analysis. In this section, we will consider an upper bound of the FETI-DP operator with the Neumann-Dirichlet preconditioner M^{-1}. First, we will construct functionals $\{f_l\}_{l=1}^6$, dual to the space $\mathbf{ker}(\varepsilon)$, which satisfy the following properties:

$$f_m(\mathbf{r}_k) = \delta_{mk}, \quad m, k = 1, \cdots, 6,$$
$$|f_m(\mathbf{w})|^2 \leq C \frac{\|\mathbf{w}\|^2_{0,\partial\Omega_i}}{H^2} \quad \text{for } \mathbf{w} \in \mathbf{L}^2(\partial\Omega_i). \tag{4.1}$$

Here $\{\mathbf{r}_k\}_{k=1}^6$ is a basis of $\mathbf{ker}$ (ε) with six rigid body motions scaled with respect to a face $F \subset \partial\Omega_i$; this means that we take $\widehat{\mathbf{x}} \in F$ and $H = \mathrm{diam}(F)$ in (2.3). Such dual functionals were first introduced by Klawonn and Widlund [23]. An arbitrary rigid body motion $\mathbf{r}$ can be represented by a linear combination of the elements of the basis $\{\mathbf{r}_k\}_{k=1}^6$

$$\mathbf{r} = \sum_{k=1}^{6} f_k(\mathbf{r})\mathbf{r}_k.$$

We will now choose six linearly independent functionals which are closely related to the primal constraints given on the face F. The functionals $\{g_l\}_{l=1}^6$ are given by

$$g_l(\mathbf{w}) = \frac{\int_F \mathbf{w} \cdot \mathbf{Q}(\mathbf{r}_l)\, ds}{\int_F \mathbf{Q}(\mathbf{r}_l) \cdot \mathbf{Q}(\mathbf{r}_l)\, ds}, \qquad \text{for } \mathbf{w} \in \mathbf{L}^2(\partial\Omega_i),\ l = 1, \cdots, 6.$$

Since these six functionals are linearly independent, they provide a basis of the dual space $(\mathbf{ker}(\varepsilon))'$. Thus there exists $\{\beta_{ml}\}_{m,l=1}^6$ such that

$$f_m = \sum_{l=1}^{6} \beta_{ml} g_l, \quad m = 1, \cdots, 6. \tag{4.2}$$

From the fact that the projection $\mathbf{Q}$ satisfies

$$\|\mathbf{Q}(\mathbf{w}) - \mathbf{w}\|^2_{0,F} \leq C h_i |\mathbf{w}|^2_{1,\Omega_i} \text{ for } \mathbf{w} \in \mathbf{H}^1(\Omega_i)$$

(see [32, Lemma 1.6]), we can show that

$$\|\mathbf{Q}(\mathbf{r}_l)\|^2_{0,F} \geq C H^2.$$

Here the constant C does not depend on any mesh parameters for sufficiently small h_i. From the above bound and Hölder's inequality, we obtain

$$|g_l(\mathbf{w})|^2 \leq C \frac{\|\mathbf{w}\|^2_{0,\partial\Omega_i}}{H^2}.$$

From this bound, (4.2), and the scaling of $\{\mathbf{r}_k\}_{k=1}^6$, the bound in (4.1) follows with the constant C independent of the mesh parameters. We denote the dual functionals described above as $\{f_l^F\}_{l=1}^6$ for the given face F. We can then express any rigid body motion $\mathbf{r} \in \mathbf{ker}(\varepsilon)$ as a linear combination using the basis:

$$\mathbf{r} = \sum_{l=1}^{6} f_l^F(\mathbf{r})\mathbf{r}_l^F.$$

In the following, we will provide several lemmas which will be used to analyze an upper bound of the FETI-DP operator. For a face $F \subset \partial\Omega_i$, the space $H_{00}^{1/2}(F)$ consists of the functions whose zero extension to the whole boundary $\partial\Omega_i$ belongs to the space $H^{1/2}(\partial\Omega_i)$ and is equipped with the norm

$$\|v\|_{H_{00}^{1/2}(F)} := \left(|v|^2_{H^{1/2}(F)} + \int_F \frac{v(x)^2}{\mathrm{dist}(x, \partial F)}\, ds \right)^{1/2}.$$

The norm can be extended to the product space $\mathbf{H}_{00}^{1/2}(F) := [H_{00}^{1/2}(F)]^3$ by using the usual product norm. Similarly, we can extend the edge and face lemmas to the product space with the product norm. The edge and face lemmas can be found in Toselli and Widlund [30, Lemma 4.24 and Lemma 4.25].

