BDDC ALGORITHMS FOR INCOMPRESSIBLE STOKES EQUATIONS

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Abstract. The purpose of this paper is to extend the BDDC (balancing domain decomposition by constraints) algorithm to saddle-point problems that arise when mixed finite element methods are used to approximate the system of incompressible Stokes equations. The BDDC algorithms are iterative substructuring methods, which form a class of domain decomposition methods based on the decomposition of the domain of the differential equations into nonoverlapping subdomains. They are defined in terms of a set of primal continuity constraints, which are enforced across the interface between the subdomains and which provide a coarse space component of the preconditioner. Sets of such constraints are identified for which bounds on the rate of convergence can be established that are just as strong as previously known bounds for the elliptic case. In fact, the preconditioned operator is effectively positive definite, which makes the use of a conjugate gradient method possible. A close connection is also established between the BDDC and FETI-DP algorithms for the Stokes case.

Key words. domain decomposition, Lagrange multipliers, FETI, preconditioners, elliptic systems, elasticity, finite elements.

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1. Introduction.

The BDDC algorithms are domain decomposition methods based on nonoverlapping subdomains into which the domain of a given partial differential equation has been divided. Introduced by Dohrmann [3, 4] and analyzed in the elliptic case by him, Mandel, and Tezaur [19, 20], these methods represent an important advance over the balancing Neumann-Neumann methods that have been used extensively in the past to solve large finite element problems; cf. [25, Section 6.2] where references to earlier work can also be found. Just as the classical balancing methods have much in common with the original one-level FETI methods, BDDC is closely related to the more recent dual-primal FETI (FETI-DP) methods. Each BDDC and FETI-DP method is defined in terms of a set of *primal* continuity constraints across the interface Γ formed by the parts of the subdomain boundaries which are common to at least two subdomains. In addition to, or instead of, point constraints, it is important to make certain averages over edges or faces of the interface the same. In some applications, we also should have certain first order moments, over edges, with common values; see [13] for a discussion of such *fully primal edges* for three-dimensional elasticity.

In an important contribution to the theory Mandel, Dohrmann, and Tezaur established that the preconditioned operators of a pair of BDDC and FETI-DP algorithms, with the same primal constraints, have the same nonzero eigenvalues; see [20]. We note that this fact was first observed experimentally by Fragakis and Papadrakakis [7] for pairs of balancing Neumann–Neumann and one-level FETI methods; these authors also discussed primal iterative substructuring methods which are close counterparts to various FETI algorithms. An important consequence of the results in [7, 20] is that these algorithms, which can be built from the same set of subprograms, have

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very similar performance. The choice of algorithm can therefore be based on other considerations.

In a recent paper [18], the authors rederived the BDDC and FETI-DP algorithms for elliptic problems and also gave a short proof of the main result in [20]. A key to these simplifications is a change of variables so that, e.g., a primal constraint on the average over an interface edge or face is represented by a single primal variable in the new coordinate system. Simultaneously, a complementary set of dual displacement variables is introduced for each of which the edge and face averages vanish; an illustrative example of how the change of variables can be carried out is given in [18]. This leads to a clear separation of the different sets of variables and the description and analysis of the algorithm is simplified considerably. This approach has also been the basis for a successful and highly accurate implementation of FETI–DP algorithms; cf. [13, 10].

In this paper, a BDDC algorithm is developed for mixed finite element approximations of the incompressible Stokes equations in a very similar way. If the set of primal constraints on the velocity across the interface satisfies a certain assumption, we are then able to show that the preconditioned saddle-point problem is positive definite on the subspace that satisfies the primal constraints and that the iterates stay in this subspace. We are then able to use a preconditioned conjugate gradient method and we can, if an additional assumption is satisfied, also prove as strong a bound on the convergence rate as for the standard elliptic case.

We note that the new algorithm has much in common with relatively recent extensions of the classical balancing Neumann-Neumann method to the Stokes equations and almost incompressible elasticity by Pavarino, Goldfeld, and the second author, see [22, 9, 8], and extensions of the FETI–DP methods developed by the first author in [15, 16, 17]. We note that all these methods, in our experience, converge quite rapidly. Just as in our earlier work, we will work with *benign subspaces*, i.e., subspaces of the mixed finite element spaces on which the saddle-point problem is positive definite. (We note that the same space of functions is called *balanced* in [25, Section 9.4.2].) We are also able to prove that any two BDDC and FETI–DP methods, with the same set of primal constraints and which satisfies our first assumption, have the same set of nonzero eigenvalues; this is the same result as given in [20, 18] for the elliptic case.

We note that Dohrmann [5], recently has developed and tested a BDDC method for the related problem of almost incompressible elasticity. We will comment further on his work in Section 7. For older references to domain decomposition algorithms for mixed finite element approximations, see [25, Chapter 9].

In addition to deriving and analyzing the algorithms, we also report on some numerical experiments in the final section.

2. Discretization of a Saddle-point Problem. Let us consider the incompressible Stokes problem on a bounded, polyhedral domain Ω, in two or three dimensions. We denote the boundary of the domain by $\partial\Omega$; for simplicity a homogeneous Dirichlet boundary condition is enforced. (Generally, in order for a divergence free extension to exist, the integral of the normal component of the velocity over the boundary of the region must vanish.) The weak solution has the following saddle-point formulation: find $\mathbf{u} \in \left(H_0^1(\Omega)\right)^d = \{\mathbf{v} \in (H^1(\Omega))^d \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$, $d = 2$ or 3, and $p \in L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}$, such that,

$$(1)\qquad \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & (\mathbf{f},\mathbf{v}), & \forall \mathbf{v} \in \left(H_0^1(\Omega)\right)^d, \\ b(\mathbf{u},q) & = & 0, & \forall q \in L_0^2(\Omega)\ , \end{cases}$$

where $b(\mathbf{u},q) = -\int_\Omega (\nabla\cdot\mathbf{u})q$, and

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} \quad \text{or} \quad a(\mathbf{u},\mathbf{v}) = 2\int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}).$$

Here the strain tensor $\epsilon(\mathbf{u})$ is defined by

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and

$$\nabla\mathbf{u} : \nabla\mathbf{v} = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} \quad \text{and} \quad \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i,j=1}^{d} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}).$$

The operator form of the Stokes problem with Dirichlet boundary conditions is the same for either choice of the bilinear form $a(\cdot,\cdot)$, but we will adopt the second which gives rise to a natural boundary condition of the form

$$2\sum_{j=1}^{d} \epsilon_{ij} n_j - p n_i = g_i \quad \text{on } \partial\Omega, \quad i = 1,\dots d. \tag{2}$$

This is the normal component of the stress field. We note that this approach is consistent with that of Quarteroni and Valli [23, Section 5.3] and the derivation of a physically relevant interface condition in Batchelor's book [1]. There is the further advantage that we will develop a theory which is equally valid for almost incompressible elasticity and that we can draw very directly on some recent results by Klawonn and the second author [13] on compressible elasticity. The following lemma, see [11, Lemma 4], [8, Lemma 1.3], and [13, Section 2], shows the equivalence between the Stokes and elasticity bilinear forms and that of H^1. Essentially, it is a variant of Korn's second inequality.

Lemma 1. *There exists a constant $c > 0$ such that*

$$c\|\nabla\mathbf{u}\|_{L^2(\Omega)} \le \|\epsilon(\mathbf{u})\|_{L^2(\Omega)} \le \|\nabla\mathbf{u}\|_{L^2(\Omega)}, \quad \forall \mathbf{u} \in (H^1(\Omega))^d,\ \mathbf{u} \perp ker(\epsilon),$$

where $ker(\epsilon)$ is the space of rigid body motions of the elasticity problem.

In our mixed finite element methods for solving the saddle-point problem (1), the velocity solution space, (or the space of displacements for the elasticity problems,) will be denoted by $\widehat{\mathbf{W}}$. It consists of vector-valued, low order piece-wise polynomial functions which are continuous across element boundaries. The pressure space $Q \subset L_0^2(\Omega)$ will consist of scalar, discontinuous functions. A characteristic diameter of the elements of the underlying triangulation is denoted by h. The finite element approximation $(\mathbf{u}, p)$ of the variational problem (1) satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}, \tag{3}$$

where the matrices A and B represent the restrictions of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the finite-dimensional space $\widehat{\mathbf{W}} \times Q$. (We will use the same notation for vectors of nodal values and the corresponding finite element functions.)

We will always assume that the chosen mixed finite element space $\widehat{\mathbf{W}} \times Q$ is inf-sup stable, i.e., that there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\widehat{\mathbf{W}}} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{H^1}} \geq \beta \|q\|_{L^2}, \qquad \forall q \in Q. \tag{4}$$

We note that we will only need this estimate for the subdomains, which we will introduce in the next section. This assumption will guarantee that the local subdomain problems, as well as the global one, are well posed.

3. Reduced Subdomain Interface Problem. The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1, 2, ..., N$, of characteristic diameter H. We assume that each of them is a union of a number of shape-regular tetrahedra (or triangles) and that there is a uniform bound on these numbers. Each subdomain is a union of shape regular elements and the nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\cup \partial\Omega_i)\backslash\partial\Omega$; the interface of an individual subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We will denote the set of nodes on Γ by Γ_h, etc. We assume, as is customary in domain decomposition theory, that the triangulation of each subdomain is quasi uniform. Our algorithms are also well defined for more irregular subdomains such as those that result from a mesh partitioner, but our theory does not fully cover such cases. The requirements on the subdomains in our full theory are discussed systematically in [25, Section 4.2].

