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Three-level BDDC in Two Dimensions

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Abstract

BDDC methods are nonoverlapping iterative substructuring domain decomposition methods for the solutions of large sparse linear algebraic systems arising from discretization of elliptic boundary value problems. They are similar to the balancing Neumann-Neumann algorithm. However, in BDDC methods, a small number of continuity constraints are enforced across the interface, and these constraints form a new coarse, global component. An important advantage of using such constraints is that the Schur complements that arise in the computation will all be strictly positive definite. The coarse problem is generated and factored by a direct solver at the beginning of the computation. However, this problem can ultimately become a bottleneck, if the number of subdomains is very large. In this paper, two three-level BDDC methods are introduced for solving the coarse problem approximately in two dimensional space, while still maintaining a good convergence rate. Estimates of the condition numbers are provided for the two three-level BDDC methods and numerical experiments are also discussed.

Key words. BDDC, three-level, domain decomposition, coarse problem, condition number, Chebyshev iteration

AMS subject classifications 65N30, 65N55

1 Introduction

In this paper, we introduce two three-level BDDC (Balancing Domain Decomposition with Constraints) methods. The BDDC algorithms, so far developed for two levels [2, 4, 5], are similar to the balancing Neumann-Neumann algorithms. However, their coarse problems, in BDDC, are given in terms of sets of primal constraints and they are generated and factored by a direct solver at the beginning of the computation. The coarse components of the preconditioners can ultimately become a bottleneck if the number of subdomains is

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very large. We will try to remove this problem by using one or several additional levels. We introduce two three-level BDDC methods. Here, we only consider two dimensional problems and vertex constraints. We also provide estimates of the condition numbers of the system with these two new preconditioners. We are currently working on the extension of our algorithms and results to the considerably much more complicated three dimensional cases.

The rest of the paper is organized as follows. We first review the two-level BDDC methods briefly in Section 2. We introduce our first three-level BDDC method and the corresponding preconditioner $\widetilde{M}^{-1}$ in Section 3. We give some auxiliary results in Section 4. In Section 5, we provide an estimate of the condition number for the system with the preconditioner $\widetilde{M}^{-1}$ which is of the form $C\left(1+\log\frac{\hat{H}}{H}\right)^2\left(1+\log\frac{H}{h}\right)^2$, where $\hat{H}$, H, and h are the diameters of the subregions, subdomains, and elements, respectively. In Section 6, we introduce a second three-level BDDC method which uses Chebyshev iterations. We denote the corresponding preconditioner by $\widehat{M}^{-1}$. We show that the condition number of the system with the preconditioner $\widehat{M}^{-1}$ is of the form $CC(k)\left(1+\log\frac{H}{h}\right)^2$, where $C(k)$ is a constant depending on the eigenvalues of the preconditioned coarse problem, the two parameters chosen for the Chebyshev iteration, and k, the number of Chebyshev iterations. $C(k)$ goes to 1 as k goes to ∞. Finally, some computational results are presented in Section 7.

2 The two-level BDDC method

We consider a second order scalar elliptic problem in a two dimensional region Ω: find $u \in H_0^1(\Omega)$, such that

$$\int_\Omega \rho\nabla u\cdot\nabla v = \int_\Omega f v \quad \forall v\in H_0^1(\Omega), \tag{1}$$

where $\rho(x) > 0$ for all $x \in \Omega$. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1,\cdots,N$, and let $H = \max_i H_i$. We then introduce a triangulation of all the subdomains. Let Γ be the interface between subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i\neq j}\partial\Omega_{i,h}\cap\partial\Omega_{j,h})\setminus\partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$.

Let $\mathbf{W}^{(i)}$ be the standard finite element space of continuous, piecewise linear functions on Ω_i. We assume that these functions vanish on $\partial\Omega$. Each $\mathbf{W}^{(i)}$ can be decomposed into a subdomain interior part and a subdomain interface part, i.e.,

$$\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma^{(i)},$$

where the subdomain interface part $\mathbf{W}_\Gamma^{(i)}$ can be further decomposed into a primal subspace $\mathbf{W}_\Pi^{(i)}$ and a dual subspace $\mathbf{W}_\Delta^{(i)}$, i.e.,

$$\mathbf{W}_\Gamma^{(i)} = \mathbf{W}_\Pi^{(i)} \bigoplus \mathbf{W}_\Delta^{(i)}.$$

We denote the associated product spaces by $\mathbf{W} := \prod_{i=1}^{N} \mathbf{W}^{(i)}$, $\mathbf{W}_\Gamma := \prod_{i=1}^{N} \mathbf{W}_\Gamma^{(i)}$, $\mathbf{W}_\Delta := \prod_{i=1}^{N} \mathbf{W}_\Delta^{(i)}$, $\mathbf{W}_\Pi := \prod_{i=1}^{N} \mathbf{W}_\Pi^{(i)}$, and $\mathbf{W}_I := \prod_{i=1}^{N} \mathbf{W}_I^{(i)}$. Correspondingly, we have

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma,$$

and

$$\mathbf{W}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta.$$

We will consider elements of a product space which are discontinuous across the interface. However, the finite element approximation of the elliptic problem is continuous across Γ. Instead, we denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$.

We further introduce an interface subspace $\widetilde{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$, for which certain primal constraints are enforced. Here, we only consider vertex constraints at the corners of each subdomain. The continuous primal subspace denoted by $\widehat{\mathbf{W}}_\Pi$ is spanned by the continuous finite element basis functions of the vertex nodes. The space $\widetilde{\mathbf{W}}_\Gamma$ can be decomposed into

$$\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta.$$

The global problem: find $(\mathbf{u}_I, \mathbf{u}_\Delta, \mathbf{u}_\Pi) \in (\mathbf{W}_I, \widehat{\mathbf{W}}_\Delta, \widehat{\mathbf{W}}_\Pi)$, such that

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & A_{\Pi I}^T \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Pi\Delta}^T \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{pmatrix}. \tag{2}$$

This problem is assembled from subdomain problems

$$\begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \\ \mathbf{u}_\Pi^{(i)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I^{(i)} \\ \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix}. \tag{3}$$

We define an operator $\widetilde{S}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \rightarrow \widetilde{\mathbf{W}}_\Gamma$, which is of the form: given $\mathbf{u}_\Gamma = \mathbf{u}_\Pi \oplus \mathbf{u}_\Delta \in \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widetilde{\mathbf{W}}_\Gamma$, find $\widetilde{S}_\Gamma \mathbf{u}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$ by eliminating the interior nodes of the system with the matrix:

$$A = \begin{pmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)^T} & \cdots & \cdots & \cdots & \cdots & A_{\Pi I}^{(1)^T} R_\Pi^{(1)} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & \cdots & \cdots & \cdots & \cdots & A_{\Pi\Delta}^{(1)^T} R_\Pi^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & A_{II}^{(i)} & A_{\Delta I}^{(i)^T} & \cdots & A_{\Pi I}^{(i)^T} R_\Pi^{(i)} \\ \cdots & \cdots & \cdots & A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & \cdots & A_{\Pi\Delta}^{(i)^T} R_\Pi^{(i)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_\Pi^{(1)^T} A_{\Pi I}^{(1)} & R_\Pi^{(1)^T} A_{\Pi\Delta}^{(1)} & \cdots & R_\Pi^{(i)^T} A_{\Pi I}^{(i)} & R_\Pi^{(i)^T} A_{\Pi\Delta}^{(i)} & \cdots & \sum_{i=1}^{N} R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)} \end{pmatrix}, \tag{4}$$

i.e.,

$$A \begin{pmatrix} \mathbf{u}_I^{(1)} \\ \mathbf{u}_\Delta^{(1)} \\ \vdots \\ \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \\ \vdots \\ \mathbf{u}_\Pi \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(1)} R_{\Gamma\Delta} \widetilde{S}_\Gamma \mathbf{u}_\Gamma \\ \vdots \\ \mathbf{0} \\ R_\Delta^{(i)} R_{\Gamma\Delta} \widetilde{S}_\Gamma \mathbf{u}_\Gamma \\ \vdots \\ R_{\Gamma\Pi} \widetilde{S}_\Gamma \mathbf{u}_\Gamma \end{pmatrix}. \tag{5}$$

Here $R_{\Gamma\Delta} : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Delta$, restricts the functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Delta$, and is of the matrix form

$$R_{\Gamma\Delta} = \begin{pmatrix} R_{\Gamma\Delta}^{(1)} & & & \\ & R_{\Gamma\Delta}^{(2)} & & \\ & & \ddots & \\ & & & R_{\Gamma\Delta}^{(N)} \end{pmatrix}, \tag{6}$$

where each $R_{\Gamma\Delta}^{(i)}$ represents the restriction from $\mathbf{W}_\Gamma^{(i)}$ to $\mathbf{W}_\Delta^{(i)}$. Furthermore, $R_\Delta^{(i)} : \mathbf{W}_\Delta \to \mathbf{W}_\Delta^{(i)}$, is the restriction matrix which extracts the subdomain part, in the space $\mathbf{W}_\Delta^{(i)}$, of a function in the space $\mathbf{W}_\Delta$, and $R_{\Gamma\Pi} : \widetilde{\mathbf{W}}_\Gamma \to \widehat{\mathbf{W}}_\Pi$, restricts the functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to $\widehat{\mathbf{W}}_\Pi$. $R_\Pi^{(i)}$ is the restriction operator from the space $\widehat{\mathbf{W}}_\Pi$ to $\mathbf{W}_\Pi^{(i)}$.