LEMMA 4.1. **(Edge lemma)** *Let E be an edge of $\partial\Omega_i$. Then for any $\mathbf{w}_i \in \mathbf{W}_i$ we have*

$$\|\mathbf{w}_i\|_{0,E} \le C \left(1 + \log \frac{H_i}{h_i}\right)^{1/2} \left(|\mathbf{w}_i|^2_{1/2,\partial\Omega_i} + \frac{1}{H_i}\|\mathbf{w}\|^2_{0,\partial\Omega_i} \right)^{1/2}.$$

For any subset $A \in \partial\Omega_i$, let us define an interpolant $I_A^i : C(\partial\Omega_i) \to W_i$

$$I_A^i(v) = \begin{cases} v(x), & \text{for } x \in A \cap N^h, \\ 0, & \text{else where.} \end{cases}$$

Here N^h denotes a set of nodes in the finite element space W_i.

LEMMA 4.2. **(Face lemma)** *Let F be a face of $\partial\Omega_i$. Then, for any $\mathbf{w}_i \in \mathbf{W}_i$, we have*

$$\|I_F^i(\mathbf{w}_i)\|_{\mathbf{H}_{00}^{1/2}(F)} \le C \left(1 + \log \frac{H_i}{h_i}\right) \left(|\mathbf{w}_i|^2_{1/2,\partial\Omega_i} + \frac{1}{H_i}\|\mathbf{w}\|^2_{0,\partial\Omega_i} \right)^{1/2}.$$

We now provide several inequalities for the mortar projection of functions. We recall that the space $\mathbf{W}_{ij}$, given on the nonmortar face of F^{ij}, the space $\mathbf{M}_{ij}$, the Lagrange multiplier space given on the face F^{ij}. The mortar projection is defined as follows.

DEFINITION 4.3. *(**Mortar projection**) The mortar projection* $\pi_{ij} : \mathbf{L}^2(F^{ij}) \to \mathbf{W}_{ij}$ *is given by*

$$\int_{F^{ij}} (\pi_{ij}(\mathbf{v}) - \mathbf{v}) \cdot \boldsymbol{\psi} \, ds = 0 \quad \forall \boldsymbol{\psi} \in \mathbf{M}_{ij}.$$

The mortar projection is continuous in both the L^2 and the $H_{00}^{1/2}$-norms.

LEMMA 4.4. *Let* $F = \partial\Omega_i \cap \partial\Omega_j$*. For* $\mathbf{w}_i \in \mathbf{W}_i$ *and* $\mathbf{w}_j \in \mathbf{W}_j$*, we have*

$$\|\pi_{ij}(\mathbf{w}_i - \mathbf{r}_i)\|^2_{H_{00}^{1/2}(F)} \le C \left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{E(\partial\Omega_i)},$$
$$\|\pi_{ij}(\mathbf{w}_j - \mathbf{r}_j)\|^2_{H_{00}^{1/2}(F)} \le C \left(1 + log\frac{H_j}{h_j}\right) \left(1 + log\frac{H_j}{h_j} + \frac{h_j}{h_i}\right) |\mathbf{w}_j|^2_{E(\partial\Omega_j)},$$

where $\mathbf{r}_i$ *and* $\mathbf{r}_j$ *are the minimizing rigid body motions of Lemma 2.3 with* $\Sigma = \partial\Omega_i$ *and* $\Sigma = \partial\Omega_j$*, respectively.*

Proof. Let us consider the first bound. We split the term into two parts

$$\pi_{ij}(\mathbf{w}_i - \mathbf{r}_i) = \pi_{ij}(I_F^i(\mathbf{w}_i - \mathbf{r}_i) + I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i)).$$

where I_F^i and $I_{\partial F}^i$ are the nodal value interpolants. From the stability of π_{ij} in $H_{00}^{1/2}$-norm and L^2-norm, and an inverse inequality, we obtain

$$\begin{aligned}
&\|\pi_{ij}(\mathbf{w}_i - \mathbf{r}_i)\|^2_{H_{00}^{1/2}(F)} \\
&\le 2\|\pi_{ij}(I_F^i(\mathbf{w}_i - \mathbf{r}_i))\|^2_{H_{00}^{1/2}(F)} + 2\|\pi_{ij}(I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i))\|^2_{H_{00}^{1/2}(F)} \\
&\le C \left(\|I_F^i(\mathbf{w}_i - \mathbf{r}_i)\|^2_{H_{00}^{1/2}(F)} + h_i^{-1}\|I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i)\|^2_{0,F} \right). \qquad (4.3)
\end{aligned}$$

Note that $I_F^i(\mathbf{w}_i - \mathbf{r}_i) \in \mathbf{H}_{00}^{1/2}(F)$ and $I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i) \in \mathbf{H}^{1/2}(F)$. From the quasi-uniformity of the triangulation T_i of the space $\mathbf{X}_i$, we get

$$\|I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i)\|^2_{0,F} \le Ch_i \|I_{\partial F}^i(\mathbf{w}_i - \mathbf{r}_i)\|^2_{0,\partial F}. \qquad (4.4)$$