We decompose the discrete velocity and pressure spaces $\widehat{\mathbf{W}}$ and Q into

$$\widehat{\mathbf{W}} = \mathbf{W}_I \bigoplus \widehat{\mathbf{W}}_\Gamma, \quad Q = Q_I \bigoplus Q_0. \tag{5}$$

$\mathbf{W}_I$ and Q_I are direct sums of subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$, and subdomain interior pressure spaces $Q_I^{(i)}$, respectively, i.e.,

$$\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_I^{(i)}, \qquad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}.$$

The elements of $\mathbf{W}_I^{(i)}$ are supported in the subdomain Ω_i and vanish on its interface Γ_i, while the elements of $Q_I^{(i)}$ are restrictions of elements in Q to Ω_i which satisfy $\int_{\Omega_i} q_I^{(i)} = 0$. $\widehat{\mathbf{W}}_\Gamma$ is the space of traces on Γ of functions in $\widehat{\mathbf{W}}$ and Q_0 is the subspace of Q with constant values $q_0^{(i)}$ in the subdomain Ω_i that satisfy $\int_\Omega q_0 dx = \sum_{i=1}^{N} q_0^{(i)} m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i.

We denote the space of interface velocity variables of the subdomain Ω_i by $\mathbf{W}_\Gamma^{(i)}$, and the associated product space by $\mathbf{W}_\Gamma = \prod_{i=1}^{N} \mathbf{W}_\Gamma^{(i)}$; generally functions in $\mathbf{W}_\Gamma$ are discontinuous across the interface. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface velocity space $\widehat{\mathbf{W}}_\Gamma$ to their subdomain components in the space $\mathbf{W}_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ.

With this decomposition of the solution space as in (5), the global saddle-point problem (3) can be written as: find $(\mathbf{u}_I, p_I, \mathbf{u}_\Gamma, p_0) \in (\mathbf{W}_I, Q_I, \widehat{\mathbf{W}}_\Gamma, Q_0)$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & \widehat{A}_{\Gamma I}^T & 0 \\ B_{II} & 0 & \widehat{B}_{I\Gamma} & 0 \\ \widehat{A}_{\Gamma I} & \widehat{B}_{I\Gamma}^T & \widehat{A}_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{bmatrix}. \tag{6}$$

In the coefficient matrix, the leading two by two block, by a symmetric permutation, can be made into a block diagonal matrix with blocks corresponding to independent subdomain problems. The lower left block is zero since the bilinear form $b(\mathbf{v}_I, q_0)$ always vanishes for any $\mathbf{v}_I \in \mathbf{W}_I$ and $q_0 \in Q_0$. The blocks related to the continuous interface velocity are assembled from the corresponding subdomain submatrices, e.g., $\widehat{A}_{\Gamma\Gamma} = \sum_{i=1}^N {R_\Gamma^{(i)}}^T A_{\Gamma\Gamma}^{(i)} R_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} R_\Gamma^{(i)}$. Correspondingly, the right hand side vector $\mathbf{f}_I$ consists of subdomain vectors $\mathbf{f}_I^{(i)}$, and $\mathbf{f}_\Gamma$ is assembled from the subdomain components $\mathbf{f}_\Gamma^{(i)}$. We denote the spaces of right hand side vectors $\mathbf{f}_I$ and $\mathbf{f}_\Gamma$ by $\mathbf{F}_I$ and $\mathbf{F}_\Gamma$, respectively.

Eliminating the independent subdomain interior variables $(\mathbf{u}_I, p_I)$ from the global problem (6), we have the global interface problem

$$
\begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}, \tag{7}
$$

where the right hand side vector $\mathbf{g}_\Gamma$ is

$$
\mathbf{g}_\Gamma = \sum_{i=1}^N {R_\Gamma^{(i)}}^T \left\{ \mathbf{f}_\Gamma^{(i)} - \left[A_{\Gamma I}^{(i)} \; {B_{I\Gamma}^{(i)}}^T \right] \begin{bmatrix} A_{II}^{(i)} & {B_{II}^{(i)}}^T \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_I^{(i)} \\ 0 \end{bmatrix} \right\}.
$$

$\widehat{S}_\Gamma$ is assembled from subdomain Stokes Schur complements $S_\Gamma^{(i)}$, which are defined by: given $\mathbf{w}_\Gamma^{(i)} \in \mathbf{W}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} \mathbf{w}_\Gamma^{(i)} \in \mathbf{F}_\Gamma^{(i)}$ such that

$$
\begin{bmatrix} A_{II}^{(i)} & {B_{II}^{(i)}}^T & {A_{\Gamma I}^{(i)}}^T \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & {B_{I\Gamma}^{(i)}}^T & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{w}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ S_\Gamma^{(i)} \mathbf{w}_\Gamma^{(i)} \end{bmatrix}. \tag{8}
$$

We see from (8), that the action of $S_\Gamma^{(i)}$ on a vector can be evaluated by solving a Dirichlet problem on the subdomain Ω_i. These Dirichlet problems are always well posed, even without the constraint that the integral of the normal component of the boundary velocity vanishes, since in (8), we have dropped the constraint associated with the constant pressure component. We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ. Then $\widehat{S}_\Gamma$ is given by

$$
\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N {R_\Gamma^{(i)}}^T S_\Gamma^{(i)} R_\Gamma^{(i)}. \tag{9}
$$

We denote the operator of the interface problem (7) by $\widehat{S}$. Since $\widehat{S}$ is symmetric and indefinite, we could use the minimal residual method, possibly with a positive definite block preconditioner, as in [25, Section 9.2], to solve problem (7). We will instead propose a different type of preconditioner and show that the preconditioned operator is positive definite, provided that a suitable set of primal constraints are chosen; cf. Assumption 1. A preconditioned conjugate gradient method can then be used.

4. A BDDC Preconditioner for Stokes Equations. When using a BDDC or FETI-DP method, we relax most, but not all, of the continuity constraints on the velocity across the interface; we will always retain sufficiently many *primal* continuity constraints to assure that we will never encounter any singular linear systems of algebraic equations. In a BDDC algorithm, full continuity is restored, at the end of each iteration step, by using an average operator, while in a FETI-DP algorithm, continuity will not be fully satisfied until the algorithm has converged. The primal constraints should also be chosen so that the rate of convergence of the iterative method is enhanced.

For our purposes, we introduce a partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$

$$\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widehat{\mathbf{W}}_\Pi \bigoplus \left(\prod_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{W}}_\Pi$ is the continuous coarse level, primal interface velocity space which is typically spanned by subdomain vertex nodal basis functions, and/or by interface edge and/or face basis functions with constant values, or with values of weight functions, on these edges or faces. These basis functions correspond to the primal interface velocity continuity constraints, which will be discussed in Section 7. We will always assume that the basis has been changed so that each primal basis function corresponds to an explicit degree of freedom. In other words, we will have explicit primal unknowns corresponding to the primal continuity constraints on edges or faces as indicated in Section 1, and further described in [18], [13, Section 6], and [10]. The primal, coarse level degrees of freedom are shared by neighboring subdomains. The complimentary space $\mathbf{W}_\Delta$ is the direct sum of the subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$, which correspond to the remaining interface velocity degrees of freedom and are spanned by basis functions which vanish at the primal degrees of freedom. Thus, an element in the space $\widetilde{\mathbf{W}}_\Gamma$ has a continuous primal velocity and typically a discontinuous dual velocity component.

We need to introduce several restriction, extension, and scaling operators between a variety of spaces. As in Section 3, $R_\Gamma^{(i)}$ is the operator which maps a function in the space $\widehat{\mathbf{W}}_\Gamma$ to its component in $\mathbf{W}_\Gamma^{(i)}$. We define $R_\Delta^{(i)}$ as the operator which maps functions in the space $\widehat{\mathbf{W}}_\Gamma$ to its dual component in the space $\mathbf{W}_\Delta^{(i)}$. $R_{\Gamma\Pi}$ is the restriction operator from the space $\widehat{\mathbf{W}}_\Gamma$ to its subspace $\widehat{\mathbf{W}}_\Pi$; $R_\Pi^{(i)}$ is the operator which maps $\widehat{\mathbf{W}}_\Pi$ into its Γ_i-component. $\widetilde{R}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_\Delta^{(i)}$, and it is a map from $\widehat{\mathbf{W}}_\Gamma$ into $\widetilde{\mathbf{W}}_\Gamma$.