R_Γ and $R_{D,\Gamma}$ are the restriction and scaled restriction operators from the space $\widehat{\mathbf{W}}_\Gamma$ onto $\widetilde{\mathbf{W}}_\Gamma$, respectively. They are of the form

$$R_\Gamma = \begin{pmatrix} R_\Gamma^{(1)} \\ R_\Gamma^{(2)} \\ \vdots \\ R_\Gamma^{(n)} \end{pmatrix}, \text{ and } \quad R_{D,\Gamma} = \begin{pmatrix} R_{D,\Gamma}^{(1)} \\ R_{D,\Gamma}^{(2)} \\ \vdots \\ R_{D,\Gamma}^{(n)} \end{pmatrix}, \tag{7}$$

where $R_\Gamma^{(i)}$ is a restriction operator mapping a vector of the space $\widehat{\mathbf{W}}_\Gamma$ to a vector of the subdomain subspace $\mathbf{W}_\Gamma^{(i)}$. Each column of $R_\Gamma^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega_{i,h} \cap \Gamma_h$, shared by the subdomain Ω_i and its neighboring subdomains. Multiplying each such column of $R_\Gamma^{(i)}$ with $\delta_i^\dagger(x)$ gives us $R_{D,\Gamma}^{(i)}$. Here, we define $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i(x) = \frac{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma}{\rho_i^\gamma}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$ and $\rho_j(x)$ is the coefficient of the scalar elliptic problem (1) at x in the subdomain Ω_j. The pseudoinverses $\delta_i^\dagger(x)$ are defined by

$$\delta_i^\dagger(x) = (\delta_i(x))^{-1}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h.$$

The reduced interface problem can be written as: find $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$ such that

$$R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma = \mathbf{g}_\Gamma, \tag{8}$$

where the operators $\widetilde{S}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{W}}_\Gamma$, and $R_\Gamma : \widehat{\mathbf{W}}_{\mathbf{\Gamma}} \to \widetilde{\mathbf{W}}_\Gamma$, are defined in Equations (5) and (7), and

$$\mathbf{g}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} \left\{ \begin{pmatrix} \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix} - \begin{pmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{pmatrix} A_{II}^{(i)^{-1}} \mathbf{f}_I^{(i)} \right\}.$$

The two-level BDDC method is of the form

$$M^{-1} R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma = M^{-1} \mathbf{g}_\Gamma,$$

where the preconditioner $M^{-1} = R_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} R_{D,\Gamma}$ has the following form:

$$M^{-1} = R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^N R_{\Gamma\Delta}^{(i)^T} \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T \right\} D_\Gamma R_\Gamma. \tag{9}$$

Here Φ is the matrix given by the coarse level basis functions with minimal energy, and is defined by

$$\Phi = R_{\Gamma\Pi}^T - \sum_{i=1}^N R_{\Gamma\Delta}^{(i)^T} \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} R_\Pi^{(i)}. \tag{10}$$

The coarse level problem matrix S_Π is determined by

$$\begin{aligned} S_\Pi &= A_{\Pi\Pi} - \begin{pmatrix} A_{\Pi I} & A_{\Pi\Delta} \end{pmatrix} \begin{pmatrix} A_{II} & A_{I\Delta} \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \end{pmatrix} \\ &= \textstyle\sum_{i=1}^N R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)}, \end{aligned} \tag{11}$$

which is obtained by assembling subdomain matrices.

We know that, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma \le C \left(1 + \log(H/h)\right)^2 \mathbf{u}_\Gamma^T R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma, \tag{12}$$

see [6], provided that the coefficient $\rho(x)$ of the scalar elliptic problem (1) varies moderately in each subdomain. We also assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. Moreover, we assume that the triangulation of each subdomain is quasi uniform.

3 A three-level BDDC method

In the three-level cases, we will not factor the coarse problem matrix S_Π defined in (11) by a direct solver. Instead, we will try to solve the coarse problem approximately by using a similar idea as for the two-level preconditioners.

We decompose Ω into N subregions Ω^j with diameters $\hat{H}^j$, $j = 1, \cdots, N$. Each subregion Ω^j has N_j subdomains Ω_i^j with diameters H_i^j. Let $\hat{H} = \max_j \hat{H}^j$ and $H = \max_{i,j} H_i^j$, for $j = 1, \cdots, N$ and $i = 1, \cdots, N_j$. We introduce the notation

$$S_\Pi^{(j)} = \sum_{i=1}^{N_j} R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)}, \qquad (13)$$

and note that the Schur complement S_Π can be assembled from the subregion matrices $S_\Pi^{(j)}$.

Let $\widehat{\Gamma}$ be the interface between the subregions. We denote the set of interior crosspoints, which contains the vertices of the subdomains in each subregion, by $\widehat{I}_H$, and the set of interface nodes, which contains the vertices of the subdomains, on the boundary of the subregions by $\widehat{\Gamma}_H$. We note that $\widehat{\Gamma} \subset \Gamma$.

We denote the vector space corresponding to the nodes of the subregion Ω^i, which are the vertices of the subdomains in Ω^i, by $\mathbf{W}_c^{(i)}$. Each $\mathbf{W}_c^{(i)}$ can be decomposed into a subregion interior part and a subregion interface part, i.e.,

$$\mathbf{W}_c^{(i)} = \mathbf{W}_{c,\widehat{I}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}}^{(i)},$$

where the subregion interface part $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$ can be further decomposed into a primal subspace $\mathbf{W}_{c,\widehat{\Pi}}^{(i)}$ and a dual subspace $\mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, i.e.,

$$\mathbf{W}_{c,\widehat{\Gamma}}^{(i)} = \mathbf{W}_{c,\widehat{\Pi}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}^{(i)}.$$

We denote the associated product spaces by $\mathbf{W}_c := \prod_{i=1}^N \mathbf{W}_c^{(i)}$, $\mathbf{W}_{c,\widehat{\Gamma}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Delta}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Pi}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Pi}}^{(i)}$, and $\mathbf{W}_{c,\widehat{I}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{I}}^{(i)}$. Correspondingly, we have

$$\mathbf{W}_c = \mathbf{W}_{c,\widehat{I}} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}},$$

and

$$\mathbf{W}_{c,\widehat{\Gamma}} = \mathbf{W}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}.$$

We denote by $\widehat{\mathbf{W}}_c$ the subspace of $\mathbf{W}_c$ of functions that are continuous across $\widehat{\Gamma}$.

We introduce an interface subspace $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \subset \mathbf{W}_{c,\widehat{\Gamma}}$, for which the primal constraints are enforced. Here, we only consider vertex constraints. The continuous primal subspace is denoted by $\widehat{\mathbf{W}}_{c,\widehat{\Pi}}$. The space $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ can be decomposed into

$$\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} = \widehat{\mathbf{W}}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}.$$

We define our three-level preconditioner $\widetilde{M}^{-1}$ by

$$\widetilde{M}^{-1} = R_{\Gamma}^{T} D_{\Gamma} \left\{ \sum_{i=1}^{N} R_{\Gamma\Delta}^{(i)^T} \left(\mathbf{0} \; R_{\Delta}^{(i)^T} \right) \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_{\Delta}^{(i)} \end{pmatrix} R_{\Gamma\Delta} + \Phi \widetilde{S}_{\Pi}^{-1} \Phi^{T} \right\} D_{\Gamma} R_{\Gamma}, \tag{14}$$

where $\widetilde{S}_{\Pi}^{-1}$ is an approximation of S_{Π}^{-1} and is defined as follows: given any $\mathbf{\Psi} \in \widehat{\mathbf{W}}_{c}$, let $\mathbf{y} = S_{\Pi}^{-1}\mathbf{\Psi}$ and $\widetilde{\mathbf{y}} = \widetilde{S}_{\Pi}^{-1}\mathbf{\Psi}$. Here $\mathbf{\Psi} = \left(\mathbf{\Psi}_{\widehat{I}}^{(1)}, \cdots, \mathbf{\Psi}_{\widehat{I}}^{(N)}, \mathbf{\Psi}_{\widehat{\Gamma}}\right)^{T}$, $\mathbf{y} = \left(\mathbf{y}_{\widehat{I}}^{(1)}, \cdots, \mathbf{y}_{\widehat{I}}^{(N)}, \mathbf{y}_{\widehat{\Gamma}}\right)^{T}$, and $\widetilde{\mathbf{y}} = \left(\widetilde{\mathbf{y}}_{\widehat{I}}^{(1)}, \cdots, \widetilde{\mathbf{y}}_{\widehat{I}}^{(N)}, \widetilde{\mathbf{y}}_{\widehat{\Gamma}}\right)^{T}$.

