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Balancing Neumann-Neumann Preconditioners for the Mixed Formulation of Almost-Incompressible Linear Elasticity

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To Thalya and our baby she is now carrying.

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Abstract

Balancing Neumann-Neumann methods are extended to the equations arising from the mixed formulation of almost-incompressible linear elasticity problems discretized with discontinuous-pressure finite elements. This family of domain decomposition algorithms has previously been shown to be effective for large finite element approximations of positive definite elliptic problems. Our methods are proved to be scalable and to depend weakly on the size of the local problems. Our work is an extension of previous work by Pavarino and Widlund on BNN methods for Stokes equation.

Our iterative substructuring methods are based on the partition of the unknowns into *interior* ones — including interior displacements and pressures with zero average on every subdomain — and *interface* ones — displacements on the geometric interface and constant-by-subdomain pressures. The restriction of the problem to the interior degrees of freedom is then a collection of decoupled local problems that are well-posed even in the incompressible limit. The interior variables are eliminated and a hybrid preconditioner of BNN type is designed for the Schur complement problem. The iterates are restricted to a *benign* subspace, on which the preconditioned operator is positive definite, allowing for the use of conjugate gradient methods.

A complete convergence analysis of the method is presented for the constant coefficient case. The algorithm is extended to handle discontinuous coefficients, but a full analysis is not provided. Extensions of the algorithm and of the analysis are also presented for problems combining pure-displacement and mixed finite elements in different subregions. An algorithm is also proposed for problems with continuous discrete pressure spaces.

All the algorithms discussed have been implemented in parallel codes that have been successfully tested on large sample problems on large parallel computers; results are presented and discussed. Implementations issues are also discussed, including a version of our main algorithm that does not require the solution of any auxiliary saddle-point problem since all subproblems of the

preconditioner can be reduced to solving symmetric positive definite linear systems.

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Chapter 1

Introduction

1.1 Overview

The process of numerically solving partial differential equations (PDEs) typically involves a discretization step, in which the original infinite-dimensional, continuous problem is approximated by a finite-dimensional one. This discretized problem usually requires the solution of a linear system of algebraic equations (or a nonlinear system, the iterative solution of which often requires a linear solve at each iteration). Therefore, the solution of linear systems of equations is one of the most ubiquitous problems in scientific computing.

How well the discretized problem approximates the continuous one usually depends on the dimension of the solution space, expressed by a mesh parameter h that measures how fine the mesh is. This is especially true when complex geometries are involved. Typically, as the dimension of the discretization space increases, the resulting problems are not only larger, but also worse conditioned.

The problem of solving linear systems is central in numerical analysis. Systems arising from the discretization of PDEs have, of course, received special attention, since they appear in many applications, such as fluid dynamics and structural mechanics. These systems often have special structure that can be exploited in the solution process. Unfortunately, the price to be paid for this exploitation is the design of problem-specific algorithms, as the one we present, as opposed to truly multi-purpose ones such as Gaussian elimination and its variants. In particular, we will present and discuss an algorithm to solve the system arising from the discretization of the mixed formulation of linear elasticity.

Nowadays, when the size of problems get into the hundreds of millions of unknowns and the largest supercomputers have thousands of processors, the issue of parallel scalability is paramount. A solution method is said to be scalable if the time required to solve a problem of size nS using nN processors is about the same as for a similar problem of size S using N processors. Our methods are scalable and their performance depends only weakly (polylogarithmically) on the size of the local problems (i.e., on S).

There are two classes of solvers for linear systems: direct methods and iterative methods. The latter have proved more successful for the solution of large problems in parallel computational environments. We now briefly describe what is arguably the most important class of iterative methods: the Krylov subspace methods. We also motivate the necessity of designing good preconditioners.

1.1.1 Krylov Subspace Methods

Let K be an invertible matrix and consider the linear system

$$Ku = f.$$

At the n-th iteration, define the error $e_n = u_* - u_n$ and the residual $r_n = Ke_n = f - Ku_n$, where $u_* = K^{-1}f$. If u_0 is an initial guess, we define the Krylov affine subspaces by

$$\mathcal{K}_n = u_0 + \text{span}\left\{r_0, \ldots, K^{n-1}r_0\right\}.$$

The iterates of a Krylov method are defined in terms of an optimization problem constrained to the Krylov subspace. In the case when K is symmetric and positive definite, we define the Conjugate Gradient method, in which the n-th iterate u_n is the solution to

$$\begin{aligned} \text{minimize} \quad & e_n^T K e_n \\ \text{subject to} \quad & u_n \in \mathcal{K}_n. \end{aligned}$$

It can be shown that

$$\|e_n\|_K \leq 2\left(\frac{\sqrt{\kappa(K)}-1}{\sqrt{\kappa(K)}+1}\right)^n \|e_0\|_K, \tag{1.1}$$

where $\kappa(K)$, the condition number of K, is the ratio of the largest and smallest eigenvalues of K (see, e.g., Golub [25] or Luenberger [37]). In each iteration of

the conjugate gradient method, the matrix K is used only in one matrix-vector product. The matrix K does not have to be explicitly available, i.e., we do not need to know the values of its individual entries. We only need to be able to compute the action of the matrix on vectors.

1.1.2 Preconditioning

The performance of Krylov and many other iterative methods depend strongly on the condition number of the matrix being "inverted", as indicated by the error bound (1.1) for the conjugate gradient method. Therefore it often is necessary to precondition the linear systems when using a iterative method: instead of solving $Ku = f$, we solve $QKu = Qf$, where Q is an invertible matrix that approximates K^{-1}, in the sense that QK is better conditioned than K. In fact, the use of an unpreconditioned conjugate gradient method for a large linear system arising in elasticity can easily result in hundreds of thousands of iterations without any visible progress towards the solution. Typically, a preconditioned Krylov method will require one application of K and one application of Q per iteration. The need of a good approximation for the inverse of K must be weighed against the requirement that multiplying a vector by Q should be computationally cheap, as compared with "multiplication by K^{-1}". In the case of preconditioned conjugate gradients, Q is also required to be symmetric positive definite and we have

$$\|e_n\|_K \leq 2 \left(\frac{\sqrt{\kappa(QK)} - 1}{\sqrt{\kappa(QK)} + 1} \right)^n \|e_0\|_K .$$

1.1.3 Domain Decomposition Methods

Domain decomposition methods are techniques to design efficient and scalable preconditioners for certain classes of linear systems arising from the discretization of PDEs. Their basic idea is to construct an approximate solution for the original problem based on the solution of a collection of smaller instances of the problem, posed on subsets of the domain called *subdomains*.

To fix ideas, let us consider a discretization of the Poisson equation with homogeneous boundary condition: *find* $u \in \mathbf{V}$ *such that*

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V,$$

where $f \in H^{-1}(\Omega)$, $V \subset H_0^1(\Omega)$ is finite-dimensional and $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$. We assume that there are subspaces $V_0, V_1, \ldots, V_N \subset V$, the sum of which

spans V,

$$V = V_0 + V_1 + \cdots + V_N,$$

and inner products $b_i : V_i \times V_i \to \mathbb{R}$ defined on them. We then define projection-like operators $T_i : V \to V_i$ by:

$$b_i(T_i u, v_i) = a(u, v_i) \ \ \forall v_i \in V_i, \ \forall u \in V. \tag{1.2}$$

We remark that the inclusion of the V_i's in V is not absolutely necessary; nonnested subspaces can also be handled with the use of restriction and interpolation operators.

If $b_i(\cdot, \cdot)$ is chosen to be the restriction of $a(\cdot, \cdot)$ to V_i, then T_i is simply the a-orthogonal projection onto V_i. The subspaces V_i are usually comprised of functions supported in small subsets of Ω, denoted by Ω_i, the subdomains. Often V_0 is reserved to be a subspace of a different nature: it represents a lower-dimensional discretization of the entire domain Ω and is called the coarse space. Such a space is typically necessary for the scalability of the method.

We define the operators (matrices) $A : V \to V'$ and $B_i : V_i \to V_i'$ by:

$$\begin{aligned} \langle Au, v \rangle &= a(u, v) \ \ \forall u, v \in V, \\ \langle B_i u_i, v_i \rangle &= b_i(u_i, v_i) \ \ \forall u_i, v_i \in V_i. \end{aligned}$$

We note that the computation of the action of T_i on a vector just requires the solution of a linear system involving B_i. We also note that we can easily compute $T_i u$, the action of T_i on the unknown solution u, since in this case the right hand side in (1.2) equals $\langle f, v_i \rangle$.

Domain decomposition methods can usually be described as the iterative solution of a problem of the form

$$Tu = g,$$

where T, the Schwarz operator, is a polynomial on the T_i's. The additive Schwarz method is then defined by

$$T_{\text{as}} = T_0 + T_1 + \cdots + T_N$$

and the multiplicative Schwarz method by

$$T_{\text{ms}} = I - (I - T_0)(I - T_1) \cdots (I - T_N).$$

Other alternatives, combining additive and multiplicative components, are also possible and give rise to hybrid methods.

The three most successful families of domain decomposition methods are overlapping Schwarz, Balancing Neumann-Neumann (BNN), and FETI methods.

Overlapping Schwarz methods are based on local spaces V_i supported on subdomains Ω_i' which overlap each other: $\Omega_i' \cap \Omega_j' \neq \emptyset$ for neighboring subdomains Ω_i' and Ω_j'. This is in contrast with the iterative substructuring class, including BNN and FETI, in which the subdomains are disjoint.

BNN methods were first developed without a coarse space; see Bourgat, Glowinski, Le Tallec and Vidrascu [5], De Roeck [13], and De Roeck and Le Tallec [14]. They were later significantly improved by the addition of a coarse level; see Dryja and Widlund [15], Le Tallec [33], and Mandel and Brezina [39]. The work in [15], as well as some recent work by Mandel and Dohrmann [40], concerns additive methods, rather than the more typical hybrid type of BNN methods, which we adopt in this work.

Unlike for the BNN family, the iterates of FETI methods are discontinuous across the interface between subdomains; continuity is only achieved upon convergence of the iteration. The iteration is written in terms of Lagrange multipliers that enforce the continuity across the interface. FETI and BNN share many algorithmic components, such as the static condensation of interior variables and the use of local solvers for both Neumann and Dirichlet problems on each subdomain. Connections between these two approaches are discussed by Klawonn and Widlund [31]. Gosselet, Rey and Rixen [26] show that with a proper initialization for FETI, FETI and BNN methods perform equally well for a class of problems.

1.2 Notation

Before proceeding, we will introduce some of the notation and conventions that we will use throughout this dissertation.

We will adopt the following convention for bilinear forms:

- inner products will be denoted by parentheses, as in $(\cdot, \cdot)_\Gamma$;
- associated with any square matrix S, we will define the bilinear form $\langle \cdot, \cdot \rangle_S$ given by $\langle \mathbf{u}, \mathbf{v} \rangle_S = \mathbf{v}^T S \mathbf{u}$.

We will use the same notation for finite-element functions and their representation in terms of a finite element basis. In the course of a proof, for instance, $\mathbf{u}$ might alternately represent an element of a finite dimensional subspace of $\left(H_0^1(\Omega)\right)^d$ or a vector in $\mathbb{R}^{n_{\text{dof}}}$, with no further warning.

1.3 Model Problems

The matrices arising from the discretization of elliptic equations or systems form one of the most studied classes of matrices and one for which a great number of preconditioners are available. In particular, Balancing Neumann-Neumann preconditioners for these problems are described in Mandel [38] and Mandel and Brezina [39].

In this dissertation, we consider instead a class of saddle-point or penalized saddle-point problems. Our main focus will be on the equations of almost-incompressible linear elasticity with a mixed formulation. In this section, however, we also introduce the related problems of Stokes equation, compressible elasticity with pure-displacement formulation, as well as mixed finite element methods for incompressible elasticity.

1.3.1 Stokes Equations

The steady Stokes equations, which model the steady-state of flows with very low Reynolds numbers, is expressed in variational terms as follows: *given* $\mathbf{f} \in (H^{-1}(\Omega))^d$ *and* $\mathbf{g} \in \left(H^{1/2}(\partial\Omega)\right)^d$ *satisfying*

$$\int_{\partial\Omega} \mathbf{g} \cdot \hat{\mathbf{n}} = 0, \tag{1.3}$$

find $\mathbf{u} \in (H^1(\Omega))^d$ *and* $p \in L^2_0(\Omega)$ *such that* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega$ *and*

$$\left\{ \begin{array}{rcccll} \nu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \langle \mathbf{f}, \mathbf{v} \rangle & \forall \mathbf{v} \in (H^1_0(\Omega))^d \\ b(\mathbf{u},q) & & & = & 0 & \forall q \in L^2_0(\Omega). \end{array} \right. \tag{1.4}$$

Here, Ω is a bounded and connected domain in $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\partial\Omega$, the bilinear forms are defined as

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}, \\ b(\mathbf{u},p) &= -\int_\Omega p \, \mathrm{div}\mathbf{u}, \end{aligned}$$

and the kinematic viscosity ν is a positive parameter.

The existence and uniqueness of the solution for problem (1.4) is well known; see, e.g., Girault and Raviart [23, Theorem I.5.1]. This result is related

to the inf-sup stability of the spaces $(H_0^1(\Omega))^d$ and $L_0^2(\Omega)$, i.e., the inequality

$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{v}\in\left(H_0^1(\Omega)\right)^d} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{H^1}\|q\|_{L^2}} = \beta(\Omega) > 0.$$

A discrete counterpart of this inequality plays an important role in the design of mixed finite-element methods.

Remark 1.1 *Even when the compatibility condition (1.3) is violated, problem (1.4) is still well-posed; in this case the solution presents constant (but not necessarily zero) divergence.*

1.3.2 Linear Elasticity

Let $\partial\Omega = \partial\Omega_D + \partial\Omega_N$ (the + sign indicates disjoint union). We assume that an elastic body Ω is subject to a body force $\mathbf{f}$ and to a surface force $\mathbf{h}$ acting on $\partial\Omega_N$, and that a displacement $\mathbf{g}$ is prescribed on $\partial\Omega_D$.

We define the Sobolev space $H^1_{\partial\Omega_D}(\Omega) = \{v \in H^1(\Omega) \mid v|_{\partial\Omega_D} = 0\}$. The compressible or almost-incompressible linear elasticity problem, in its *pure-displacement* formulation, is then of the form: *given* $\mathbf{f} \in (H^{-1}(\Omega))^d$, $\mathbf{g} \in (H^{1/2}(\partial\Omega_D))^d$ *and* $\mathbf{h} \in (H^{-1/2}(\partial\Omega_N))^d$, *find* $\mathbf{u} \in (H^1(\Omega))^d$ *such that* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega_D$ *and*

$$2\mu \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \lambda \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v} = \langle \mathbf{F}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \left(H^1_{\partial\Omega_D}(\Omega)\right)^d, \qquad (1.5)$$

where

$$\begin{aligned} \langle \mathbf{F}, \mathbf{v}\rangle &= \langle \mathbf{f}, \mathbf{v}\rangle + \langle \mathbf{h}, \mathbf{v}|_{\partial\Omega_N}\rangle, \\ \varepsilon_{ij}(\mathbf{u}) &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ and} \\ \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) &= \sum_{i=1}^d \sum_{j=1}^d \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}). \end{aligned}$$

Here the positive parameters λ and μ are the Lamé constants and they are related to the Poisson ratio ν and Young's modulus E by the following formulas:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

As the Poisson ratio approaches 0.5 (i.e., μ/λ approaches zero), corresponding to almost-incompressible materials, finite element discretizations of this pure displacement formulation suffer from *locking* (see, e.g., [6, 9]). As a result, the stiffness matrices get increasingly ill-conditioned and the quality of the discrete solution deteriorates. A well-known remedy is the *mixed formulation* obtained by the introduction of the pressure $p = -\lambda \mathrm{div}\mathbf{u}$ as a new, additional variable (see, e.g., [10]): *given* $\mathbf{f} \in (H^{-1}(\Omega))^d$, $\mathbf{g} \in (H^{1/2}(\partial\Omega_D))^d$ *and* $\mathbf{h} \in (H^{-1/2}(\partial\Omega_N))^d$, *find* $\mathbf{u} \in (H^1(\Omega))^d$ *and* $p \in L^2(\Omega)$ *such that* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega_D$ *and*

$$\left\{ \begin{array}{rcrcll} \mu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \langle \mathbf{F}, \mathbf{v} \rangle & \forall \mathbf{v} \in H^1_{\partial\Omega_D}(\Omega) \\ b(\mathbf{u},q) & - & \frac{1}{\lambda} c(p,q) & = & 0 & \forall q \in L^2(\Omega). \end{array} \right. \tag{1.6}$$

Here

$$\begin{array}{rcl} a(\mathbf{u},\mathbf{v}) & = & 2\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}), \\ c(p,q) & = & \int_\Omega pq, \end{array} \tag{1.7}$$

and $b(\cdot,\cdot)$ and $\langle \mathbf{F}, \cdot \rangle$ are defined as before.

When $\partial\Omega_D$ has positive measure, the existence and uniqueness of the solution to problems (1.5) and (1.6) is guaranteed by the ellipticity of the form $a(\cdot,\cdot)$, expressed by Korn's inequality (see, e.g., [9, Corollary 9.2.22]). We will also need to consider problems with natural boundary conditions on the entire border (i.e., $\partial\Omega_N = \partial\Omega$). In this case, the bilinear form $a(\cdot,\cdot)$ has a nontrivial nullspace, comprised by the rigid-body modes (rotations and translations; a three-dimensional space in $\mathbb{R}^2$ and six-dimensional space in $\mathbb{R}^3$). A solution exists only when the following compatibility condition is satisfied:

$$\langle \mathbf{F}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \ker(a).$$

An ideally incompressible material has $\lambda = \infty$ and is modelled by replacing $\frac{1}{\lambda} c(p,q)$ with 0 in (1.6). The existence and uniqueness results are the same as above, except for the case $\partial\Omega_D = \partial\Omega$, when the pressure is defined only up to an additive constant and a solution exists only when

$$\int_{\partial\Omega} \mathbf{g} \cdot \hat{\mathbf{n}} = 0.$$

Remark 1.2 *Existence and uniqueness of the solution are guaranteed with no compatibility condition, even in the incompressible case, if* $\partial\Omega_D$ *has positive measure and the pressure space is restricted to* $L^2_0(\Omega)$*, rather than* $L^2(\Omega)$*; cf. Remark 1.1.*

The following useful result shows the equivalence between the Stokes and mixed elasticity bilinear forms.

Lemma 1.3 *There exists a constant $c > 0$ such that*

$$c\|\nabla \mathbf{u}\|_{L^2(\Omega)} \leq \|\epsilon(\mathbf{u})\|_{L^2(\Omega)} \leq \|\nabla \mathbf{u}\|_{L^2(\Omega)}, \qquad \forall \mathbf{u} \in (H^1(\Omega))^d,\ \mathbf{u} \perp \mathit{ker}(a).$$

Here $\|\epsilon(\mathbf{u})\|^2_{L^2(\Omega)} = \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{u}) dx$.

The requirement $\mathbf{u} \perp \mathit{ker}(a)$ *is not necessary for the upper bound.*

Proof The lower bound is a version of Korn's inequality (see Klawonn and Widlund [32, Lemma 4]). The upper bound follows from elementary estimates:

$$\begin{aligned}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 &\leq \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{1}{2}\left(\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_j}{\partial x_i}\right)^2\right) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} \left(\frac{\partial u_i}{\partial x_j}\right)^2.\end{aligned}$$

□

1.3.3 The Incompressible Limit

The focus of this dissertation is on problems of almost-incompressible elasticity. We will often explicitly use the fact that λ is finite in proofs and algorithms. We feel that this approach is justifiable and does not limit the application of the methods described here, for a variety of reasons.

The algorithm we propose is built of components that are well defined in the incompressible limit. Only minimal modifications are required to handle $\lambda = \infty$ and the theory can accommodate the ideally incompressible problem, as well; cf. Pavarino and Widlund [46] on the related incompressible Stokes problem.

Real materials are somewhat compressible; perfectly incompressible materials are just a mathematical idealization.

Most important, the preconditioners developed for almost-incompressible problems can be used to precondition the matrix of the perfectly incompressible problem (and vice-versa), since the two problems are spectrally equivalent; see Pavarino and Widlund [45, Section 4].

1.4 Some Solution Methods for Saddle-Point Problems

Before we proceed with the presentation of our method, we briefly discuss some other approaches for the solution of saddle-point problems; we will not provide an exhaustive discussion.

The matrices that concern us, arising from Stokes, penalized Stokes and (almost-)incompressible elasticity, are of the form

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix},$$

where the block C might be zero (for incompressible problems). In Uzawa's algorithms, the $(1,1)$-block A is eliminated, creating a reduced system for the pressure unknowns only, with a matrix $C + B^T A^{-1} B$. An iterative procedure based on the gradient method or conjugate directions is then applied. The multiplication by A^{-1} is avoided in inexact Uzawa's methods, with a preconditioner applied instead. Used in conjuction with penalty methods, Uzawa's algorithm gives rise to the Augmented Lagrangian method; see, e.g., Fortin and Glowinski [21].

Bramble and Pasciak [8] introduced an inner-product that transforms the indefinite problem into a positive definite one. Conjugate gradients can then be applied.

Block preconditioners are considered by Klawonn in [27, 28]. It is shown that the condition number of the preconditioned system is bounded independently of discretization and penalty parameters, provided that the preconditioners of A and C are of a sufficiently good quality. Other works on block preconditioners include Elman and Silvester [16], Elman, Silvester and Wathen [17], Pavarino [43, 44], Rusten and Winther [50], and Silvester and Wathen [51].

Klawonn and Pavarino [29] introduced an overlapping Schwarz method, in which the local problems are restrictions of the original saddle point problem to the overlapping subdomains and the coarse problem is a saddle point problem associated with a coarse grid given by the subdomains. An analysis of this method is still missing. See also [30]. Other overlapping Schwarz methods have been considered by Fischer [18], Fischer, Miller and Tufo [19], Gervasio [22], and Rønquist [49].

Bramble and Pasciak [7] introduced an iterative substructuring method for the Stokes equation. The substructuring strategy they adopted is the same as our algorithm (cf. Chapter 3), but they suggested the use of a simpler block-

diagonal preconditioner to precondition the resulting Schur complement problem. Pavarino and Widlund [46] presented a Balancing Neumann-Neumann preconditioner for the incompressible Stokes equation, which can be viewed as the starting point for our current work. Li [35, 36] designed and analyzed a dual-primal FETI preconditioner for Stokes equation. His bounds for the condition number of the preconditioned operator are similar to ours and depend polylogarithmically on the size of the local problems. They also depend on the inf-sup constants of the finite element discretization and of the coarse space. Other iterative substructuring methods for Stokes have been studied by Ainsworth and Sherwin [1], Casarin [12], Fischer and Rønquist [20], Le Tallec and Patra [34], Marini and Quarteroni [41], Pasciak [42], Pavarino and Widlund [45], Quarteroni [47], and Rønquist [48].

1.5 Structure of this Dissertation

The remainder of this dissertation is organized as follows. In Chapter 2, we discuss the discretization of saddle-point problems by mixed finite elements and review some main results concerning inf-sup stability. In Chapter 3, we discuss the partition of the mixed space into interior and interface subspaces and define the Schur complement problem and a saddle-point extension operator necessary for the iterative substructuring processing. We then present in detail our Balancing Neumann-Neumann algorithm for saddle-point problems in Chapter 4, provide a full analysis of its spectral bounds in Chapter 5, and extend the algorithm to the case of heterogeneous coefficients with jumps in Chapter 6. Some implementation issues are discussed in Chapter 7, most importantly an implementation of our algorithm that relies only on the solution of positive definite subproblems. In Chapter 8, we extend our algorithm to handle the combined use of mixed and displacement-only formulations in different subregions of the domain and in Chapter 9 we propose a preconditioner for the continuous pressure case. Finally, in Chapter 10, we present a large set of numerical experiments that supports our theoretical results and illustrates the applicability of our methods.