In order to define certain scaling operators, which will be used in the definition of the BDDC preconditioner, see Equation (15), we introduce a positive scaling factor $\delta_i^\dagger(x)$ for the nodes x on the interface Γ_i of each subdomain Ω_i. For the incompressible Stokes problems, with $\mathcal{I}_x$ the set of indices of the subdomains which have x on their boundaries, we will only need to use inverse counting functions defined by $\delta_i^\dagger(x) = 1/card(\mathcal{I}_x), x \in \Gamma_{i,h}$, where $card(\mathcal{I}_x)$ is the number of the subdomains to which x belongs. It is then easy to see that

$$\sum_{j \in \mathcal{I}_x} R_\Gamma^{(i)T} \delta_j^\dagger(x) = 1, \quad x \in \Gamma_{i,h}. \tag{10}$$

Given the scaling factors at the subdomain interface nodes, we can define scaled restriction operators $R_{D,\Delta}^{(i)}$. We first note that each row of $R_\Delta^{(i)}$ has only one nonzero entry, which corresponds to a node $x \in \Gamma_{i,h}$. Multiplying each such element with the scaling factor $\delta_i^\dagger(x)$ gives us $R_{D,\Delta}^{(i)}$. The scaled operator $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. For elasticity problems, these scaling factors should depend on the first Lamé constant μ, which can be allowed to change across the interface between neighboring subdomains; see [25, Section 8.5.1] and [13].

The interface velocity Schur complement $\widetilde{S}_\Gamma$ is defined on the partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ by: given $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$, $\widetilde{S}_\Gamma \mathbf{w}_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ satisfies

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)^T} & A_{\Delta I}^{(1)^T} & & \widetilde{A}_{\Pi I}^{(1)^T} \\ B_{II}^{(1)} & 0 & B_{I\Delta}^{(1)} & & \widetilde{B}_{I\Pi}^{(1)} \\ A_{\Delta I}^{(1)} & B_{I\Delta}^{(1)^T} & A_{\Delta\Delta}^{(1)} & & \widetilde{A}_{\Pi\Delta}^{(1)^T} \\ & & & \ddots & \vdots \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{I\Pi}^{(1)^T} & \widetilde{A}_{\Pi\Delta}^{(1)} & \dots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(1)} \\ p_I^{(1)} \\ \mathbf{w}_\Delta^{(1)} \\ \vdots \\ \mathbf{w}_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Delta^{(1)} \\ \vdots \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Pi \end{bmatrix}. \tag{11}$$

Here $\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}$, $\widetilde{A}_{\Pi I}^{(i)} = R_\Pi^{(i)^T} A_{\Pi I}^{(i)}$, $\widetilde{A}_{\Pi\Delta}^{(i)} = R_\Pi^{(i)^T} A_{\Pi\Delta}^{(i)}$, $\widetilde{B}_{I\Pi}^{(i)} = B_{I\Pi}^{(i)} R_\Pi^{(i)}$.

From the definition of $\widetilde{S}_\Gamma$, we see that it can be obtained from the subdomain Schur complements $S_\Gamma^{(i)}$ by assembling only the primal interface velocity part, i.e., as

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{12}$$

Here $\overline{R}_\Gamma$ maps the space $\widetilde{\mathbf{W}}_\Gamma$ into the product space $\mathbf{W}_\Gamma$ associated with the set of subdomains. We recall that the global interface Schur operator $\widehat{S}_\Gamma$ is obtained by fully assembling the $S_\Gamma^{(i)}$ across the subdomain interface, cf. (9). $\widehat{S}_\Gamma$ can therefore also be obtained from $\widetilde{S}_\Gamma$ by further assembling the dual interface velocity part, i.e., we have $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma$. Correspondingly, we define an operator $\widetilde{B}_{0\Gamma}$, which maps the partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ into F_0, the space of right hand sides corresponding to Q_0, and it is obtained from the subdomain operators $B_{0\Gamma}^{(i)}$ by assembling the primal interface velocity part. The operator $\widehat{B}_{0\Gamma}$ can then be obtained from $\widetilde{B}_{0\Gamma}$ by assembling the dual interface velocity part on the subdomain interfaces, i.e., $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma} \widetilde{R}_\Gamma$. We can therefore write $\widehat{S}$, the operator of the global interface problem (7), as

$$\widehat{S} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma & \widetilde{R}_\Gamma^T \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} \widetilde{R}_\Gamma & 0 \end{bmatrix} = \widetilde{R}^T \widetilde{S} \widetilde{R}, \tag{13}$$

where we use the notation

$$\widetilde{R} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix}, \quad \widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}. \tag{14}$$

The preconditioner for solving the global interface saddle-point problem (7) is

$$M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D, \tag{15}$$

and the preconditioned BDDC problem is of the form: find $(\mathbf{u}_\Gamma, p_0) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \widehat{S} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}. \tag{16}$$

Here $\widetilde{R}_D$ is of the same form as $\widetilde{R}$ in (14), except that $\widetilde{R}_\Gamma$ is replaced by the scaled operator $\widetilde{R}_{D,\Gamma}$. We will show that this preconditioned problem is symmetric, in the bilinear form defined by $\widehat{S}$, and positive definite with respect to a *benign* subspace of $\widehat{\mathbf{W}}_\Gamma \times Q_0$, under a certain assumption; see Definition 1 and Sections 5 and 7.

To determine $\widetilde{S}^{-1}\mathbf{g}$ for any given $\mathbf{g} = (\mathbf{g}_\Gamma, g_0) \in \widetilde{\mathbf{F}}_\Gamma \times F_0$, we need to solve the linear system

$$\begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ g_0 \end{bmatrix}. \tag{17}$$

Given the definition of $\widetilde{S}_\Gamma$ in Equation (11), we know that solving (17) is equivalent to solving

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)^T} & A_{\Delta I}^{(1)^T} & & \widetilde{A}_{\Pi I}^{(1)^T} & \\ B_{II}^{(1)} & 0 & B_{I\Delta}^{(1)} & & \widetilde{B}_{I\Pi}^{(1)} & \\ A_{\Delta I}^{(1)} & B_{I\Delta}^{(1)^T} & A_{\Delta\Delta}^{(1)} & & \widetilde{A}_{\Pi\Delta}^{(1)^T} & B_{0\Delta}^{(1)^T} \\ & & & \ddots & \vdots & \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{I\Pi}^{(1)^T} & \widetilde{A}_{\Pi\Delta}^{(1)} & \dots & \widetilde{A}_{\Pi\Pi} & \widetilde{B}_{0\Pi}^T \\ & & B_{0\Delta}^{(1)} & & \widetilde{B}_{0\Pi} & \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(1)} \\ p_I^{(1)} \\ \mathbf{u}_\Delta^{(1)} \\ \vdots \\ \mathbf{u}_\Pi \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \mathbf{g}_\Delta^{(1)} \\ \vdots \\ \mathbf{g}_\Pi \\ g_0 \end{bmatrix},$$

where $\widetilde{B}_{0\Pi} = \sum_{i=1}^N B_{0\Pi}^{(i)} R_\Pi^{(i)}$. As in [18], we obtain by using a block factorization,

$$\widetilde{S}^{-1} = \sum_{i=1}^N \left[0\ 0\ R_{\Delta,i}^T\right] \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Delta I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ R_{\Delta,i} \end{bmatrix} + \Phi S_{CC}^{-1} \Phi^T,$$

where $R_{\Delta,i}$ maps $\widetilde{\mathbf{F}}_\Gamma \times F_0$ into $\mathbf{F}_\Delta^{(i)}$, the set of right hand sides corresponding to $\mathbf{W}_\Delta^{(i)}$. Furthermore,

$$\begin{aligned} S_{CC} \;=\; \sum_{i=1}^N R_C^{(i)^T} \Bigg\{ & \begin{bmatrix} A_{\Pi\Pi}^{(i)} & B_{0\Pi}^{(i)^T} \\ B_{0\Pi}^{(i)} & 0 \end{bmatrix} - \begin{bmatrix} A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} & A_{\Pi\Delta}^{(i)} \\ 0 & 0 & B_{0\Delta}^{(i)^T} \end{bmatrix} \\ & \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Delta I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ B_{I\Pi}^{(i)} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \end{bmatrix} \Bigg\} R_C^{(i)}, \end{aligned}$$

where

$$R_C^{(i)} = \begin{bmatrix} R_\Pi^{(i)} & \\ & I \end{bmatrix},$$

which maps $\widehat{\mathbf{W}}_\Pi \times Q_0$ into $\mathbf{W}_\Pi^{(i)} \times Q_0$. The matrix Φ is defined by

$$\Phi = R_{\Pi 0}^T - \sum_{i=1}^N \left[0\ 0\ R_{\Delta,i}^T\right] \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Delta I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ B_{I\Pi}^{(i)} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \end{bmatrix} R_C^{(i)},$$

where $R_{\Pi 0}$ maps the space $\widetilde{\mathbf{F}}_\Gamma \times F_0$ into $\widehat{\mathbf{F}}_\Pi \times F_0$.