Combining (4.3) and (4.4), and Lemmas 4.1 and 4.2, we obtain

$$\begin{aligned}
&\|\pi_{ij}(\mathbf{w}_i - \mathbf{r}_i)\|^2_{H_{00}^{1/2}(F)} \\
&\le C \left(1 + \log\frac{H_i}{h_i}\right)^2 \left(|\mathbf{w}_i - \mathbf{r}_i|^2_{1/2,\partial\Omega_i} + \frac{1}{H_i}\|\mathbf{w}_i - \mathbf{r}_i\|^2_{0,\partial\Omega_i} \right).
\end{aligned}$$

Then using Lemmas 2.2 and 2.3, the first bound is shown.

For the second bound, we use the nodal interpolants I_F^j and $I_{\partial F}^j$. We then get a factor h_j in (4.4) instead of h_i. Arguing as before, we obtain the bound for the second term. ☐

The following lemma is a simple modification of Dryja, Smith, and Widlund [9, Lemma 4.4].

Lemma 4.5. *Let $F \subset \partial\Omega_i$. For a linear function ϕ, we have*

$$\|I_F^i(\phi)\|^2_{H_{00}^{1/2}(F)} \leq C\left(1 + log\frac{H_i}{h_i}\right) H_i\|\phi\|^2_{\infty,F}.$$

Lemma 4.6. *For the basis $\{\mathbf{r}_m^{F^{lk}}\}_{m=1}^6$ of $\mathbf{ker}(\varepsilon)$ scaled with respect to the face $F^{lk} = \partial\Omega_l \cap \partial\Omega_k$, we have*

$$\|\pi_{ij}(\mathbf{r}_m^{F^{lk}})\|^2_{H_{00}^{1/2}(F^{ij})} \leq C\left(1 + log\frac{H_i}{h_i}\right) H_i\|\mathbf{r}_m^{F^{lk}}\|^2_{\infty,F^{ij}}.$$

Proof. Since each component of the function $\mathbf{r}_m^{F^{lk}}$ is linear, we can decompose the function into

$$\mathbf{r}_m^{F^{lk}} = I^i_{F^{ij}}(\mathbf{r}_m^{F^{lk}}) + I^i_{\partial F^{ij}}(\mathbf{r}_m^{F^{lk}}). \tag{4.5}$$

From the identity (4.5), an inverse inequality, the continuity of π_{ij} in both the L^2 and $H_{00}^{1/2}$-norms, and Lemma 4.5, we then obtain

$$\begin{aligned}\|\pi_{ij}(\mathbf{r}_m^{F^{lk}})\|^2_{H_{00}^{1/2}(F^{ij})} &\leq C\left(\|I_F^i(\mathbf{r}_m^{F^{lk}})\|^2_{H_{00}^{1/2}(F^{ij})} + h_i^{-1}\|I^i_{\partial F^{ij}}(\mathbf{r}_m^{F^{lk}})\|^2_{0,F^{ij}}\right)\\ &\leq C\left(\left(1 + \log\frac{H_i}{h_i}\right) H_i\|\mathbf{r}_m^{F^{lk}}\|^2_{\infty,F^{ij}} + \|I^i_{\partial F^{ij}}(\mathbf{r}_m^{F^{lk}})\|^2_{0,\partial F^{ij}}\right),\end{aligned} \tag{4.6}$$

where we have used

$$h_i^{-1}\|I^i_{\partial F}(\mathbf{r}_m^{F^{lk}})\|^2_{0,F^{ij}} \leq C\|I^i_{\partial F^{ij}}(\mathbf{r}_m^{F^{lk}})\|^2_{0,\partial F^{ij}},$$

which follows from the quasi-uniformity of the triangulation T_i. By employing Lemma 4.1 for the edges $E \subset \partial F^{ij}$, we get

$$\begin{aligned}\|I^i_{\partial F^{ij}}(\mathbf{r}_m^{F^{lk}})\|^2_{0,\partial F^{ij}} &\leq C\left(1 + \log\frac{H_i}{h_i}\right)\left(|\mathbf{r}_m^{F^{lk}}|^2_{1/2,F^{ij}} + \frac{1}{H_i}\|\mathbf{r}_m^{F^{lk}}\|^2_{0,F^{ij}}\right)\\ &\leq C\left(1 + \log\frac{H_i}{h_i}\right)\left(|\mathbf{r}_m^{F^{lk}}|^2_{1,\Omega_i} + \frac{1}{H_i}\|\mathbf{r}_m^{F^{lk}}\|^2_{0,F^{ij}}\right)\\ &\leq C\left(1 + \log\frac{H_i}{h_i}\right)\left(\frac{H_i^3}{H_{lk}^2} + \frac{H_{ij}^2}{H_i}\|\mathbf{r}_m^{F^{lk}}\|^2_{\infty,F^{ij}}\right).\end{aligned} \tag{4.7}$$