Chapter 2

Mixed Finite Element Discretization

2.1 Continuous Formulation of Saddle-Point Problems

We now discuss the discretization of the problems introduced in Section 1.3 by mixed finite element methods and present some key results on existence, uniqueness, and stability of the solution. We first discuss the incompressible case and then examine the almost-incompressible case as a penalized problem.

Let $\mathbf{V}$ and U be Hilbert spaces, $a(\cdot,\cdot)$ a continuous bilinear form on the space $\mathbf{V} \times \mathbf{V}$ and $b(\cdot,\cdot)$ a continuous bilinear form on $\mathbf{V} \times U$. We define the linear transformations $A : \mathbf{V} \to \mathbf{V}'$, $B : \mathbf{V} \to U'$ and $B^T : U \to \mathbf{V}'$ by

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{v}\rangle &= a(\mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ \langle B\mathbf{u}, p\rangle &= b(\mathbf{u}, p) \quad \forall\, \mathbf{u} \in \mathbf{V}, p \in U, \\ \left\langle \mathbf{u}, B^T p\right\rangle &= b(\mathbf{u}, p) \quad \forall\, \mathbf{u} \in \mathbf{V}, p \in U. \end{aligned}$$

Then, given $\mathbf{f} \in \mathbf{V}'$ and $g \in U'$, the problem

$$\left\{ \begin{array}{rcccll} a(\mathbf{u}, \mathbf{v}) & + & b(\mathbf{v}, p) & = & \langle \mathbf{f}, \mathbf{v}\rangle & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) & & & = & \langle g, q\rangle & \forall q \in U \end{array} \right. \tag{2.1}$$

can be expressed as

$$\left\{ \begin{array}{rcccll} A\mathbf{u} & + & B^T p & = & \mathbf{f} & \text{in } \mathbf{V}' \\ B\mathbf{u} & & & = & g & \text{in } U'. \end{array} \right.$$

When $\mathbf{V}$ and U are finite dimensional, we have the matrix representation

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix}.$$

In the theory of mixed methods, a parameter of great importance is the *inf-sup condition* defined by

$$\beta = \inf_{q \in U} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}} \|q\|_U} > 0. \tag{2.2}$$

This condition is also known as *LBB condition*, named after Ladyzhenskaya, Babuška, and Brezzi. For the discretized problem, we will be interested in pairs of spaces for which the *inf-sup constant* β is uniformly bounded away from zero for any mesh size.

We have the following result (see, e.g., [6, Theorem III.4.3]).

Theorem 2.1 *If the bilinear form $a(\cdot, \cdot)$ is elliptic on* $\ker B$*, i.e.,*

$$a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|_{\mathbf{V}}^2 \quad \forall\, \mathbf{v} \in \ker B$$

and the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (2.2), then for any $\mathbf{f} \in \mathbf{V}'$ and $g \in U'$ the saddle point problem (2.1) has a unique solution, which satisfies the following stability estimates:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{V}} &\leq \frac{1}{\alpha}\|\mathbf{f}\|_{\mathbf{V}'} + \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{U'}, \\ \|p\|_U &\leq \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|\mathbf{f}\|_{\mathbf{V}'} + \frac{\|a\|}{\beta^2}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{U'}, \end{aligned}$$

where $\|a\| = \sup_{\mathbf{v} \in \mathbf{V}} \dfrac{a(\mathbf{v}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{V}}^2}$.

Almost-incompressible problems can be viewed as penalized versions of (2.1) and take the form

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) - t^2 c(p, q) = \langle g, q \rangle & \forall q \in U, \end{cases} \tag{2.3}$$

where t is a small parameter and $c(\cdot, \cdot)$ is a continuous bilinear form on $U \times U$. When considering almost-incompressible problems, we can use the following result (see [6, Theorem III.4.11]).

Theorem 2.2 *Suppose that the hypotheses of Theorem 2.1 are satisfied. In addition, assume $c(\cdot,\cdot)$ is coercive. Then, for any $\mathbf{f} \in \mathbf{V}'$ and $g \in U'$, problem (2.3) has a unique solution. Moreover, there is a constant C such that*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_U \leq C(\|\mathbf{f}\|_{\mathbf{V}'} + \|g\|_{U'}) \quad \forall \ \mathbf{f} \in \mathbf{V}', \ g \in U', \ 0 \leq t \leq 1.$$

What is essential in the theorem above is that the solution of the penalized problem is bounded *uniformly* on t. In problems of mixed almost-incompressible elasticity, the small parameter t will be replaced by the ratio μ/λ.

In Lemma 2.10, we will derive explicit bounds for the stability estimate above, in the case of finite-dimensional U and $\mathbf{V}$.

2.2 Mixed Finite Elements

Let h be the characteristic size of our finite element triangulation τ_h. Among the many choices of mixed finite elements available for our class of saddle-point problems, we consider the following:

- $Q_2 - P_1$

  The displacement space is composed of continuous, piecewise bi-quadratic (or tri-quadratic) functions, while the pressure space is discontinuous and consists of piecewise linear functions. This pair satisfies the inf-sup condition uniformly in h; see [10, Example VI.3.9].

- Taylor-Hood and *Broken* Taylor-Hood

  Taylor-Hood elements for quadrilaterals are the pair $Q_2 - Q_1$: the displacement space is composed of continuous, piecewise bi-quadratic functions, while the pressure space consists of continuous bilinear functions. This pair also satisfies the inf-sup condition uniformly on h; see [23, Corollary II.4.1].

  As we show in Chapter 3, our main algorithm requires that the characteristic function of each subdomain, which is obviously discontinuous, be part of the pressure space. To that end, we drop the continuity requirement only across the interface between subdomains. We will call the resulting space the *broken* Taylor-Hood finite element space. It is shown in [11] that this space is inf-sup stable as long as the pressure space is restricted to the functions that have zero average on every subdomain. We will use this broken space in Chapter 9, when considering

an extension of our preconditioner to problems with continuous pressure discretizations.

We note that while finite element methods based on hexahedra and quadrilaterals enjoy popularity, our theory applies equally well to stable mixed methods based on tetrahedra or triangles. Spectral elements $Q_n - Q_{n-2}$ could also be used; the methods and the theory discussed in this dissertation can be easily extended to that case; see [46], [24].

2.3 Some auxiliary results

The following result provides a useful characterization of the inf-sup constant as the smallest eigenvalue of a generalized eigenvalue problem. We consider the Stokes matrix

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

and note that $|\mathbf{u}|_{H^1}^2 = \mathbf{u}^T A \mathbf{u}$. Let C be the mass matrix for the pressures, i.e., $\|p\|_{L^2}^2 = p^T C p$. If we redefine the inf-sup constant using the H^1-seminorm instead of the H^1-norm for the displacements (due to Friedrichs inequality, they are equivalent), i.e.,

$$\beta = \inf_{q \in U} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{|\mathbf{v}|_{H^1} \|q\|_{L^2}},$$

we have the following result.

Lemma 2.3 $\beta^2 = \lambda_{\min}\left(C^{-1}(BA^{-1}B^T)\right)$.

Proof We have

$$\begin{aligned} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)^2}{|\mathbf{v}|_{H^1}^2} &= \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q^T B \mathbf{v})^2}{\mathbf{v}^T A \mathbf{v}} \\ &= \sup_{\tilde{\mathbf{v}} \in \mathbf{V}} \frac{\left((A^{-1/2} B^T q)^T \tilde{\mathbf{v}}\right)^2}{\tilde{\mathbf{v}}^T \tilde{\mathbf{v}}} \\ &= \|A^{-1/2} B^T q\|_{l^2}^2 \\ &= q^T B A^{-1} B^T q \end{aligned}$$

and therefore

$$\beta^2 = \inf_{q \in U} \frac{q^T B A^{-1} B^T q}{q^T C q},$$

which completes the proof. □

When dealing with saddle-point problems, we will often find it convenient to restrict our attention to a subspace of the solution space on which the second equation is satisfied. This motivates the definition of what we call the *benign* subspace.

Definition 2.4 *Associated with the problem*

$$\left\{ \begin{array}{rcrcll} \mu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \langle \mathbf{f},\mathbf{v}\rangle & \forall \mathbf{v}\in\mathbf{V} \\ b(\mathbf{u},q) & - & \frac{1}{\lambda}c(p,q) & = & 0 & \forall q\in U, \end{array} \right.$$

we define the benign subspace

$$(\mathbf{V}\times U)_B = \left\{ \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in \mathbf{V}\times U \;\middle|\; b(\mathbf{u},q) = \frac{1}{\lambda}c(p,q)\ \ \forall q\in U \right\}.$$

When dealing with incompressible problems, the second equation does not involve the pressures and we define

$$\mathbf{V}_B = \{\mathbf{v}\in\mathbf{V} \mid b(\mathbf{u},q) = 0\ \ \forall q\in U\}.$$

We now prove a series of results that later will be necessary for the analysis of our algorithm. We start with the observation that restricted to the space of benign functions, the saddle-point bilinear form is actually positive definite.

Lemma 2.5 *If K is a block matrix of the form $K = \left[\begin{array}{cc} \mu A & B^T \\ B & -\frac{1}{\lambda}C \end{array} \right]$, where C is symmetric and $\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right]$ and $\left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right]$ are benign, i.e., $B\mathbf{u} = \frac{1}{\lambda}Cp$ and $B\mathbf{v} = \frac{1}{\lambda}Cq$, then*

$$\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right]^T K \left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] = \mu\mathbf{u}^T A\mathbf{v} + \frac{1}{\lambda}p^T Cq.$$

Proof Indeed,

$$\begin{aligned} \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right]^T K \left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] &= \mu\mathbf{u}^T A\mathbf{v} + \mathbf{u}^T B^T q + p^T B\mathbf{v} - \frac{1}{\lambda}p^T Cq \\ &= \mu\mathbf{u}^T A\mathbf{v} + (B\mathbf{u})^T q + p^T(B\mathbf{v}) - \frac{1}{\lambda}p^T Cq \\ &= \mu\mathbf{u}^T A\mathbf{v} + \frac{1}{\lambda}p^T Cq \end{aligned}$$

□

The following result shows that the saddle-point problem can be reformulated as a positive definite one.

Lemma 2.6 *Let A, B and C be as in the previous lemma and assume also that C is positive definite. Let* $(\mathbf{V}\times U)_B = \left\{ \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in \mathbf{V}\times U \;\middle|\; B\mathbf{u} = \frac{1}{\lambda} Cp \right\}$, $a(\mathbf{u},\mathbf{v}) = \mathbf{v}^T A\mathbf{u}$, $b(\mathbf{u},p) = p^T B\mathbf{u}$, *and* $c(p,q) = p^T Cq$. *Then, the following two problems are equivalent:*

- *find* $\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in \mathbf{V}\times U$ *such that*

$$\left\{ \begin{array}{rcrcl} \mu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & f(\mathbf{v}) \\ b(\mathbf{u},q) & - & \frac{1}{\lambda} c(p,q) & = & 0 \end{array} \right. \quad \forall \left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] \in \mathbf{V}\times U \tag{2.4}$$

- *find* $\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in (\mathbf{V}\times U)_B$ *such that*

$$\mu a(\mathbf{u},\mathbf{v}) + \frac{1}{\lambda} c(p,q) = f(\mathbf{v}) \quad \forall \left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] \in (\mathbf{V}\times U)_B. \tag{2.5}$$

Proof By using Lemma 2.5, it is evident that any solution to (2.4) also solves (2.5).

Now assume that $\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right]$ solves (2.5). For an arbitrary $\left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] \in \mathbf{V}\times U$, let $\tilde{q} = \lambda C^{-1} B\mathbf{v}$. Then $\left[\begin{array}{c} \mathbf{v} \\ \tilde{q} \end{array} \right] \in (\mathbf{V}\times U)_B$ and again by Lemma 2.5,

$$\left\{ \begin{array}{rcrcl} \mu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & f(\mathbf{v}) \\ b(\mathbf{u},\tilde{q}) & - & \frac{1}{\lambda} c(p,\tilde{q}) & = & 0. \end{array} \right.$$

We note that the second equation above is also satisfied if $\tilde{q}$ is replaced by q, since $\left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in (\mathbf{V}\times U)_B$, and that the first equation does not involve $\tilde{q}$. We conclude that

$$\left\{ \begin{array}{rcrcl} \mu a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & f(\mathbf{v}) \\ b(\mathbf{u},q) & - & \frac{1}{\lambda} c(p,q) & = & 0 \end{array} \right. \quad \forall \left[\begin{array}{c} \mathbf{v} \\ q \end{array} \right] \in \mathbf{V}\times U.$$

□

Remark 2.7 *A similar result can be proved with $\lambda = \infty$ if $\mathbf{V} \times U$ is inf-sup stable.*

Remark 2.8 *There are no assumptions on A. In particular, the result still holds when A is singular. This will allow us to apply this result to local Neumann problems as in equation (4.6).*

We will need the following two results, which give an explicit formula for the solution of a saddle point problem with a penalty term and a stability result for its solution. We note that the result in Lemma 2.10 is of the same form as the ones in Theorems 2.1 and 2.2, but it is sharper, in the sense that it reveals the interplay of the inf-sup constant β and the ratio μ/λ.

Lemma 2.9 *Let A and C be positive definite matrices and, if $\lambda = \infty$, let B have full row rank. Then,*

$$\begin{bmatrix} \mu A & B^T \\ B & -\frac{1}{\lambda}C \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\mu}(A^{-1} - A^{-1}B^T S^{-1} B A^{-1}) & A^{-1}B^T S^{-1} \\ S^{-1}BA^{-1} & -\mu S^{-1} \end{bmatrix}, \quad (2.6)$$

where $S = BA^{-1}B^T + \dfrac{\mu}{\lambda}\, C$.

Proof The proof follows from a direct computation. □

Lemma 2.10 *Consider the discrete saddle point problem*

$$\begin{bmatrix} \mu A & B^T \\ B & -\frac{1}{\lambda}C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix},$$

where A and C are positive definite and, if $\lambda = \infty$, B has full row rank. Let the positive scalar α and the positive-definite matrix Δ satisfy $\mathbf{v}^T A \mathbf{v} \leq \alpha \mathbf{v}^T \Delta \mathbf{v}$ $\forall \mathbf{v}$ and let $\beta \geq 0$ be the inf-sup constant such that

$$p^T B \Delta^{-1} B^T p \geq \beta^2 p^T C p \quad \forall p \quad (2.7)$$

(cf. Lemma 2.3). Then,

$$\|\mathbf{u}\|_A \leq \frac{1}{\mu}\|\mathbf{f}\|_{A^{-1}} + \frac{1}{\sqrt{\frac{\beta^2}{\alpha} + \frac{\mu}{\lambda}}}\|g\|_{C^{-1}} \quad \textit{and} \quad (2.8)$$

$$\|p\|_C \leq \frac{1}{\sqrt{\frac{\beta^2}{\alpha} + \frac{\mu}{\lambda}}}\|\mathbf{f}\|_{A^{-1}} + \frac{\mu}{\frac{\beta^2}{\alpha} + \frac{\mu}{\lambda}}\|g\|_{C^{-1}}. \quad (2.9)$$

Remark 2.11 *In our mixed elasticity problem, A will essentially be some discretization of the bilinear form $a(\cdot,\cdot)$. Nevertheless, the inf-sup constant is usually defined using the $(H^1(\Omega))^d$-norm for the displacements. This is the reason for introducing Δ, which can be a discretization of the $(H^1(\Omega))^d$-inner-product.*

Proof [Lemma 2.10] First note that

$$p^T B A^{-1} B^T p \geq \frac{\beta^2}{\alpha} p^T C p \quad \forall p. \tag{2.10}$$

By the explicit formula (2.6) for the inverse of an invertible saddle point problem, we have

$$\begin{aligned} \mathbf{u} &= \frac{1}{\mu}(A^{-1} - A^{-1}B^T S^{-1} B A^{-1})\mathbf{f} + A^{-1}B^T S^{-1} g, & (2.11)\\ p &= S^{-1} B A^{-1}\mathbf{f} - \mu S^{-1} g, & (2.12) \end{aligned}$$

and from (2.10), we have

$$S = B A^{-1} B^T + \frac{\mu}{\lambda} C \geq \left(\frac{\beta^2}{\alpha} + \frac{\mu}{\lambda}\right) C.$$

(Here and in the following an inequality between matrices means an inequality between the associated quadratic forms.) We note that $\|\mathbf{u}\|_A = \|A^{1/2}\mathbf{u}\|_{l^2}$ and that $\|p\|_C = \|C^{1/2}p\|_{l^2}$; moreover $\|\mathbf{f}\|_{A^{-1}} = \|A^{-1/2}\mathbf{f}\|_{l^2}$ and $\|g\|_{C^{-1}} = \|C^{-1/2}g\|_{l^2}$ are the matrix representations of the dual norms of $\mathbf{f}$ and g, respectively. Indeed,

$$\sup_{\mathbf{v}} \frac{(\mathbf{f}^T\mathbf{v})^2}{\mathbf{v}^T A \mathbf{v}} = \sup_{\mathbf{w}} \frac{(\mathbf{f}^T A^{-1/2}\mathbf{w})^2}{\mathbf{w}^T\mathbf{w}} = \frac{(\mathbf{f}^T A^{-1/2} A^{-1/2}\mathbf{f})^2}{\mathbf{f}^T A^{-1}\mathbf{f}} = \mathbf{f}^T A^{-1}\mathbf{f},$$

and similarly for $\|g\|_{C^{-1}}$.

By using (2.11), the $A-$norm of the displacement component is estimated by

$$\begin{aligned} \|A^{1/2}\mathbf{u}\|_{l^2} \leq\ & \frac{1}{\mu}\,\|(I - A^{-1/2}B^T S^{-1} B A^{-1/2})A^{-1/2}\mathbf{f}\|_{l^2} \\ & + \|A^{-1/2}B^T S^{-1} g\|_{l^2}. \end{aligned} \tag{2.13}$$

The first term in (2.13) is bounded by $\frac{1}{\mu}\|A^{-1/2}\mathbf{f}\|_{l^2}$ because, from $S^{-1} \leq (BA^{-1}B^T)^{-1}$, it follows that

$$0 \leq A^{-1/2}B^T S^{-1} B A^{-1/2} \leq A^{-1/2}B^T (BA^{-1}B^T)^{-1} B A^{-1/2} \leq I,$$

since the next to last expression is an orthogonal projection. The square of the second term in (2.13) is estimated similarly by

$$\begin{aligned} \|A^{-1/2}B^TS^{-1}g\|_{l^2}^2 &= g^TS^{-1}BA^{-1}B^TS^{-1}g \le g^TS^{-1}g \\ &\le \frac{1}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\, g^TC^{-1}g = \frac{1}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\, \|g\|_{C^{-1}}^2, \end{aligned}$$

and (2.8) follows.

From (2.12), the $C-$norm of the pressure component is estimated by

$$\|C^{1/2}p\|_{l^2} \le \|C^{1/2}S^{-1}BA^{-1}\mathbf{f}\|_{l^2} + \mu\|C^{1/2}S^{-1}g\|_{l^2}. \tag{2.14}$$

The first term on the right in (2.14) is bounded by $\dfrac{1}{\sqrt{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}}\,\|A^{-1/2}\mathbf{f}\|_{l^2}$ because

$$\begin{aligned} &\|C^{1/2}S^{-1}BA^{-1}\mathbf{f}\|_{l^2}^2 \\ &= \mathbf{f}^TA^{-1}B^TS^{-1}CS^{-1}BA^{-1}\mathbf{f} \\ &\le \frac{1}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\,\mathbf{f}^TA^{-1}B^TS^{-1}BA^{-1}\mathbf{f} \\ &\le \frac{1}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\,\mathbf{f}^TA^{-1}B^T(BA^{-1}B^T)^{-1}BA^{-1}\mathbf{f} \\ &\le \frac{1}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\,\mathbf{f}^TA^{-1}\mathbf{f}; \end{aligned}$$

we have again used that the matrix $A^{-1/2}B^T(BA^{-1}B^T)^{-1}BA^{-1/2}$ is an orthogonal projection. The square of the second term on the right in (2.14) is estimated by

$$\begin{aligned} \mu^2\,\|C^{1/2}S^{-1}g\|_{l^2}^2 &= \mu^2\, g^TS^{-1}CS^{-1}g \\ &\le \frac{\mu^2}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}} g^TS^{-1}g \\ &\le \Big(\frac{\mu}{\frac{\beta^2}{\alpha}+\frac{\mu}{\lambda}}\Big)^2 g^TC^{-1}g, \end{aligned}$$

and (2.9) follows. □

Chapter 3

Substructuring

In this chapter, we carry out the substructuring of the problems described in Section 1.3, both on the continuous and discrete level. The basic idea is as follows: associated with a nonoverlapping partition of the domain Ω into N subdomains,

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j,$$

we define a decomposition of the solution space $\mathbf{W}$ into a direct sum of $N+1$ subspaces, $\mathbf{W}_\Gamma$ and $\mathbf{W}_i$, $i = 1, \ldots, N$, where the elements of $\mathbf{W}_i$ have compact support in $\overline{\Omega_i}$. This decomposition is done so as to allow the solution of the original problem to be divided into two parts: we first solve N decoupled local problems, one on each subdomain, and then solve a global problem in the space $\mathbf{W}_\Gamma$ (which is usually much smaller than $\mathbf{W}$).

For positive definite problems, such as Poisson's equation or the pure displacement formulation of elasticity, the definition of these subspaces is closely related to the geometry of the decomposition. We define Γ as the interface between the subdomains, i.e., $\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i\right) \setminus \partial\Omega$ (see Figure 3.1), and let $\mathbf{W}_\Gamma$ be the space of the traces on Γ of functions of $\mathbf{W}$ (or, more exactly, extensions of these traces); the spaces $\mathbf{W}_i$ contain the functions which are supported in $\overline{\Omega_i}$.

For the saddle-point problems we are interested in, the decomposition is more involved. Nevertheless, we can borrow much of the notation from the positive definite case and we will refer to the subspace $\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_i$ as the *interior* subspace and to $\mathbf{W}_\Gamma$ as the *interface* subspace (even though the latter will include pressure functions that are not directly related to Γ.)

In the sequel, we will assume that the subdomains define a quadrilateral

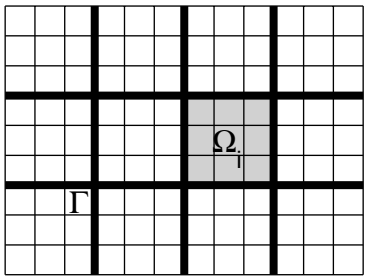

Figure 3.1: Subdomains Ω_i and interface Γ.

(or hexahedral) finite element mesh τ_H of characteristic size H, which is shape regular but not necessarily quasi uniform, and that this coarse triangulation is further refined into a fine quadrilateral (or hexahedral) finite element triangulation τ_h of characteristic size h. We note that this requirement that the subdomains be quadrilateral (or hexahedral) can be relaxed. The only point where our algorithm actually relies on this fact is in the definition of the enrichment of the coarse space, but alternative definitions could be devised. In particular, an enrichment based on bubble functions for the edges (or faces), similar to $\mathbf{V}_H^1$ (see Section 4.1), could easily be defined for more general subdomain geometries.