5. Benign Subspaces. The subdomain Schur complements $S_\Gamma^{(i)}$ are symmetric, positive semi-definite. This is a consequence of a well-known result on the inertia of Schur complements. We know, e.g., that the number of negative eigenvalues of a symmetric, two-by-two block matrix equals the sum of the number of negative eigenvalues of the leading block and those of the Schur complement formed by eliminating the variables of the leading block. We find,

LEMMA 2. *The subdomain Schur complements $S_\Gamma^{(i)}$, defined in (8), are symmetric, positive semi-definite, and are singular for any subdomain with a boundary that does not intersect $\partial\Omega$.*

The $S_\Gamma^{(i)}-$ and $S_\Gamma-$seminorms are defined by

$$|\mathbf{w}_\Gamma^{(i)}|^2_{S_\Gamma^{(i)}} = {\mathbf{w}_\Gamma^{(i)}}^T S_\Gamma^{(i)} \mathbf{w}_\Gamma^{(i)}, \quad |\mathbf{w}_\Gamma|^2_{S_\Gamma} = \mathbf{w}_\Gamma^T S_\Gamma \mathbf{w}_\Gamma = \sum_{i=1}^N |\mathbf{w}_\Gamma^{(i)}|^2_{S_\Gamma^{(i)}}.$$

We define a $|\cdot|_{\mathbf{E}(\Gamma_i)}-$seminorm on the space $\mathbf{W}_\Gamma^{(i)}$ by

$$|\mathbf{w}_\Gamma^{(i)}|_{\mathbf{E}(\Gamma_i)} = \inf_{\substack{\mathbf{v}^{(i)} \in (H^1(\Omega_i))^d \\ \mathbf{v}^{(i)}|_{\Gamma_i} = \mathbf{w}_\Gamma^{(i)}}} \|\epsilon(\mathbf{v}^{(i)})\|_{L^2(\Omega_i)},$$

and $|\cdot|_{\mathbf{E}(\Gamma)}$ is defined on the space $\mathbf{W}_\Gamma$ by $|\mathbf{w}_\Gamma|^2_{\mathbf{E}(\Gamma)} = \sum_{i=1}^N |\mathbf{w}_\Gamma^{(i)}|^2_{\mathbf{E}(\Gamma_i)}$.

The following lemma shows the equivalence of the $|\cdot|_{S_\Gamma}-$ and $|\cdot|_{\mathbf{E}(\Gamma)}-$ seminorms. It can essentially be found in Bramble and Pasciak [2, Theorem 4.1], or Pavarino and Widlund [22, Lemma 3.1], for incompressible Stokes problem. This same result is also valid for the incompressible elasticity problems, cf. Lemma 1 and [13].

LEMMA 3. *There exists a positive constant c, which is independent of H, h, and the shape of the subdomains, such that*

$$c\beta^2 |\mathbf{w}_\Gamma^{(i)}|^2_{S_\Gamma^{(i)}} \le |\mathbf{w}_\Gamma^{(i)}|^2_{\mathbf{E}(\Gamma_i)} \le |\mathbf{w}_\Gamma^{(i)}|^2_{S_\Gamma^{(i)}}, \qquad \forall \mathbf{w}_\Gamma^{(i)} \in \mathbf{W}_\Gamma^{(i)},$$

where β is the inf-sup stability constant defined in Equation (4).

The operators $\widehat{S}_\Gamma$ and $\widetilde{S}_\Gamma$, given in (9) and (12), are both symmetric, positive definite because of the Dirichlet boundary conditions on $\partial\Omega$ and the fact that sufficiently many primal constraints are always chosen. We can then define the $\widehat{S}_\Gamma-$ and $\widetilde{S}_\Gamma-$ norms on the spaces $\widehat{\mathbf{W}}_\Gamma$ and $\widetilde{\mathbf{W}}_\Gamma$, respectively, by

$$\|\mathbf{w}_\Gamma\|^2_{\widehat{S}_\Gamma} = \mathbf{w}_\Gamma^T R_\Gamma^T S_\Gamma R_\Gamma \mathbf{w}_\Gamma = |R_\Gamma \mathbf{w}_\Gamma|^2_{S_\Gamma}, \quad \forall \mathbf{w}_\Gamma \in \widehat{\mathbf{W}}_\Gamma, \tag{18}$$

$$\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = \mathbf{w}_\Gamma^T \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma \mathbf{w}_\Gamma = |\overline{R}_\Gamma \mathbf{w}_\Gamma|^2_{S_\Gamma}, \quad \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma. \tag{19}$$

The interface velocity subspaces $\widehat{\mathbf{W}}_{\Gamma,B}$ and $\widetilde{\mathbf{W}}_{\Gamma,B}$ are defined in

DEFINITION 1.

$$\begin{aligned} \widehat{\mathbf{W}}_{\Gamma,B} &= \{\mathbf{w}_\Gamma \in \widehat{\mathbf{W}}_\Gamma \mid \widehat{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\}, \\ \widetilde{\mathbf{W}}_{\Gamma,B} &= \{\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \mid \widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\}. \end{aligned}$$

We will call $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ the *benign subspaces* of $\widehat{\mathbf{W}}_\Gamma \times Q_0$ and $\widetilde{\mathbf{W}}_\Gamma \times Q_0$, respectively. The interface problem operator $\widehat{S}$ of Equation (7) is indefinite on the space $\widehat{\mathbf{W}}_\Gamma \times Q_0$. But restricted to the subspace $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$, it is positive semi-definite, which follows from the fact that, for any $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$,

$$\mathbf{w}^T \widehat{S} \mathbf{w} = \begin{bmatrix} \mathbf{w}_\Gamma^T & q_0^T \end{bmatrix} \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \mathbf{w}_\Gamma^T \widehat{S}_\Gamma \mathbf{w}_\Gamma = \|\mathbf{w}_\Gamma\|^2_{\widehat{S}_\Gamma} \geq 0.$$

The same is also true for the operator $\widetilde{S}$ on the space $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$. The $\widehat{S}-$ and $\widetilde{S}-$ seminorms are defined on the benign subspaces by

$$|\mathbf{w}|^2_{\widehat{S}} = \mathbf{w}^T \widehat{S} \mathbf{w} = \|\mathbf{w}_\Gamma\|^2_{\widehat{S}_\Gamma}, \qquad \forall \mathbf{w} = (\mathbf{w}_\Gamma,\ q_0) \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0, \tag{20}$$

$$|\mathbf{w}|^2_{\widetilde{S}} = \mathbf{w}^T \widetilde{S} \mathbf{w} = \|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma}, \qquad \forall \mathbf{w} = (\mathbf{w}_\Gamma,\ q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0. \tag{21}$$

Since both $\widehat{S}$ and $\widetilde{S}$ are nonsingular, they are isomorphisms from $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ onto $\widehat{\mathbf{F}}_\Gamma \times 0$ and $\widetilde{\mathbf{F}}_\Gamma \times 0$, respectively. Here $\widehat{\mathbf{F}}_\Gamma$ and $\widetilde{\mathbf{F}}_\Gamma$ are the spaces of right hand sides corresponding to $\widehat{\mathbf{W}}_\Gamma$ and $\widetilde{\mathbf{W}}_\Gamma$, respectively.

6. Condition Number Bounds. We first define an average operator $E_D = \widetilde{R}\widetilde{R}_D^T$, which maps $\widetilde{\mathbf{W}}_\Gamma \times Q_0$, with generally discontinuous interface velocities, to elements with continuous interface velocities in the same space. For any $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_\Gamma \times Q_0$,

$$E_D \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} E_{D,\Gamma}\mathbf{w}_\Gamma \\ q_0 \end{bmatrix} \in \widetilde{\mathbf{W}}_\Gamma \times Q_0, \tag{22}$$

where $E_{D,\Gamma} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$, provides the average of the interface velocities across the interface Γ. We note that, restricted to the space of vectors with continuous interface velocity, E_D is an identity operator; this follows from (10), cf. [25, Section 6.2.1]. Denoting the primal and dual parts of $\mathbf{w}_\Gamma$ by $\mathbf{w}_\Pi$ and $\mathbf{w}_\Delta$, we can write $E_{D,\Gamma}\mathbf{w}_\Gamma$ as the direct sum of $\mathbf{w}_\Pi$ and $E_{D,\Delta}\mathbf{w}_\Delta$, where $E_{D,\Delta}\mathbf{w}_\Delta$ is the dual part of the averaged vector.

The following two assumptions will be needed in the condition number bound of the preconditioned operator; recipes will be provided in Section 7 for which the assumptions hold.

Assumption 1. *For any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, $\int_{\partial\Omega_i} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n} = 0$ and $\int_{\partial\Omega_i} (E_{D,\Delta}\mathbf{w}_\Delta)^{(i)} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit outward normal of $\partial\Omega_i$. Or equivalently, $B_{0\Delta}^{(i)}\mathbf{w}_\Delta^{(i)} = 0$ and $B_{0\Delta}^{(i)} (E_{D,\Delta}\mathbf{w}_\Delta)^{(i)} = 0$.*

Our second assumption is quite similar to those of [21, 14, 13], for standard elliptic problems. It concerns the stability of the average operator $E_{D,\Gamma}$ on the space $\widetilde{\mathbf{W}}_\Gamma$.

Assumption 2. *There exists a positive constant C, which is independent of H, h, and the number of subdomains, such that*

$$|\overline{R}_\Gamma \left(E_{D,\Gamma}\mathbf{w}_\Gamma\right)|_{\mathbf{E}(\Gamma)} \leq C\left(1 + \log\frac{H}{h}\right) |\overline{R}_\Gamma \mathbf{w}_\Gamma|_{\mathbf{E}(\Gamma)}, \quad \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma.$$

With Assumptions 1 and 2, we can prove the following lemmas.