Here H_{kl} and H_{ij} denote the diameter of the face F^{lk} and the face F^{ij}, respectively. The shape regularity of the subdomain partition implies that the diameters of neighbors are comparable. The required bound follows by combining (4.6) and (4.7). ∎

Remark 4.7. *In Lemma 4.6, we may use the interpolants $I^j_{F^{ij}}$ and $I^j_{\partial F^{ij}}$ for the nodal set of the finite element space X_j instead of X_i. In this case, we obtain the following bound*

$$\|\pi_{ij}(\mathbf{r}_m^{F^{lk}})\|^2_{H_{00}^{1/2}(F)} \leq C\left(1 + log\frac{H_j}{h_j}\right) H_j\frac{h_j}{h_i}\|\mathbf{r}_m^{F^{lk}}\|^2_{\infty,F}. \tag{4.8}$$

LEMMA 4.8. *Let $F^{ij}(=\partial\Omega_i\cap\partial\Omega_j)$ be a primal face with $G_i \le G_j$. For $\mathbf{w} = (\mathbf{w}_1,\cdots,\mathbf{w}_N)\in\widetilde{\mathbf{W}}$, we have*

$$G_i\|\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)\|^2_{H^{1/2}_{00}(F^{ij})} \le C\left\{\left(1+log\frac{H_i}{h_i}\right)^2|\mathbf{w}_i|^2_{S_i} + \frac{G_i}{G_j}\left(1+log\frac{H_j}{h_j}\right)\left(1+log\frac{H_j}{h_j}+\frac{h_j}{h_i}\right)|\mathbf{w}_j|^2_{S_j}\right\},$$

where $|\mathbf{w}_l|^2_{S_l} = \langle S_l\mathbf{w}_l,\mathbf{w}_l\rangle$ for $l=i,j$.

Proof. Let $\{\mathbf{r}^{ij}_m\}^6_{m=1}$ be a basis of $\mathbf{ker}(\varepsilon)$ scaled with respect to the face F^{ij}. Since $\mathbf{w}=(\mathbf{w}_1,\cdots,\mathbf{w}_N)\in\widetilde{\mathbf{W}}$ satisfies face constraints across the primal face F^{ij}, the following identity holds

$$\sum_{m=1}^6 f_m(\mathbf{w}_i)\mathbf{r}^{ij}_m = \sum_{m=1}^6 f_m(\mathbf{w}_j)\mathbf{r}^{ij}_m.$$

We then have

$$\|\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)\|^2_{H^{1/2}_{00}(F^{ij})} \le 2\left\|\pi_{ij}\left(\mathbf{w}_i-\sum_{m=1}^6 f_m(\mathbf{w}_i)\mathbf{r}^{ij}_m\right)\right\|^2_{H^{1/2}_{00}(F^{ij})} + 2\left\|\pi_{ij}\left(\mathbf{w}_j-\sum_{m=1}^6 f_m(\mathbf{w}_j)\mathbf{r}^{ij}_m\right)\right\|^2_{H^{1/2}_{00}(F^{ij})}.$$

We now estimate

$$\begin{aligned}&\left\|\pi_{ij}\left(\mathbf{w}_i-\sum_{m=1}^6 f_m(\mathbf{w}_i)\mathbf{r}^{ij}_m\right)\right\|^2_{H^{1/2}_{00}(F^{ij})}\\ &=\left\|\pi_{ij}\left(\mathbf{w}_i-\mathbf{r}_i-\sum_{m=1}^6 f_m(\mathbf{w}_i-\mathbf{r}_i)\mathbf{r}^{ij}_m\right)\right\|^2_{H^{1/2}_{00}(F^{ij})}\\ &\le C\left(\|\pi_{ij}(\mathbf{w}_i-\mathbf{r}_i)\|^2_{H^{1/2}_{00}(F^{ij})}+\sum_{m=1}^6|f_m(\mathbf{w}_i-\mathbf{r}_i)|^2\|\pi_{ij}(\mathbf{r}^{ij}_m)\|^2_{H^{1/2}_{00}(F^{ij})}\right),\end{aligned}$$

where $\mathbf{r}_i\in$ **ker** (ε) satisfies $(\mathbf{w}_i-\mathbf{r}_i,\mathbf{r})_{\partial\Omega_i}=0\ \forall\,\mathbf{r}\in$ **ker** (ε).