3.1 Substructuring in Variational Form

We partition our mixed finite element space $\mathbf{W} = \mathbf{V} \times U \subset (H_0^1(\Omega))^d \times L^2(\Omega)$ into a direct sum of subspaces:

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Gamma = \left(\bigoplus_{i=1}^{N} \mathbf{V}_i \times U_{0,i} \right) \oplus (\mathbf{V}_\Gamma \times U_\Gamma) \,. \tag{3.1}$$

Here $\mathbf{V}_i = \mathbf{V} \cap (H_0^1(\Omega_i))^d$, $U_{0,i} = U \cap L_0^2(\Omega_i)$ and $U_\Gamma = \bigoplus_{i=1}^N U_{\Gamma,i}$, where $U_{\Gamma,i} = \text{span}\{\chi_{\Omega_i}\}$. We also define $U_i = U_{0,i} \oplus U_{\Gamma,i} = U|_{\Omega_i}$. We note that the requirement that U be discontinuous across the interface is necessary for U_Γ to be a subspace of U. At times, we will regard the local spaces $\mathbf{V}_i$, U_i and $U_{\Gamma,i}$ as subspaces of $\mathbf{V}$, U and U_Γ, respectively, by extending the functions by zero. The space $\mathbf{V}_\Gamma$ of saddle-point harmonic extensions is defined below.

We define the saddle-point harmonic extension operator

$$\begin{aligned} \mathcal{SH}: \quad \mathbf{V}|_\Gamma &\rightarrow \mathbf{V} \times U_I \\ \mathbf{u}_\Gamma &\mapsto \mathcal{SH}\mathbf{u}_\Gamma = \begin{bmatrix} \mathcal{SH}^u \mathbf{u}_\Gamma \\ \mathcal{SH}^p \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathcal{SH}^{u_I} \mathbf{u}_\Gamma \\ u_\Gamma \end{bmatrix} \\ \mathcal{SH}^p \mathbf{u}_\Gamma \end{bmatrix}, \end{aligned}$$

where $U_I = \bigoplus_{i=1}^N U_{0,i}$ and $\mathcal{SH}^u$, $\mathcal{SH}^{u_I}$ and $\mathcal{SH}^p$ act on the following spaces:

$$\begin{aligned} \mathcal{SH}^u: \quad & \mathbf{V}|_\Gamma \rightarrow \mathbf{V}, \\ \mathcal{SH}^{u_I}: \quad & \mathbf{V}|_\Gamma \rightarrow \mathbf{V}_I \quad \text{and} \\ \mathcal{SH}^p: \quad & \mathbf{V}|_\Gamma \rightarrow U_I. \end{aligned}$$

$\mathcal{SH}\mathbf{u}_\Gamma$ is the solution of the problem: *find* $\mathcal{SH}^u\mathbf{u}_\Gamma \in \mathbf{V}$ *and* $\mathcal{SH}^p\mathbf{u}_\Gamma \in U_I$ *such that* $\mathcal{SH}^u\mathbf{u}_\Gamma|_\Gamma = \mathbf{u}_\Gamma$ *and, for* $i = 1, \ldots, N$,

$$\begin{cases} \mu a_i(\mathcal{SH}^u\mathbf{u}_\Gamma, \mathbf{v}) + b_i(\mathbf{v}, \mathcal{SH}^p\mathbf{u}_\Gamma) = 0 & \forall \mathbf{v} \in \mathbf{V}_i \\ b_i(\mathcal{SH}^u\mathbf{u}_\Gamma, q) - \frac{1}{\lambda} c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, q) = 0 & \forall q \in U_{0,i}. \end{cases} \tag{3.2}$$

Here $a_i: \mathbf{V}|_{\Omega_i} \times \mathbf{V}|_{\Omega_i} \rightarrow \mathbb{R}$ is defined by replacing Ω with Ω_i in the definition of $a(\cdot,\cdot)$ (see Section 1.3). At times, we will use $a_i(\mathbf{u}, \mathbf{v})$, with $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, in which case we actually mean $a_i(\mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i})$. The same applies to $b_i(\cdot,\cdot)$ and $c_i(\cdot,\cdot)$.

We point out that each problem (3.2) has a unique solution for an arbitrary $\mathbf{u}_\Gamma$, even in the incompressible case when $\lambda = \infty$ (cf. Remark 1.2). If we would have used U_i instead of $U_{0,i}$ in the definition of problem (3.2), we would have a well-defined extension operator only for finite λ and we should expect ill-conditioning when approaching the incompressible limit. This is the reason for our choice of decomposition (3.1), instead of treating all pressures as interior variables. In fact, we will return later to this idea of eliminating all the pressure variables. That will lead to a very practical *positive definite implementation* of our algorithm; see Section 7.2. We regard this approach as just an implementation artifact, though, and believe that the natural way to understand and analyze our method is by using the decomposition (3.1).

We define the space $\mathbf{V}_\Gamma$ of saddle-point harmonic extensions as $\mathbf{V}_\Gamma = \mathcal{SH}^u(\mathbf{V}|_\Gamma)$. We note that the functions of $\mathbf{V}_\Gamma$ are defined in the entire domain and should not be confused with $\mathbf{V}|_\Gamma$, the functions of which are defined only on Γ.

We define an inner product on the space of interface displacements $\mathbf{V}_\Gamma$ by

$$s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \mu a(\mathcal{SH}^u \mathbf{u}_\Gamma, \mathcal{SH}^u \mathbf{v}_\Gamma) + \frac{1}{\lambda} c(\mathcal{SH}^p \mathbf{u}_\Gamma, \mathcal{SH}^p \mathbf{v}_\Gamma). \tag{3.3}$$

Analogously, we define local interface inner products by

$$s_i(\mathbf{u}_{\Gamma,i}, \mathbf{v}_{\Gamma,i}) = \mu a_i(\mathcal{SH}_i^u \mathbf{u}_{\Gamma,i}, \mathcal{SH}_i^u \mathbf{v}_{\Gamma,i}) + \frac{1}{\lambda} c_i(\mathcal{SH}_i^p \mathbf{u}_{\Gamma,i}, \mathcal{SH}_i^p \mathbf{v}_{\Gamma,i}),$$

where $\mathcal{SH}_i \mathbf{u}_{\Gamma,i} = \begin{bmatrix} \mathcal{SH}_i^u \mathbf{u}_{\Gamma,i} \\ \mathcal{SH}_i^p \mathbf{u}_{\Gamma,i} \end{bmatrix}$ is defined in the same way as in (3.2).

We also recall the definition of the discrete harmonic extension operator $\mathcal{H}$: *given* $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$, *we define* $\mathcal{H}\mathbf{u}_\Gamma \in \mathbf{V}$ *to satisfy* $\mathcal{H}\mathbf{u}_\Gamma|_\Gamma = \mathbf{u}_\Gamma$ *and*

$$\int_{\Omega_i} \nabla \mathcal{H}\mathbf{u}_\Gamma \cdot \nabla \mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathbf{V}_i, \quad i = 1, \ldots, N.$$

The following comparison of the energy of the discrete saddle-point harmonic extension operator and the discrete harmonic extensions $\mathcal{H}$ is a generalization of the analogous result in the Stokes case (see [7], [23]). In the proof below, we make use of the compressibility of the material (i.e., we assume $\lambda < \infty$).

Lemma 3.1 *Given* $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$, *let* $\mathcal{H}\mathbf{u}_\Gamma$ *be its discrete harmonic extension. For any* $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$ *such that* $\mathcal{SH}^u \mathbf{u}_\Gamma \perp \ker(a_i)$, *we have*

$$\left(2(1+\sigma)^2\right)^{-1} s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \leq \mu \left\|\nabla \mathcal{H}\mathbf{u}_\Gamma\right\|^2_{L^2(\Omega_i)} \leq C s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma)$$

where $\sigma = \sqrt{\frac{d}{\frac{\beta^2}{2} + \frac{\mu}{\lambda}}}$, β *is the inf-sup constant of the local mixed finite element space* $\mathbf{V}_i \times U_{0,i}$ *and the constant* C *does not depend on* $\mathbf{u}_\Gamma$.

The hypothesis that $\mathcal{SH}\mathbf{u}_\Gamma \perp \ker(a_i)$ *is not necessary for the lower bound.*

Proof The second inequality is an easy consequence of the minimal property of the discrete harmonic extension and the lower bound of Lemma 1.3. Indeed,

$$\begin{aligned} \mu \left\|\nabla \mathcal{H}\mathbf{u}_\Gamma\right\|^2_{L^2(\Omega_i)} &\leq \mu \left\|\nabla \mathcal{SH}\mathbf{u}_\Gamma\right\|^2_{L^2(\Omega_i)} \\ &\leq C\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) \\ &\leq C s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma). \end{aligned}$$

In order to prove the first inequality, we choose $\mathbf{v} = (\mathcal{SH}\mathbf{u}_\Gamma - \mathcal{H}\mathbf{u}_\Gamma)|_{\Omega_i}$ in (3.2) and obtain

$$\begin{aligned} &\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) + b_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}^p \mathbf{u}_\Gamma) \\ &\quad = \mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) + b_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{SH}^p \mathbf{u}_\Gamma). \end{aligned}$$

By applying Cauchy-Schwarz and using the facts that $b_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{u}_\Gamma) = \frac{1}{\lambda}c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{u}_\Gamma)$ and that

$$\begin{aligned}(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{u})_{L^2(\Omega_i)} &= \int_{\Omega_i}\left(\sum_{i=1}^d \frac{\partial u_i}{\partial x_i}\right)^2 \leq d\int_{\Omega_i}\sum_{i=1}^d\left(\frac{\partial u_i}{\partial x_i}\right)^2 \\ &\leq d\int_{\Omega_i}\sum_{i=1}^d\sum_{j=1}^d\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right)^2 \\ &= \frac{d}{2}a_i(\mathbf{u},\mathbf{u}),\end{aligned} \tag{3.4}$$

we obtain

$$\begin{aligned}&\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) + \tfrac{1}{\lambda}c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{u}_\Gamma) \\ &\quad\leq \left(\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma)^{1/2} + \sqrt{\tfrac{d}{2}}\,\|\mathcal{SH}^p\mathbf{u}_\Gamma\|_{L^2(\Omega_i)}\right) a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)^{1/2}.\end{aligned} \tag{3.5}$$

We estimate $\|\mathcal{SH}^p\mathbf{u}_\Gamma\|_{L^2(\Omega_i)}$ by applying Lemma 2.10 to the saddle-point problem with homogeneous boundary conditions satisfied by $\begin{bmatrix}\mathcal{SH}\mathbf{u}_\Gamma - \mathcal{H}\mathbf{u}_\Gamma \\ \mathcal{SH}^p\mathbf{u}_\Gamma\end{bmatrix}$. From (3.2), we find that on each Ω_i

$$\begin{cases} \mu a_i(\mathcal{SH}\mathbf{u}_\Gamma - \mathcal{H}\mathbf{u}_\Gamma, \mathbf{v}) + b_i(\mathbf{v}, \mathcal{SH}^p\mathbf{u}_\Gamma) = -\mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathbf{v}) & \forall \mathbf{v}\in\mathbf{V}_i \\ b_i(\mathcal{SH}\mathbf{u} - \mathcal{H}\mathbf{u}_\Gamma, q) - \frac{1}{\lambda}c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, q) = -b_i(\mathcal{H}\mathbf{u}_\Gamma, q) & \forall q\in U_{0,i}. \end{cases}$$

Taking into account that $a_i(\mathbf{u},\mathbf{u}) \leq 2\|\nabla\mathbf{u}\|_{L^2(\Omega_i)}$ (cf. Lemma 1.3), Lemma 2.10 yields

$$\|\mathcal{SH}^p\mathbf{u}_\Gamma\|_{L^2(\Omega_i)} \leq \frac{1}{\sqrt{\frac{\beta^2}{2}+\frac{\mu}{\lambda}}}\sup_{\mathbf{v}\in\mathbf{V}_i}\frac{\mu a_i(\mathcal{H}\mathbf{u}_\Gamma,\mathbf{v})}{a_i(\mathbf{v},\mathbf{v})^{1/2}} + \frac{\mu}{\frac{\beta^2}{2}+\frac{\mu}{\lambda}}\sup_{q\in U_{0,i}}\frac{b_i(\mathcal{H}\mathbf{u}_\Gamma,q)}{\|q\|_{L^2(\Omega_i)}},$$

and again by Cauchy-Schwarz and (3.4),

$$\|\mathcal{SH}^p\mathbf{u}_\Gamma\|_{L^2(\Omega_i)} \leq \left(\frac{1}{\sqrt{\frac{\beta^2}{2}+\frac{\mu}{\lambda}}} + \frac{\sqrt{d/2}}{\frac{\beta^2}{2}+\frac{\mu}{\lambda}}\right)\mu a_i(\mathcal{H}\mathbf{u}_\Gamma,\mathcal{H}\mathbf{u}_\Gamma)^{1/2}.$$

It follows from (3.5) that

$$\begin{aligned}&\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) + \frac{1}{\lambda}\,c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{u}_\Gamma) \\ &\quad\leq \mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma)^{1/2} a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)^{1/2} + \left(\frac{\sigma}{\sqrt{2}} + \frac{\sigma^2}{2}\right)\mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \\ &\quad\leq \frac{1}{2}\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) + \left(\frac{1}{2} + \frac{\sigma}{\sqrt{2}} + \frac{\sigma^2}{2}\right)\mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)\end{aligned}$$

and therefore

$$\begin{aligned}
&\mu a_i(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) + 1/\lambda \; c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{u}_\Gamma) \\
&\quad \leq \quad (1+\sigma)^2 \mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \\
&\quad = \quad 2(1+\sigma)^2 \mu \left\| \varepsilon\left(\mathcal{H}\mathbf{u}_\Gamma\right)\right\|^2_{L^2(\Omega_i)} .
\end{aligned}$$

We complete the proof by using the upper bound of Lemma 1.3. □

3.2 Substructuring in Matrix Form

In order to eliminate the interior degrees of freedom, we reorder the vector of unknowns as

$$\left[\begin{array}{l} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_\Gamma \end{array}\right] \quad \begin{array}{l} \text{interior displacements,} \\ \text{interior pressures with zero average,} \\ \text{interface displacements, and} \\ \text{constant pressures in each } \Omega_i. \end{array} \tag{3.6}$$

Then, after using the same permutation, the discrete system matrix can be written as

$$\left[\begin{array}{c|c} K_{II} & K_{\Gamma I}^T \\ \hline K_{\Gamma I} & K_{\Gamma\Gamma} \end{array}\right] = \left[\begin{array}{cc|cc} \mu A_{II} & B_{II}^T & \mu A_{I\Gamma} & 0 \\ B_{II} & -\frac{1}{\lambda} C_{II} & B_{I\Gamma} & 0 \\ \hline \mu A_{\Gamma I} & B_{I\Gamma}^T & \mu A_{\Gamma\Gamma} & B_{\Gamma\Gamma}^T \\ 0 & 0 & B_{\Gamma\Gamma} & -\frac{1}{\lambda} C_{\Gamma\Gamma} \end{array}\right],$$

where the zero blocks are due to the interior displacements having zero flux across the subdomain boundaries and the interior pressures having a zero average.

Eliminating the interior unknowns $\mathbf{u}_I$ and p_I by static condensation, we obtain the saddle-point Schur complement system

$$S \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_\Gamma \end{array}\right] = \left[\begin{array}{c} \tilde{\mathbf{b}}_\Gamma \\ 0 \end{array}\right], \tag{3.7}$$

where

$$
\begin{aligned}
S &= K_{\Gamma\Gamma} - K_{\Gamma I} K_{II}^{-1} K_{\Gamma I}^T \\
&= \begin{bmatrix} \mu A_{\Gamma\Gamma} & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & -\frac{1}{\lambda} C_{\Gamma\Gamma} \end{bmatrix} \\
&\quad - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -\frac{1}{\lambda} C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{I\Gamma} & 0 \\ B_{I\Gamma} & 0 \end{bmatrix} \qquad (3.8) \\
&= \begin{bmatrix} S_\Gamma & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & -\frac{1}{\lambda} C_{\Gamma\Gamma} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{bmatrix} \tilde{\mathbf{b}}_\Gamma \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\Gamma \\ 0 \end{bmatrix} - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -\frac{1}{\lambda} C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix}.
$$

By using a second permutation that reorders the interior displacements and pressures subdomain by subdomain, we find that K_{II}^{-1} represents the solution of N decoupled saddle-point problems, one for each subdomain and all uniquely solvable, with Dirichlet data given on $\partial\Omega_i$:

$$
K_{II}^{-1} = \begin{bmatrix} {K_{II}^{(1)}}^{-1} & & 0 \\ & \ddots & \\ 0 & & {K_{II}^{(N)}}^{-1} \end{bmatrix}.
$$

The Schur complement S does not need to be explicitly assembled since only its action $S\mathbf{w}$ on a vector $\mathbf{w}$ is needed in a Krylov iteration. This operation essentially only requires the action of K_{II}^{-1} on a vector, i.e., the solution of N decoupled saddle-point problems. In other words, the action of S is computed by subassembling the actions of the subdomain Schur complements $S^{(i)}$ defined

for Ω_i by

$$\begin{aligned} S^{(i)} &= K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)}(K_{II}^{(i)})^{-1}{K_{\Gamma I}^{(i)}}^T \\ &= \begin{bmatrix} \mu A_{\Gamma\Gamma}^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda}C_{\Gamma\Gamma}^{(i)} \end{bmatrix} \\ &\quad - \begin{bmatrix} \mu A_{\Gamma I}^{(i)} & {B_{I\Gamma}^{(i)}}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II}^{(i)} & {B_{II}^{(i)}}^T \\ B_{II}^{(i)} & -\frac{1}{\lambda}C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{I\Gamma}^{(i)} & 0 \\ B_{I\Gamma}^{(i)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} S_{\Gamma}^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda}C_{\Gamma\Gamma}^{(i)} \end{bmatrix}. \end{aligned} \tag{3.9}$$

Once $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ is known, $\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix}$ can be found by back substitution:

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix} = \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -\frac{1}{\lambda}C_{II} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix} - \begin{bmatrix} \mu A_{I\Gamma} & 0 \\ B_{I\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \right).$$

The following lemma and its corollary provide the matricial counterpart of the inner-product $s(\cdot,\cdot)$ defined in (3.3).

Lemma 3.2 *For $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$, we have*

$$s_i(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = {\mathbf{u}_\Gamma^{(i)}}^T S_\Gamma^{(i)} \mathbf{v}_\Gamma^{(i)}.$$

Proof The definition of $\mathcal{SH}$ in equation (3.2) can be translated to matrix notation as follows:

$$\begin{bmatrix} \mu A_{II}^{(i)} & {B_{II}^{(i)}}^T & \mu A_{I\Gamma}^{(i)} \\ B_{II}^{(i)} & -\frac{1}{\lambda}C_{II}^{(i)} & B_{I\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathcal{SH}_i^{u_I}\mathbf{u}_\Gamma \\ \mathcal{SH}_i^{p}\mathbf{u}_\Gamma \\ \mathbf{u}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.10}$$

Therefore we have

$$s_i(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \mu a_i(\mathcal{SH}^u\mathbf{u}_\Gamma, \mathcal{SH}^u\mathbf{v}_\Gamma) + \frac{1}{\lambda}c_i(\mathcal{SH}^p\mathbf{u}_\Gamma, \mathcal{SH}^p\mathbf{v}_\Gamma)$$

$$
\begin{aligned}
&= \mathcal{SH}_i^u \mathbf{u}_\Gamma{}^T \begin{bmatrix} \mu A_{II}^{(i)} & \mu A_{I\Gamma}^{(i)} \\ \mu A_{\Gamma I}^{(i)} & \mu A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \mathcal{SH}_i^u \mathbf{v}_\Gamma + \frac{1}{\lambda} \mathcal{SH}_i^p \mathbf{u}_\Gamma{}^T C_{II}^{(i)} \mathcal{SH}_i^p \mathbf{v}_\Gamma \\
&= \mathcal{SH}_i^u \mathbf{u}_\Gamma{}^T \begin{bmatrix} \mu A_{II}^{(i)} & \mu A_{I\Gamma}^{(i)} \\ \mu A_{\Gamma I}^{(i)} & \mu A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \mathcal{SH}_i^u \mathbf{v}_\Gamma - \frac{1}{\lambda} \mathcal{SH}_i^p \mathbf{u}_\Gamma{}^T C_{II}^{(i)} \mathcal{SH}_i^p \mathbf{v}_\Gamma \\
&\quad + \mathcal{SH}_i^u \mathbf{u}_\Gamma{}^T \begin{bmatrix} B_{II}^{(i)T} \\ B_{I\Gamma}^{(i)T} \end{bmatrix} \mathcal{SH}_i^p \mathbf{v}_\Gamma + \mathcal{SH}_i^p \mathbf{u}_\Gamma{}^T \begin{bmatrix} B_{II}^{(i)} & B_{I\Gamma}^{(i)} \end{bmatrix} \mathcal{SH}_i^u \mathbf{v}_\Gamma \\
&= \begin{bmatrix} \mathcal{SH}_i^u \mathbf{u}_\Gamma \\ \mathcal{SH}_i^p \mathbf{u}_\Gamma \end{bmatrix}^T \left[\begin{array}{cc|c} \mu A_{II}^{(i)} & \mu A_{I\Gamma}^{(i)} & B_{II}^{(i)T} \\ \mu A_{\Gamma I}^{(i)} & \mu A_{\Gamma\Gamma}^{(i)} & B_{I\Gamma}^{(i)T} \\ \hline B_{II}^{(i)} & B_{I\Gamma}^{(i)} & -\frac{1}{\lambda} C_{II}^{(i)} \end{array}\right] \begin{bmatrix} \mathcal{SH}_i^u \mathbf{v}_\Gamma \\ \mathcal{SH}_i^p \mathbf{v}_\Gamma \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{SH}_i^{u_I} \mathbf{u}_\Gamma \\ \mathcal{SH}_i^p \mathbf{u}_\Gamma \\ \mathbf{u}_\Gamma^{(i)} \end{bmatrix}^T \begin{bmatrix} \mu A_{II}^{(i)} & B_{II}^{(i)T} & \mu A_{I\Gamma}^{(i)} \\ B_{II}^{(i)} & -\frac{1}{\lambda} C_{II}^{(i)} & B_{I\Gamma}^{(i)} \\ \mu A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)T} & \mu A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathcal{SH}_i^{u_I} \mathbf{v}_\Gamma \\ \mathcal{SH}_i^p \mathbf{v}_\Gamma \\ \mathbf{v}_\Gamma^{(i)} \end{bmatrix},
\end{aligned}
$$

where we have used the following easy consequences of (3.10):

$$
\begin{aligned}
\mathcal{SH}_i^u \mathbf{u}_\Gamma{}^T \begin{bmatrix} B_{II}^{(i)T} \\ B_{I\Gamma}^{(i)T} \end{bmatrix} &= \tfrac{1}{\lambda} \mathcal{SH}_i^p \mathbf{u}_\Gamma{}^T C_{II}^{(i)} \quad \text{and} \\
\begin{bmatrix} B_{II}^{(i)} & B_{I\Gamma}^{(i)} \end{bmatrix} \mathcal{SH}_i^u \mathbf{v}_\Gamma &= \tfrac{1}{\lambda} C_{II}^{(i)} \mathcal{SH}_i^p \mathbf{v}_\Gamma.
\end{aligned} \tag{3.11}
$$

The proof is now completed by using the identity

$$
\begin{bmatrix} \mathcal{SH}_i^{u_I} \mathbf{u}_\Gamma \\ \mathcal{SH}_i^p \mathbf{u}_\Gamma \\ \mathbf{u}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} \mu A_{II}^{(i)} & B_{II}^{(i)T} \\ B_{II}^{(i)} & -\frac{1}{\lambda} C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{I\Gamma}^{(i)} \\ B_{I\Gamma}^{(i)} \end{bmatrix} \\ I \end{bmatrix} \mathbf{u}_\Gamma^{(i)},
$$

which is easily derived from (3.10), and a similar expression for $\mathbf{v}_\Gamma$. □

Corollary 3.3 *For $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$, we have*

$$
s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \mathbf{u}_\Gamma^T S_\Gamma \mathbf{v}_\Gamma.
$$

Chapter 4

Balancing Neumann–Neumann Preconditioners

We will solve the saddle-point Schur complement problem

$$S \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} S_\Gamma & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & -\frac{1}{\lambda} C_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}}_\Gamma \\ 0 \end{bmatrix} \tag{4.1}$$

by a preconditioned Krylov space method such as GMRES or PCG. The latter can be applied to this indefinite problem because we will start and keep the iterates in the subspace of benign functions (see Definition 2.4), on which the preconditioned operator T, defined below, is positive definite (see Theorem 5.4).