To solve $S_{\Pi}\mathbf{y} = \mathbf{\Psi}$ by block factorization, we write

$$\begin{pmatrix} S_{\Pi_{\widehat{I}\widehat{I}}}^{(1)} & \cdots & \cdots & \cdots & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)^T} \widehat{R}_{\widehat{\Gamma}}^{(i)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)} & \cdots & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T} \widehat{R}_{\widehat{\Gamma}}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)} & \cdots & \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} & \cdots & \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{y}_{\widehat{I}}^{(1)} \\ \vdots \\ \mathbf{y}_{\widehat{I}}^{(i)} \\ \vdots \\ \mathbf{y}_{\widehat{\Gamma}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Psi}_{\widehat{I}}^{(1)} \\ \vdots \\ \mathbf{\Psi}_{\widehat{I}}^{(i)} \\ \vdots \\ \mathbf{\Psi}_{\widehat{\Gamma}} \end{pmatrix}, \tag{15}$$

where $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ is the restriction operator mapping a vector of the space $\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$ to a vector of the subregion subspace $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$. Each column of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ with a nonzero entry corresponds to a subregion interface node, $x \in \partial\Omega^{i} \cap \partial\Omega^{j}$, shared by the subregion Ω^{i} and its neighboring subregions.

We have

$$\mathbf{y}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \mathbf{y}_{\widehat{\Gamma}} \right), \tag{16}$$

and

$$\left(\sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} \left(S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}\right) \widehat{R}_{\widehat{\Gamma}}^{(i)} \right) \mathbf{y}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}. \tag{17}$$

We introduce an operator $\widetilde{T} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \rightarrow \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ by

$$\widehat{R}_{\widehat{\Gamma}}^{T} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} = \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} \left(S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}\right) \widehat{R}_{\widehat{\Gamma}}^{(i)}, \tag{18}$$

and denote by

$$\mathbf{h}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}. \tag{19}$$

Then,

$$\widehat{R}_{\widehat{\Gamma}}^{T}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\mathbf{y}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}. \tag{20}$$

We denote by $\widehat{R}_{\widehat{\Gamma}}$ and $\widehat{R}_{\widehat{D},\widehat{\Gamma}}$ the restriction and scaled restriction operators from the space $\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$ onto $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, respectively. They are of the form

$$\widehat{R}_{\widehat{\Gamma}} = \begin{pmatrix} \widehat{R}_{\widehat{\Gamma}}^{(1)} \\ \widehat{R}_{\widehat{\Gamma}}^{(2)} \\ \vdots \\ \widehat{R}_{\widehat{\Gamma}}^{(N)} \end{pmatrix}, \text{ and } \quad \widehat{R}_{\widehat{D},\widehat{\Gamma}} = \begin{pmatrix} \widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(1)} \\ \widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(2)} \\ \vdots \\ \widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(N)} \end{pmatrix}. \tag{21}$$

Multiplying each such column of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ with $\widehat{\delta}_i^{\dagger}(x)$ gives us $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, where $x \in \partial\Omega_H^i \cap \widehat{\Gamma}_H$. Here, we define $\widehat{\delta}_i^{\dagger}(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\widehat{\delta}_i(x) = \frac{\sum_{j\in\mathcal{N}_x} \hat{\rho}_j^{\gamma}}{\hat{\rho}_i^{\gamma}}, \qquad x \in \partial\Omega_H^i \cap \widehat{\Gamma}_H,$$

where $\mathcal{N}_x$ is the set of indices j of the subregions such that $x \in \partial\Omega_H^j$ and $\hat{\rho}_j$ is the coefficient of the scalar elliptic problem (1). In our theory, we assume $\hat{\rho}_i$ is a constant in the subregion Ω^i. The pseudoinverses $\widehat{\delta}_i^{\dagger}(x)$ are defined by

$$\widehat{\delta}_i^{\dagger}(x) = (\widehat{\delta}_i(x))^{-1}, \quad x \in \partial\Omega_H^i \cap \widehat{\Gamma}_H.$$

When using the three-level preconditioner $\widetilde{M}^{-1}$, we do not solve (20) exactly. Instead, we replace $\mathbf{y}_{\widehat{\Gamma}}$ by

$$\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \widehat{R}_{\widehat{D},\widehat{\Gamma}}^{T}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\mathbf{h}_{\widehat{\Gamma}}. \tag{22}$$

We will maintain the same relation between $\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)}$ and $\widetilde{\mathbf{y}}_{\widehat{\Gamma}}^{(i)}$, i.e.,

$$\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \right). \tag{23}$$

4 Some auxiliary results

In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants c and C, independent of any parameters, such that $cA \leq B \leq CA$, where $C < \infty$ and $c > 0$. For the definition of discrete harmonic function, see [7, Section 4.4].

Lemma 1 *Let $\mathcal{D}$ be a square with vertices $A = (0,0)$, $B = (H,0)$, $C = (H,H)$, and $D = (0,H)$, with a quasi-uniform triangulation of mesh size h. Then, there exists a discrete harmonic function v defined on $\mathcal{D}$ such that $\|v\|_{L^\infty(\mathcal{D})} = v(A) \approx 1 + \log(\frac{H}{h})$, $v(B) = v(C) = v(D) = 0$ and $|v|^2_{H^1(\mathcal{D})} \approx 1 + \log(\frac{H}{h})$.*

Proof: This lemma follows from [1, Lemma 4.2]: let N be an integer and G_N be the function defined on $(0,1)$ by

$$G_N(x) = \sum_{n=1}^{N} \left(\frac{1}{4n-3} \sin\left((4n-3)\pi x\right) \right). \tag{24}$$

G_N is symmetric with respect to the midpoint of $(0,1)$, where it attains its maximum in absolute value. Moreover, we have:

$$|G_N|^2_{H^{1/2}_{00}(0,1)} \approx 1 + \log N, \tag{25}$$

and

$$\|G_N\|_{L^\infty(0,1)} = G_N(1/2) \approx 1 + \log N; \tag{26}$$

see [1, Lemma 3.2] .

Let P_h be the nodal interpolation operator. Let $[-H,0]$ and $[0,H]$ have the mesh inherited from the quasi-uniform mesh on DA and AB respectively and let $g_h(x) = P_h\left(G_N(\frac{x+H}{2H})\right)$. Then we have

$$|g_h|^2_{H^{1/2}_{00}(-H,H)} \approx 1 + \log \frac{H}{h}, \tag{27}$$

and

$$\|g_h\|_{L^\infty(0,1)} \approx 1 + \log \frac{H}{h}. \tag{28}$$

See [1, Corollary 3.6]. We point out that in [1, Corollary 3.6], a uniform mesh is used. But in the proof of the bound for $|\cdot|_{H^{1/2}_{00}(-H,H)}$, we only need the interpolation error estimate theorem and the fact that $H^{1/2}_{00}(-H,H)$ is the interpolation space halfway between $L^2(-H,H)$ and $H^1_0(-H,H)$. Therefore the result is still valid for a quasi-uniform mesh.

We can define v as 0 on the line segments CD and CB and by

$$v(x,0) = g_h(x), \qquad \text{for} \quad 0 \le x \le H, \tag{29}$$

and

$$v(0,y) = g_h(y), \qquad \text{for} \quad 0 \le y \le H. \tag{30}$$

Since v is discrete harmonic function in $\mathcal{D}$, we have,

$$|v|^2_{H^1(\mathcal{D})} = |v|^2_{H^{1/2}(\partial\mathcal{D})} \approx |g_h|^2_{H^{1/2}_{00}(-H,H)} \approx 1 + \log \frac{H}{h}. \tag{31}$$

$\square$

Remark: In Lemma 1, we have constructed the function v for the square $\mathcal{D}$. By using similar ideas, we can easily construct a function v for other shape-regular polygons which satisfies the same properties.