From Lemma 4.4, the first term of the above expression is bounded by

$$\|\pi_{ij}(\mathbf{w}_i-\mathbf{r}_i)\|^2_{H^{1/2}_{00}(F^{ij})} \le C\frac{1}{G_i}\left(1+\log\frac{H_i}{h_i}\right)^2|\mathbf{w}_i|^2_{S_i},$$

and from (4.1), Lemmas 4.6 and 2.3, the second term is bounded by

$$\begin{aligned} &|f_m(\mathbf{w}_i - \mathbf{r}_i)|^2 \; \|\pi_{ij}(\mathbf{r}_m^{ij})\|^2_{H^{1/2}_{00}(F^{ij})} \\ &\le C \frac{\|\mathbf{w}_i - \mathbf{r}_i\|^2_{0,\partial\Omega_i}}{H_i^2} \left(1 + \log\frac{H_i}{h_i}\right) H_i \|\mathbf{r}_m^{ij}\|^2_{\infty, F^{ij}} \\ &\le C \frac{1}{G_i} \left(1 + \log\frac{H_i}{h_i}\right) G_i |\mathbf{w}_i|^2_{E(\partial\Omega_i)} \\ &\le C \frac{1}{G_i} \left(1 + \log\frac{H_i}{h_i}\right) |\mathbf{w}_i|^2_{S_i}. \end{aligned}$$

Similarly, we obtain

$$\|\pi_{ij}(\mathbf{w}_j - \mathbf{r}_j)\|^2_{H^{1/2}_{00}(F)} \le C \frac{1}{G_j} \left(1 + \log\frac{H_j}{h_j}\right) \left(1 + \log\frac{H_j}{h_j} + \frac{h_j}{h_i}\right) |\mathbf{w}_j|^2_{S_j},$$

and

$$|f_m(\mathbf{w}_j - \mathbf{r}_j)|^2 \; \|\pi_{ij}(\mathbf{r}_m^{ij})\|^2_{H^{1/2}_{00}(F^{ij})} \le C \frac{1}{G_j} \frac{h_j}{h_i} \left(1 + \log\frac{H_j}{h_j}\right) |\mathbf{w}_j|^2_{S_j}.$$

Here $\mathbf{r}_j \in \mathbf{ker}(\varepsilon)$ satisfies $(\mathbf{w}_j - \mathbf{r}_j, \mathbf{r})_{\partial\Omega_i} = 0$ for all $\mathbf{r} \in \mathbf{ker}(\varepsilon)$. In the above bound, we have used the bound (4.8) for the term $\|\pi_{ij}(\mathbf{r}_m^{ij})\|_{H^{1/2}_{00}(F^{ij})}$. □

Lemma 4.9. *Let $F(= \partial\Omega_i \cap \partial\Omega_j)$ be a non-primal face with $G_i \le G_j$ and $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$ be an acceptable face path. Then, for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \widetilde{\mathbf{W}}$ we have*

$$\begin{aligned} G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F)} \le C \Bigg\{ &\left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i} \\ &+ L * \sum_{l=1}^{n} \left(1 + log\frac{H_i}{h_i}\right) \frac{G_i}{G_{k_l}} |\mathbf{w}_{k_l}|^2_{S_{k_l}} \\ &+ \frac{G_i}{G_j} \left(1 + log\frac{H_j}{h_j}\right) \left(1 + log\frac{H_j}{h_j} + \frac{h_j}{h_i}\right) |\mathbf{w}_j|^2_{S_j} \Bigg\}, \end{aligned}$$

where the constant L is the number of subdomains on the acceptable face path.

Proof. Let $\{\mathbf{r}_m^{ik_1}\}$, $\{\mathbf{r}_m^{k_1k_2}\}$, $\cdots$, $\{\mathbf{r}_m^{k_nj}\}$ be bases of $\mathbf{ker}(\varepsilon)$ scaled with respect to the primal faces F^{ik_1}, $F^{k_1k_2}$, $\cdots$, F^{k_nj} on the acceptable face path, respectively. We then have

$$\begin{aligned} \mathbf{w}_i - \mathbf{w}_j = \mathbf{w}_i &- \sum_{m=1}^{6} f_m^{ik_1}(\mathbf{w}_i)\mathbf{r}_m^{ik_1} + \sum_{m=1}^{6} f_m^{ik_1}(\mathbf{w}_{k_1})\mathbf{r}_m^{ik_1} - \sum_{m=1}^{6} f_m^{k_1k_2}(\mathbf{w}_{k_1})\mathbf{r}_m^{k_1k_2} \\ &+ \sum_{m=1}^{6} f_m^{k_1k_2}(\mathbf{w}_{k_2})\mathbf{r}_m^{k_1k_2} - \sum_{m=1}^{6} f_m^{k_2k_3}(\mathbf{w}_{k_2})\mathbf{r}_m^{k_2k_3} + \cdots \\ &+ \sum_{m=1}^{6} f_m^{k_nj}(\mathbf{w}_j)\mathbf{r}_m^{k_nj} - \mathbf{w}_j. \end{aligned} \tag{4.9}$$