Lemma 4. *Let Assumption 1 hold. Then, $\widetilde{R}_D^T\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$, for any $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.*

Proof: We need to show that, given $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_{\Gamma,B}$, $\widehat{B}_{0\Gamma}\widetilde{R}^T_{D,\Gamma}\mathbf{w}_\Gamma = 0$. With $\mathbf{w}_\Gamma = \mathbf{w}_\Pi \oplus \mathbf{w}_\Delta$, we have

$$\widehat{B}_{0\Gamma}\widetilde{R}^T_{D,\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Gamma}\widetilde{R}_\Gamma\widetilde{R}^T_{D,\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Gamma}E_{D,\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi + B_{0\Delta}E_{D,\Delta}\mathbf{w}_\Delta = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi,$$

where the last step is a result of Assumption 1. Since $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_{\Gamma,B}$ and hence $\widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = 0$, we have $\widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi + B_{0\Delta}\mathbf{w}_\Delta = 0$. From Assumption 1, we also know that $B_{0\Delta}\mathbf{w}_\Delta = 0$. Therefore $\widetilde{B}_{0\Pi}\mathbf{w}_\Pi = 0$ and $\widehat{B}_{0\Gamma}\widetilde{R}^T_{D,\Gamma}\mathbf{w}_\Gamma = 0$.

□

LEMMA 5. *Let Assumptions 1 and 2 hold. There then exists a positive constant C, which is independent of H, h, and the number of subdomains, such that*

$$|E_D\mathbf{w}|_{\widetilde{S}} \le C\frac{1}{\beta}\left(1+\log\frac{H}{h}\right)|\mathbf{w}|_{\widetilde{S}}, \quad \forall \mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B}\times Q_0,$$

where β is the inf-sup stability constant of Equation (4).

Proof: Given any $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B}\times Q_0$, we know, from Lemma 4, that $\widetilde{R}^T_D\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$. Therefore, $E_D\mathbf{w} = \widetilde{R}\widetilde{R}^T_D\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B}\times Q_0$. We have, from the definition of the $\widetilde{S}-$seminorm in (21), that

$$|E_D\mathbf{w}|^2_{\widetilde{S}} = \|E_{D,\Gamma}\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = |\overline{R}_\Gamma\left(E_{D,\Gamma}\mathbf{w}_\Gamma\right)|^2_{\overline{S}_\Gamma} \le C\frac{1}{\beta^2}|\overline{R}_\Gamma\left(E_{D,\Gamma}\mathbf{w}_\Gamma\right)|^2_{\mathbf{E}(\Gamma)}, \tag{23}$$

where the last inequality follows from Lemma 3.

We have, from Assumption 2, Lemma 3, and (19),

$$|\overline{R}_\Gamma\left(E_{D,\Gamma}\mathbf{w}_\Gamma\right)|^2_{\mathbf{E}(\Gamma)} \le C\left(1+\log\frac{H}{h}\right)^2|\overline{R}_\Gamma\mathbf{w}_\Gamma|^2_{\mathbf{E}(\Gamma)}$$

$$\le C\left(1+\log\frac{H}{h}\right)^2|\overline{R}_\Gamma\mathbf{w}_\Gamma|^2_{\overline{S}_\Gamma} \le C\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma}. \tag{24}$$

Then from Equations (23), (24), and (21), we have

$$|E_D\mathbf{w}|^2_{\widetilde{S}} \le C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)^2|\mathbf{w}|^2_{\widetilde{S}}.$$

□

We are now ready to prove the condition number bound of the preconditioned operator $\widetilde{R}^T_D\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}$, on the benign space $\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$. We know that $M^{-1}\widehat{S}$ is indefinite on the space $\widehat{\mathbf{W}}_\Gamma \times Q_0$, since both $\widehat{S}$ and $\widetilde{S}$ are indefinite. However, we know from Section 5, that both $\widehat{S}$ and $\widetilde{S}$ are positive semi-definite, when restricted to the benign subspaces $\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B}\times Q_0$, respectively. We also know, from Lemma 4, that $M^{-1}\widehat{S}$ maps $\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$ into itself and that $M^{-1}\widehat{S}$ is symmetric with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$. In the following, we will prove that $M^{-1}\widehat{S}$ is positive definite, when restricted to the benign subspace $\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$. Therefore a preconditioned conjugate gradient method can be used.

LEMMA 6. *Any vector of the form $\mathbf{u} = (\mathbf{0}, p_0) \in \widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$ is an eigenvector of the preconditioned operator $M^{-1}\widehat{S}$ with eigenvalue equal to 1.*

Proof: From Assumption 1, we know that for any $\mathbf{w}^{(i)}_\Delta \in \mathbf{W}^{(i)}_\Delta$, $B^{(i)}_{0\Delta}\mathbf{w}^{(i)}_\Delta = 0$. Since each column of the matrix $R^{(i)}_\Delta$ is an element of the space $\mathbf{W}^{(i)}_\Delta$, we have $B^{(i)}_{0\Delta}R^{(i)}_\Delta = 0$.

Therefore $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma}\widetilde{R}_{\Gamma} = \widetilde{B}_{0\Pi}R_{\Gamma\Pi} + \sum_{i=1}^{N} B_{0\Delta}^{(i)}R_{\Delta}^{(i)} = \widetilde{B}_{0\Pi}R_{\Gamma\Pi}$. Then for any $p_0 \in Q_0$, $\widehat{B}_{0\Gamma}^T p_0 = R_{\Gamma\Pi}^T \widetilde{B}_{0\Pi}^T p_0$, i.e., the dual part of $\widehat{B}_{0\Gamma}^T p_0$ is always zero and its primal part equals $\widetilde{B}_{0\Pi}^T p_0$. In the same way, we can also show that $\widetilde{B}_{0\Gamma}^T p_0 = \widetilde{R}_{\Gamma} R_{\Gamma\Pi}^T \widetilde{B}_{0\Pi}^T p_0$, which equals $\widetilde{R}_{D,\Gamma} R_{\Gamma\Pi}^T \widetilde{B}_{0\Pi}^T p_0$, since its dual part is zero and $\widetilde{R}_{D,\Gamma}$ does not change its primal part. Therefore, for any $\mathbf{u} = (\mathbf{0}, p_0)$, we have, from the form of $\widehat{S}$ in Equation (7), the definition of M^{-1}, and the expressions of $\widehat{B}_{0\Gamma}^T p_0$ and $\widetilde{B}_{0\Gamma}^T p_0$, that

$$M^{-1}\widehat{S}\mathbf{u} = M^{-1}\begin{bmatrix} \widehat{B}_{0\Gamma}^T p_0 \\ 0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \begin{bmatrix} R_{\Gamma\Pi}^T \widetilde{B}_{0\Pi}^T p_0 \\ 0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \begin{bmatrix} \widetilde{B}_{0\Gamma}^T p_0 \\ 0 \end{bmatrix}.$$

From the definition of $\widetilde{S}$ in Equations (17), we know that the right hand side equals $(\mathbf{0}, p_0)$, and therefore $M^{-1}\widehat{S}\mathbf{u} = \mathbf{u}$.

□

THEOREM 1. *Let Assumptions 1 and 2 hold. The preconditioned operator $M^{-1}\widehat{S}$ is then symmetric, positive definite with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\widehat{S}}$ on the benign space $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$. Its minimum eigenvalue is 1 and its maximum eigenvalue is bounded by*

$$C\frac{1}{\beta^2}\left(1 + \log\frac{H}{h}\right)^2.$$

Here, C is a constant which is independent of H, h, and the number of subdomains and β is the inf-sup stability constant defined in Equation (4).

Proof: We know from Lemma 6, that any vector of the form $\mathbf{u} = (\mathbf{0}, p_0) \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ is an eigenvector of the preconditioned operator $M^{-1}\widehat{S}$ with an eigenvalue equal to 1. It is then sufficient to find lower and upper bounds of the quotient $\left\langle M^{-1}\widehat{S}\mathbf{u}, \mathbf{u} \right\rangle_{\widehat{S}} / \langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}}$, for any $\mathbf{u} = (\mathbf{u}_\Gamma, p_0) \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$, where $\mathbf{u}_\Gamma$ is nonzero and therefore $\langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}} > 0$.

Lower bound: Given $\mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$, let

$$\mathbf{w} = \widetilde{S}^{-1}\widetilde{R}_D\widehat{S}\mathbf{u} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0. \tag{25}$$

We have, from the fact that $\widetilde{R}^T\widetilde{R}_D = \widetilde{R}_D^T\widetilde{R} = I$,

$$\langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}} = \mathbf{u}^T\widehat{S}\widetilde{R}_D^T\widetilde{R}\mathbf{u} = \mathbf{u}^T\widehat{S}\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{S}\widetilde{R}\mathbf{u} = \left\langle \mathbf{w}, \widetilde{R}\mathbf{u} \right\rangle_{\widetilde{S}}. \tag{26}$$

From the Cauchy-Schwarz inequality and the fact that $\widehat{S} = \widetilde{R}^T\widetilde{S}\widetilde{R}$, we find that

$$\left\langle \mathbf{w}, \widetilde{R}\mathbf{u} \right\rangle_{\widetilde{S}} \leq \langle \mathbf{w}, \mathbf{w} \rangle_{\widetilde{S}}^{1/2} \left\langle \widetilde{R}\mathbf{u}, \widetilde{R}\mathbf{u} \right\rangle_{\widetilde{S}}^{1/2} = \langle \mathbf{w}, \mathbf{w} \rangle_{\widetilde{S}}^{1/2} \langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}}^{1/2}. \tag{27}$$

Therefore, from (26) and (27),

$$\langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}} \leq \langle \mathbf{w}, \mathbf{w} \rangle_{\widetilde{S}}. \tag{28}$$

Since,

$$\langle \mathbf{w}, \mathbf{w} \rangle_{\widetilde{S}} = \mathbf{u}^T\widehat{S}\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{S}\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}\mathbf{u} = \left\langle \mathbf{u}, \widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}} = \left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}}, \tag{29}$$

we obtain, from Equations (28) and (29), that $\langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}} \leq \left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}}$, which gives a lower bound of 1 for the eigenvalues. Then from Lemma 6, we know that 1 is the minimum eigenvalue of the preconditioned operator.