Lemma 2 *Let V^H and V_j^h, $j = 1, \cdots, N_i$, be the standard continuous piecewise linear finite element function spaces in a subregion Ω^i with respect to the quasi-uniform coarse mesh with mesh size H and in a subdomain Ω_j^i with respect to the quasi-uniform fine mesh with mesh size h, respectively. Moreover, each subdomain is a union of coarse triangles with vertices on the boundary of the subdomain. Given $u \in V^H$, let $\hat{u} \in V^H$ interpolate u at each coarse node and let $\hat{u}$ be the discrete V_j^h-harmonic extension in each subdomain Ω_j^i with the values given at the vertices of Ω_j^i, $j = 1, \cdots, N_i$. Then, there exist two positive constants C_1 and C_2, which are independent of $\hat{H}$, H and h, such that*

$$C_1 \left(1 + \log(\frac{H}{h})\right) \left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega_j^i)}\right) \le |u|^2_{H^1(\Omega^i)} \le C_2 \left(1 + \log(\frac{H}{h})\right) \left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega_j^i)}\right). \quad (32)$$

Proof: Without loss of generality, we assume that the subdomains are quadrilaterals. Denote the vertices of the subdomain Ω_j^i by a_j, b_j, c_j, and d_j, and denote the nodal values of u at these four crosspoints by $u(a_j)$, $u(b_j)$, $u(c_j)$, and $u(d_j)$, respectively. Since u is a piecewise linear function, we have,

$$|u|^2_{H^1(\Omega^i)} = \sum_{j=1}^{N_i} |u|^2_{H^1(\Omega_j^i)}, \quad (33)$$

and

$$|u|^2_{H^1(\Omega_j^i)} = |u - u(a_j)|^2_{H^1(\Omega_j^i)} \approx C \left(\sum_{m=b,c,d} \left((u(m_j) - u(a_j))^2\right)\right). \quad (34)$$

According to Lemma 1, we can construct three discrete harmonic functions ϕ_b, ϕ_c, and ϕ_d on Ω_j^i such that

$$\phi_b(b_j) = (u(b_j) - u(a_j))\,(1 + \log(\frac{H}{h})), \quad \phi_b(a_j) = \phi_b(c_j) = \phi_b(d_j) = 0,$$

$$\phi_c(c_j) = (u(c_j) - u(a_j))\,(1 + \log(\frac{H}{h})), \quad \phi_c(a_j) = \phi_c(b_j) = \phi_c(d_j) = 0,$$

$$\phi_d(d_j) = (u(d_j) - u(a_j))\,(1 + \log(\frac{H}{h})), \quad \phi_d(a_j) = \phi_d(b_j) = \phi_d(c_j) = 0,$$

and with

$$|\phi_m|^2_{H^1(\Omega_j^i)} \approx (u(m_j) - u(a_j))^2\,(1 + \log(\frac{H}{h})), \quad m = b, c, d. \quad (35)$$

Let $v_j = \frac{1}{1+\log(\frac{H}{h})}(\phi_b + \phi_c + \phi_d) + u(a_j)$; we then have $v_j(m_j) = u(m_j)$, $m = a, b, c, d$, and

$$\begin{aligned} |v_j|^2_{H^1(\Omega_j^i)} &= |\frac{1}{1+\log(\frac{H}{h})}(\phi_b + \phi_c + \phi_d) + u(a_j)|^2_{H^1(\Omega_j^i)} \\ &= \left(\frac{1}{1+\log(\frac{H}{h})}\right)^2 |\phi_b + \phi_c + \phi_d|^2_{H^1(\Omega_j^i)} \end{aligned}$$

$$
\begin{aligned}
&\le 3\left(\frac{1}{1+\log(\frac{H}{h})}\right)^2 \sum_{m=b,c,d} |\phi_m|^2_{H^1(\Omega^i_j)} \\
&\le \left(\frac{1}{c^{1/2}(1+\log(\frac{H}{h}))}\right)^2 \left(1+\log(\frac{H}{h})\right) \sum_{m=b,c,d} (u(m_j)-u(a_j))^2 \\
&\le \frac{1}{C_1(1+\log(\frac{H}{h}))} |u|^2_{H^1(\Omega^i_j)}. \qquad (36)
\end{aligned}
$$

Here, we have used (34) and (35) for the last two inequalities.

By the definition of $\hat{u}$, we have,

$$
|\hat{u}|^2_{H^1(\Omega^i_j)} \le |v|^2_{H^1(\Omega^i_j)} \le \frac{1}{C_1(1+\log(\frac{H}{h}))} |u|^2_{H^1(\Omega^i_j)}. \qquad (37)
$$

Summing over all the subdomains in the subregion Ω^i, we have,

$$
C_1\left(1+\log(\frac{H}{h})\right)\left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega^i_j)}\right) \le \sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^i_j)} = |u|^2_{H^1(\Omega^i)}. \qquad (38)
$$

This proves the first inequality.

We prove the second inequality as follows:

$$
\begin{aligned}
|u|^2_{H^1(\Omega^i)} &= \sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^i_j)} \\
&= \sum_{j=1}^{N_i} |u-u(a_j)|^2_{H^1(\Omega^i_j)} \\
&\le C_2\left(\sum_{j=1}^{N_i} \max_{m=b,c,d}\left((u(m_j)-u(a_j))^2\right)\right) \\
&\le C_2\left(\sum_{j=1}^{N_i} \|\hat{u}-u(a_j)\|^2_{L^\infty(\Omega^i_j)}\right) \\
&\le C_2\left(1+\log(\frac{H}{h})\right)\left(\sum_{j=1}^{N_i} |\hat{u}|^2_{H^1(\Omega^i_j)}\right). \qquad (39)
\end{aligned}
$$

Here, we have used a standard finite element Sobolev inequality, see [7, Lemma 4.15].

□

We next list several results for the two-level BDDC methods. To be fully rigorous, we assume that each subregion is a union of shape-regular coarse triangles and the number of such triangles forming an individual subregion is uniformly bounded. Thus, there is a quasi-uniform coarse triangulation of each subregion. Similarly, each subdomain is a union of shape-regular coarse triangles with the vertices on the boundary of the subdomain.

Moreover the fine triangulation of each subdomain is quasi uniform. We can then get uniform constants C_1 and C_2 in Lemma 2, which work for all the subregions.

We define the interface averages operator $\widehat{E}_{\widehat{D}} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \rightarrow \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$, by

$$\widehat{E}_{\widehat{D}} = \widehat{R}_{\widehat{\Gamma}} \widehat{R}^T_{\widehat{D},\widehat{\Gamma}}, \tag{40}$$

which computes the averages across the subregion interface $\widehat{\Gamma}$ and then distributes the averages to the boundary points of the subregion.

The interface average operator $\widehat{E}_{\widehat{D}}$ has the following property:

Lemma 3

$$\widehat{E}_{\widehat{D}} \mathbf{w}_{\widehat{\Gamma}} = \widehat{R}^T_{\widehat{\Gamma}} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} = \mathbf{w}_{\widehat{\Gamma}},$$

for any $\mathbf{w}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$.

Proof: See [7, Section 6.2].

$\square$

Moreover, we have the following estimate for $\widehat{E}_{\widehat{D}}$:

Lemma 4 *Consider the two-level BDDC, and let $\widehat{K}^{(i)}$ be the stiffness matrix for the subregion Ω^i and $\widehat{S}^{(i)}$ be the Schur complement of $\widehat{K}^{(i)}$ with respect to the interior nodes in the subregion Ω^i and let*

$$\widehat{S} = \begin{pmatrix} \widehat{S}^{(1)} & & & \\ & \widehat{S}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{S}^{(N)} \end{pmatrix}. \tag{41}$$

Then,

$$|\widehat{E}_{\widehat{D}} \mathbf{u}_{\widehat{\Gamma}}|^2_{\widehat{S}} \leq C \left(1 + \log(\frac{\hat{H}}{H})\right)^2 |\mathbf{u}_{\widehat{\Gamma}}|^2_{\widehat{S}}, \quad \forall u \in \widetilde{W}_{\widehat{\Gamma}}, \tag{42}$$

where $\widetilde{W}_{\widehat{\Gamma}}$, which corresponds to a mesh with size H, is analogous to $\widetilde{W}_{\Gamma}$, which corresponds to a mesh with size h.