For the first and last terms in the above equation, the following bounds are given in the proof of Lemma 4.8:

$$\|\pi_{ij}(\mathbf{w}_i - \sum_{m=1}^{6} f_m^{ik_1}(\mathbf{w}_i)\mathbf{r}_m^{ik_1})\|^2_{H^{1/2}_{00}(F)} \le C\frac{1}{G_i}\left(1+\log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i},$$

$$\|\pi_{ij}(\sum_{m=1}^{6} f_m^{k_nj}(\mathbf{w}_j)\mathbf{r}_m^{k_nj} - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F)}$$
$$\le C\frac{1}{G_j}\left(1+\log\frac{H_j}{h_j}\right)\left(1+\log\frac{H_j}{h_j}+\frac{h_j}{h_i}\right)|\mathbf{w}_j|^2_{S_j}.$$

We now consider

$$\sum_{m=1}^{6} f_m^{k_1k_2}(\mathbf{w}_{k_2})\mathbf{r}_m^{k_1k_2} - \sum_{m=1}^{6} f_m^{k_2k_3}(\mathbf{w}_{k_2})\mathbf{r}_m^{k_2k_3},$$
$$= \sum_{m=1}^{6} f_m^{k_1k_2}(\mathbf{w}_{k_2}-\mathbf{r}_{k_2})\mathbf{r}_m^{k_1k_2} - \sum_{m=1}^{6} f_m^{k_2k_3}(\mathbf{w}_{k_2}-\mathbf{r}_{k_2})\mathbf{r}_m^{k_2k_3},$$

where $\mathbf{r}_{k_2} \in \mathbf{ker}(\varepsilon)$ satisfies $(\mathbf{w}_{k_2}-\mathbf{r}_{k_2},\mathbf{r})_{\partial\Omega_{k_2}} = 0 \ \forall \mathbf{r} \in \mathbf{ker}(\varepsilon)$. From (4.1) and Lemma 2.3, we obtain

$$|f_m^{k_1k_2}(\mathbf{w}_{k_2}-\mathbf{r}_{k_2})|^2 \ \|\pi_{ij}(\mathbf{r}_m^{k_1k_2})\|^2_{H^{1/2}_{00}(F)}$$
$$\le C\frac{\|\mathbf{w}_{k_2}-\mathbf{r}_{k_2}\|^2_{0,\partial\Omega_{k_2}}}{H^2_{k_2}}\left(1+\log\frac{H_i}{h_i}\right)H_i\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F}$$
$$\le C * L * \frac{1}{G_{k_2}}\left(1+\log\frac{H_i}{h_i}\right)|\mathbf{w}_{k_2}|^2_{S_{k_2}},$$

where the constant L is the number of subdomains on the acceptable face path. The rigid body motion $\mathbf{r}_m^{k_1k_2}$ is scaled with respect to the face $F^{k_1k_2}$. From the regularity of the subdomain partition, we may assume that the subdomain partition is locally quasi uniform. Hence we can bound the term $\|\mathbf{r}_m^{k_1k_2}\|^2_{\infty,F}$ by the face path length L. The remaining terms in (4.9) can be bounded in a similar way leading to the required bound of $G_i\|\mathbf{w}_i-\mathbf{w}_j\|^2_{H^{1/2}_{00}(F)}$. ☐

From the bounds of Lemmas 4.8 and 4.9, we have learned that we need an assumption on the mesh sizes to remove the factor $(G_i/G_j)(h_j/h_i)$ in the bound.

ASSUMPTION 4.10. *For the subdomains Ω_i and Ω_j which have a common face F with $G_i \le G_j$, the mesh sizes h_i and h_j satisfy*

$$\frac{h_j}{h_i} \le C\left(\frac{G_j}{G_i}\right)^{\gamma} \text{ for some } 0 \le \gamma \le 1. \tag{4.10}$$

REMARK 4.11. *Let $F(=\partial\Omega_i \cap \partial\Omega_j)$ be a face with $G_i \le G_j$. Then from the assumption on the mesh sizes, we have*

$$\frac{G_i}{G_j}\frac{h_j}{h_i} \le C\left(\frac{G_i}{G_j}\right)^{1-\gamma} \le C. \tag{4.11}$$

Moreover the acceptable face path assumption gives

$$\left(1+log\frac{H_i}{h_i}\right)\frac{G_i}{G_{k_l}} \leq TOL * \left(1+log\frac{H_{k_l}}{h_{k_l}}\right)^2. \tag{4.12}$$

Combining Lemmas 4.8 and 4.9 with (4.11) *and* (4.12)*, we obtain the following bound for both the primal and non-primal cases*

$$G_i\|\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)\|^2_{H^{1/2}_{00}(F)} \leq C(TOL,L)\sum_{l\in N_{ij}}\left(1+log\frac{H_l}{h_l}\right)^2|\mathbf{w}_l|^2_{S_l}, \tag{4.13}$$

where N_{ij} is the set of subdomain indices which appear on the acceptable face path. The constant C depends on TOL and L but not on any mesh parameters and not on the coefficients G_i.