Upper bound: Given $\mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$, take $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ as in Equation (25). We have, $\widetilde{R}_D^T \mathbf{w} = M^{-1}\widehat{S}\mathbf{u}$. Since $\widehat{S} = \widetilde{R}^T \widetilde{S} \widetilde{R}$ and by using Lemma 5, we have

$$\begin{aligned} \left\langle M^{-1}\widehat{S}\mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}} &= \left\langle \widetilde{R}_D^T\mathbf{w}, \widetilde{R}_D^T\mathbf{w} \right\rangle_{\widehat{S}} = \left\langle \widetilde{R}\widetilde{R}_D^T\mathbf{w}, \widetilde{R}\widetilde{R}_D^T\mathbf{w} \right\rangle_{\widetilde{S}} \\ &= |E_D\mathbf{w}|_{\widetilde{S}}^2 \leq C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)^2 |\mathbf{w}|_{\widetilde{S}}^2. \end{aligned}$$

Therefore, from Equation (29), we have

$$\left\langle M^{-1}\widehat{S}\mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}} \leq C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)^2 \left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}}. \tag{30}$$

Using the Cauchy-Schwarz inequality and Equation (30), we have

$$\begin{aligned} \left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}} &\leq \langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}}^{1/2} \left\langle M^{-1}\widehat{S}\mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}}^{1/2} \\ &\leq C\frac{1}{\beta}\left(1+\log\frac{H}{h}\right) \langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}}^{1/2} \left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}}^{1/2}. \end{aligned}$$

This gives,

$$\left\langle \mathbf{u}, M^{-1}\widehat{S}\mathbf{u} \right\rangle_{\widehat{S}} \leq C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)^2 \langle \mathbf{u}, \mathbf{u} \rangle_{\widehat{S}},$$

and the upper bound of the theorem.

□

7. Satisfying the Assumptions: Choosing Primal Constraints. Assumptions 1 and 2 can be satisfied with appropriate choices of the primal continuity constraints on the interface velocity variables. We first describe a recipe for two-dimensional problems, and then one for the three-dimensional case.

For two-dimensional problems, it is natural to make all subdomain vertices primal, i.e, make both components of the velocity continuous at those nodes. In order to satisfy Assumption 1, some extra edge constraints are necessary. For each interface edge Γ^{ij}, which is shared by a pair of subdomains Ω_i and Ω_j, we make

$$\int_{\Gamma^{ij}} \mathbf{w}_\Gamma^{(i)} \cdot \mathbf{n}_{ij} = \int_{\Gamma^{ij}} \mathbf{w}_\Gamma^{(j)} \cdot \mathbf{n}_{ij}, \tag{31}$$

for a fixed selection of the normal $\mathbf{n}_{ij}$ of Γ^{ij}. After changing the variables, the dual interface velocity component will vanish at the subdomain vertices and its normal component will have a weighted zero average over each Γ^{ij}, i.e., $\int_{\Gamma^{ij}} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n}_{ij} = \int_{\Gamma^{ij}} \mathbf{w}_\Delta^{(j)} \cdot \mathbf{n}_{ij} = 0$. For each edge, the weights in the average, i.e., the weights for the nodal values on the edge, are determined by the integrals of the normal components of the nodal finite element basis functions on that edge. Assumption 1 can then be confirmed, given that the average interface velocity $E_{D,\Delta}\mathbf{w}_\Delta$ equals $\frac{1}{2}(\mathbf{w}_\Delta^{(i)} + \mathbf{w}_\Delta^{(j)})$

on each edge for the two-dimensional case and hence $\int_{\Gamma^{ij}} (E_{D,\Delta}\mathbf{w}_\Delta)^{(i)} \cdot \mathbf{n}_{ij} = 0$. Assumption 2 is also satisfied; in fact only the vertex constraints are required, cf. [21].

For three-dimensional problems, the interface Γ is composed of subdomain faces, denoted by $\mathcal{F}^l$, which are shared by two subdomains, edges $\mathcal{E}^k$, which are shared by more than two subdomains, and vertices which are the end points of the edges. For each face $\mathcal{F}^l$, we denote by $\theta_{\mathcal{F}^l}$ the finite element cut-off function on the face $\mathcal{F}^l$, which equals 1 at the interior nodes of the face $\mathcal{F}^l$ and equals 0 at its boundary nodes; for each edge $\mathcal{E}^k$ of the face $\mathcal{F}^l$, we denote $\theta_{\mathcal{E}^k(\mathcal{F}^l)}$ the finite element cut-off function on the face $\mathcal{F}^l$ which equals 1 at all the nodes of $\mathcal{E}^k$ and 0 at all the other nodes of $\mathcal{F}^l$. For each edge $\mathcal{E}^k$, we denote the set of subdomains which have this edge in common by $\mathcal{N}_{\mathcal{E}^k}$, and the set of faces which share this edge by $\mathcal{M}_{\mathcal{E}^k}$. We also denote by $\mathcal{F}^{ij}$ the face shared by a pair of subdomains Ω_i and Ω_j. The selected normals of $\mathcal{F}^l$ and $\mathcal{F}^{ij}$ are denoted by $\mathbf{n}_l$ and $\mathbf{n}_{ij}$, respectively.

Let us now consider a recipe to satisfy Assumption 1 in three dimensions. First of all, we make all subdomain vertices primal. For any dual velocity component $\mathbf{w}_\Delta$, we then have

$$\int_{\mathcal{F}^{ij}} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n}_{ij} = \int_{\mathcal{F}^{ij}} (\theta_{\mathcal{F}^{ij}} \mathbf{w}_\Delta^{(i)}) \cdot \mathbf{n}_{ij} + \sum_{\mathcal{E}^k \subset \mathcal{F}^{ij}} \int_{\mathcal{F}^{ij}} \left(\theta_{\mathcal{E}^k(\mathcal{F}^{ij})} \mathbf{w}_\Delta^{(i)}\right) \cdot \mathbf{n}_{ij}, \tag{32}$$

and

$$\begin{aligned}\int_{\mathcal{F}^{ij}} (E_{D,\Delta}\mathbf{w})_\Delta^{(i)} \cdot \mathbf{n}_{ij} \quad &= \quad \frac{1}{2} \int_{\mathcal{F}^{ij}} \left(\theta_{\mathcal{F}^{ij}} (\mathbf{w}_\Delta^{(i)} + \mathbf{w}_\Delta^{(j)})\right) \cdot \mathbf{n}_{ij} \\ &+ \quad \sum_{\mathcal{E}^k \subset \mathcal{F}^{ij}} \sum_{m \in \mathcal{N}_{\mathcal{E}^k}} \frac{1}{card(\mathcal{N}_{\mathcal{E}^k})} \int_{\mathcal{F}^{ij}} \left(\theta_{\mathcal{E}^k(\mathcal{F}^{ij})} \mathbf{w}_\Delta^{(m)}\right) \cdot \mathbf{n}_{ij}.\end{aligned} \tag{33}$$

Here we note that the averaged interface vector, at any nodal point x on an edge $\mathcal{E}^k$, depend on the values of the $\mathbf{w}_\Delta^{(m)}(x)$ on all the subdomains Ω_m which have this edge in common. Assumption 1 will be satisfied if all the integrals of the dual velocity component $\mathbf{w}_\Delta$, on the right sides of Equations (32) and (33), vanish. This can be achieved by enforcing the following primal continuity constraints: for each face $\mathcal{F}^{ij}$,

$$\int_{\mathcal{F}^{ij}} (\theta_{\mathcal{F}^{ij}} \mathbf{w}_\Gamma^{(i)}) \cdot \mathbf{n}_{ij} = \int_{\mathcal{F}^{ij}} (\theta_{\mathcal{F}^{ij}} \mathbf{w}_\Gamma^{(j)}) \cdot \mathbf{n}_{ij}, \tag{34}$$

and for each edge $\mathcal{E}^k$, on each face $\mathcal{F}^l$, $l \in \mathcal{M}_{\mathcal{E}^k}$, that

$$\int_{\mathcal{F}^l} \left(\theta_{\mathcal{E}^k(\mathcal{F}^l)} \mathbf{w}_\Gamma^{(m)}\right) \cdot \mathbf{n}_l, \tag{35}$$

are the same for all $m \in \mathcal{N}_{\mathcal{E}^k}$. We see that only one primal variable need to be introduced to enforce the face constraint (34), while the number of primal variables that results from the constraints (35), for each edge, equals the number of faces which share that edge. The primal basis functions for the edge constraints (35) are determined by the integrals of the normal components of the edge nodal finite element basis functions on the corresponding faces. It can easily happen that these primal basis functions, of the same edge, are linearly dependent, e.g., this happens in the case when the subdomains are cubes and an uniform mesh is used. In general, we must make sure that the primal basis functions maintain linear independence for each edge separately. This can be done by using a singular value decomposition. This idea for eliminating the linearly dependent coarse level primal constraints has previously been applied for both FETI-DPH and BDDC algorithms; see [6, 5]. This computation is carried out on each edge independently.