The matrix form of the preconditioner is

$$Q = Q_H + (I - Q_H S) \sum_{i=1}^{N} Q_i (I - S Q_H),$$

where the coarse operator Q_H and local operators Q_i are defined below. The preconditioned operator — usually referred to as the Schwarz operator — is then

$$T = QS = T_H + (I - T_H) \sum_{i=1}^{N} T_i (I - T_H),$$

where $T_H = Q_H S$ and $T_i = Q_i S$. We note that Q can also be written as a three-step preconditioner as in [46]. For simplicity, we will use the same symbol (for example $\mathbf{v}_\Gamma$) for both the interface vector and the function of $\mathbf{V}_\Gamma$ obtained by extension inside each subdomain using the discrete saddle-point

harmonic extension operator $\mathcal{SH}$. In addition, we will not write finite element interpolants explicitly; therefore, when we write a product of functions, e.g. $\delta_i \mathbf{v}_\Gamma$, we will mean the finite element function with nodal values equal to those of the product of the two functions δ_i and $\mathbf{v}_\Gamma$.

This balancing Neumann-Neumann preconditioner T is associated with further decomposing the interface space $\mathbf{V}_\Gamma \times U_\Gamma$ into

$$\mathbf{V}_\Gamma \times U_\Gamma = \mathbf{V}_H \times U_H + \sum_{i=1}^{N} \mathbf{V}_{\Gamma,i} \times U_{\Gamma,i}.$$

This decomposition is *not* a direct sum. Here, the coarse pressure space U_H is chosen to be the same as the interface pressure space, i.e., $U_H = U_\Gamma$. The coarse displacement space $\mathbf{V}_H$ is defined below; it must include the scaled nullspace of $a(\cdot,\cdot)$ in order to ensure solvability of the local problems, but other factors, such as inf-sup stability, also play an important role in its design. The local spaces $\mathbf{V}_{\Gamma,i}$ are defined by:

$$\mathbf{V}_{\Gamma,i} = \{\mathbf{v} \in \mathbf{V}_\Gamma \mid \mathbf{v} \equiv 0 \text{ on } \Gamma \setminus \partial\Omega_i\}.$$

We now describe the coarse and local problems in detail.

4.1 The Coarse Problem

Given a residual vector $\mathbf{r}$, the coarse term $Q_H\mathbf{r}$ is the solution of a coarse, global saddle-point problem with a few displacement degrees of freedom and one constant pressure per subdomain Ω_i:

$$Q_H = R_H^T S_H^{-1} R_H,$$

where

$$R_H = \begin{bmatrix} L_H^T & 0 \\ 0 & I \end{bmatrix},$$

and

$$S_H = R_H S R_H^T = \begin{bmatrix} L_H^T S_\Gamma L_H & L_H^T B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} L_H & -\frac{1}{\lambda} C_{\Gamma\Gamma} \end{bmatrix}. \tag{4.2}$$

The columns of the matrix L_H span the coarse space $\mathbf{V}_H$ and in order to define them, we need to define the Neumann-Neumann counting functions $\delta_i \in \mathbf{V}_\Gamma$, associated with each subdomain Ω_i, and their pseudoinverses $\delta_i^\dagger$. Let Γ_h be the set of nodes lying on Γ and $\partial\Omega_{i,h}$ be the set of nodes lying on $\partial\Omega_i$. δ_i and $\delta_i^\dagger$ are completely defined by their nodal values on Γ_h:

- δ_i is zero at all nodes on $\Gamma_h \setminus \partial\Omega_{i,h}$ while its value at any node x on $\partial\Omega_{i,h}$ equals the number of subdomains to which x belongs;
- the pseudoinverse $\delta_i^\dagger$ is defined as

$$\delta_i^\dagger(x) = \begin{cases} \dfrac{1}{\delta_i(x)} & \text{if } x \in \Gamma_h \cap \partial\Omega_{i,h}, \\ 0 & \text{if } x \in \Gamma_h \setminus \partial\Omega_{i,h}. \end{cases}$$

The space $\mathbf{V}_H$ is defined as the range of L_H. We discuss a few alternative choices:

L_H^0: its columns contain the inverse counting functions $\delta_i^\dagger$ multiplied by the functions of $\ker(a)$ (i.e., the rigid body modes);

L_H^1: it includes, in addition to the columns of L_H^0, for $d = 2$ ($d = 3$), one bubble function per edge (face) of the interface, representing a quadratic function in the normal direction;

L_H^2: it includes, in addition to the columns of L_H^0, columns representing the continuous piecewise d-linear functions on the coarse mesh τ_H;

L_H^3: it includes, in addition to the columns of L_H^0, columns representing the continuous piecewise d-quadratic functions on the coarse mesh τ_H.

The choice $\mathbf{V}_H = \mathbf{V}_H^0$ corresponds to the standard choice for second order scalar elliptic problems and it provides a quite minimal coarse displacement space. It turns out not to be uniformly inf-sup stable (see numerical experiments in [46]) and it therefore leads to a nonscalable algorithm in the incompressible case. However, in the compressible case where λ/μ is bounded, it still leads to a scalable algorithm (see Theorem 5.4). In Lemmas 5.2 and 5.3, we show that $\mathbf{V}_H^1$ and $\mathbf{V}_H^3$ are inf-sup stable uniformly in the number of subdomains N and in μ/λ. Numerical experiments indicate that the choice $\mathbf{V}_H^2$ is also uniformly inf-sup stable.

In order to avoid linearly dependent $\delta_i^\dagger$ functions, and hence a singular coarse problem, we might have to drop all of the components of these functions for one subdomain; this depends on the coarse triangulation.

In variational terms, the coarse problem is defined as follows. Given $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ in $\mathbf{V}_\Gamma \times U_\Gamma$, define $\begin{bmatrix} \mathbf{w}_\Gamma \\ r_\Gamma \end{bmatrix} = T_H \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \in \mathbf{V}_H \times U_H$ as the solution of the

coarse saddle-point problem: $\forall \mathbf{v} \in \mathbf{V}_H$ *and* $\forall q \in U_H$,

$$\left\{ \begin{array}{rcrcrcl} s(\mathbf{w}_\Gamma, \mathbf{v}) & + & b_\Gamma(\mathbf{v}, r_\Gamma) & = & s(\mathbf{u}_\Gamma, \mathbf{v}) & + & b_\Gamma(\mathbf{v}, p_\Gamma) \\ b_\Gamma(\mathbf{w}_\Gamma, q) & - & \frac{1}{\lambda} c_\Gamma(r_\Gamma, q) & = & b_\Gamma(\mathbf{u}_\Gamma, q) & - & \frac{1}{\lambda} c_\Gamma(p_\Gamma, q) \end{array} \right. \tag{4.3}$$

or equivalently,

$$\left\{ \begin{array}{rcrcl} s(\mathbf{w}_\Gamma - \mathbf{u}_\Gamma, \mathbf{v}) & + & b_\Gamma(\mathbf{v}, r_\Gamma - p_\Gamma) & = & 0 \\ b_\Gamma(\mathbf{w}_\Gamma - \mathbf{u}_\Gamma, q) & - & \frac{1}{\lambda} c_\Gamma(r_\Gamma - p_\Gamma, q) & = & 0. \end{array} \right. \tag{4.4}$$

Lemma 4.2, below, lists some important properties of the operator T_H. Before proving this lemma, we will need the following auxiliary result:

Lemma 4.1 *If $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form and T is an operator such that $\langle (I - T)u, Tz \rangle = 0 \ \ \forall u, z$, then T is symmetric with respect to $\langle \cdot, \cdot \rangle$.*

Proof

$$\begin{array}{rcl} \langle Tu, z \rangle & = & \langle Tu, z \rangle + \langle (I - T)u, Tz \rangle = \langle Tu, z \rangle + \langle u, Tz \rangle - \langle Tu, Tz \rangle \\ & = & \langle u, Tz \rangle + \langle Tu, (I - T)z \rangle = \langle u, Tz \rangle , \end{array}$$

which implies the symmetry of T with respect to $\langle \cdot, \cdot \rangle$. □

Lemma 4.2 *For T_H as defined in (4.3) and (4.4), we have:*

- *T_H is a projection, i.e. $T_H^2 = T_H$;*
- *$(I - T_H) \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ does not depend on p_Γ, i.e.,*

$$(I - T_H) \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = (I - T_H) \begin{bmatrix} \mathbf{u}_\Gamma \\ q_\Gamma \end{bmatrix}$$

  for all $p_\Gamma, q_\Gamma \in U_\Gamma$;
- *T_H is symmetric with respect to the bilinear form $\langle \cdot, \cdot \rangle_S$;*
- *$Range(I - T_H) \subset (\mathbf{V}_\Gamma, U_\Gamma)_B$ (see Definition 2.4) and*
- *$T_H \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \in (\mathbf{V}_\Gamma, U_\Gamma)_B$ for all $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \in (\mathbf{V}_\Gamma, U_\Gamma)_B$.*

Proof It is clear that if $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \in \mathbf{V}_H \times U_H$, then $\begin{bmatrix} \mathbf{w}_\Gamma \\ r_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ satisfies (4.3), which shows that $T_H^2 = T_H$. In particular, since $U_H = U_\Gamma$, we have that for any $p_\Gamma \in U_\Gamma$, $\begin{bmatrix} 0 \\ p_\Gamma \end{bmatrix} \in \mathbf{V}_H \times U_H$ and therefore $T_H \begin{bmatrix} 0 \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ p_\Gamma \end{bmatrix}$ and $(I - T_H) \begin{bmatrix} 0 \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This proves the second statement.

By choosing $\begin{bmatrix} \mathbf{v} \\ q \end{bmatrix} = T_H \begin{bmatrix} \mathbf{z}_\Gamma \\ s_\Gamma \end{bmatrix}$ in (4.4), we conclude that

$$\left\langle (I - T_H) \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}, T_H \begin{bmatrix} \mathbf{z}_\Gamma \\ s_\Gamma \end{bmatrix} \right\rangle_S = 0 \quad \forall \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}, \begin{bmatrix} \mathbf{z}_\Gamma \\ s_\Gamma \end{bmatrix} \text{ in } \mathbf{V}_\Gamma \times U_\Gamma.$$

Lemma 4.1 now implies the symmetry of T_H with respect to S.

From the second equation of (4.4), we see that $\begin{bmatrix} \mathbf{w}_\Gamma - \mathbf{u}_\Gamma \\ r_\Gamma - p_\Gamma \end{bmatrix}$ is balanced, i.e., $Range(I - T_H) \subset (\mathbf{V}_\Gamma, U_\Gamma)_B$. Since $T_H = I - (I - T_H)$, the last statement follows. □

Remark 4.3 *Since T_H is a projection and is S-symmetric, it is tempting to believe that T_H is an S-orthogonal projection. This is* not *the case, since S is not positive definite. However, when restricted to the benign subspace $(\mathbf{V}_\Gamma, U_\Gamma)_B$, T_H is indeed an S-orthogonal projection.*

4.2 Local Problems

Each local operator Q_i is based on the solution of a local saddle-point problem on Ω_i with a natural boundary condition on $\partial\Omega_i \backslash \partial\Omega$. This local problem is singular for any *floating* subdomain, i.e., those subdomains whose boundaries do not intersect the Dirichlet boundary $\partial\Omega$. That is why we use a pseudoinverse in the matrix description of Q_i below:

$$Q_i = R_i^T \begin{bmatrix} D_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_\Gamma^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda} C_{\Gamma\Gamma}^{(i)} \end{bmatrix}^\dagger \begin{bmatrix} D_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} R_i. \tag{4.5}$$

Here R_i are $0, 1$ restriction matrices mapping $\mathbf{V}_\Gamma \times U_\Gamma$ into $\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i}$ and D_i are diagonal matrices representing multiplication by the counting functions δ_i.

For a floating subdomain Ω_i, the nullspace of $S^{(i)}$ is spanned by the rigid-body modes (rotations and translations). It is easy to check, though, that

$$\text{range}\left(\begin{bmatrix} D_i^{-1} & 0 \\ 0 & \star \end{bmatrix} R_i S(I - T_H)\right) \subset \text{range}\left(S^{(i)}\right),$$

as long as $\mathbf{V}_H \supset \mathbf{V}_H^0$, where $\star$ can be any scalar. Also

$$(I - T_H)R_i^T \begin{bmatrix} D_i^{-1} & 0 \\ 0 & \star \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\Gamma,i} \\ 0 \end{bmatrix} = 0$$

for any rigid-body mode vector $\mathbf{v}_{\Gamma,i}$. Therefore, the particular choice of the pseudoinverse will not affect the algorithm; this fact will allow for an easier implementation of Q_i. In the sequel, we will fix this choice by assuming that $\text{range}\left(S^\dagger\right) = \text{range}\,(S)$. We note that the 1's in (4.5) could equally well be replaced by zero.

In preparation for writing the local problems in variational form, we define the operator $\widetilde{T}_i : \mathbf{V}_\Gamma \times U_\Gamma \to \mathbf{V}_{\Gamma,i} \times U_{\Gamma,i}$ as $\widetilde{T}_i = R_i T_i$, and note that $R_i^T \widetilde{T}_i = T_i$. The local problems are now defined in variational terms: for $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$, $\widetilde{T}_i \mathbf{w} = \begin{bmatrix} \widetilde{T}_i^u \mathbf{w} \\ \widetilde{T}_i^p \mathbf{w} \end{bmatrix}$ is the solution of the following local saddle-point problem with natural boundary conditions: $\forall \mathbf{v} \in \mathbf{V}_{\Gamma,i}, \forall q \in U_{\Gamma,i}$,

$$\begin{cases} s_i(\delta_i \widetilde{T}_i^u \mathbf{u}_\Gamma, \delta_i \mathbf{v}) + b_{\Gamma,i}(\delta_i \mathbf{v}, \widetilde{T}_i^p \mathbf{u}_\Gamma) = s(\mathbf{u}_\Gamma, \mathbf{v}) + b_\Gamma(\mathbf{v}, p_\Gamma) \\ b_{\Gamma,i}(\delta_i \widetilde{T}_i^u \mathbf{u}_\Gamma, q) - \frac{1}{\lambda} c_{\Gamma,i}(\widetilde{T}_i^p \mathbf{u}_\Gamma, q) = b_\Gamma(\mathbf{u}_\Gamma, q) - \frac{1}{\lambda} c_\Gamma(p_\Gamma, q). \end{cases} \tag{4.6}$$

Analogously to Definition 2.4, we introduce the concept of *local benign spaces.*

Definition 4.4 *We define the local benign space* $(\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i})_B$ *by*

$$\begin{aligned} &(\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i})_B \\ &\quad = \left\{ \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} \in \mathbf{V}_{\Gamma,i} \times U_{\Gamma,i} \;\middle|\; b_{\Gamma,i}(\delta_i \mathbf{u}, q) = \tfrac{1}{\lambda} c_{\Gamma,i}(p, q) \quad \forall q \in U_{\Gamma,i} \right\}. \end{aligned} \tag{4.7}$$

We note that whenever $\mathbf{w} \in \mathit{Range}(I - T_0) \subset (\mathbf{V}_\Gamma \times U_\Gamma)_B$, the right-hand side of the second equation in (4.6) equals zero and in this case we can use

Lemma 2.6 and restate the definition of $\widetilde{T}_i$ as follows: $\widetilde{T}_i\mathbf{w} \in (\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i})_B$ *and satisfies,* $\forall \begin{bmatrix} \mathbf{v} \\ q \end{bmatrix} \in (\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i})_B$,

$$s_i(\delta_i \widetilde{T}_i^u \mathbf{w}, \delta_i \mathbf{v}) + \frac{1}{\lambda} c_{\Gamma,i}(\widetilde{T}_i^p \mathbf{w}, q) = s(\mathbf{u}_\Gamma, \mathbf{v}) + b_\Gamma(\mathbf{v}, p_\Gamma), \tag{4.8}$$

or,

$$\left(\begin{bmatrix} \delta_i \widetilde{T}_i^u \mathbf{w} \\ \widetilde{T}_i^p \mathbf{w} \end{bmatrix}, \begin{bmatrix} \delta_i \mathbf{v} \\ q \end{bmatrix} \right)_{\Gamma,i} = \left\langle \mathbf{w}, \begin{bmatrix} \mathbf{v} \\ q \end{bmatrix} \right\rangle_S . \tag{4.9}$$

Here the inner-product $(\cdot,\cdot)_{\Gamma,i}$ is defined by

$$\left(\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ q \end{bmatrix} \right)_{\Gamma,i} = s_i(\delta_i \mathbf{u}, \delta_i \mathbf{v}) + \frac{1}{\lambda} c_{\Gamma,i}(p, q)$$

Chapter 5

Analysis of the Method

5.1 Auxiliary Results

We will work with the Γ-inner product

$$\left(\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ q_\Gamma \end{bmatrix} \right)_\Gamma = s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) + \frac{1}{\lambda} c_\Gamma(p_\Gamma, q_\Gamma).$$

On the benign subspace $(\mathbf{V}_\Gamma \times U_\Gamma)_B$ (see Definition 2.4), this inner product coincides with the bilinear form $\langle \cdot, \cdot \rangle_S$; cf. Lemma 2.5.

In the proof of our main result, we need a bound on the norm of the coarse correction operator. We note that this operator has norm 1 when restricted to the space of benign functions but that it is applied to more general functions in our algorithm.

Lemma 5.1 *The coarse correction operator $I - T_H$ satisfies the stability estimate*

$$\|I - T_H\|_\Gamma \leq 1 + \sqrt{2\left(2 + \frac{d}{\frac{\beta_H^2}{2(1+\sigma)^2} + \frac{\mu}{\lambda}}\right)},$$

where σ is as in Lemma 3.1 and β_H is the inf-sup constant of the coarse space, defined as

$$\beta_H = \inf_{q \in U_H} \sup_{\mathbf{v} \in \mathbf{V}_H} \frac{b(\mathbf{v}, q)}{\|\nabla(\mathcal{H}\mathbf{v}|_\Gamma)\|_{L^2} \|q\|_{L^2}}. \tag{5.1}$$

We note that the bound above is less than or equal to $C\sqrt{1 + \lambda/\mu}$ whatever the value of β_H. We will also establish such a bound by a direct argument in the general case discussed in Chapter 6.

Proof We want to bound $\left\|(I-T_H)\begin{bmatrix}\mathbf{u}_\Gamma\\ p_\Gamma\end{bmatrix}\right\|_\Gamma$ in terms of $\left\|\begin{bmatrix}\mathbf{u}_\Gamma\\ p_\Gamma\end{bmatrix}\right\|_\Gamma$. In view of Lemma 4.2, we have

$$\begin{aligned}\left\|(I-T_H)\begin{bmatrix}\mathbf{u}_\Gamma\\ p_\Gamma\end{bmatrix}\right\|_\Gamma &= \left\|(I-T_H)\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix}\right\|_\Gamma \\ &\le \left\|\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix}\right\|_\Gamma + \left\|T_H\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix}\right\|_\Gamma \qquad (5.2)\\ &\le \left\|\begin{bmatrix}\mathbf{u}_\Gamma\\ p_\Gamma\end{bmatrix}\right\|_\Gamma + \left\|T_H\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix}\right\|_\Gamma\end{aligned}$$

We define

$$\widetilde{T}_H\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix} = \begin{bmatrix}\widetilde{T}_H^u\mathbf{u}_\Gamma\\ \widetilde{T}_H^p\mathbf{u}_\Gamma\end{bmatrix} = \begin{bmatrix}L_H^TS_\Gamma L_H & L_H^TB_{\Gamma\Gamma}^T\\ B_{\Gamma\Gamma}L_H & -\frac{1}{\lambda}C_{\Gamma\Gamma}\end{bmatrix}^{-1}\begin{bmatrix}L_H^TS_\Gamma\mathbf{u}_\Gamma\\ B_{\Gamma\Gamma}\mathbf{u}_\Gamma\end{bmatrix},$$

so that

$$T_H\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix} = \begin{bmatrix}T_H^u\mathbf{u}_\Gamma\\ T_H^p\mathbf{u}_\Gamma\end{bmatrix} = \begin{bmatrix}L_H & 0\\ 0 & I\end{bmatrix}\widetilde{T}_H\begin{bmatrix}\mathbf{u}_\Gamma\\ 0\end{bmatrix}.$$

In view of Lemma 3.1, we find that

$$\frac{1}{\mu}\tilde{\mathbf{v}}^TL_H^TS_\Gamma L_H\tilde{\mathbf{v}} \le 2(1+\sigma)^2\tilde{\mathbf{v}}^TL_H^T\Delta_\Gamma L_H\tilde{\mathbf{v}},$$

where Δ_Γ is such that $\mathbf{v}|_\Gamma^T\Delta_\Gamma\mathbf{v}|_\Gamma = \|\nabla(\mathcal{H}\mathbf{v}|_\Gamma)\|_{L^2}^2$. Let $\omega = \frac{\beta_H^2}{2(1+\sigma)^2} + \frac{\mu}{\lambda}$. Then Lemma 2.10 gives

$$\begin{aligned}\|\widetilde{T}_H^u\mathbf{u}_\Gamma\|^2_{\frac{1}{\mu}L_H^TS_\Gamma L_H} &\le \tfrac{2}{\mu^2}\|L_H^TS_\Gamma\mathbf{u}_\Gamma\|^2_{(\frac{1}{\mu}L_H^TS_\Gamma L_H)^{-1}} + \tfrac{2}{\omega}\|B_{\Gamma\Gamma}\mathbf{u}_\Gamma\|^2_{C_{\Gamma\Gamma}^{-1}},\\ \|\widetilde{T}_H^p\mathbf{u}_\Gamma\|^2_{C_{\Gamma\Gamma}} &\le \tfrac{2}{\omega}\|L_H^TS_\Gamma\mathbf{u}_\Gamma\|^2_{(\frac{1}{\mu}L_H^TS_\Gamma L_H)^{-1}} + \tfrac{2\mu^2}{\omega^2}\|B_{\Gamma\Gamma}\mathbf{u}_\Gamma\|^2_{C_{\Gamma\Gamma}^{-1}}.\end{aligned}$$

We proceed to estimate each of the terms above:

$$\begin{aligned}\|\widetilde{T}_H^u\mathbf{u}_\Gamma\|^2_{\frac{1}{\mu}L_H^TS_\Gamma L_H} &= \frac{1}{\mu}\mathbf{u}_\Gamma^T{\widetilde{T}_H^u}^TL_H^TS_\Gamma L_H\widetilde{T}_H^u\mathbf{u}_\Gamma = \frac{1}{\mu}s(T_H^u\mathbf{u}_\Gamma, T_H^u\mathbf{u}_\Gamma)\\ \|\widetilde{T}_H^p\mathbf{u}_\Gamma\|^2_{C_{\Gamma\Gamma}} &= c_\Gamma(T_H^p\mathbf{u}_\Gamma, T_H^p\mathbf{u}_\Gamma)\\ \|L_H^TS_\Gamma\mathbf{u}_\Gamma\|^2_{(\frac{1}{\mu}L_H^TS_\Gamma L_H)^{-1}} &= \sup_{\tilde{\mathbf{v}}}\frac{(\tilde{\mathbf{v}}^TL_H^TS_\Gamma\mathbf{u}_\Gamma)^2}{\frac{1}{\mu}\tilde{\mathbf{v}}^TL_H^TS_\Gamma L_H^T\tilde{\mathbf{v}}} \le \mu s(\mathbf{u}_\Gamma,\mathbf{u}_\Gamma)\\ \|B_{\Gamma\Gamma}\mathbf{u}_\Gamma\|^2_{C_{\Gamma\Gamma}^{-1}} &= \sup_q\frac{(q^TB_{\Gamma\Gamma}\mathbf{u}_\Gamma)^2}{q^TC_{\Gamma\Gamma}q} = \sup_q\frac{(\mathrm{div}\mathcal{S}\mathcal{H}\mathbf{u}_\Gamma, q)^2_{L^2}}{\|q\|^2_{L_2}}\\ &\le \|\mathrm{div}\mathcal{S}\mathcal{H}\mathbf{u}_\Gamma\|^2_{L^2} \le \frac{1}{\mu}\frac{d}{2}s(\mathbf{u}_\Gamma,\mathbf{u}_\Gamma)\end{aligned}$$

In the last estimate above, we have used (3.4).