Lemma 4.12. *Assume that the mesh sizes satisfy the assumption* (4.10) *and that every non-primal face satisfies the acceptable face path condition with given TOL and L. We then obtain*

$$\langle F_{DP}\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle^2 = \max_{\mathbf{w}\in\widetilde{\mathbf{W}}}\frac{\langle B\mathbf{w},\boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w},\mathbf{w}\rangle} \leq C(TOL,L)\max_{i=1,\cdots,N}\left\{\left(1+log\frac{H_i}{h_i}\right)^2\right\}\langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle,$$

where the constant C depends on the TOR and L but not on any mesh parameters and not on the coefficients G_i.

Proof. We consider

$$\begin{aligned}\langle B\mathbf{w},\boldsymbol{\lambda}\rangle^2 &= \left(\sum_{i=1}^N\sum_{j\in m_i}\int_{F^{ij}}(\mathbf{w}_i-\mathbf{w}_j)\cdot\boldsymbol{\lambda}\,ds\right)^2\\ &= \left(\sum_{i=1}^N\sum_{j\in m_i}\int_{F^{ij}}\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)\cdot\boldsymbol{\lambda}\,ds\right)^2.\end{aligned}$$

Let $\mathbf{z}_n\in\mathbf{W}_n$ such that $\mathbf{z}_n|_{F^{ij}}=\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)$. On a primal face F^{ij}, $\mathbf{w}$ satisfies

$$\int_{F^{ij}}(\mathbf{w}_i-\mathbf{w}_j)\cdot\mathbf{Q}(\mathbf{r}_l)\,ds=0,\quad l=1,\cdots,6.$$

This implies

$$\int_{F^{ij}}\pi_{ij}(\mathbf{w}_i-\mathbf{w}_j)\cdot\mathbf{Q}(\mathbf{r}_l)\,ds=0,\quad l=1,\cdots,6,$$

so that $\mathbf{z}_n$ belongs to $\widetilde{\mathbf{W}}_n$. By the definition of M, we get

$$\begin{aligned}\langle B\mathbf{w},\boldsymbol{\lambda}\rangle^2 &= \langle B\mathbf{z},\boldsymbol{\lambda}\rangle^2\\ &\leq \langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle\langle S\mathbf{z},\mathbf{z}\rangle,\end{aligned}$$

where $\mathbf{z}=E(\mathbf{z}_n)\in\widetilde{\mathbf{W}}$ is the zero extension of $\mathbf{z}_n\in\widetilde{\mathbf{W}}_n$.

It suffices to show that

$$\langle S\mathbf{z}, \mathbf{z}\rangle \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log\frac{H_i}{h_i}\right)^2 \right\} \langle S\mathbf{w}, \mathbf{w}\rangle. \tag{4.14}$$

We now consider

$$\begin{aligned} \langle S\mathbf{z}, \mathbf{z}\rangle &= \sum_{i=1}^{N} \langle S_i \mathbf{z}_i, \mathbf{z}_i\rangle \\ &\leq C \sum_{i=1}^{N} \sum_{j\in m_i} G_i |\mathcal{H}^i \left(\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\right)|^2_{1,\Omega_i} \\ &\leq C \sum_{i=1}^{N} \sum_{j\in m_i} G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F^{ij})}. \end{aligned}$$

Here $\mathcal{H}^i$ is the discrete harmonic extension into X_i. From the bound (4.13), we obtain

$$G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F^{ij})} \leq C(TOL, L) \max_{l\in N_{ij}} \left\{ \left(1 + \log\frac{H_l}{h_l}\right)^2 \right\} \sum_{l\in N_{ij}} |\mathbf{w}_l|^2_{S_l}. \tag{4.15}$$

Here N_{ij} contains the indices of the subdomains that appear on the acceptable face path $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$. Assuming that the length of the acceptable face path is bounded by some number L, and by summing up the term in (4.15) over the whole faces F^{ij}, we obtain (4.14) with a constant $C(TOL, L)$ depending only on the TOL and L, the maximum length of the acceptable face paths. □

The lower bound in (3.4) and the upper bound in Lemma 4.12 lead to the following condition number bound.