Proof: We can use a result by Mandel and Tezaur for the FETI-DP algorithm in [6] to estimate $(\widehat{E}_{\widehat{D}} - I)\mathbf{u}_{\widehat{\Gamma}}$. We find:

$$|\widehat{E}_{\widehat{D}} \mathbf{u}_{\widehat{\Gamma}}|^2_{\widehat{S}} = |(\widehat{E}_{\widehat{D}} - I)\mathbf{u}_{\widehat{\Gamma}} + \mathbf{u}_{\widehat{\Gamma}}|^2_{\widehat{S}} \leq C \left(1 + \log(\frac{\hat{H}}{H})\right)^2 |\mathbf{u}_{\widehat{\Gamma}}|^2_{\widehat{S}}. \tag{43}$$

$\square$

We also have a result for the condition number of the two-level BDDC, see [4].

Lemma 5 *The condition number for the operator with the two-level preconditioner M^{-1} is bounded by $C\left(1+\log\frac{H}{h}\right)^2$.*

In addition, we have:

Lemma 6

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}},$$

for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, where C is a positive constant independent of $\hat{H}$, H and h. Remind that $\widetilde{T}$ is defined in (18).

Proof: Denote by $\mathcal{H}^i$ the discrete harmonic extension in the subregion Ω^i with respect to $S_\Pi^{(i)}$, given by the values on the boundary of Ω^i, i.e., $\mathcal{H}^i$ satisfies:

$$|\mathcal{H}^i(w)|_{S_\Pi^{(i)}} = \min_{v\in\widetilde{\mathbf{W}}_c^{(i)}, v=w \text{ on } \partial\Omega^i} |v|_{S_\Pi^{(i)}}, \quad w \in \widetilde{\mathbf{W}}^{(i)}_{c,\widehat{\Gamma}}.$$

Let $\hat{\mathcal{H}}^i$ satisfy:

$$|\hat{\mathcal{H}}^i(w)|_{H^1(\Omega^i)} = \min_{v\in\widetilde{\mathbf{W}}_c^{(i)}, v=w \text{ on } \partial\Omega^i} |v|_{H^1(\Omega^i)}, \quad w \in \widetilde{\mathbf{W}}^{(i)}_{c,\widehat{\Gamma}}.$$

Denote by $\tilde{\mathcal{H}}^i_j$ the discrete harmonic extension in each subdomain Ω^i_j, with respect to the fine mesh with mesh size h, given the crosspoint nodal values, where $i = 1,\cdots,N$, and $j = 1,\cdots,N_i$.

We have

$$\begin{aligned}
|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} &= \sum_{i=1}^{N} |\mathcal{H}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})|^2_{S_\Pi^{(i)}} \\
&\leq \sum_{i=1}^{N} |\hat{\mathcal{H}}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})|^2_{S_\Pi^{(i)}} \\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{N_i} |\tilde{\mathcal{H}}^i_j\left(\hat{\mathcal{H}}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})\right)|^2_{H^1(\Omega^i_j)}\right).
\end{aligned}$$

Here we have used the definitions of $\mathcal{H}$, $\hat{\mathcal{H}}$, $\tilde{\mathcal{H}}$, and $S_\Pi^{(i)}$.

By Lemma 2,

$$\begin{aligned}
|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} &\leq \sum_{i=1}^{N}\left(\sum_{j=1}^{N_i} |\tilde{\mathcal{H}}^i_j\left(\hat{\mathcal{H}}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})\right)|^2_{H^1(\Omega^i_j)}\right) \\
&\leq \frac{1}{C_1(1+\log\frac{H}{h})}\sum_{i=1}^{N}\left(|\hat{\mathcal{H}}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}})|^2_{H^1(\Omega^i)}\right) \\
&= \frac{1}{C_1(1+\log\frac{H}{h})}|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widehat{S}}.
\end{aligned}$$

Using Lemma 4, we obtain

$$
\begin{aligned}
|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} &\leq \frac{1}{C_1(1+\log\frac{H}{h})}|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widehat{S}} \\
&\leq \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widehat{S}} \\
&= \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N}|\hat{\mathcal{H}}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})|^2_{H^1(\Omega^i)}\right) \\
&\leq \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N}|\mathcal{H}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})|^2_{H^1(\Omega^i)}\right).
\end{aligned}
$$

Here we have used the definition of $\mathcal{H}$ and $\hat{\mathcal{H}}$ again.

By Lemma 2 and the definition of $\mathcal{H}$, we have

$$
\begin{aligned}
|\widehat{E}_{\widehat{D}}\mathbf{w}_{\Gamma}|^2_{\widetilde{T}} &\leq \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N}|\mathcal{H}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})|^2_{H^1(\Omega^i)}\right) \\
&\leq \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 C_2\left(1+\log\frac{H}{h}\right)\left(\sum_{i=1}^{N}\sum_{j=1}^{N_i}\left(|\tilde{\mathcal{H}}^i_j\left(\mathcal{H}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})\right)|^2_{H^1(\Omega^i_j)}\right)\right) \\
&= \frac{CC_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N}|\mathcal{H}^i(\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})|^2_{S^{(i)}_{\Pi}}\right) \\
&= \frac{CC_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}}. \qquad (44)
\end{aligned}
$$

$\square$

Lemma 7 *Given any $\mathbf{u}_{\mathbf{\Gamma}} \in \widehat{\mathbf{W}}_{\Gamma}$, let $\mathbf{\Psi} = \Phi^T D_{\Gamma} R_{\Gamma}\mathbf{u}_{\Gamma}$. We have,*

$$
\mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi} \leq \mathbf{\Psi}^T \widetilde{S}_{\Pi}^{-1}\mathbf{\Psi}^T \leq C\left(1+\log(\frac{\hat{H}}{H})\right)^2 \mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi}. \qquad (45)
$$

Proof: Using (16), (19), and (20), we have

$$
\begin{aligned}
\mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi} &= \sum_{i=1}^{N}\Psi^{(i)^T}_{\widehat{I}}\mathbf{y}^{(i)}_{\widehat{I}} + \mathbf{\Psi}^T_{\widehat{\Gamma}}\mathbf{y}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^{N}\mathbf{\Psi}^{(i)^T}_{\widehat{I}}\left(S^{(i)^{-1}}_{\Pi_{\widehat{I}\widehat{I}}}(\mathbf{\Psi}^{(i)}_{\widehat{I}} - S^{(i)}_{\Pi_{\widehat{I}\widehat{\Gamma}}}\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{y}_{\widehat{\Gamma}})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^{N}\widehat{R}^{(i)^T}_{\widehat{\Gamma}}S^{(i)}_{\Pi_{\widehat{\Gamma}I}}S^{(i)^{-1}}_{\Pi_{\widehat{I}\widehat{I}}}\mathbf{\Psi}^{(i)}\right)^T \mathbf{y}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^{N}\mathbf{\Psi}^{(i)^T}_{\widehat{I}}S^{(i)^{-1}}_{\Pi_{\widehat{I}\widehat{I}}}\mathbf{\Psi}^{(i)}_{\widehat{I}} + \mathbf{h}^T_{\widehat{\Gamma}}\mathbf{y}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^{N}\mathbf{\Psi}^{(i)^T}_{\widehat{I}}S^{(i)^{-1}}_{\Pi_{\widehat{I}\widehat{I}}}\mathbf{\Psi}^{(i)}_{\widehat{I}} + \mathbf{h}^T_{\widehat{\Gamma}}\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}. \qquad (46)
\end{aligned}
$$

Using (23), (19), and (22), we also have

$$
\begin{aligned}
\mathbf{\Psi}^T \widetilde{S}_\Pi^{-1} \mathbf{\Psi} &= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} \widetilde{y}_{\widehat{I}^{(i)}} + \mathbf{\Psi}_{\widehat{\Gamma}}^T \widetilde{g}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} \left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} (\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \widetilde{\mathbf{y}}_{\widehat{\Gamma}}) \right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^N \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)} \right)^T \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}. \qquad (47)
\end{aligned}
$$

We only need to compare $\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}}$ and $\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}$ for any $\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$.