Using the previous results, we conclude that

$$\begin{aligned}
\left\| T_H \begin{bmatrix} \mathbf{u}_\Gamma \\ 0 \end{bmatrix} \right\|_\Gamma^2 &= \left\| \begin{bmatrix} T_H^u \mathbf{u}_\Gamma \\ T_H^p \mathbf{u}_\Gamma \end{bmatrix} \right\|_\Gamma^2 \\
&= s(T_H^u \mathbf{u}_\Gamma, T_H^u \mathbf{u}_\Gamma) + \frac{1}{\lambda} c_\Gamma(T_H^p \mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma) \\
&\leq \left(1 + \frac{\mu/\lambda}{\omega}\right)\left(2 + \frac{d}{\omega}\right) s(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \\
&\leq 2\left(2 + \frac{d}{\omega}\right) s(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \\
&\leq 2\left(2 + \frac{d}{\omega}\right) \left\| \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \right\|_\Gamma^2
\end{aligned}$$

From (5.2), we finally conclude that

$$\left\| (I - T_H) \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \right\|_\Gamma \leq \left(1 + \sqrt{2\left(2 + \frac{d}{\omega}\right)}\right) \left\| \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} \right\|_\Gamma .$$

Since $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ is arbitrary, the result follows. □

In the following two lemmas, we ensure that for appropriate choices of coarse space for the displacement, we have good bounds for β_H, defined in (5.1). First we show that we can reduce the problem to the incompressible Stokes case.

Lemma 5.2

$$\inf_{q \in U_H} \sup_{\mathbf{v} \in \mathbf{V}_H} \frac{b(\mathbf{v}, q)}{\|\nabla(\mathcal{H}\mathbf{v}|_\Gamma)\|_{L^2} \|q\|_{L^2}} \geq \inf_{q \in U_H} \sup_{\mathbf{v} \in \mathbf{V}_H} \frac{b(\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma, q)}{\|\nabla(\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma)\|_{L^2} \|q\|_{L^2}},$$

where $\widetilde{\mathcal{SH}}$ *represents the* Stokes *harmonic extension.*

Proof Because $\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma$ and $\mathbf{v}$ coincide on Γ, the divergence theorem assures that the numerators are the same in both sides of the inequality. As for the denominator, the minimal property of the harmonic extension guarantees that $\|\nabla(\mathcal{H}\mathbf{v}|_\Gamma)\|_{L^2} \leq \|\nabla(\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma)\|_{L^2}$. □

Now we can use the following result, which is proved in [46, Lemma 5.2].

Lemma 5.3 *The coarse spaces $V_H^1 \times U_H$ and $V_H^3 \times U_H$ satisfy the inf-sup condition*

$$\beta_H = \inf_{q \in U_H} \sup_{\mathbf{v} \in \mathbf{V}_H} \frac{b(\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma, q)}{\|\nabla(\widetilde{\mathcal{SH}}\mathbf{v}|_\Gamma)\|_{L^2}\|q\|_{L^2}} > 0,$$

where β_H is independent of the number of subdomains N. Moreover, for the space $V_H^3 \times U_H$, β_H is also independent of the size of the local problems, measured by H/h, while for $V_H^1 \times U_H$ we have

$$\beta_H \geq \frac{C}{\sqrt{1 + \log \frac{H}{h}}}.$$

Numerical results in [24] and [46] indicate that a uniform inf-sup condition does not hold for the coarse space $\mathbf{V}_H^0 \times U_H$. The results for the coarse space $\mathbf{V}_H^2 \times U_H$ are quite satisfactory although we do not have a full theory. We note that the $Q_1 - Q_0$ elements by themselves are not inf-sup stable but that we are using a richer velocity space which also includes the $\delta_i^\dagger$ functions times basis elements for the space of rigid body modes. We also work in the somewhat different context of saddle-point harmonic extensions of traces on Γ.

5.2 Main Result

We are now ready to formulate our main theorem.

Theorem 5.4 *On the benign subspace $(\mathbf{V}_\Gamma \times U_\Gamma)_B$ the balancing Neumann-Neumann operator T is symmetric, positive definite with respect to the inner product $(\cdot,\cdot)_\Gamma$, and its condition number $\kappa(T)$ is bounded by*

$$\kappa(T) \leq C \left(1 + \log \frac{H}{h}\right)^2 \left(1 + \left(\beta^2 + \frac{\mu}{\lambda}\right)^{-1}\right) \left(1 + \left(\frac{\beta_H^2}{1+\sigma^2} + \frac{\mu}{\lambda}\right)^{-1}\right), \quad (5.3)$$

where σ is as in Lemma 3.1 and β_H and β are the inf-sup constants of the coarse problem and the original discrete saddle-point problem, respectively.

Proof Let $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix}$ be benign. Then, $T_H\mathbf{w}$ as well as $(I - T_H)\mathbf{w}$ are benign and we can use either $(\cdot,\cdot)_\Gamma$ or $\langle\cdot,\cdot\rangle_S$ in our formulas. Since T_H is a

$(\cdot,\cdot)_\Gamma$-orthogonal projection when restricted to the benign subspace (cf. Lemma 4.2), we find that

$$\begin{aligned}
&(T\mathbf{w},\mathbf{w})_\Gamma \\
&\quad = (T_H\mathbf{w},\mathbf{w})_\Gamma + ((I-T_H)\textstyle\sum_i T_i(I-T_H)\mathbf{w},\mathbf{w})_\Gamma \\
&\quad = \|T_H\mathbf{w}\|_\Gamma^2 + \langle\textstyle\sum_i T_i(I-T_H)\mathbf{w},(I-T_H)\mathbf{w}\rangle_S \\
&\quad = \|\mathbf{w}\|_\Gamma^2 - \|(I-T_H)\mathbf{w}\|_\Gamma^2 + \langle\textstyle\sum_i T_i(I-T_H)\mathbf{w},(I-T_H)\mathbf{w}\rangle_S .
\end{aligned} \tag{5.4}$$

Our goal is to find both lower and upper bounds for this expression in terms of $\|\mathbf{w}\|_\Gamma^2$.

5.2.1 Lower Bound

Define $\tilde{\mathbf{w}} = \begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_\Gamma \end{bmatrix} = (I-T_H)\mathbf{w}$. Since the pseudo inverses $\delta_i^\dagger$ of the counting functions define a partition of unity, we have $\tilde{\mathbf{u}}_\Gamma = \sum_i \tilde{\mathbf{u}}_i$ with $\tilde{\mathbf{u}}_i = \delta_i^\dagger\tilde{\mathbf{u}}_\Gamma \in \mathbf{V}_{\Gamma,i}$. Let $\tilde{q}_i$ be such that $\begin{bmatrix} \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \in (\mathbf{V}_{\Gamma,i}\times U_{\Gamma,i})_B$ (see Definition 4.4). From the definition of the local problems (4.8), we have

$$\begin{aligned}
s(\tilde{\mathbf{u}}_\Gamma,\tilde{\mathbf{u}}_\Gamma) &= \sum_i s(\tilde{\mathbf{u}}_\Gamma,\tilde{\mathbf{u}}_i) \\
&= \sum_i s_i(\delta_i\widetilde{T}_i^u\tilde{\mathbf{w}},\delta_i\tilde{\mathbf{u}}_i) + \sum_i \frac{1}{\lambda}c_{\Gamma,i}(\widetilde{T}_i^p\tilde{\mathbf{w}},\tilde{q}_i) - \sum_i b_\Gamma(\tilde{\mathbf{u}}_i,\tilde{p}_\Gamma).
\end{aligned}$$

But $\sum_i b_\Gamma(\tilde{\mathbf{u}}_i,\tilde{p}_\Gamma) = b_\Gamma(\tilde{\mathbf{u}}_\Gamma,\tilde{p}_\Gamma) = \frac{1}{\lambda}c_\Gamma(\tilde{p}_\Gamma,\tilde{p}_\Gamma)$ because $\begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_\Gamma \end{bmatrix}$ is benign. Then,

$$\begin{aligned}
\|\tilde{\mathbf{w}}\|_\Gamma^2 &= s(\tilde{\mathbf{u}}_\Gamma,\tilde{\mathbf{u}}_\Gamma) + \tfrac{1}{\lambda}c_\Gamma(\tilde{p}_\Gamma,\tilde{p}_\Gamma) \\
&= \textstyle\sum_{i=1}^N s_i(\delta_i\widetilde{T}_i^u\tilde{\mathbf{w}},\delta_i\tilde{\mathbf{u}}_i) + \tfrac{1}{\lambda}c_{\Gamma,i}(\widetilde{T}_i^p\tilde{\mathbf{w}},\tilde{q}_i) \\
&= \textstyle\sum_{i=1}^N \left(\begin{bmatrix} \delta_i\widetilde{T}_i^u\tilde{\mathbf{w}} \\ \widetilde{T}_i^p\tilde{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \delta_i\tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right)_{\Gamma,i} \\
&\le \left(\textstyle\sum_{i=1}^N \left\| \begin{bmatrix} \delta_i\widetilde{T}_i^u\tilde{\mathbf{w}} \\ \widetilde{T}_i^p\tilde{\mathbf{w}} \end{bmatrix} \right\|_{\Gamma,i}^2 \right)^{\frac{1}{2}} \left(\textstyle\sum_{i=1}^N \left\| \begin{bmatrix} \delta_i\tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right\|_{\Gamma,i}^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{5.5}$$

We note that $\delta_i\tilde{\mathbf{u}}_i = \delta_i\delta_i^\dagger\tilde{\mathbf{u}}_\Gamma = \tilde{\mathbf{u}}_\Gamma|_{\partial\Omega_i}$. From the definition of $(\mathbf{V}_{\Gamma,i}\times U_{\Gamma,i})_B$, we have that $b_{\Gamma,i}(\delta_i\tilde{\mathbf{u}}_i,r_i) = \frac{1}{\lambda}c_{\Gamma,i}(\tilde{q}_i,r_i)$ for all $r_i \in U_{\Gamma,i}$. Summing over i and

recalling that $\begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_\Gamma \end{bmatrix}$ is benign, we conclude that $\tilde{q}_i = \tilde{p}_\Gamma|_{\Omega_i}$. The square of the second factor in (5.5) is therefore

$$\sum_{i=1}^{N} \left\| \begin{bmatrix} \delta_i \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right\|_{\Gamma,i}^2 = \left\| \begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_\Gamma \end{bmatrix} \right\|_{\Gamma}^2 = \|\tilde{\mathbf{w}}\|_\Gamma^2 . \tag{5.6}$$

The square of the first factor in (5.5) is estimated by using the definition of the local problems (4.9):

$$\begin{aligned} \sum_i \left\| \begin{bmatrix} \delta_i \widetilde{T}_i^u \tilde{\mathbf{w}} \\ \widetilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\|_{\Gamma,i}^2 &= \sum_i \left(\begin{bmatrix} \delta_i \widetilde{T}_i^u \tilde{\mathbf{w}} \\ \widetilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \delta_i \widetilde{T}_i^u \tilde{\mathbf{w}} \\ \widetilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right)_{\Gamma,i} \\ &= \sum_i \left\langle \tilde{\mathbf{w}}, \begin{bmatrix} \widetilde{T}_i^u \tilde{\mathbf{w}} \\ \widetilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\rangle_S \\ &= \langle (I - T_H)\mathbf{w}, \textstyle\sum_i T_i (I - T_H)\mathbf{w} \rangle_S . \end{aligned} \tag{5.7}$$

Putting (5.5), (5.6), and (5.7) together, we obtain

$$\|(I - T_H)\mathbf{w}\|_\Gamma^2 \leq \left\langle \sum_i T_i (I - T_H)\mathbf{w}, (I - T_H)\mathbf{w} \right\rangle_S . \tag{5.8}$$

Finally, from (5.4) and (5.8),

$$(T\mathbf{w}, \mathbf{w})_\Gamma \geq \|\mathbf{w}\|_\Gamma^2 .$$

5.2.2 Upper bound

We recall that T_H restricted to the benign subspace is an orthogonal projection with respect to $(\cdot,\cdot)_\Gamma$. Therefore, the only term we have to control in (5.4) is $\langle \sum_i T_i (I - T_H)\mathbf{w}, (I - T_H)\mathbf{w} \rangle_S$. This expression will be bounded from above in terms of the square of the norm of $\mathbf{w}$. Since the norm of $(I - T_H)\mathbf{w}$ is less than or equal to that of $\mathbf{w}$, we will assume, henceforth, that $\mathbf{w} \in \mathit{Range}(I - T_H)$.

Then, by using Lemma 4.2,

$$\begin{aligned}
&\left\langle \sum_i T_i(I-T_H)\mathbf{w}, (I-T_H)\mathbf{w} \right\rangle_S \\
&= \langle \mathbf{w}, (I-T_H)\textstyle\sum_i T_i\mathbf{w}\rangle_S \\
&= \left\langle \mathbf{w}, (I-T_H)\textstyle\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ T_i^p\mathbf{w} \end{bmatrix} \right\rangle_S \\
&= \left\langle \mathbf{w}, (I-T_H)\textstyle\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ 0 \end{bmatrix} \right\rangle_S \\
&= \left(\mathbf{w}, (I-T_H)\textstyle\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ 0 \end{bmatrix} \right)_\Gamma \\
&\leq \|I-T_H\|_\Gamma \, \|\mathbf{w}\|_\Gamma \left\| \textstyle\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ 0 \end{bmatrix} \right\|_\Gamma
\end{aligned} \tag{5.9}$$

and we are left with bounding the third factor from above. We remark that each $T_i^u\mathbf{w}$ is supported in Ω_i and the subdomains adjacent to it. By a standard coloring argument, it suffices to bound the Γ-norm of just one term, $\left\| \begin{bmatrix} T_i^u\mathbf{w} \\ 0 \end{bmatrix} \right\|_\Gamma$, of the sum.

By the comparison of the energy of the discrete saddle-point and harmonic extensions in Lemma 3.1, we have

$$\left\| \begin{bmatrix} T_i^u\mathbf{w} \\ 0 \end{bmatrix} \right\|_\Gamma^2 = s(T_i^u\mathbf{w}, T_i^u\mathbf{w}) \leq 2(1+\sigma)^2\mu \|\nabla\mathcal{H}(T_i^u\mathbf{w})\|_{L^2(\Omega)}^2. \tag{5.10}$$

We then apply to each scalar component of $\mathcal{H}(T_i^u\mathbf{w})$ the decomposition lemma for the scalar Neumann-Neumann algorithm (see Dryja and Widlund [15, lemma 4]) and obtain

$$\mu\|\nabla\mathcal{H}(T_i^u\mathbf{w})\|_{L^2(\Omega)}^2 \leq C\alpha\mu \, \|\nabla\mathcal{H}(\delta_i T_i^u\mathbf{w})\|_{L^2(\Omega_i)}^2 ,$$

where $\alpha = (1+\log(H/h))^2$. By using Lemma 3.1 and equation (4.9), we have:

$$\begin{aligned}
&\mu \left\| \nabla \mathcal{H}(\delta_i T_i^u \mathbf{w}) \right\|^2_{L^2(\Omega_i)} \\
&\quad \leq \; C s_i(\delta_i T_i^u \mathbf{w}, \delta_i T_i^u \mathbf{w}) \\
&\quad \leq \; C \left\| \begin{bmatrix} \delta_i T_i^u \mathbf{w} \\ T_i^p \mathbf{w} \end{bmatrix} \right\|^2_{\Gamma,i} \\
&\quad = \; C \left\langle \mathbf{w}, T_i \mathbf{w} \right\rangle_S \\
&\quad = \; C \left\langle \mathbf{w}, (I - T_H) T_i \mathbf{w} \right\rangle_S \\
&\quad = \; C \left(\mathbf{w}, (I - T_H) \begin{bmatrix} T_i^u \mathbf{w} \\ 0 \end{bmatrix} \right)_\Gamma \\
&\quad \leq \; C \left\| I - T_H \right\|_\Gamma \; \left\| \mathbf{w} \right\|_\Gamma \; \left\| \begin{bmatrix} T_i^u \mathbf{w} \\ 0 \end{bmatrix} \right\|_\Gamma
\end{aligned} \tag{5.11}$$

From (5.9), (5.10) and (5.11), we obtain

$$\left\langle \sum_i T_i (I - T_H) \mathbf{w}, (I - T_H) \mathbf{w} \right\rangle_S \leq C \alpha (1+\sigma)^2 \left\| I - T_H \right\|^2_\Gamma \; \left\| \mathbf{w} \right\|^2_\Gamma \tag{5.12}$$

From (5.4) and (5.12), after using Lemma 5.1, we obtain

$$(T\mathbf{w}, \mathbf{w})_\Gamma \leq C \alpha (1+\sigma)^2 \left(1 + 2 \left(2 + \frac{d}{\frac{\beta_H^2}{2(1+\sigma)^2} + \frac{\mu}{\lambda}} \right) \right) \left\| \mathbf{w} \right\|^2_\Gamma .$$

Since the lower bound for $\frac{(T\mathbf{w},\mathbf{w})_\Gamma}{\|\mathbf{w}\|^2_\Gamma}$ is 1, the upper bound above is also the upper bound for $\kappa(T)$. The leading constant, as it appears in the statement of this theorem, is then obtained by elementary inequalities. $\square$

Chapter 6

Discontinuous Coefficients with Large Jumps

6.1 The algorithm

Our algorithm can be extended to handle problems with heterogeneous materials with different Lamé constants λ_i and μ_i in the different subdomains Ω_i. Such problems can be formulated as follows:

$$\begin{cases} \sum_{i=1}^{N} \mu_i a_i(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \langle \mathbf{F},\mathbf{v}\rangle & \forall \mathbf{v}\in\mathbf{V} \\ b(\mathbf{u},q) & - & \sum_{i=1}^{N} \frac{1}{\lambda_i} c_i(p,q) & = & 0 & \forall q\in U; \end{cases}$$

cf. equation (1.6). The global stiffness matrix K is constructed by subassembling the local contributions from the individual substructures,

$$K^{(i)} = \begin{bmatrix} \mu_i A^{(i)} & {B^{(i)}}^T \\ B^{(i)} & -\frac{1}{\lambda_i} C^{(i)} \end{bmatrix}.$$

A saddle-point Schur complement matrix can similarly be assembled from the matrices

$$S^{(i)} = \begin{bmatrix} S_\Gamma^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda_i} C_{\Gamma\Gamma}^{(i)} \end{bmatrix},$$

which are obtained from the $K^{(i)}$'s by static condensation.

The balancing Neumann-Neumann preconditioner Q for S has the same form as before, but uses modified local and coarse spaces and bilinear forms. As in the scalar elliptic case, the jumps in the coefficients μ_i are accounted for by appropriately scaling the special counting functions δ_i and their pseudoinverses $\delta_i^\dagger$. As in [39], we now use the definition

$$\delta_i^\dagger(x) = \frac{\mu_i^\gamma(x)}{\sum_{j\in N_x} \mu_j^\gamma(x)}, \tag{6.1}$$

where $\gamma \in [1/2, \infty)$ and N_x is the set of indices of all the subdomains that have the node x on their boundaries. The new δ_i is the pseudoinverse of $\delta_i^\dagger$. As before, both δ_i and $\delta_i^\dagger$ vanish at all interface nodes outside $\partial\Omega_i$ and are extended inside each subdomain by discrete saddle-point harmonic extensions. The pseudoinverses $\delta_i^\dagger$ still form a partition of unity. We have chosen $\gamma = 1$ in our numerical experiments. The local and coarse problems are then defined as before, but using the modified functions δ_i and $\delta_i^\dagger$.

Our balancing Neumann-Neumann preconditioner is therefore well defined also in the case of variable coefficients and our numerical experiments, reported in Chapter 10, indicate that indeed our preconditioner retains its excellent convergence rate also for heterogeneous materials.

6.2 Analysis

Unfortunately, we have not been able to completely extend our analysis to the case of variable coefficients. While it is straightforward to check that all other parts of the proof still works, we have not been able to extend Lemma 5.1 to the general case with variable coefficients. We note that we do not know how to prove the uniform inf-sup stability for the underlying finite element discretization or for the continuous problem for arbitrary heterogeneous coefficients and that is at the heart of our difficulties.