REMARK 1. *A modified BDDC algorithm was introduced by Dohrmann in [5] for solving nearly incompressible elasticity problems. Zero divergence constraints were used on the substructure corrections to keep the volume change of each substructure relatively small in the presence of nearly incompressible materials. For two-dimensional problems, our constraints (31) are the same as those in [5]; these type of constraints have also been used in FETI-DP algorithms for Stokes problems, cf. [15]. For three-dimensional problems, our vertex and face constraints are the same as Dohrmann's. But for each edge* $\mathcal{E}^k$*, Dohrmann requires that on each subdomain* $\Omega_i \in \mathcal{N}_{\mathcal{E}^k}$ *the integrals*

$$\sum_{\mathcal{F}^l \subset \partial\Omega_i} \int_{\mathcal{F}^l} \left(\theta_{\mathcal{E}^k(\mathcal{F}^l)} \mathbf{w}_\Gamma^{(m)} \right) \cdot \mathbf{n}_l, \tag{36}$$

be the same for all $m \in \mathcal{N}_{\mathcal{E}^k}$*. While either set of edge constraints (35) and (36), together with the face and vertex constraints, give rise to a zero divergence constraint on the substructures, we have adopted the form (35) to facilitate the analysis.*

We also have to make sure that we have the right type of constraints so as to guarantee a stable E_D operator as in Assumption 2. We will rely on recent results of Klawonn and the second author [13]. By examining Section 8 of that paper, we find that Assumption 2 will be satisfied if all the faces of the interface Γ are fully primal, cf. [13]; here we have used that the coefficients of the Stokes problem are the same for each subdomain and that all vertices are primal. For the definition of fully primal faces, see [13, Section 5]. To decide if a face is fully primal, we have to test if the set of constraints that are active on that face, excluding the vertex constraints, is rich enough to make sure that if they all vanish for an arbitrary rigid body mode, then the rigid body mode must vanish. We can check this condition numerically easily for each face, but we can also provide some insights a priori.

We recall that the space of rigid body modes on each subdomain Ω_i is spanned by the three translations

$$\mathbf{r}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{r}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{r}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{37}$$

and the three rotations

$$\mathbf{r}_4 := \frac{1}{H_i} \begin{bmatrix} x_2 - \hat{x}_2 \\ -x_1 + \hat{x}_1 \\ 0 \end{bmatrix}, \mathbf{r}_5 := \frac{1}{H_i} \begin{bmatrix} -x_3 + \hat{x}_3 \\ 0 \\ x_1 - \hat{x}_1 \end{bmatrix}, \mathbf{r}_6 := \frac{1}{H_i} \begin{bmatrix} 0 \\ x_3 - \hat{x}_3 \\ -x_2 + \hat{x}_2 \end{bmatrix}. \tag{38}$$

Here $\hat{\mathbf{x}} \in \Omega_i$ and H_i denotes the diameter of Ω_i. (The shift of the origin makes the basis for the space of rigid body modes well conditioned and the scaling and shift make the $L_2(\Omega_i)-$norms of these six functions scale in the same way with H_i.)

Let us consider a face which is part of the $x_1 - x_2$ plane. Since we have weighted edge average constraints for the third component over all, i.e., at least three edges of the face, we can conclude that the third component of the rigid body modes must vanish at at least three points which are not colinear. Since this third component is a linear combination of the third component of the three basis elements $\mathbf{r}_3$, $\mathbf{r}_5$, and $\mathbf{r}_6$, the rigid body mode cannot have any component involving these three basis elements. The remaining part is a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_4$, i.e, effectively a rigid body mode in two dimensions. It has the form of a first order Nédélec element on the face,

$$\mathbf{r} = \begin{bmatrix} a_1 + bx_2 \\ a_2 - bx_1 \\ 0 \end{bmatrix},$$

where a_1, a_2, and b are the three remaining degrees of freedom of the rigid body mode for this two-dimensional surface. We will now consider the other edge constraints to see if we can conclude that $a_1 = a_2 = b = 0$.

It is easy to see that we have at least one constraint per edge with a vanishing weighted average of the component of the velocity projected onto the plane of the face and normal to the edge. If three of these edge constraints are linearly independent, when restricted to this three-dimensional space of the rigid body modes, we can conclude that the rigid body mode on this two-dimensional surface vanish and therefore that the face is fully primal. However, it is easy to see that this does not always hold; consider a face with three edges only and with constant weights. Then, by the divergence theorem and the fact that the rigid body modes are divergence free, we have linear dependence. A simple computation reveals that the rank is also two for a rectangular face. In such cases, some extra primal continuity constraints need to be added on such faces to make them fully primal.

We know that the Nédélec elements have a constant tangential component on each edge. If there are three vanishing edge tangential components on each face, where the three edges are in different directions, then the Nédélec element on that face will be zero, and therefore the rigid body mode will vanish. In fact, combined with the normal edge constraints, only two tangential edge constraints on two adjacent edges are needed to have three linearly independent constraints. Therefore we can always guarantee that a face be fully primal by, in addition, requiring for two adjacent edges that $\int_{\mathcal{E}^k} \mathbf{w}_\Gamma^{(m)} \cdot \mathbf{t}_{\mathcal{E}^k}$ takes the same value for all $m \in \mathcal{N}_{\mathcal{E}^k}$, where $\mathbf{t}_{\mathcal{E}^k}$ is a unit vector tangent to $\mathcal{E}^k$. We note that only one primal variable will result from such a tangential edge constraint. We recall that these extra edge constraints are only necessary for the faces which are not fully primal with normal edge constraints only.

For a further discussion of the choices of primal constraints for satisfying Assumption 2, see [13, 10].

8. Connections with the FETI-DP Algorithms. In the FETI-DP algorithms developed in [15] for incompressible Stokes equations, the subdomain problems are also assembled only at the coarse level, primal velocity degrees of freedom, which are shared by neighboring subdomains. Lagrange multipliers are then introduced on the interface to enforce the continuity of the dual velocity variables, by requiring that $B_\Delta \mathbf{u}_\Delta = \sum_{i=1}^N B_\Delta^{(i)} \mathbf{u}_\Delta^{(i)} = 0$. Here, the subdomain matrices $B_\Delta^{(i)}$ have elements chosen from the set $\{0, 1, -1\}$. The original problem is reduced to a linear system for the Lagrange multipliers by eliminating the other variables, cf. [15]. The FETI-DP operator for the Lagrange multipliers λ is $B_\Delta \widetilde{S}_\Delta^{-1} B_\Delta^T$, where the operator $\widetilde{S}_\Delta$ is defined by $\widetilde{S}_\Delta^{-1} = R_\Delta \widetilde{S}^{-1} R_\Delta^T$ and R_Δ is the restriction map from $\widetilde{\mathbf{W}}_\Gamma \times Q_0$ to $\mathbf{W}_\Delta$.

The preconditioner used in [15] for the FETI-DP algorithm is $B_{D,\Delta} S_\Delta B_{D,\Delta}^T$ where $B_{D,\Delta}$ is constructed from the subdomain operators $B_{D,\Delta}^{(i)}$ in the same way as B_Δ from the $B_\Delta^{(i)}$. Each $B_{D,\Delta}^{(i)}$ is defined as follows: each nonzero element of $B_\Delta^{(i)}$ corresponds to a Lagrange multiplier connecting the subdomain Ω_i to a neighboring subdomain Ω_j at a point $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$. Multiplying each such element with the positive scaling factor $\delta_j^\dagger(x)$ gives us $B_{D,\Delta}^{(i)}$. S_Δ is the direct sum of subdomain Schur operators $S_\Delta^{(i)}$, which are defined on the dual subdomain velocity space $\mathbf{W}_\Delta^{(i)}$, as $S_\Gamma^{(i)}$ in Equation (8) except that the operator is restricted to the dual interface velocity variables; S_Δ can be written as the restriction of the operator $\widetilde{S}$ to the space $\mathbf{W}_\Delta$, i.e, $S_\Delta = R_\Delta \widetilde{S} R_\Delta^T$.