THEOREM 4.13. *Under the assumption that the mesh sizes satisfy* (4.10) *and that every non-primal face satisfies the acceptable face path condition with a given TOL and L, we obtain the condition number bound*

$$\kappa(M^{-1}F_{DP}) \leq C(TOL, L) \max_{i=1,\cdots,N} \left\{ \left(1 + log\frac{H_i}{h_i}\right)^2 \right\}.$$

Here the constant C is independent of the mesh parameters and the coefficients G_i, but depends on TOL and L, the maximum face path length.

5. An algorithm for selecting primal faces. We now introduce an algorithm which selects a quite small number of primal faces for an arbitrary distribution of $\{G_i\}_{i=1}^N$. First we select an initial set of primal faces and put it in the set P of primal faces. We then determine non-primal faces based on the set P. We then visit the remaining undetermined faces in a certain order and add them one by one to the set P. Whenever we add an undetermined face to the set P, we determine the current set of non-primal faces based on the updated primal set P. We repeat this process

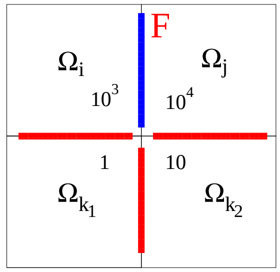

FIG. 1. *Essentially primal face* $F = \partial\Omega_i \cap \partial\Omega_j$ *when* $TOL = 10$, $\frac{H_l}{h_l} = 4$

until every face is determined. In order to be able to choose a small initial primal set P, we introduce the concept of an essentially primal face.

DEFINITION 5.1. *(**Essentially primal face**) A face $F = \partial\Omega_i \cap \partial\Omega_j$ is essentially primal, if there is no acceptable face path for (Ω_i, Ω_j) when all faces except F are chosen to be primal.*

For example, the face F in Figure 1 is essentially primal for the given TOL, coefficient distribution, and mesh size. The essentially primal faces are the faces that have to be chosen as primal faces given the coefficient distribution.

We will now explain the algorithm in detail. For a given TOL and L, we determine the essentially primal faces and add them to the set P of primal faces. Based on this set P, we determine the non-primal faces. For the remaining undetermined faces, we order them according to decreasing ratio of the coefficients between the two subdomain Ω_i and Ω_j. If we have more than one face having the same coefficient ratio then we order the faces according to the number of neighbors of the subdomain Ω_i and Ω_j which intersect on the face. We then add an undetermined face to the set P and determine the non-primal faces of this updated set P. We repeat this until every face is determined. The ordering of the undetermined faces increases our chances that there will exist acceptable face paths for other faces undetermined at this time.

Algorithm (TOL, L, $\{G_i\}$, $\{H_i\}, \{h_i\}$ given)

Step 1. Determine essentially primal faces F and add them to the primal face set P.

Step 2. Determine non-primal faces based on the set P.

Step 3. For the remaining undetermined faces F, order them decreasingly according to the ratio of the coefficients. If there are more than two faces with the same ratio then order them decreasingly according to the number of neighbors of the subdomains which intersect the current face F.

Step 4. **Do** until every undetermined face F determined

- Add a current undetermined face F to the primal face set P
- Then determine non-primal faces based on the updated primal face set P

End

N^3	**Total**	**Min**	**Const**	**Random**
2^3	12	7	7	8
4^3	144	63	68	89
6^3	540	215	246	322
8^3	1344	511	646	804
10^3	2700	999	1300	1598

TABLE 1

Number of primal faces from the algorithm: N^3 *(number of subdomains),* **Total** *(number of faces over the subdomain partition),* **Min** *(number of primal faces with no limit on* TOL *and* L*),* **Const** *(number of primal faces for the constant coefficient case with* $TOL = 10$ *and* $L = 6$*),* **Random** *(number of primal faces for the discontinuous coefficient case with* $TOL = 10$ *and* $L = 6$*)*

We have tested the algorithm for both constant and variable coefficient cases. The domain $\Omega = [0,1]^3$ is partitioned into N^3 hexagonal subdomains. For the case of constant coefficient, we take $G(x) = 1$, and for the case of discontinuous coefficient we distribute the values 1, 10, 10^2 and 10^3 randomly over the subdomain partition. In Table 5, we give the number of primal faces when increasing the number of subdomains with $TOL = 10$, $L = 6$, and the same number of nodes (H_i/h_i) for all subdomains. Here **Total** means the total number of faces in the subdomain partition, **Min** denotes the number of primal faces what we obtain from the algorithm with no limit on TOL and L. The columns **Const** and **Random** show the number of primal faces for the constant coefficient case and the discontinuous coefficient case, respectively. Comparing these two columns, we see that this algorithm gives a quite small number of primal faces for the case with the discontinuous coefficients.

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