However, if we assume that in each subdomain the Poisson ratio is bounded away from 0.5, i.e., that λ/μ is uniformly bounded, then we can still prove an upper bound for $\|I - T_H\|_\Gamma$. Looking at the proof of Lemma 5.1, we see that what we need is to bound $\left\| T_H \begin{bmatrix} \mathbf{u}_\Gamma \\ 0 \end{bmatrix} \right\|_\Gamma = \left\| \begin{bmatrix} T_H^u \mathbf{u}_\Gamma \\ T_H^p \mathbf{u}_\Gamma \end{bmatrix} \right\|_\Gamma$ in terms of $\sum_i s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma)$. By making $p_\Gamma = 0$, $\mathbf{v} = T_H^u \mathbf{u}_\Gamma$ and $q = T_H^p \mathbf{u}_\Gamma$ in (4.3), we have

$$\sum_i s_i(T_H^u \mathbf{u}_\Gamma, T_H^u \mathbf{u}_\Gamma) + \sum_i \frac{1}{\lambda_i} c_{\Gamma,i}(T_H^p \mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma)$$

$$= \sum_i s_i(\mathbf{u}_\Gamma, T_H^u \mathbf{u}_\Gamma) - b_\Gamma(\mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma).$$

The first term on the right can be bounded by

$$\sum_i s_i(\mathbf{u}_\Gamma, T_H^u \mathbf{u}_\Gamma) \leq \frac{1}{2} \sum_i s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) + \frac{1}{2} \sum_i s_i(T_H^u \mathbf{u}_\Gamma, T_H^u \mathbf{u}_\Gamma).$$

For the second term, we have

$$\begin{aligned}
& b_\Gamma(\mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma) \\
&= \sum_i \int_{\Omega_i} \sqrt{\lambda_i} \mathrm{div} \mathcal{S}\mathcal{H}\mathbf{u}_\Gamma \frac{T_H^p \mathbf{u}_\Gamma}{\sqrt{\lambda_i}} \\
&\leq \frac{1}{2} \sum_i \lambda_i \|\mathrm{div} \mathcal{S}\mathcal{H}\mathbf{u}_\Gamma\|^2_{L^2(\Omega_i)} + \frac{1}{2} \sum_i \frac{1}{\lambda_i} \|T_H^p \mathbf{u}_\Gamma\|^2_{L^2(\Omega_i)} \\
&\leq \frac{1}{2}\frac{d}{2} \sum_i \frac{\lambda_i}{\mu_i} \mu_i a_i(\mathcal{S}\mathcal{H}\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}\mathbf{u}_\Gamma) + \frac{1}{2} \sum_i \frac{1}{\lambda_i} c_{\Gamma,i}(T_H^p \mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma) \\
&\leq \frac{1}{2}\frac{d}{2} \max_i \left(\frac{\lambda_i}{\mu_i}\right) \sum_i s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) + \frac{1}{2} \sum_i \frac{1}{\lambda_i} c_{\Gamma,i}(T_H^p \mathbf{u}_\Gamma, T_H^p \mathbf{u}_\Gamma).
\end{aligned}$$

We conclude that

$$\left\| \begin{bmatrix} T_H^u \mathbf{u}_\Gamma \\ T_H^p \mathbf{u}_\Gamma \end{bmatrix} \right\|^2_\Gamma \leq \left(1 + \frac{d}{2} \max_i \left(\frac{\lambda_i}{\mu_i}\right)\right) \sum_i s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma).$$

We stress once again that our numerical results in Chapter 10 do *not* indicate an actual dependence of the convergence rate on the parameter $\max_i \left(\frac{\lambda_i}{\mu_i}\right)$, which strongly indicates that the result above is not sharp.

Chapter 7

Some Implementation Issues

7.1 Saddle-Point Version of the Algorithm

7.1.1 Avoiding a special basis

In our discussion, we have assumed that the basis functions for the pressure degrees of freedom can be divided into two sets: functions with zero average and functions that are constant in each subdomain Ω_i; see (3.6). Although our method requires a pressure space that admits such a partition, it still can be implemented using a standard nodal basis for the pressure.

In our actual implementation, we generate a stiffness matrix $\widetilde{K}$ using a standard nodal basis, that does not separate zero-average pressures and constant-by-subdomain pressures. Furthermore, we never assemble the entire matrix $\widetilde{K}$. Instead, we work with local stiffness matrices:

$$\widetilde{K}^{(i)} = \begin{bmatrix} \mu A^{(i)} & \widetilde{B}^{(i)^T} & 0 \\ \widetilde{B}^{(i)} & -\frac{1}{\lambda}\,\widetilde{C}^{(i)} & w^{(i)} \\ 0 & w^{(i)^T} & 0 \end{bmatrix}. \tag{7.1}$$

We note that these local matrices include, besides the block $\begin{bmatrix} \mu A^{(i)} & \widetilde{B}^{(i)^T} \\ \widetilde{B}^{(i)} & -\frac{1}{\lambda}\,\widetilde{C}^{(i)} \end{bmatrix}$ from which the matrix $\widetilde{K}$ is subassembled, also blocks associated with a Lagrange multiplier. The multiplier is used to enforce zero average for the pressures when solving the local Dirichlet problems. Here, the matrices $\widetilde{B}^{(i)}$ and $\widetilde{C}^{(i)}$ differ from $B^{(i)}$ and $C^{(i)}$ of Section 3.2, since a standard basis for the

pressure is being used. The entries of the vector $w^{(i)}$ are the integrals of the pressure basis functions.

In each of the local matrices $\widetilde{K}^{(i)}$, we eliminate the interior velocities, all the pressures and the Lagrange multiplier. This corresponds to taking the Schur complement with respect to the (2,2)-block in the following matrix, which is a reordering of (7.1):

$$\left[\begin{array}{c|c} \widetilde{K}_{DD}^{(i)} & \widetilde{K}_{D\Gamma}^{(i)} \\ \hline \widetilde{K}_{\Gamma D}^{(i)} & \widetilde{K}_{\Gamma\Gamma}^{(i)} \end{array}\right] = \left[\begin{array}{ccc|c} \mu A_{II}^{(i)} & \widetilde{B}_I^{(i)^T} & 0 & \mu A_{I\Gamma}^{(i)} \\ \widetilde{B}_I^{(i)} & -\frac{1}{\lambda}\,\widetilde{C}^{(i)} & w^{(i)} & \widetilde{B}_\Gamma^{(i)} \\ 0 & w^{(i)^T} & 0 & 0 \\ \hline \mu A_{\Gamma I}^{(i)} & \widetilde{B}_\Gamma^{(i)^T} & 0 & \mu A_{\Gamma\Gamma}^{(i)} \end{array}\right].$$

We can show that the result of this static condensation, $\widetilde{K}_{\Gamma\Gamma}^{(i)} - \widetilde{K}_{\Gamma D}^{(i)} {\widetilde{K}_{DD}^{(i)}}^{-1} \widetilde{K}_{D\Gamma}^{(i)}$, is equal to $S_\Gamma^{(i)}$, the $(1,1)$-block of $S^{(i)}$, as defined in (3.9). We note that while the ${w^{(i)}}^T$ in the third row is responsible for enforcing a zero-average pressure, the $w^{(i)}$ in the second row allows for the divergence equation to be satisfied only up to an additive constant, and therefore $\widetilde{K}_{DD}^{(i)}$ is invertible even in the incompressible limit.

The remaining blocks of $S^{(i)}$, the vector $B_{\Gamma\Gamma}^{(i)}$ and the scalar $-\frac{1}{\lambda}C_{\Gamma\Gamma}^{(i)}$, are computed using the formula:

$$\left[\begin{array}{cc} \mu A_{\Gamma\Gamma}^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda}C_{\Gamma\Gamma}^{(i)} \end{array}\right] = \left[\begin{array}{c} I \\ {e^{(i)}}^T \end{array}\right] \left[\begin{array}{cc} \mu A_{\Gamma\Gamma}^{(i)} & {\widetilde{B}_\Gamma^{(i)}}^T \\ \widetilde{B}_\Gamma^{(i)} & -\frac{1}{\lambda}\,\widetilde{C}^{(i)} \end{array}\right] \left[\begin{array}{cc} I & e^{(i)} \end{array}\right].$$

Here the entries of the vector $e^{(i)}$ are the coefficients that express the constant pressure on subdomain Ω_i in terms of the standard basis functions, i.e.,

$$\sum_{k=1}^{\widetilde{n}_p} e_k^{(i)} \widetilde{\psi}_k = \chi_{\Omega_i},$$

where $\left\{\widetilde{\psi}_k\right\}_{k=1,\ldots,\tilde{n}_p}$ is the regular pressure basis and χ_{Ω_i} is the characteristic function of the subdomain Ω_i.

7.1.2 Solution of the Local Problems

We described our local problems in terms of the operators ${S^{(i)}}^\dagger$, which, thanks to the coarse corrrection $(I - T_H)$, are applied solely to vectors in range($\mathrm{S}^{(i)}$).

We adopt the following definition of pseudoinverse: ${S^{(i)}}^\dagger$ is any linear transformation satisfying

$$S^{(i)} {S^{(i)}}^\dagger S^{(i)} \begin{bmatrix} \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix} = S^{(i)} \begin{bmatrix} \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix} \quad \forall \begin{bmatrix} \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix} \in \mathbf{V}_{\Gamma,i} \times U_{\Gamma,i}.$$

Then it is easy to show that $\forall \begin{bmatrix} \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix} \in \text{range}(S^{(i)})$,

$$\begin{aligned} &\begin{bmatrix} -\begin{bmatrix} \mu A_{II}^{(i)} & {B_{II}^{(i)}}^T \\ B_{II}^{(i)} & -\frac{1}{\lambda} C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{I\Gamma}^{(i)} & 0 \\ B_{I\Gamma}^{(i)} & 0 \end{bmatrix} \\ I \end{bmatrix} {S^{(i)}}^\dagger \begin{bmatrix} \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \mu A_{II}^{(i)} & {B_{II}^{(i)}}^T & \mu A_{I\Gamma}^{(i)} & 0 \\ B_{II}^{(i)} & -\frac{1}{\lambda} C_{II}^{(i)} & B_{I\Gamma}^{(i)} & 0 \\ \mu A_{\Gamma I}^{(i)} & {B_{I\Gamma}^{(i)}}^T & \mu A_{\Gamma\Gamma}^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ 0 & 0 & B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda} C_{\Gamma\Gamma}^{(i)} \end{bmatrix}^\dagger}_{{K_{NN}^{(i)}}^\dagger} \begin{bmatrix} 0 \\ 0 \\ \mathbf{v}_\Gamma^{(i)} \\ q_\Gamma^{(i)} \end{bmatrix}, \end{aligned}$$

in the sense that once a choice of pseudo-inverse for $S^{(i)}$ is fixed, there exists a choice for ${K_{NN}^{(i)}}^\dagger$ that satisfies the identity above and vice-versa. Therefore we will be able, once again, to compute the action of ${S^{(i)}}^\dagger$ on a vector without explicitly assembling the matrix $S^{(i)}$.

The matrices $K_{NN}^{(i)}$ are invertible except for floating subdomains Ω_i. For these subdomains, a naive attempt to factor $K_{NN}^{(i)}$ will fail. Our approach, which is very convenient in terms of implementation and which has proven to be very robust in all our numerical experiments, is inspired in the following observation.

Lemma 7.1 *Let S be a real-symmetric matrix. Then $(S + \varepsilon I)^{-1}$ is a good approximation for $S^\dagger$, in the sense that*

$$\left\| S(S + \varepsilon I)^{-1} S - S \right\| = \mathcal{O}(\varepsilon).$$

Proof Let $U\Lambda U^T = S$ be the eigenvalue decomposition of S, where U is orthogonal and

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}.$$

is diagonal. Then

$$S(S+\varepsilon I)^{-1}S - S = -\varepsilon U \begin{bmatrix} \frac{\lambda_1}{\lambda_1+\varepsilon} & & & & & \\ & \ddots & & & & \\ & & \frac{\lambda_r}{\lambda_r+\varepsilon} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} U^T$$

and the result follows from elementary estimates. □

Remark 7.2 *The same result does* not *hold for nonsymmetric matrices. For instance, if* $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$*, then it is trivial to verify that*

$$\left\| S(S+\varepsilon I)^{-1}S - S \right\| = \mathcal{O}(1).$$

In our implementation of our algorithm, we simply add a small multiple of the identity to $\mu A^{(i)}$, when solving the Neumann problems involving $K_{NN}^{(i)}$.

7.2 Positive Definite Implementation of the Algorithm

Recall the block structure of S,

$$S = \begin{bmatrix} S_\Gamma & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & -\frac{1}{\lambda}C_{\Gamma\Gamma} \end{bmatrix}.$$

For compressible materials, the pressure variables can be eliminated by static condensation. We define

$$\widetilde{S} = S_\Gamma + \lambda B_{\Gamma\Gamma}^T C_{\Gamma\Gamma}^{-1} B_{\Gamma\Gamma} \tag{7.2}$$

and we note that $\widetilde{S}$ is positive definite. The solution to (3.7) is then given by

$$\begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} I \\ \lambda C_{\Gamma\Gamma}^{-1} B_{\Gamma\Gamma} \end{bmatrix} \tilde{\mathbf{u}}_\Gamma,$$

where $\tilde{\mathbf{u}}_\Gamma$ solves $\widetilde{S}\tilde{\mathbf{u}}_\Gamma = \tilde{\mathbf{b}}_\Gamma$.

We can formulate a *positive definite implementation* of our algorithm, in which we construct a preconditioner $\widetilde{Q}$ for $\widetilde{S}$ and iteratively solve the system $\widetilde{Q}\widetilde{S}\tilde{\mathbf{u}}_\Gamma = \widetilde{Q}\tilde{\mathbf{b}}_\Gamma$. We will show that this algorithm is fully equivalent to the one already described. When the discrete pressure space is discontinuous across elements, as it is the case for $Q_2 - P_1$ or $Q_2 - Q_0$, this positive definite implementation is easier to program and is computationally less costly, since the elimination of the pressures can be done on the element level. If the element-level stiffness matrix is originally defined as

$$K_{\text{el}} = \begin{bmatrix} \mu A_{\text{el}} & B_{\text{el}}^T \\ B_{\text{el}} & -\frac{1}{\lambda} C_{\text{el}} \end{bmatrix},$$

we now define

$$\widetilde{K}_{\text{el}} = \mu A_{\text{el}} + \lambda B_{\text{el}}^T C_{\text{el}}^{-1} B_{\text{el}}.$$

Then, by subassembly, we form matrices $\widetilde{K}^{(i)}$ and $\widetilde{K}$, which are related to their original, saddle-point counterparts $K^{(i)}$ and K by

$$\widetilde{K}^{(i)} = \mu A^{(i)} + \lambda {B^{(i)}}^T {C^{(i)}}^{-1} B^{(i)} \quad \text{and} \quad \widetilde{K} = \mu A + \lambda B^T C^{-1} B.$$

After reordering, $\widetilde{K}$ can be written as

$$\widetilde{K} = \begin{bmatrix} \widetilde{K}_{II} & \widetilde{K}_{I\Gamma} \\ \widetilde{K}_{\Gamma I} & \widetilde{K}_{\Gamma\Gamma} \end{bmatrix}$$

and the action of $\widetilde{S}$, defined by (7.2), can be computed from

$$\widetilde{S} = \widetilde{K}_{\Gamma\Gamma} - \widetilde{K}_{\Gamma I} \widetilde{K}_{II}^{-1} \widetilde{K}_{I\Gamma}.$$

We define the preconditioner $\widetilde{Q}$ as

$$\widetilde{Q} = \widetilde{Q}_H + (I - \widetilde{Q}_H \widetilde{S}) \sum_{i=1}^{N} \widetilde{Q}_i (I - \widetilde{S}\widetilde{Q}_H),$$

where

$$\widetilde{Q}_H = L_H (L_H^T \widetilde{S} L_H)^{-1} L_H^T$$

and

$$\widetilde{Q}_i = \widetilde{R}_i^T \begin{bmatrix} 0 & D_i^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{K}_{II}^{(i)} & \widetilde{K}_{I\Gamma}^{(i)} \\ \widetilde{K}_{\Gamma I}^{(i)} & \widetilde{K}_{\Gamma\Gamma}^{(i)} \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ D_i^{-1} \end{bmatrix} \widetilde{R}_i.$$

Here, L_H and D_i are defined as in Sections 4.1 and 4.2 and the restriction matrices $\widetilde{R}_i$ map from the global interface displacement space $\mathbf{V}|_\Gamma$ to the local interface displacement space $\mathbf{V}|_{\Gamma \cap \partial\Omega_i}$.

The equivalence of this positive definite version of the algorithm and the original one can be summarized by the following relation, the proof of which we omit:

$$\begin{bmatrix} I \\ \lambda C_{\Gamma\Gamma}^{-1} B_{\Gamma\Gamma} \end{bmatrix} \widetilde{T} = T \begin{bmatrix} I \\ \lambda C_{\Gamma\Gamma}^{-1} B_{\Gamma\Gamma} \end{bmatrix}. \tag{7.3}$$

We point out that, despite of the fact that we iterate on the space of displacements only, the underlying finite-element formulation is mixed and our results will not present locking. Also, even though the matrix $\widetilde{S}$ gets increasingly ill-conditioned as λ/μ increases, the equivalence of the algorithms allows us to prove a bound on $\kappa(\widetilde{T})$ that is uniform in λ/μ.

The implementation of the method just described is very similar to the implementation of the standard Balancing Neumann-Neumann method for the pure-displacement formulation (1.5). The differences are in the stiffness matrices — we note that the matrix $\lambda B^T C^{-1} B$ can be regarded as a special discretization of the form $\lambda \int \mathrm{div}(\cdot)\mathrm{div}(\cdot)$ — and in the use of a richer coarse space ($\mathbf{V}_H^1$, $\mathbf{V}_H^2$ or $\mathbf{V}_H^3$, instead of $\mathbf{V}_H^0$). This similarity also simplifies the implementation of a code that combines mixed and pure-displacement formulations in different subdomains (see Chapter 8).

The local matrices that need to be factored are now somewhat smaller (since the pressure variables have been eliminated) and, most importantly, positive definite, which allows for the use of off-the-shelf reordering algorithms and of Cholesky solvers. Due to these factors, we can have a significant gain in performance.

7.2.1 A Positive Definite Theory?

The variational counterpart of solving $\widetilde{S}\tilde{\mathbf{u}}_\Gamma = \tilde{\mathbf{b}}_\Gamma$ is the following problem: *find* $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma$ *such that*

$$\tilde{s}(\mathbf{u}_\Gamma, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle \ \ \forall \mathbf{v} \in \mathbf{V}_\Gamma,$$

where

$$\tilde{s}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \mu a(\widetilde{\mathcal{SH}}^u \mathbf{u}_\Gamma, \widetilde{\mathcal{SH}}^u \mathbf{v}_\Gamma) + \frac{1}{\lambda} c(\widetilde{\mathcal{SH}}^p \mathbf{u}_\Gamma, \widetilde{\mathcal{SH}}^p \mathbf{v}_\Gamma) = \mathbf{u}_\Gamma^T \widetilde{S} \mathbf{v}_\Gamma.$$

Here, $\widetilde{\mathcal{SH}}$ is defined by the problem: *find* $\widetilde{\mathcal{SH}}^u \mathbf{u}_\Gamma \in \mathbf{V}$ *and* $\widetilde{\mathcal{SH}}^p \mathbf{u}_\Gamma \in U$ *such that* $\widetilde{\mathcal{SH}}^u \mathbf{u}_\Gamma \Big|_\Gamma = \mathbf{u}_\Gamma$ *and, for* $i = 1, \ldots, N$,

$$\left\{ \begin{array}{rcrcll} \mu a_i(\widetilde{\mathcal{SH}}^u \mathbf{u}, \mathbf{v}) & + & b_i(\mathbf{v}, \widetilde{\mathcal{SH}}^p \mathbf{u}_\Gamma) & = & 0 & \forall \mathbf{v} \in \mathbf{V}_i \\ b_i(\widetilde{\mathcal{SH}}^u \mathbf{u}, q) & - & \frac{1}{\lambda} c_i(\widetilde{\mathcal{SH}}^p \mathbf{u}_\Gamma, q) & = & 0 & \forall q \in U_i. \end{array} \right. \tag{7.4}$$

The only difference between (3.2) and (7.4) is that in the latter we allow q to vary in the entire space of local pressures U_i, which includes constant pressures.

Notice that the pair of spaces $\mathbf{V}_i \times U_i$ is *not* inf-sup stable (as opposed to the pair $\mathbf{V}_i \times U_{0,i}$). We do not know how to prove, in this case, a result in the spirit of Lemma 3.1. We have a positive definite algorithm, but so far we do not have a positive definite proof. The way in which we can prove the convergence of this algorithm is by estabilishing its equivalence to the saddle-point one, via (7.3), and relying on our results for that case. In particular, we stress that the use of an enriched coarse space, i.e., the use of L_H^1, L_H^2 or L_H^3, and not of L_H^0, is still a requirement when λ/μ is large. Clear numerical evidence indicates that an underlying inf-sup stable coarse space in necessary in order to assure a good convergence of the positive definite algorithm.

Chapter 8

Combining Mixed and Pure-Displacement Formulations

Consider a domain Ω, portions of which are comprised of almost-incompressible material and portions of which of fairly compressible material. This kind of configuration is actually encountered in practice, for instance in a mechanical structure that combines rubber and steel.

The pure-displacement formulation is not suitable for discretizing the problem on the entire domain, due to locking. One could, of course, use a mixed formulation for the entire domain. In this chapter, we instead discuss how to combine both formulations in one problem, using the pure-displacement formulation in the compressible regions and mixed formulation in the (almost-)incompressible.

8.1 Definition of the Problem and Substructuring

We partition the N subdomains into two groups, by defining the disjoint sets $\mathcal{D}$ and $\mathcal{M}$, such that $\mathcal{D} \cup \mathcal{M} = \{1, \ldots, N\}$, containing the indices of the displacement-only and mixed formulation subdomains, respectively. In this

context, we define the space $\mathbf{W} = \mathbf{V} \times U$ as

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Gamma = \Big(\underbrace{\bigoplus_{i=1}^{N} \mathbf{V}_i}_{\mathbf{V}_I} \times \underbrace{\bigoplus_{i=1}^{N} U_{0,i}}_{U_I} \Big) \oplus \Big(\underbrace{\bigoplus_{i=1}^{N} \mathbf{V}_{\Gamma,i}}_{\mathbf{V}_\Gamma} \times \underbrace{\bigoplus_{i=1}^{N} U_{\Gamma,i}}_{U_\Gamma} \Big).$$

Here, $\mathbf{V}_i$ is defined, for all i, in the same way as in Section 3.1. $U_{0,i}$ and $U_{\Gamma,i}$ are the same as before for $i \in \mathcal{M}$, but for $i \in \mathcal{D}$ they are defined as the trivial space containing only the zero function. The spaces $\mathbf{V}_{\Gamma,i}$ are again defined as the ranges of the operators $\mathcal{SH}_i^u$, the definition of which is given below.

The forms $a_i(\cdot,\cdot)$, $i = 1, \ldots, N$, are as before and the forms $b_i(\cdot,\cdot)$ and $c_i(\cdot,\cdot)$ are only defined for $i \in \mathcal{M}$, in which case they are as before. We need to define, for $i \in \mathcal{D}$, a new bilinear form,

$$g_i(\mathbf{u}, \mathbf{v}) = \langle \mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v} \rangle_{L^2(\Omega_i)}.$$

The variational formulation of our problem is now: *find* $\mathbf{u} \in \mathbf{V}$ *and* $p \in U$ *such that, for all* $\mathbf{v} \in \mathbf{V}$ *and* $q \in U$*,*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \mu_i a_i(\mathbf{u}, \mathbf{v}) + \sum_{i \in \mathcal{D}} \lambda_i g_i(\mathbf{u}, \mathbf{v}) \;+\; \sum_{i \in \mathcal{M}} b_i(\mathbf{v}, p) \;=\; \langle \mathbf{F}, \mathbf{v} \rangle \\ \displaystyle\sum_{i \in \mathcal{M}} b_i(\mathbf{u}, q) \;-\; \sum_{i \in \mathcal{M}} \frac{1}{\lambda_i} c_i(p, q) \;=\; 0. \end{cases}$$

The local stiffness matrices $K^{(i)}$ are now of the form

$$K^{(i)} = \begin{cases} \begin{bmatrix} \mu_i A^{(i)} & {B^{(i)}}^T \\ B^{(i)} & -\frac{1}{\lambda_i} C^{(i)} \end{bmatrix}, & i \in \mathcal{M} \\ \mu_i A^{(i)} + \lambda_i G^{(i)}, & i \in \mathcal{D}. \end{cases}$$

The global stiffness matrix is generated by subassembly, as usual.