Therefore, the preconditioned FETI-DP operator can be written as

$$B_{D,\Delta} R_\Delta \widetilde{S} R_\Delta^T B_{D,\Delta}^T B_\Delta R_\Delta \widetilde{S}^{-1} R_\Delta^T B_\Delta^T. \tag{39}$$

Since the diagonal blocks, corresponding to the dual interface velocity part in $\mathbf{W}_\Delta$, of the matrices $\widetilde{S}$ and $\widetilde{S}^{-1}$, are positive definite, both $R_\Delta \widetilde{S} R_\Delta^T$ and $R_\Delta \widetilde{S}^{-1} R_\Delta^T$ are positive definite. When non-redundant Lagrange multipliers are used, the matrices B_Δ^T and $B_{D,\Delta}^T$ are of full rank and the FETI–DP operator (39) is therefore a product of two positive definite matrices; cf. [25, Section 6.3]. If we use redundant Lagrange multipliers, see [12, 25], then B_Δ^T will not be of full rank. But this does not matter; the Lagrange multiplier λ is always restricted to **range** (B_Δ), which is orthogonal to the null space of B_Δ^T, cf. [12]. We can therefore assume that a set of non-redundant Lagrange multipliers is used in the FETI–DP algorithm; Theorem 2, of this section, applies equally well to the case of redundant Lagrange multipliers.

Since both R_Δ^T and B_Δ^T are of full rank, the preconditioned FETI–DP operator (39) has the same nonzero eigenvalues as the operator

$$R_\Delta^T B_\Delta^T B_{D,\Delta} R_\Delta \widetilde{S} R_\Delta^T B_{D,\Delta}^T B_\Delta R_\Delta \widetilde{S}^{-1}. \tag{40}$$

We now introduce the operator $P_D = R_\Delta^T B_{D,\Delta}^T B_\Delta R_\Delta$, which maps the space $\widetilde{\mathbf{W}}_\Gamma \times Q_0$ into itself. It computes the jump across the subdomain interface of the dual interface velocity component, and maps any element in the primal space $\widehat{\mathbf{W}}_\Pi \times Q_0$ to zero; cf. [18]. The operator (40) can then be written as

$$P_D^T \widetilde{S} P_D \widetilde{S}^{-1}. \tag{41}$$

The preconditioned BDDC operator in Equations (16) is

$$\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \widetilde{R}^T \widetilde{S} \widetilde{R}. \tag{42}$$

When Assumption 1 is satisfied, this preconditioned operator is symmetric with respect to the bilinear form $< \cdot, \cdot >_{\widehat{S}}$ and all its eigenvalues are real and positive, cf. Theorem 1. Since $\widetilde{R}$ is of full column rank, the nonzero eigenvalues of the preconditioned BDDC operator (42) are the same as those of

$$E_D \widetilde{S}^{-1} E_D^T \widetilde{S}, \tag{43}$$

where the average operator E_D is defined in Equation (22). It can be verified that both E_D and P_D are projectors with $E_D + P_D = I$ and $E_D P_D = P_D E_D = 0$. We can then prove, just as in the elliptic case, see [18], that the operators (41) and (43) have the same nonzero eigenvalues for the same set of primal constraints. We obtain,

THEOREM 2. *Let Assumption 1 hold. The preconditioned FETI–DP and BDDC operators, given by (39) and (42), respectively, then have the same nonzero eigenvalues.*

Table 1

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different number of subdomains, for $H/h = 8$ and a primal space spanned by both corner and normal edge basis functions.

Num. of subs	BDDC			FETI–DP		
$n_x \times n_y$	λ_{min}	λ_{max}	*iter.*	λ_{min}	λ_{max}	*iter.*
4×4	1.00	3.14	11	1.00	3.14	11
8×8	1.00	3.88	12	1.00	3.88	12
12×12	1.00	4.02	12	1.00	4.02	13
16×16	1.00	4.06	12	1.00	4.07	13
20×20	1.00	4.08	12	1.00	4.08	13

9. Numerical Experiments. We solve a lid-driven-cavity problem on the domain $\Omega = [0,1] \times [0,1]$ with Dirichlet boundary condition, where the velocity is $(1,0)$ on the upper side, and vanishes on the other three sides. We use a uniform mesh, as in Figure 1. The mixed finite elements is also indicated in Figure 1; the velocity is continuous and linear in each element and the pressure is constant on macro elements which are unions of four triangles. The inf-sup stability of this mixed finite elements can easily be proved by using the macro element technique developed in [24].

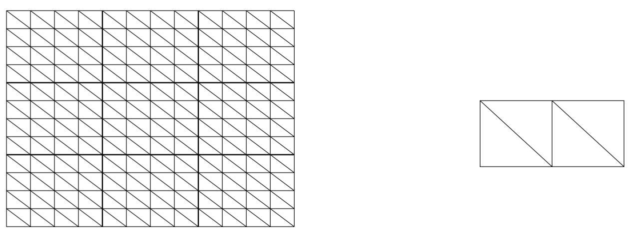

Fig. 1. *The mesh and the mixed finite elements.*

Both the BDDC and FETI–DP algorithms, as in (16) and (39), have been tested. The preconditioned conjugate gradient method is used and the iteration is halted when the L_2-norm of the residual has been reduced by a factor 10^{-6}. In our experiments, we have used three different sets of primal constraints. The first two satisfy both Assumptions 1 and 2 and we see that both the BDDC and FETI–DP operators are positive definite and that the results are fully consistent with our theory. Our third choice violates Assumption 1 and the BDDC operator is then no longer positive definite.

In the first case, the primal velocity space is spanned by the subdomain vertex nodal basis functions for both components and by a constant vector in the direction normal to the edge for each interface edge as in (31). From Tables 1 and 2, we see that the preconditioned BDDC and FETI–DP operators are both positive definite and quite well-conditioned as established in Theorems 1 and 2. We observe that the extreme eigenvalues and the iteration counts of the BDDC and FETI–DP algorithms match very well, and that the condition numbers of both algorithms are independent of the number of subdomains, and increases only slowly with the number of elements across each subdomain, all as predicted by the theory. In our experiments, the extreme eigenvalues are estimated by using the tridiagonal Lanczos matrix generated by the preconditioned conjugate gradient method.

Table 2

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different H/h, for 4×4 subdomains and a primal space spanned by both corner and normal edge basis functions.

	BDDC			FETI–DP		
H/h	λ_{min}	λ_{max}	$iter.$	λ_{min}	λ_{max}	$iter.$
4	1.00	2.17	8	1.00	2.17	9
8	1.00	3.14	11	1.00	3.14	11
16	1.00	4.22	13	1.00	4.22	12
32	1.00	5.42	14	1.00	5.42	14

Table 3

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different number of subdomains, for $H/h = 8$ and a primal space spanned by both corner and two edge basis functions for each edge.

Num. of subs	BDDC			FETI–DP		
$n_x \times n_y$	λ_{min}	λ_{max}	$iter.$	λ_{min}	λ_{max}	$iter.$
4×4	1.00	2.32	8	1.00	2.32	9
8×8	1.00	2.58	9	1.00	2.58	9
12×12	1.00	2.63	9	1.00	2.63	10
16×16	1.00	2.65	9	1.00	2.65	10
20×20	1.00	2.65	9	1.00	2.65	10

Table 4

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different H/h, for 4×4 subdomains and a primal space spanned by both corner and two edge basis functions for each edge.

	BDDC			FETI–DP		
H/h	λ_{min}	λ_{max}	$iter.$	λ_{min}	λ_{max}	$iter.$
4	1.00	1.66	7	1.00	1.65	7
8	1.00	2.32	8	1.00	2.32	9
16	1.00	3.07	10	1.00	3.07	10
32	1.00	3.93	11	1.00	3.93	12

In the experiments of Tables 3 and 4, the integral of both velocity components are required to have common values across each interface edge. The subdomain corner degrees of freedom are also chosen as primal variables as in the first case. Both Assumptions 1 and 2 are again satisfied and we observe similar, slightly faster convergence compared with the first experiments since the coarse level problem has been enlarged.

In Tables 5 and 6, the primal velocity space is spanned only by the corner basis functions; Assumption 1 then does not hold. In this case, the preconditioned BDDC operator (16) is no longer positive definite and the iterates will no longer stay in the benign space of the saddle-point problem. However, the FETI–DP operator (39) is still positive definite. The interface problems of both the BDDC and the FETI–DP algorithms are solved by a preconditioned conjugate gradient method, but the residual norm of the BDDC methods is no longer strictly decreasing. We see that the iteration counts of the BDDC and FETI–DP algorithms still match very well, but that for both algorithms, this count will now depend on both the number of subdomains as well as

Table 5

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different number of subdomains, for $H/h = 8$ and a primal space spanned only by the corner basis functions.

Num. of subs	BDDC			FETI–DP		
$n_x \times n_y$	λ_{min}	λ_{max}	*iter.*	λ_{min}	λ_{max}	*iter.*
4×4	N/A	N/A	17	0.49	3.61	16
8×8			21	0.37	4.01	21
12×12			21	0.33	4.08	23
16×16			21	0.31	4.10	22
20×20			22	0.29	4.10	24

Table 6

Spectral bounds and iteration counts for a pair of BDDC and FETI–DP algorithms, with different H/h, for 4×4 subdomains and for a primal space spanned only by the corner basis functions.

	BDDC			FETI–DP		
H/h	λ_{min}	λ_{max}	*iter.*	λ_{min}	λ_{max}	*iter.*
4	N/A	N/A	13	0.51	2.34	13
8			17	0.49	3.61	16
16			19	0.48	5.13	19
32			21	0.48	6.99	21

on the number of elements across each subdomain. These results are less satisfactory than those of the previous two choices.

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