We have,

$$
\begin{aligned}
& \mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \\
= \; & \mathbf{h}_{\widehat{\Gamma}}^T \widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} = \| \widetilde{T}^{-1/2} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} \|_2^2, \\
= \; & \max_{\mathbf{v}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{v}_{\widehat{\Gamma}}, \widetilde{T}^{-1/2} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}})^2}{\|\mathbf{v}_{\widehat{\Gamma}}\|_2^2} = \max_{\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{w}_{\widehat{\Gamma}}, \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}})^2}{\|\mathbf{w}_{\widehat{\Gamma}}\|_{\widetilde{T}}^2}. \qquad (48)
\end{aligned}
$$

In Equation (48), we make the substitution in $\mathbf{w}_{\widehat{\Gamma}} = \widehat{R}_{\widehat{\Gamma}} \mathbf{v}_{\widehat{\Gamma}}$, for any $\mathbf{v}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$, and we have

$$
\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \geq \max_{\mathbf{v}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\widehat{R}_{\widehat{\Gamma}} \mathbf{v}_{\widehat{\Gamma}}, \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}})^2}{\|\widehat{R}_{\widehat{\Gamma}} \mathbf{v}_{\widehat{\Gamma}}\|_{\widetilde{T}}^2} = \max_{\mathbf{v}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{v}_{\widehat{\Gamma}}, \widehat{R}_{\widehat{\Gamma}}^T \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}})^2}{\|\widehat{R}_{\widehat{\Gamma}} \mathbf{v}_{\widehat{\Gamma}}\|_{\widetilde{T}}^2}.
$$

We have, from Lemma 3, that $\widehat{R}_{\widehat{\Gamma}}^T \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} = \mathbf{w}_{\widehat{\Gamma}}$, for any $\mathbf{w}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$, and therefore,

$$
\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \geq \max_{\mathbf{v}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{h}_{\widehat{\Gamma}}, \mathbf{v}_{\widehat{\Gamma}})^2}{\|\widehat{R}_{\widehat{\Gamma}} \mathbf{v}_{\widehat{\Gamma}}\|_{\widetilde{T}}^2} = \max_{\mathbf{v}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{h}_{\widehat{\Gamma}}, \mathbf{v}_{\widehat{\Gamma}})^2}{(\mathbf{v}_{\widehat{\Gamma}}, \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{v}_{\widehat{\Gamma}})}. \qquad (49)
$$

Taking $\mathbf{v}_{\widehat{\Gamma}} = \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}}$ in the above equation, we have

$$
\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \geq \mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}}.
$$

Since, by Lemma 6, $|\widehat{E}_{\widehat{D}} \mathbf{w}_{\widehat{\Gamma}}|_{\widetilde{T}}^2 \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 |\mathbf{w}_{\widehat{\Gamma}}|_{\widetilde{T}}^2$, for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, we have, from (48),

$$
\mathbf{h}_{\widehat{\Gamma}}^T \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \leq C \left(1 + \log \frac{\hat{H}}{H} \right)^2 \max_{\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{h}_{\widehat{\Gamma}}, \widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{w}_{\widehat{\Gamma}})^2}{\|\widehat{R}_{\widehat{\Gamma}} \widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{w}_{\widehat{\Gamma}}\|_{\widetilde{T}}^2}
$$

$$
\begin{aligned}
&= C\left(1+\log\frac{\hat{H}}{H}\right)^2 \max_{\mathbf{w}_{\widehat{\Gamma}}\in\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{h}_{\widehat{\Gamma}}, \widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})^2}{(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}, \left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}})} \\
&= C\left(1+\log\frac{\hat{H}}{H}\right)^2 \max_{\mathbf{v}_{\widehat{\Gamma}}\in\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}} \frac{(\mathbf{h}_{\widehat{\Gamma}}, \mathbf{v}_{\widehat{\Gamma}})^2}{(\mathbf{v}_{\widehat{\Gamma}}, \left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\mathbf{v}_{\widehat{\Gamma}})} \\
&= C\left(1+\log\frac{\hat{H}}{H}\right)^2 \mathbf{h}^T_{\widehat{\Gamma}}\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}.
\end{aligned}
\tag{50}
$$

$\square$

5 Condition number estimate for the new preconditioner

In order to estimate the condition number for the system with the new preconditioner $\widetilde{M}^{-1}$, we compare it to the system with the preconditioner M^{-1}.

Lemma 8 *Given any* $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$
\mathbf{u}^T_\Gamma M^{-1}\mathbf{u}_\Gamma \le \mathbf{u}^T_\Gamma \widetilde{M}^{-1}\mathbf{u}_\Gamma \le \left(1+\log\left(\frac{\hat{H}}{H}\right)\right)^2 \mathbf{u}^T_\Gamma M^{-1}\mathbf{u}_\Gamma. \tag{51}
$$

Proof: We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$
\begin{aligned}
&\mathbf{u}^T_\Gamma M^{-1}\mathbf{u}_\Gamma \\
&= \mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta}\left(\mathbf{0}\ R^{(i)^T}_\Delta\right)\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0} \\ R^{(i)}_\Delta\end{pmatrix} R_{\Gamma\Delta} + \Phi S^{-1}_\Pi \Phi^T \right\} D_\Gamma R_\Gamma \mathbf{u}_\Gamma \\
&= \mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta}\left(\mathbf{0}\ R^{(i)^T}_\Delta\right)\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0} \\ R^{(i)}_\Delta\end{pmatrix} R_{\Gamma\Delta} \right\} \mathbf{u}_\Gamma \\
&+\mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \Phi S^{-1}_\Pi \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma,
\end{aligned}
\tag{52}
$$

and

$$
\begin{aligned}
&\mathbf{u}^T_\Gamma \widetilde{M}^{-1}\mathbf{u}_\Gamma \\
&= \mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta}\left(\mathbf{0}\ R^{(i)^T}_\Delta\right)\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0} \\ R^{(i)}_\Delta\end{pmatrix} R_{\Gamma\Delta} + \Phi \widetilde{S}^{-1}_\Pi \Phi^T \right\} D_\Gamma R_\Gamma \mathbf{u}_\Gamma \\
&= \mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta}\left(\mathbf{0}\ R^{(i)^T}_\Delta\right)\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0} \\ R^{(i)}_\Delta\end{pmatrix} R_{\Gamma\Delta} \right\} \mathbf{u}_\Gamma \\
&+\mathbf{u}^T_\Gamma R^T_\Gamma D_\Gamma \Phi \widetilde{S}^{-1}_\Pi \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma.
\end{aligned}
\tag{53}
$$

We get our result by using Lemma 7.

□

Theorem 1 *The condition number for the system with the three-level preconditioner* $\widetilde{M}^{-1}$ *is bounded by* $C(1+\log(\frac{\hat{H}}{H}))^2(1+\log(\frac{H}{h}))^2$.

Proof: Combining the condition number bound, given in Lemma 5, for the two-level BDDC method and Lemma 8, we find that the condition number for the three-level method is bounded by $C(1+\log(\frac{\hat{H}}{H}))^2(1+\log(\frac{H}{h}))^2$.

□

6 Using Chebyshev iterations

Another approach to the three-level BDDC methods is to use an iterative method with a preconditioner to solve (20). Here, we consider a Chebyshev method with a fixed number of iterations and use $\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}$ as a preconditioner.

Thus, we do not solve (20) directly. Instead, we replace $\mathbf{y}_{\widehat{\Gamma}}$ by $\mathbf{y}_{\widehat{\Gamma},k}$, where $\mathbf{y}_{\widehat{\Gamma},k}$ is the approximation of $\mathbf{y}_{\widehat{\Gamma}}$ given by a k-step Chebyshev iteration.

We will maintain the same relation between $\mathbf{y}^{(i)}_{\widehat{I},k}$ and $\mathbf{y}^{(i)}_{\widehat{\Gamma},k}$ as in (16), i.e.,

$$\mathbf{y}^{(i)}_{\widehat{I},k} = S^{(i)^{-1}}_{\Pi\widehat{I}\widehat{I}}\left(\mathbf{\Psi}^{(i)}_{\widehat{I}} - S^{(i)}_{\Pi\widehat{I}\widehat{\Gamma}}\widehat{R}^{(i)}_{\widehat{\Gamma}}\mathbf{y}_{\widehat{\Gamma},k}\right). \tag{54}$$

Let $\mathbf{y}_k = \left(\mathbf{y}^{(1)}_{\widehat{I},k},\cdots,\mathbf{y}^{(N)}_{\widehat{I},k},\mathbf{y}_{\widehat{\Gamma},k}\right)^T$, and denote the corresponding new coarse problem matrix by $\widehat{S}_\Pi$. Then,

$$\widehat{S}_\Pi\mathbf{y}_k = \Psi, \tag{55}$$

and the new preconditioner $\widehat{M}^{-1}$ is defined by:

$$\widehat{M}^{-1} = R^T_\Gamma D_\Gamma\left\{\sum_{i=1}^N R^T_{\Gamma\Delta}\left(\mathbf{0}\ R^{(i)^T}_\Delta\right)\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0} \\ R^{(i)}_\Delta\end{pmatrix}R_{\Gamma\Delta} + \Phi\widehat{S}^{-1}_\Pi\Phi^T\right\}D_\Gamma R_\Gamma. \tag{56}$$

6.1 Algorithm

We need two input parameters l and u for the Chebyshev iteration, where l and u are estimates for the smallest and largest eigenvalues of $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$, respectively. From our analysis above, we know that $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$ has a smallest eigenvalue 1 and a largest eigenvalue bounded by $C(1+\log(\frac{\hat{H}}{H}))^2(1+\log(\frac{H}{h}))$. We can use the conjugate gradient method to get an estimate for the largest eigenvalue at the beginning of the computation and to choose a proper u.