Eliminating interior variables in each subdomain (i.e., interior displacements for $i \in \mathcal{D}$ and interior displacements and zero-average pressures for $i \in \mathcal{M}$), we define the matrices $S^{(i)}$. For $i \in \mathcal{M}$,

$$S^{(i)} = \begin{bmatrix} S_\Gamma^{(i)} & {B_{\Gamma\Gamma}^{(i)}}^T \\ B_{\Gamma\Gamma}^{(i)} & -\frac{1}{\lambda} C_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \text{ as before,}$$

and, for $i \in \mathcal{D}$,

$$\begin{aligned} S^{(i)} &= S_\Gamma^{(i)} \\ &= \left(\mu_i A_{\Gamma\Gamma}^{(i)} + \lambda_i G_{\Gamma\Gamma}^{(i)}\right) \\ &\quad - \left(\mu_i A_{\Gamma I}^{(i)} + \lambda_i G_{\Gamma I}^{(i)}\right) \left(\mu_i A_{II}^{(i)} + \lambda_i G_{II}^{(i)}\right)^{-1} \left(\mu_i A_{I\Gamma}^{(i)} + \lambda_i G_{I\Gamma}^{(i)}\right). \end{aligned}$$

We have then (cf. Lemma 3.2)

$$\begin{aligned} s_i(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) &= \mathbf{u}_\Gamma^T S_\Gamma^{(i)} \mathbf{v}_\Gamma \\ &= \begin{cases} \mu_i a_i(\mathcal{SH}_i^u \mathbf{u}_\Gamma, \mathcal{SH}_i^u \mathbf{v}_\Gamma) + \frac{1}{\lambda_i} c_i(\mathcal{SH}_i^p \mathbf{u}_\Gamma, \mathcal{SH}_i^p \mathbf{v}_\Gamma) & i \in \mathcal{M} \\ \mu_i a_i(\mathcal{SH}_i^u \mathbf{u}_\Gamma, \mathcal{SH}_i^u \mathbf{v}_\Gamma) + \lambda_i g_i(\mathcal{SH}_i^u \mathbf{u}_\Gamma, \mathcal{SH}_i^u \mathbf{v}_\Gamma) & i \in \mathcal{D}, \end{cases} \end{aligned}$$

where $\mathcal{SH}_i$ is defined, for $i \in \mathcal{M}$, by: *find* $\mathcal{SH}_i^u \mathbf{u}_\Gamma \in \mathbf{V}|_{\Omega_i}$ *and* $\mathcal{SH}_i^p \mathbf{u}_\Gamma \in U_{0,i}$ *such that* $\mathcal{SH}_i^u \mathbf{u}_\Gamma|_{\Gamma\cap\partial\Omega_i} = \mathbf{u}_\Gamma$ *and*

$$\begin{cases} \mu_i a_i(\mathcal{SH}^u \mathbf{u}_\Gamma, \mathbf{v}) + b_i(\mathbf{v}, \mathcal{SH}^p \mathbf{u}_\Gamma) = 0 & \forall \mathbf{v} \in \mathbf{V}_i \\ b_i(\mathcal{SH}^u \mathbf{u}_\Gamma, q) - \frac{1}{\lambda_i} c_i(\mathcal{SH}^p \mathbf{u}_\Gamma, q) = 0 & \forall q \in U_{0,i} \end{cases} \tag{8.1}$$

and, for $i \in \mathcal{D}$, by: *find* $\mathcal{SH}_i^u \mathbf{u}_\Gamma \in \mathbf{V}|_{\Omega_i}$ *such that* $\mathcal{SH}_i^u \mathbf{u}_\Gamma|_{\Gamma\cap\partial\Omega_i} = \mathbf{u}_\Gamma$ *and*

$$\mu_i a_i(\mathcal{SH}^u \mathbf{u}_\Gamma, \mathbf{v}) + \lambda_i g_i(\mathcal{SH}^u \mathbf{u}_\Gamma, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_i. \tag{8.2}$$

8.2 The Preconditioner

The definition of the preconditioner is very much as before. For $i \in \mathcal{M}$, we define

$$Q_i = R_i^T \begin{bmatrix} D_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} S^{(i)\dagger} \begin{bmatrix} D_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} R_i$$

and for $i \in \mathcal{D}$,

$$Q_i = R_i^T D_i^{-1} S^{(i)\dagger} D_i^{-1} R_i.$$

Here R_i maps $\mathbf{V}_\Gamma \times U_\Gamma$ into $\mathbf{V}_{\Gamma,i} \times U_{\Gamma,i}$ for $i \in \mathcal{M}$ and into $\mathbf{V}_{\Gamma,i}$ for $i \in \mathcal{D}$.

The coarse component of the preconditioner is defined by

$$Q_H = \begin{bmatrix} L_H & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} L_H^T & 0 \\ 0 & I \end{bmatrix} S \begin{bmatrix} L_H & 0 \\ 0 & I \end{bmatrix} \right)^{-1} \begin{bmatrix} L_H^T & 0 \\ 0 & I \end{bmatrix}.$$

We note that the identity blocks above are now $(\#\mathcal{M})$-by-$(\#\mathcal{M})$, where $\#\mathcal{M}$ denotes the number of elements in the index set $\mathcal{M}$. The matrix L_H is defined as before. We will see that it can be somewhat reduced in some cases; we will discuss this point later.

8.3 Analysis

The analysis of this algorithm is very similar to the one carried out in Chapter 5. In this section, we highlight only the differences.

First, we need a result to replace Lemma 3.1 for the subdomains in $\mathcal{D}$.

Lemma 8.1 *Given $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$, let $\mathcal{H}\mathbf{u}_\Gamma$ be its componentwise discrete harmonic extension. For any $\mathbf{u}_\Gamma \in \mathbf{V}|_\Gamma$ such that $\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma \perp \ker(a_i)$,*

$$cs_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \leq \mu_i \left\|\nabla \mathcal{H}\mathbf{u}_\Gamma\right\|^2_{L^2(\Omega_i)} \leq Cs_i(\mathcal{S}\mathcal{H}\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}\mathbf{u}_\Gamma)$$

where c depends only on λ_i/μ_i. The hypothesis that $\mathcal{S}\mathcal{H}\mathbf{u}_\Gamma \perp \ker(a_i)$ is not necessary for the lower bound.

Proof The upper bound is obtained just as in Lemma 3.1.

We note that $\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma - \mathcal{H}\mathbf{u}_\Gamma\right)|_{\Omega_i} \in \mathbf{V}_i$ and therefore, by (8.2) and (3.4),

$$\begin{aligned}
&s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \\
&\quad = \mu_i a_i\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma\right) + \lambda_i \left\langle \operatorname{div}(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma), \operatorname{div}(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma)\right\rangle_{L^2(\Omega_i)} \\
&\quad = \mu_i a_i\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma\right) + \lambda_i \left\langle \operatorname{div}(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma), \operatorname{div}(\mathcal{H}\mathbf{u}_\Gamma)\right\rangle_{L^2(\Omega_i)} \\
&\quad \leq \mu_i a_i\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma\right)^{1/2} a_i\left(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma\right)^{1/2} \\
&\qquad + \lambda_i \left\langle \operatorname{div}(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma), \operatorname{div}(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma)\right\rangle^{1/2}_{L^2(\Omega_i)} \left\langle \operatorname{div}(\mathcal{H}\mathbf{u}_\Gamma), \operatorname{div}(\mathcal{H}\mathbf{u}_\Gamma)\right\rangle^{1/2}_{L^2(\Omega_i)} \\
&\quad \leq \mu_i \left(1 + \frac{\lambda_i}{\mu_i}\frac{d}{2}\right) a_i\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma\right)^{1/2} a_i\left(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma\right)^{1/2} \\
&\quad \leq \frac{\mu_i}{2} a_i\left(\mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma, \mathcal{S}\mathcal{H}^u\mathbf{u}_\Gamma\right) + \frac{\mu_i}{2}\left(1 + \frac{\lambda_i}{\mu_i}\frac{d}{2}\right)^2 a_i\left(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma\right).
\end{aligned}$$

Therefore, using Lemma 1.3,

$$\begin{aligned}
&s_i(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \\
&\quad \leq \mu_i \left(1 + \frac{\lambda_i}{\mu_i}\frac{d}{2}\right)^2 a_i\left(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma\right) \\
&\quad \leq 2\left(1 + \frac{\lambda_i}{\mu_i}\frac{d}{2}\right)^2 \mu_i \left\|\nabla \mathcal{H}\mathbf{u}_\Gamma\right\|^2_{L^2(\Omega_i)}.
\end{aligned}$$

□

Remark 8.2 *We can assume that λ_i/μ_i is bounded, since the displacement-only formulation is used only on subdomains that are compressible enough.*

The other step that deserves to be reexamined, in this new context, is the bound on the Γ-norm of $I - T_H$, Lemma 5.1. Unfortunately, we don't know how to extend this result for the case with heterogeneous coefficients (cf. Chapter 6). So we assume, for a moment, that we have constant coefficients: $\lambda_i = \lambda$ and $\mu_i = \mu$, $i = 1, \ldots, N$. Of course this is an unreasonable assumption, since the only motivation for combining the two formulations is to handle highly heterogeneous coefficients. Once again, the actual performance of the method in all our numerical experiments indicates that this limitation is probably an artifact of our proof and not intrinsic to the algorithm.

Assuming constant coefficients, the proof of Lemma 5.1 requires no changes. We only remark that if L_H is reduced so as not to include any enrichment supported on $\bigcup_{i \in \mathcal{D}} \Omega_i$, then β_H will not be affected. Depending on the geometry of the problem, this observation may significantly reduce the dimension of the coarse problem.

Chapter 9

Continuous Pressure Spaces

Our main algorithm, in Chapter 4, requires the use of a discrete pressure space that includes the discontinuous functions χ_{Ω_i}, which are 1 on Ω_i and zero elsewhere. That, of course, precludes the use of *Taylor-Hood* or *MINI* finite elements, or any other elements that use a continuous approximation for the pressure (see, e.g., [10] or [23]). In this chapter, we present an extension of our algorithm which is suitable for this kind of discretization. The numerical experiments, reported in Chapter 10, have shown fairly good convergence away from the incompressible limit, but we have not been able to provide a supporting theory.

When continuous pressures are used, there are pressure degrees of freedom that are directly associated with the geometrical interface Γ. Trying to include them in the group of interface variables associated with $\mathbf{W}_\Gamma$ does not appear to be a promising approach: imposing essential boundary conditions for the pressure would result in non-physical problems, since, on the continuous level, pressures are only in L^2 and their traces are not defined.

Our approach, instead, is to construct a preconditioner for the continuous model based on our preconditioner for the discontinuous model. The latter is applied to a *broken* problem, defined by dropping the pressure continuity requirement only across the interface Γ.

Let $K_C^{(i)}$ be the local stiffness matrices associated with the continuous finite element model. The global stiffness matrix K_C is then given by

$$K_C = \sum_{i=1}^{N} {R_C^{(i)}}^T K_C^{(i)} R_C^{(i)} = R_C^T \begin{bmatrix} K_C^{(1)} & & \\ & \ddots & \\ & & K_C^{(N)} \end{bmatrix} R_C,$$

where $R_C^{(i)}$ is a 0,1 restriction matrix mapping local degrees of freedom into their global representation and $R_C^T = \left[\begin{array}{ccc} R_C^{(1)^T} & \cdots & R_C^{(N)^T} \end{array} \right]$. We note that each column of R_C associated with a pressure degree of freedom has as many 1's as the number of subdomains sharing that degree of freedom (two or more on the interface, one in the interior of subdomains).

We can define a *broken* stiffness matrix by subassembling the same local blocks $K_C^{(i)}$ with different restriction and extension matrices,

$$K_B = \sum_{i=1}^{N} R_B^{(i)^T} K_C^{(i)} R_B^{(i)} = R_B^T \left[\begin{array}{ccc} K_C^{(1)} & & \\ & \ddots & \\ & & K_C^{(N)} \end{array} \right] R_B,$$

where $R_B^T = \left[\begin{array}{ccc} R_B^{(1)^T} & \cdots & R_B^{(N)^T} \end{array} \right]$. The matrix K_B represents a model in which the pressure is still continuous *inside* each subdomain, but is discontinuous across the interface. The difference between $R_C^{(i)}$ and $R_B^{(i)}$ is that the former uses a single global representation for a pressure degree of freedom lying on the interface, therefore enforcing continuity, while the latter adopts multiple representations of these degrees of freedom, allowing jumps. Each column of R_B associated with a pressure degree of freedom is a column of the identity matrix, i.e., has only one 1.

Let R_B^C be a 0,1 matrix that takes broken pressure residuals into continuous pressure ones, by adding up multiple values for pressure degrees of freedom on the interface Γ; the values of the remaining pressure and displacement degrees of freedom are unchanged. The rows of R_B^C associated with interface pressure degrees of freedom contain as many 1's as the number of subdomains sharing that degree of freedom, while the other rows are rows of the identity. We note that

$$K_C = R_B^C K_B R_B^{C^T}.$$

We also define $D_C = R_B^C R_B^{C^T}$, which is a diagonal matrix with 1's on its diagonal except for the entries associated with interface pressure degrees of freedom, which contain the number of subdomains sharing that degree of freedom. This is analogous to the matrices D_i in Section 4.2, which were defined in terms of counting functions δ_i.

The preconditioner Q_C that we propose for K_C is now

$$Q_C = D_C^{-1} R_B^C Q_B R_B^{C^T} D_C^{-1},$$

where Q_B is a preconditioner for K_B. The application of Q_C is therefore composed of three steps: each component of the continuous residual is evenly

partitioned among all subdomains that contain it (corresponding to the term $R_B^{C^T} D_C^{-1}$); the broken preconditioner is then applied and, finally, the broken correction vector is averaged in order to obtain a continuous correction (corresponding to the term $D_C^{-1} R_B^C$).

Remark 9.1 *We expect that a different choice for D_C (something in the lines of (6.1)) should be necessary when dealing with discontinuous coefficients with large jumps. We have not performed experiments on such problems.*

We know how to construct a preconditioner Q for the matrix S, the Schur complement of K_B (see Chapters 3 and 4). Q_B can be constructed from Q using the following observation (see [52, page 141]). The LDL^T factorization of $K_B = \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{\Gamma I} & K_{\Gamma\Gamma} \end{bmatrix}$ is

$$K_B = \begin{bmatrix} I & 0 \\ K_{\Gamma I} K_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} K_{II} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & K_{II}^{-1} K_{I\Gamma} \\ 0 & I \end{bmatrix},$$

where $S = K_{\Gamma\Gamma} - K_{\Gamma I} K_{II}^{-1} K_{I\Gamma}$. By defining

$$Q_B = \begin{bmatrix} I & -K_{II}^{-1} K_{I\Gamma} \\ 0 & I \end{bmatrix} \begin{bmatrix} K_{II}^{-1} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ -K_{\Gamma I} K_{II}^{-1} & I \end{bmatrix}, \tag{9.1}$$

we have

$$Q_B K_B = \begin{bmatrix} I & -K_{II}^{-1} K_{I\Gamma} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & QS \end{bmatrix} \begin{bmatrix} I & K_{II}^{-1} K_{I\Gamma} \\ 0 & I \end{bmatrix}.$$

$\sigma(Q_B K_B)$, the spectrum of $Q_B K_B$, is therefore $\{1\} \cup \sigma(QS)$, which proves that Q_B is a good preconditioner for K_B, provided that Q is a good preconditioner for S.

As mentioned in Subsection 7.1.1, in our actual implementation, we have not used a special basis for the pressure. Instead, we have enforced the zero average through Lagrange multipliers. Similarly, $\widetilde{Q}_B$ should be defined as

$$\begin{aligned} \widetilde{Q}_B \;=\; & \begin{bmatrix} R_B^{D^T} & R_B^{S^T} \end{bmatrix} \begin{bmatrix} I & -\widetilde{K}_{DD}^{-1} \widetilde{K}_{D\Gamma} R_S^\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{K}_{DD}^{-1} & 0 \\ 0 & Q \end{bmatrix} \\ & \begin{bmatrix} I & 0 \\ -R_S^{\Gamma^T} \widetilde{K}_{\Gamma D} \widetilde{K}_{DD}^{-1} & I \end{bmatrix} \begin{bmatrix} R_B^D \\ R_B^S \end{bmatrix}, \end{aligned} \tag{9.2}$$

where

$$\begin{aligned}
\widetilde{K}_{DD} &= \begin{bmatrix} \widetilde{K}_{DD}^{(1)} & & \\ & \ddots & \\ & & \widetilde{K}_{DD}^{(N)} \end{bmatrix}, \\
\widetilde{K}_{\Gamma D} &= \begin{bmatrix} \widetilde{K}_{\Gamma D}^{(1)} & \cdots & \widetilde{K}_{\Gamma D}^{(N)} \end{bmatrix}, \\
\widetilde{K}_{D\Gamma} &= \widetilde{K}_{\Gamma D}^{T}
\end{aligned}$$

and $\widetilde{K}_{DD}^{(i)}$ and $\widetilde{K}_{\Gamma D}^{(i)}$ are defined as in Subsection 7.1.1. Here, R_B^D is the 0,1 restriction matrix that extracts from a residual vector in the broken space those of its components which are relevant to the Dirichlet problem: all pressures and the interior displacements. The entries associated with the Lagrange multipliers are set to zero. R_S^Γ extracts, from a vector containing interface displacements and constant pressures, only the interface displacement entries. Finally, ${R_B^S}^T$ maps from the space of interface displacements and constant pressures into the entire broken space: interface displacement are simply copied; constant pressures are mapped into their representation in terms of the nodal basis for the pressures.

It can be shown that if Q is replaced with S^{-1} in (9.2), then $\widetilde{Q}_B = \widetilde{K}_B^{-1}$.

Chapter 10

Numerical Experiments

The algorithms described in this dissertation were implemented in C, using the parallel numerical library *PETSc* (see [2, 3, 4]). PETSc provides fundamental parallel objects and methods, handles all message passing, and also ensures great code portability.

Our code is currently limited to two-dimensional rectangular domains, partitioned into rectangular subdomains. In all our experiments, the domain is the unit square $[0,1] \times [0,1]$, decomposed into $\sqrt{N} \times \sqrt{N}$ square subdomains. We impose Dirichlet conditions on the entire boundary. The direct solver used for the local problems is PETSc's LU, with PETSc's nested dissection for ordering. Except where otherwise stated, in each experiment the domain is uniformly triangulated into square $Q_2 - P_1$ elements (or simply Q_2, where only displacements are involved). The right-hand side of the Schur-complement problem is a random vector and the initial guess is chosen so that the initial error is in the range of $I - T_H$. The Krylov method we use is PCG and the stopping criterion is $\|r_n\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_n is the residual at the $n-$th iterate. The coarse space is the one based on the matrix L_H^2 (see Section 4.1).

All the experiments reported here were run on *Seaborg*, an IBM SP RS/6000 of NERSC (the National Energy Research Scientific Computing Center of the Department of Energy's Office of Science). This is a distributed memory machine with 6080 compute processors distributed among 380 compute nodes with 16 processors each. Each node has at least 16 GBytes of memory, shared among its processors.

A. Fixed number of subdomains: $N = 4 \times 4$					
mesh size	local prob.	dofs	iter.	$\lambda_{\max}$	time (sec.)
160×160	40×40	282,882	17	7.21	11.6
240×240	60×60	635,522	19	8.30	30.6
320×320	80×80	1,128,962	20	9.12	64.1
400×400	100×100	1,763,202	20	9.78	110.5
480×480	120×120	2,538,242	20	10.34	174.6
B. Fixed local prob.: 80×80 elements (71,042 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	4×4	1,128,962	20	9.12	64.1
640×640	8×8	4,510,722	21	9.36	67.7
960×960	12×12	10,145,282	23	9.44	72.8
1280×1280	16×16	18,032,642	23	9.48	75.9

Table 10.1: Numerical results for the saddle-point implementation applied to an almost-incompressible problem ($\nu = 0.499$).

10.1 Saddle-Point Implementation

Our first implementation of our algorithm (both in the order of this presentation and historically), which we refer to as the saddle-point implementation, is one closely following the description in Chapter 4, with the modification discussed in Subsection 7.1.1 in order to avoid the use of a special basis for the pressure. The coarse and local problems require the solution of saddle-point systems.

Table 10.1 and Figure 10.1 collect the results of a set of experiments on an almost-incompressible problem: the Lamé parameters are chosen to be $\mu = 1.0$ and $\lambda = 499.0$, to yield a Poisson ratio of 0.499. In part A of the table, we show a series of experiments in which the number of subdomains is held constant, 4×4, and the size H/h of the local problems varies. The column "mesh size" lists the total number of elements; "local prob.", the size of the local problems, i.e., the number of elements in each subdomain; "dofs", the total number of degrees of freedom; "iter.", the number of PCG iterations;

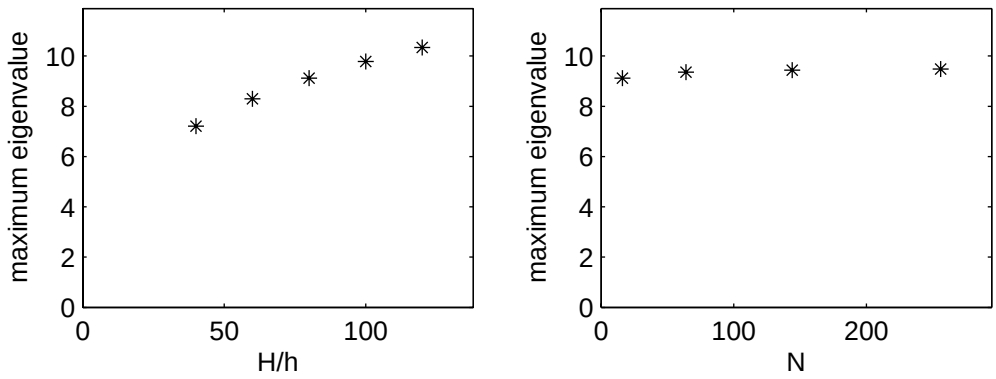

Figure 10.1: Condition number of the preconditioned operator, as a function of H/h for fixed $N = 4 \times 4$ (left; cf. Table 10.1, part A) and as a function of N for fixed $H/h = 80$ (right; cf. Table 10.1, part B).

"$\lambda_{\max}$", an estimate of the maximum eigenvalue of the preconditioned operator T and finally "time (sec.)", the CPU time spent for the experiments. We note that the estimate for the smallest eigenvalue of the preconditioned operator is not reported, since in all our experiments we found it to be very close to 1.0, the theoretical lower bound.

We can observe a weak growth in the condition number of T as the size of the local problems increase (see also Figure 10.1, left). This is in accordance with our main result, equation (5.3).

Part B of Table 10.1 is a parallel scalability experiment: the size of the local problems is held constant (80×80 elements, 71,042 degrees of freedom) and the number of subdomains N varies. The condition number of the preconditioned operator stays close to constant, as it can also be seen in the right in Figure 10.1. This also agrees with the bound in (5.3).

Similar results are obtained when this code is applied to perfectly incompressible problems, compressible problems or problems with discontinuous coefficients (see also results in [24]). We do not report these here since we prefer, instead, the positive definite implementation of the algorithm.