Let $\alpha = \frac{2}{l+u}$ and $\mu = \frac{u+l}{u-l}$. Let c_k be the value of the k^{th} Chebyshev polynomial evaluated at μ, i.e.,

$$c_{k+1} = 2\mu c_k - c_{k-1}, \quad k = 1, 2, \cdots, \tag{57}$$

with

$$c_0 = 1, \text{ and } c_1 = \mu. \tag{58}$$

We set the initial guess:

$$\mathbf{y}_{\widehat{\Gamma},0} = 0. \tag{59}$$

The Chebyshev acceleration is defined by, see [3],

$$\mathbf{y}_{\widehat{\Gamma},1} = \mathbf{y}_{\widehat{\Gamma},0} + \alpha \mathbf{z}_0, \tag{60}$$

$$\mathbf{y}_{\widehat{\Gamma},k+1} = \mathbf{y}_{\widehat{\Gamma},k-1} + \omega_{k+1}(\alpha \mathbf{z}_k + \mathbf{y}_{\widehat{\Gamma},k} - \mathbf{y}_{\widehat{\Gamma},k-1}), \quad k = 1, 2, \cdots, \tag{61}$$

where

$$\mathbf{r}_k = \mathbf{h}_{\widehat{\Gamma}} - \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right) \mathbf{y}_{\widehat{\Gamma},k}, \tag{62}$$

$$\mathbf{z}_k = \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right) \mathbf{r}_k, \tag{63}$$

and

$$\omega_{k+1} = 2\mu \frac{c_k}{c_{k+1}}. \tag{64}$$

6.2 Error analysis

Let $\mathbf{e}_k = \mathbf{y}_{\widehat{\Gamma}} - \mathbf{y}_{\widehat{\Gamma},k}$. Using (59), (60), and (61), we obtain

$$\mathbf{e}_{k+1} = \omega_{k+1} Q \mathbf{e}_k + (1 - \omega_{k+1}) \mathbf{e}_{k-1}, \tag{65}$$

with

$$\mathbf{e}_0 = \mathbf{y}_{\widehat{\Gamma}}, \text{ and } \mathbf{e}_1 = Q \mathbf{e}_0, \tag{66}$$

where

$$Q = I - \alpha \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right) \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right). \tag{67}$$

The symmetrized operator $\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right) \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}}$ has the following eigenvalue decomposition:

$$\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right) \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} = P \Lambda P^T, \tag{68}$$

where Λ is a diagonal matrix and the eigenvalues $\{\lambda_j\}$ of $\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} \left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right) \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}}$ are its diagonal entries. P is an orthogonal matrix, and P^T is its transpose.

Let

$$P_1 = \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} P. \tag{69}$$

We note that

$$\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right) \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right) = P_1 \Lambda P_1^{-1}. \tag{70}$$

Then, we have,

$$Q = P_1 \Sigma P_1^{-1}, \tag{71}$$

where Σ is a diagonal matrix with the eigenvalues $\{\sigma_j\}$ of Q on the diagonal and

$$\sigma_j = 1 - \alpha \lambda_j. \tag{72}$$

Let

$$\mathbf{f}_k = c_k P_1^{-1} \mathbf{e}_k. \tag{73}$$

If we substitute (73) into (65) and (66), we then obtain a diagonal system of difference equations by using (57), (58), (64), and (71):

$$\mathbf{f}_{k+1} = 2\mu \Sigma \mathbf{f}_k - \mathbf{f}_{k-1}, \quad k = 1, 2, \cdots, \tag{74}$$

with

$$\mathbf{f}_1 = \mu \Sigma \mathbf{f}_0, \text{ and } \mathbf{f}_0 = P_1^{-1} \mathbf{y}_{\widehat{\Gamma}}. \tag{75}$$

Solving this system, see [3], we obtain

$$\mathbf{f}_k = \Theta P_1^{-1} \mathbf{y}_{\widehat{\Gamma}}, \quad k = 1, 2, \cdots, \tag{76}$$

where Θ is a diagonal matrix with $\cosh\left(k \cosh^{-1}(\mu \sigma_j)\right)$ on its diagonal.

Using (73), we obtain:

$$\mathbf{e}_k = \left(P_1 \Theta P_1^{-1}\right) \frac{\mathbf{y}_{\widehat{\Gamma}}}{c_k}, \quad k = 1, 2, \cdots. \tag{77}$$

Using the definition of $\mathbf{e}_k$, our approximate solution after k Chebyshev iterations is given by

$$\mathbf{y}_{\widehat{\Gamma},k} = P_1 J P_1^{-1} \mathbf{y}_{\widehat{\Gamma}}, \tag{78}$$

where J is a diagonal matrix with $1 - \frac{\cosh\left(k \cosh^{-1}(\mu \sigma_j)\right)}{c_k}$ on its diagonal.

Using (57) and (58), we obtain

$$c_k = \cosh\left(k \cosh^{-1}(\mu)\right).$$

Therefore, we have $1 - \frac{\cosh\left(k \cosh^{-1}(\mu \sigma_j)\right)}{\cosh\left(k \cosh^{-1}(\mu)\right)}$ as the diagonal entries of the matrix J.

From (77), we know that the Chebyshev iteration converges if and only if $|\sigma_j| < 1$, i.e., $0 < \lambda_j < l + u$. Since $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right) \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right)$ has a smallest eigenvalue 1, we can guarantee that $0 < \lambda_j$ and choose $l = 1$. From our analysis above, we get an upper bound for λ_j and can choose u to guarantee that $\lambda_j < l + u$.

Since we choose u such that $\lambda_j < l + u$, we find that $1 - \frac{\cosh\left(k\cosh^{-1}(\mu\sigma_j)\right)}{\cosh\left(k\cosh^{-1}(\mu)\right)} > 0$, i.e., J has positive diagonal elements.

6.3 Condition number estimate for the second new preconditioner

We begin with a lemma.

Lemma 9 *Given any $\mathbf{u_\Gamma} \in \widehat{\mathbf{W}}_\Gamma$, let $\mathbf{\Psi} = \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma$. If we choose u such that $\lambda_j < u+l$, then there exist two constants $C_1(k)$ and $C_2(k)$ that*

$$C_1(k)\mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi} \le \mathbf{\Psi}^T \widehat{S}_\Pi^{-1}\mathbf{\Psi}^T \le C_2(k)\mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi}, \tag{79}$$

where

$$C_1(k) = \min_j \left(1 - \frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right), \tag{80}$$

and

$$C_2(k) = \max_j \left(1 - \frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right). \tag{81}$$

Proof: Using (54), (19), and (20), we have,

$$\begin{aligned}
\mathbf{\Psi}^T \widehat{S}_\Pi^{-1}\mathbf{\Psi} &= \sum_{i=1}^N \Psi_{\widehat{I}}^{(i)^T}\mathbf{y}_{\widehat{I},k}^{(i)} + \mathbf{\Psi}_{\widehat{\Gamma}}^T \mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)}\mathbf{y}_{\widehat{\Gamma},k})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^N \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}I}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}^{(i)}\right)^T \mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\mathbf{y}_{\widehat{\Gamma},k} \\
&= \sum_{i=1}^N \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\mathbf{y}_{\widehat{\Gamma},k}.
\end{aligned} \tag{82}$$

Comparing (82) with (46), we only need to compare $\mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\mathbf{y}_{\widehat{\Gamma}}$ and $\mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\mathbf{y}_{\widehat{\Gamma},k}$.

Using (78) and (69), we obtain

$$\begin{aligned}
&\mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\mathbf{y}_{\widehat{\Gamma},k} \\
=\ &\mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)P_1 J P_1^{-1}\mathbf{y}_{\widehat{\Gamma}} \\
=\ &\mathbf{y}_{\widehat{\Gamma}}^T\left(\widehat{R}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{\frac{1}{2}} PJP^{-1}\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)^{-\frac{1}{2}}\mathbf{y}_{\widehat{\Gamma}}.
\end{aligned} \tag{83}$$

Let $\mathbf{Y}_{\widehat{\Gamma}} = P^T \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{-\frac{1}{2}} \mathbf{y}_{\widehat{\Gamma}}$. Using (68), we have,

$$\begin{aligned}
& \mathbf{y}^T_{\widehat{\Gamma}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} P^T \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} P \mathbf{Y}_{\widehat{\Gamma}} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} P^T P \Lambda P^T P \mathbf{Y}_{\widehat{\Gamma}} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} \Lambda \mathbf{Y}_{\widehat{\Gamma}},
\end{aligned} \tag{84}$$

and, using (83), we have

$$\begin{aligned}
& \mathbf{y}^T_{\widehat{\Gamma}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} P^T \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}} \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \right)^{\frac{1}{2}} P J \mathbf{Y}_{\widehat{\Gamma}} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} P^T P \Lambda P^T P J \mathbf{Y}_{\widehat{\Gamma}} \\
= \;& \mathbf{Y}^T_{\widehat{\Gamma}} \Lambda J \mathbf{Y}_{\widehat{\Gamma}}.
\end{aligned} \tag{85}$$

Under our assumption, J is a diagonal matrix with positive diagonal entries $\left(1 - \frac{\cosh(k \cosh^{-1}(\mu\sigma_j))}{\cosh(k \cosh^{-1}(\mu))} \right)$. Thus, we have,

$$C_1(k) \mathbf{y}^T_{\widehat{\Gamma}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}} \le \mathbf{y}^T_{\widehat{\Gamma}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma},k} \le C_2(k) \mathbf{y}^T_{\widehat{\Gamma}} \left(\widehat{R}^T_{\widehat{\Gamma}} \widetilde{T} \widehat{R}_{\widehat{\Gamma}} \right) \mathbf{y}_{\widehat{\Gamma}}. \tag{86}$$

$\square$

Lemma 10 *Given any* $\mathbf{u}_{\boldsymbol{\Gamma}} \in \widehat{\mathbf{W}}_\Gamma$*,*

$$C_1(k) \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \widehat{M}^{-1} \mathbf{u}_\Gamma \le C_2(k) \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma, \tag{87}$$

where $C_1(k)$ *and* $C_2(k)$ *are defined in (80) and (81), respectively.*

Proof: We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\begin{aligned}
& \mathbf{u}_\Gamma^T \widehat{M}^{-1} \mathbf{u}_\Gamma \\
= \;& \mathbf{u}_\Gamma^T R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta} \begin{pmatrix} \mathbf{0} & R^{(i)^T}_\Delta \end{pmatrix} \begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R^{(i)}_\Delta \end{pmatrix} R_{\Gamma\Delta} + \Phi \widehat{S}_\Pi^{-1} \Phi^T \right\} D_\Gamma R_\Gamma \mathbf{u}_\Gamma \\
= \;& \mathbf{u}_\Gamma^T R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^N R^T_{\Gamma\Delta} \begin{pmatrix} \mathbf{0} & R^{(i)^T}_\Delta \end{pmatrix} \begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R^{(i)}_\Delta \end{pmatrix} R_{\Gamma\Delta} \right\} \mathbf{u}_\Gamma \\
& + \mathbf{u}_\Gamma^T R_\Gamma^T D_\Gamma \Phi \widehat{S}_\Pi^{-1} \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma.
\end{aligned} \tag{88}$$

Comparing this expression with (52), we obtain the result by using Lemma 9.

Table 1: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$

Num. of Subregions	Iterations	Condition number
4×4	12	3.04
8×8	15	3.45
12×12	17	3.53
16×16	17	3.56
20×20	17	3.57

Table 2: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the number of subdomains, 4×4 subregions and $\frac{H}{h} = 4$

$\frac{\hat{H}}{H}$	Iterations	Condition number
4	12	3.04
8	13	4.17
12	13	4.96
16	14	5.57
20	15	6.08

$\square$

Theorem 2 *The condition number using the three-level preconditioner $\widehat{M}^{-1}$is bounded by $C\frac{C_2(k)}{C_1(k)}(1+\log(\frac{H}{h}))^2$, where $C_1(k)$ and $C_2(k)$ are defined in (80) and (81), respectively.*

Proof: Combining the condition number bound, given in Lemma 5, for the two-level BDDC method and Lemma 10, we find that the condition number for the system with the three-level preconditioner $\widehat{M}^{-1}$ is bounded by $C\frac{C_2(k)}{C_1(k)}(1+\log(\frac{H}{h}))^2$.

$\square$

7 Numerical experiments

We have applied our two three-level BDDC algorithms to the model problem (1), where $\Omega = [0,1]^2$. We decompose the unit square into $\widehat{N} \times \widehat{N}$ subregions with the sidelength $\hat{H} = 1/\widehat{N}$ and each subregion into $N \times N$ subdomains with the sidelength $H = \hat{H}/N$. Equation (1) is discretized, in each subdomain, by conforming piecewise linear elements with a finite element diameter h. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of 10^{-8}.

Table 3: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the size of subdomain problems, 4×4 subregions and 4×4 subdomains

$\frac{\hat{H}}{h}$	Iterations	Condition number
4	12	3.04
8	15	4.08
12	16	4.80
16	17	5.36
20	19	5.83

Table 4: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$

Num. of Subregions	Iterations	Condition number
4×4	11	1.81
8×8	11	1.82
12×12	12	1.82
16×16	12	1.82
20×20	12	1.82

Table 5: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the number of subdomains, 4×4 subregions and $\frac{H}{h} = 4$

$\frac{\hat{H}}{H}$	Iterations	Condition number
4	11	1.81
8	12	1.85
12	12	1.88
16	12	1.89
20	12	1.91

Table 6: Eigenvalue bounds and iteration counts for the preconditioner $\widetilde{M}$ with a change of the size of subdomain problems, 4×4 subregions and 4×4 subdomains

$\frac{\hat{H}}{h}$	Iterations	Condition number
4	11	1.81
8	14	2.50
12	16	3.00
16	17	3.35
20	18	3.65

Table 7: Eigenvalue bounds and iteration counts for the preconditioner $\widehat{M}$, $u = 3.2$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{\hat{H}}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	20	0.4762	0.4829	2.7110	5.6141
2	13	0.8410	0.8540	1.8820	2.2038
3	11	0.9548	0.9981	1.9061	1.9098
4	11	0.9872	1.0019	1.8663	1.8629
5	11	0.9964	1.0006	1.8551	1.8541

Table 8: Eigenvalue bounds and iteration counts for the preconditioner $\widehat{M}$, $u = 4$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{\hat{H}}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	22	0.4000	0.4053	2.3027	5.6821
2	14	0.7805	0.7909	1.9687	2.4892
3	12	0.9260	0.9781	1.9382	1.9816
4	11	0.9753	1.0028	1.8891	1.8837
5	11	0.9918	1.0026	1.8787	1.8739

Table 9: Eigenvalue bounds and iteration counts for the preconditioner $\widehat{M}$, $u = 6$, 4×4 subregions, $\frac{\hat{H}}{H} = 16$ and $\frac{\hat{H}}{h} = 4$

k	Iterations	$C_1(k)$	λ_{min}	λ_{max}	Condition number
1	24	0.2857	0.2899	1.8287	6.3086
3	16	0.6575	0.6670	2.3435	3.5134
3	12	0.8524	0.9286	1.9628	3.1136
4	12	0.9377	0.9795	1.9850	2.0266
5	12	0.9738	0.9983	1.9403	1.9437

We have carried out three different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of the experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $\rho \equiv 1$. Table 1 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 2 gives the results with a change of the number of subdomains. Table 3 gives the results with a change of the size of the subdomain problems.

In the second set of the experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $\rho = 1$ in one subregion and $\rho = 101$ in the neighboring subregions, i.e., in a checkerboard pattern. Table 4 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 5 gives the results with a change of number of subdomains. Table 6 gives the results with a change of the size of the subdomain problems.

In the third set of the experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $\rho \equiv 1$. We use the PCG to estimate the largest eigenvalue of $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$ which is approximately 3.2867. And if we have 64 subdomains and $\frac{H}{h} = 4$ for the two-level BDDC, we have a condition number estimate of 1.8380. We select different values of u and k to see how the condition number changes. We take $u = 3.2$ in Table 7. We also give an estimate for $C_1(k)$ for $k = 1, 2, 3, 4, 5$. From Table 7, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and the condition number estimate becomes closer to 1.8380, the value in the two-level case, as k increases.

We take $u = 4$ in Table 8 and $u = 6$ in Table 9. From these two tables, we see that if we can get more precise estimate for the largest eigenvalue of $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$, we need fewer Chebyshev iterations to get a condition number, similar to that of the two-level case. However, the iteration count is not very sensitive to the choice of u.

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