10.2 Positive Definite Implementation

A second implementation of our algorithm, which involves only the solution of positive definite local and coarse problems, was used in the experiments

	saddle-point	positive-definite
iter.	12	12
$\lambda_{\max}$	9.12	9.12
time (sec.)	91.7	87.1
error	1.03×10^{-5}	1.03×10^{-5}
diff.	1.35×10^{-10}	

Table 10.2: Comparison of saddle-point and positive definite implementations.

reported in this section (cf. Section 7.2). We recall that, in this formulation, we only have displacement degrees of freedom after the early elimination of all the pressures. Nevertheless, the stiffness matrix depends on the choice of finite element space for the pressures. The pair $Q_2 - P_1$ was used here.

10.2.1 Validating the Positive Definite Implementation

In Section 7.2, we mentioned that we can prove the equivalence of the saddle-point implementation and the positive definite one. Here we also present some numerical evidence of this fact.

We consider again an almost incompressible problem with $\mu = 1.0$ and $\lambda = 499.0$. The right hand-side and boundary condition are now chosen to reflect the discretization of a continuous problem with the solution $\begin{bmatrix} \sin(x+y) \\ \cos(x-y) \end{bmatrix}$. We use a mesh of 320×320 elements divided into 4×4 subdomains. We compare the solutions, after 12 PCG iterations, of both implementations (see Table 10.2). The l^∞-norm of the difference between the numerical solution and the nodal interpolant of the continuous solution is given in the row "error". The row "diff." shows the l^∞-norm of the difference between the two numerical solutions. The fact that the latter is much smaller than the former indicates that both methods actually produce the same iterates (up to round-off and related effects). We also observe that the CPU time for the positive definite implementation is smaller. This is to be expected, since the local problems in the positive definite implementation have fewer unknowns and, most importantly, are positive definite and therefore easier to solve by direct methods. However, if one compares Tables 10.1 and 10.3, one finds both examples and counterexamples of this trend. We believe that upon accurate time profiling

A. Fixed number of subdomains: $N = 4 \times 4$					
mesh size	local prob.	dofs	iter.	$\lambda_{\max}$	time (sec.)
160×160	40×40	206,082	17	7.21	10.5
240×240	60×60	462,722	18	8.30	26.3
320×320	80×80	821,762	19	9.12	65.6
400×400	100×100	1,283,202	19	9.78	96.2
480×480	120×120	1,847,042	19	10.34	188.8
B. Fixed local prob.: 80×80 elements (51,842 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	4×4	821,762	19	9.12	65.6
640×640	8×8	3,281,922	20	9.33	69.5
960×960	12×12	7,380,482	21	9.44	72.1
1280×1280	16×16	13,117,442	21	9.48	73.8
1600×1600	20×20	20,492,802	21	9.49	75.5

Table 10.3: Numerical results for the positive definite implementation applied to an almost-incompressible problem ($\nu = 0.499$).

and a careful tuning of the direct solvers, the positive definite implementation will prove itself consistently and considerably faster. It is also much easier to code.

10.2.2 Almost-Incompressible problems

We consider once again the same almost-incompressible problem, now solved using the positive definite implementation of the algorithm. The results are summarized in Table 10.3 and Figure 10.2, which, not surprisingly, closely resemble Table 10.1 and Figure 10.1. The same observations about the scalability of the method and its polylogarithmic dependence on the size of the local problems apply to this series of experiments as well.

It is interesting to notice that, in our estimate (5.3), the inf-sup constant

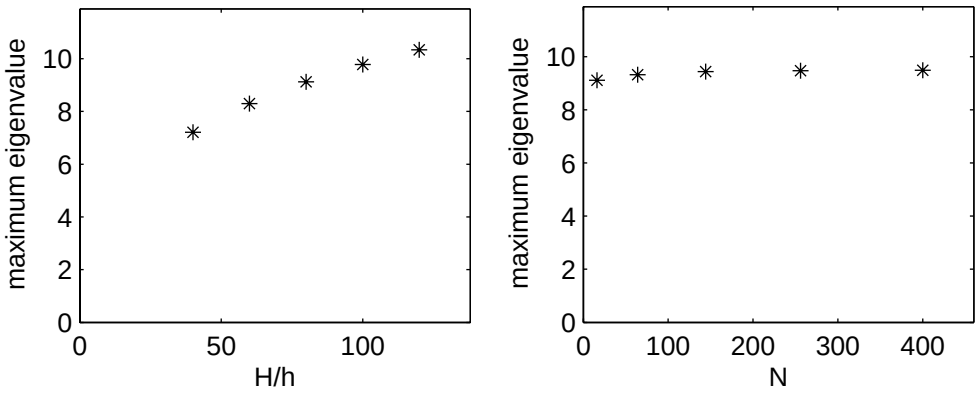

Figure 10.2: Condition number of the preconditioned operator, as a function of H/h for fixed $N = 4 \times 4$ (left; cf. Table 10.3, part A) and as a function of N for fixed $H/h = 80$ (right; cf. Table 10.3, part B).

of the coarse space, β_H, appears in the form

$$\frac{\beta_H^2}{1+\sigma^2} + \frac{\mu}{\lambda}.$$

Assuming that the local mixed finite element spaces are inf-sup stable (and therefore σ is uniformly bounded), this expression indicates that the inf-sup stability of the coarse problem will play an important role when (and only when) λ/μ is large. That is what we verify in the next two experiments.

In table 10.4 we present again a scalability experiment for an almost-incompressible problem. This time, though, we use a minimal coarse space, based on L_H^0, which we believe not to be inf-sup stable (see [46]). Unlike the previous examples, the method is no longer scalable with this poor coarse space (see also the left side of Figure 10.3).

The same poor coarse space, when used in a compressible problem, performs quite satisfactorily. Table 10.5 summarizes the results for a scalability test of the algorithm with a poor coarse space when solving a problem with steel ($\nu = 0.275$). See also the right side of Figure 10.3.

Fixed local prob.: 80×80 elements (51,842 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	4×4	821,762	20	13.13	65.4
640×640	8×8	3,281,922	27	35.01	77.3
960×960	12×12	7,380,482	34	57.24	86.4

Table 10.4: Numerical results for the positive definite implementation applied to an almost-incompressible problem, $\nu = 0.499$. A poor coarse space, based on the matrix L_H^0, is used and scalability is lost.

Fixed local prob.: 80×80 elements (51,842 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	4×4	821,762	17	11.55	61.3
640×640	8×8	3,281,922	20	12.17	67.9
960×960	12×12	7,380,482	20	12.36	69.2
1280×1280	16×16	13,117,442	19	12.35	68.4

Table 10.5: Numerical results for the positive definite implementation applied to a compressible problem, $\nu = 0.275$. A poor coarse space, based on the matrix L_H^0, is used, but scalability is preserved.

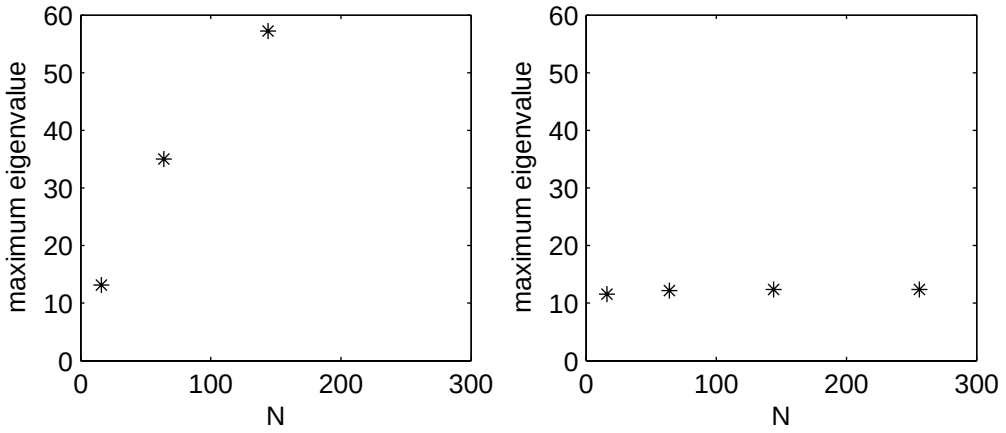

Figure 10.3: Condition number of the preconditioned operator, as a function of N for fixed $H/h = 80$. In this set of experiments, a poor coarse space has been used. At the left, the results for an almost-incompressible problem; cf. Table 10.4. At the right, for a compressible one; cf. Table 10.5.

10.2.3 Heterogeneous Problems with Jumps in the Coefficients

In this subsection, we present the results of numerical experiments on a heterogeneous problem with discontinuous coefficients. Our model problem is a composite material comprised of steel, aluminum and rubber in a checkerboard-like arrangement, as depicted in Figure 10.4.

The results, collected in Table 10.6 and graphically represented in Figure 10.5, present the same features as the ones obtained for a homogeneous domain. This is strong evidence that our inability to extend Theorem 5.4 to the heterogeneous case is only an artifact of the structure of our proof and not an intrinsic limitation of the method.

We also note that the last row of Table 10.6 reports an experiment involving almost 100 million variables and 2,000 subdomains, showing that our method is scalable when applied to really large problems.

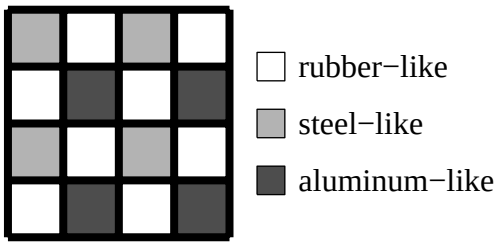

Figure 10.4: Heterogeneous domain with large coefficient jumps. The Lamé parameters for the aluminum-like material, $\mu = 2.6$ and $\lambda = 5.6$, and for the steel-like material, $\mu = 8.2$ and $\lambda = 10$, are typical; the values for the rubber-like material, $\mu = 0.01$ and $\lambda = 0.99$, were "invented" so as to produce a Poisson ratio of 0.495. (The values for μ and λ are expressed in 10^5kg/cm^2.)

A. Fixed number of subdomains: $N = 8 \times 8$					
mesh size	local prob.	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	40×40	821,762	20	8.60	12.5
480×480	60×60	1,847,042	22	10.08	31.2
640×640	80×80	3,281,922	23	11.22	72.3
800×800	100×100	5,126,402	23	12.14	115.6
B. Fixed local prob.: 80×80 elements (51,842 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
640×640	8×8	3,281,922	23	11.22	72.3
960×960	12×12	7,380,482	22	10.67	71.7
1280×1280	16×16	13,117,442	22	10.78	74.4
1600×1600	20×20	20,492,802	22	10.76	76.2
2560×2560	32×32	52,449,282	22	10.70	88.7
3520×3520	44×44	99,151,362	21	10.64	102.9

Table 10.6: Numerical results for the positive definite implementation applied to a problem with heterogeneous coefficients with large jumps (see Figure 10.4).

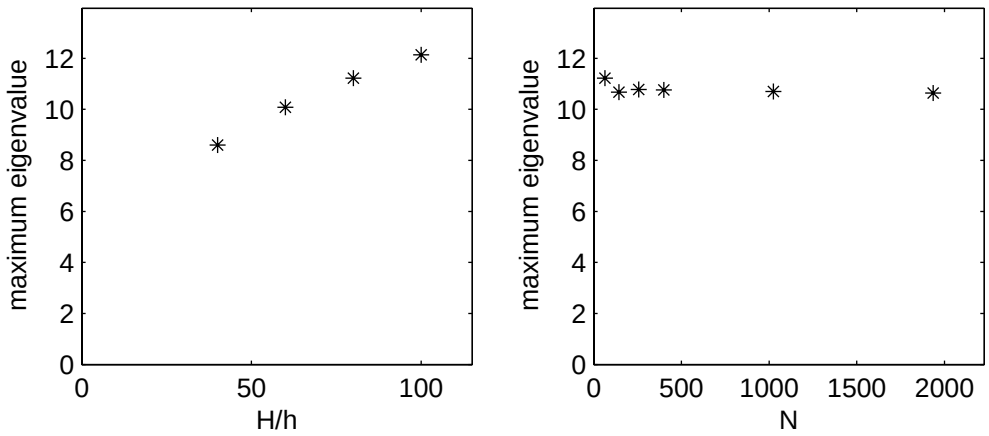

Figure 10.5: Condition number of the preconditioned operator, as a function of H/h for fixed $N = 8 \times 8$ (left; cf. Table 10.6, part A) and as a function of N for fixed $H/h = 80$ (right; cf. Table 10.6, part B).

10.3 Combined Mixed and Displacement-Only Formulations

We now report another set of experiments on the same heterogeneous problem described at the end of the previous section. This time, we use different finite element formulations for the almost incompressible regions (rubber) and the compressible ones (aluminum and steel): mixed and pure-displacement formulations, respectively; see Chapter 8.

We note that the results in Table 10.7 and Figure 10.6 are almost identical to those on Table 10.6 and Figure 10.5. This shows that combining different formulations in one problem is a perfectly viable alternative for our method.

In our implementation of the combined approach, we used the positive-definite implementation for the subdomains with mixed formulation, because of its greater simplicity (see Section 7.2). Nevertheless, the use of the saddle-point implementation would also be possible and is expected to produce identical results (cf. Subsection 10.2.1).

A. Fixed number of subdomains: $N = 8 \times 8$					
mesh size	local prob.	dofs	iter.	$\lambda_{\max}$	time (sec.)
320×320	40×40	821,762	20	8.61	12.2
480×480	60×60	1,847,042	22	10.10	32.0
640×640	80×80	3,281,922	23	11.23	75.2
800×800	100×100	5,126,402	23	12.15	111.0
B. Fixed local prob.: 80×80 elements (51,842 degrees of freedom)					
mesh size	N	dofs	iter.	$\lambda_{\max}$	time (sec.)
640×640	8×8	3,281,922	23	11.23	75.2
960×960	12×12	7,380,482	23	10.68	75.5
1280×1280	16×16	13,117,442	22	10.79	82.0
1600×1600	20×20	20,492,802	22	10.76	92.9

Table 10.7: Combined mixed and pure-displacement implementation applied to a problem with heterogeneous coefficients with large jumps.

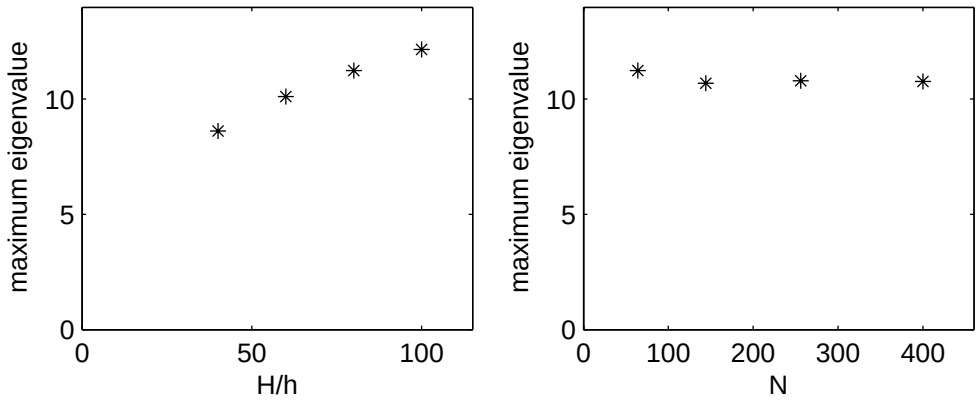

Figure 10.6: Condition number of the preconditioned operator, as a function of H/h for fixed $N = 8 \times 8$ (left; cf. Table 10.6, part A) and as a function of N for fixed $H/h = 80$ (right; cf. Table 10.6, part B).

A. Fixed number of subdomains: $N = 4 \times 4$									
	broken run			continuous run					
$\sqrt{\text{l.p.}}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
40	22	8.98	10.4	21	0.73	8.98	1.6e-5	11.1	29.8
60	23	10.36	28.2	22	0.73	10.36	1.6e-5	30.0	81.6
80	24	11.39	55.5	23	0.73	11.39	2.4e-5	59.2	170.0
100	25	12.24	96.8	23	0.73	12.24	3.0e-5	98.2	305.9
B. Fixed local prob.: 80×80 elements									
	broken run			continuous run					
$\sqrt{N}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
4	24	11.39	55.5	23	0.73	11.39	2.4e-5	59.2	170.0
8	31	12.07	74.0	28	0.73	12.07	7.1e-5	74.5	206.1
12	31	12.29	75.4	29	0.73	12.29	9.6e-5	78.2	212.4

Table 10.8: Continuous pressure implementation applied to a compressible problem (steel, $\nu = 0.275$).

10.4 Continuous Pressures

In this section, we present the results of our numerical experiments for the method described in Chapter 9, to which we will refer as the continuous preconditioner. In this set of experiments, we used the Taylor-Hood pair of finite elements; cf. Section 2.2.

Tables 10.8, 10.9, and 10.10 list the results for a compressible problem with a Poisson ratio $\nu = 0.275$ (typical of steel), and two almost-incompressible problems: one with $\nu = 0.485$ (typical of rubber) and the other with $\nu = 0.499$. Part A of each table shows a series of experiments in which the number of subdomains is fixed and the size of the local problem increases; the first column, "$\sqrt{\text{l.p.}}$", lists the square root of the number of elements in the local problems for each experiment. Part B shows the results for experiments with increasing number of subdomains N and a local problem of fixed size.

Each numerical experiment in this section (corresponding to one row in a table) involves two solves. The first one is for the solution of a broken Taylor-

A. Fixed number of subdomains: $N = 4 \times 4$									
	broken run			continuous run					
$\sqrt{\text{l.p.}}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
40	21	7.39	10.0	31	0.066	7.39	9.6e-6	16.3	34.4
60	22	8.48	26.9	33	0.066	8.48	1.4e-5	44.5	94.9
80	22	9.31	51.6	33	0.067	9.31	7.6e-5	85.2	191.1
100	23	9.98	88.6	34	0.067	9.98	8.9e-5	143.6	341.6
B. Fixed local prob.: 80×80 elements									
	broken run			continuous run					
$\sqrt{N}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
4	22	9.31	51.6	33	0.067	9.31	7.6e-5	85.2	191.1
8	27	9.58	63.9	43	0.067	9.60	1.2e-4	113.3	233.9
12	28	9.68	67.1	44	0.066	9.69	8.9e-5	118.2	243.5

Table 10.9: Continuous pressure implementation applied to an almost-incompressible problem (rubber, $\nu = 0.485$).

A. Fixed number of subdomains: $N = 4 \times 4$									
	broken run			continuous run					
$\sqrt{\text{l.p.}}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
40	22	7.39	10.4	31	0.0048	7.39	2.8e-2	16.3	35.0
60	23	8.48	27.6	33	0.0048	8.48	4.3e-2	44.0	95.2
80	24	9.30	55.1	32	0.0048	9.30	2.6e-1	81.2	190.1
100	24	9.97	58.9	29	0.0048	9.97	2.9e-1	62.9	324.5
B. Fixed local prob.: 80×80 elements									
	broken run			continuous run					
$\sqrt{N}$	iter.	$\lambda_{\max}$	t_{broken}	iter.	$\lambda_{\min}$	$\lambda_{\max}$	drop	t_{cont}	t_{total}
4	24	9.30	55.1	32	0.0048	9.30	2.6e-1	81.2	190.1
8	29	9.61	68.5	44	0.0048	9.61	4.0e-1	115.5	241.1
12	30	9.69	72.5	45	0.0048	9.69	6.6e-1	122.0	254.4

Table 10.10: Continuous pressure implementation applied to an almost-incompressible problem ($\nu = 0.499$).

Hood problem using the saddle-point implementation of our main algorithm; we remark that the positive definite implementation would not be of practical use here, since the continuous pressure cannot be eliminated at the element level. This broken solve is used as a benchmark for measuring the performance of the continuous preconditioner. We note that its setup phase is also part of the setup phase for the continuous preconditioner, since the latter requires the application of our main algorithm to the *broken* problem; cf. Chapter 9. The second solve is, of course, the one for the problem with fully continuous pressure.

For the broken solve, we list the number of PCG iterations required for the norm of the residual to drop to 10^{-8} of its original value (column "iter."), an estimate of the maximum eigenvalue of the preconditioned operator ("$\lambda_{\max}$"), and the time, in seconds, spent in the iteration ("t_{broken}"). For the continuous solve, the columns "iter.", "$\lambda_{\max}$" and "t_{cont}" are the quantities analogous to those just described, except for the fact that the iteration count refers to GMRES iterations. We include the estimated minimum eigenvalue of the preconditioned operator ("$\lambda_{\min}$"); unlike in all experiments reported so far, the value of the minimum eigenvalue is now not close to 1. The convergence of GMRES is monitored in terms of the norm of the *preconditioned* residual (the iteration count in "iter." reflects the number of GMRES iterations required for the norm of the *preconditioned* residual to drop to 10^{-8} of its original value.) We now include, in the column "drop", the ratio of final and initial *unpreconditioned* residuals. Once again, similar information is not provided in other tables because the drops of the preconditioned and unpreconditioned residuals have always been of the same order of magnitude, elsewhere.

Finally, we also include in these tables the total time spent for the assembly of the problem, the setup of the preconditioners and the two iterative solves. This is listed under the header "t_{total}".

Table 10.8 shows our results for a compressible material such as steel. We note that the minimum eigenvalue of the preconditioned operator is always the same, 0.73, for all the continuous runs and that the maximum eigenvalue for the continuous run is the same as the one for the broken run in every row of the table. Unexpectedly, the number of iterations is slightly smaller for the continuous runs, but we point out that the norm of the residuals dropped only 4 to 5 orders of magnitude for the continuous runs, while it dropped by 8 for the broken one. The time required for a continuous run turns out to be just slightly over the one required for the corresponding broken run.

Table 10.9, which lists results for an almost-incompressible material such as rubber, shows similar features. The main difference is that the minimum

eigenvalue for the continuous problem is now one order of magnitude smaller and the number of iterations grows accordingly.

The experiments with $\nu = 0.499$, reported in Table 10.10, show a minimum eigenvalue still an order of magnitude smaller for the continuous runs. Although the iteration counts are similar to those of Table 10.9, the unpreconditioned residual has dropped only one or two orders of magnitude in this set of experiments.

We can conclude from these experiments that the proposed preconditioner is efficient for compressible problems and even for rubber, but that, unlike our methods for discontinuous pressures, its performance deteriorates as we approach the incompressible limit. We also note that some of the patterns that these tables exhibit — in each table, the values in the two columns named "$\lambda_{\max}$" are almost identical, as well as all the entries in column "$\lambda_{\min}$" — are worth our attention. We consider unlikely that such patterns emerge by chance; the understanding of their theoretical justification might lead to an improved algorithm.

10.5 A Few Remarks on the Experiments

The results presented in this chapter show that our method is suitable for the solution of large elasticity problems. The method is scalable and produces extremely well-conditioned operators, even in the presence of large jumps in the coefficients. All the variants of the method perform satisfactorily for the discountinuous pressure problems. The preconditioner for the continuous pressure problem shows a narrower range of applicability, but it might still be competitive away from the incompressible limit.

We notice that the computational effort required for the solution of a mixed elasticity problem by this method is not much greater than one required to solve a Poisson problem of similar dimension with the balancing Neumann-Neumann preconditioner.

In the previous tables, we reported the CPU time required for each experiment. These are not precise numbers: there are many issues involved in accurate profiling, which we did not attempt to tackle; see, e.g., [3, Chapter 12]. Also, these running times could certainly be improved by fine tuning of the code and, in particular, by a careful selection of the direct solvers used for the local problems. Despite all these limitations, and with this warning, we have included this information in order to provide the reader a first order approximation of the computational effort involved in each experiment.

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