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DUAL-PRIMAL FETI METHODS FOR INCOMPRESSIBLE STOKES AND LINEARIZED NAVIER-STOKES EQUATIONS

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Abstract. In this paper, a dual-primal FETI method is developed for solving incompressible Stokes equations approximated by mixed finite elements with discontinuous pressures in three dimensions. The domain of the problem is decomposed into non-overlapping subdomains, and the continuity of the velocity across the subdomain interface is enforced by introducing Lagrange multipliers. By a Schur complement procedure, the indefinite Stokes problem is reduced to a symmetric positive definite problem for the dual variables, i.e., the Lagrange multipliers. This dual problem is solved by a Krylov space method with a Dirichlet preconditioner. At each step of the iteration, both subdomain problems and a coarse problem on a coarse subdomain mesh are solved by a direct method. It is proved that the condition number of this preconditioned dual problem is independent of the number of subdomains and bounded from above by the product of the inverse of the inf-sup constant of the discrete problem and the square of the logarithm of the number of unknowns in the individual subdomain problems. Illustrative numerical results are presented by solving lid driven cavity problems. This algorithm is also extended to solving linearized non-symmetric Navier-Stokes equation.

Key words. domain decomposition, Stokes, FETI, dual-primal methods

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction.

The finite element tearing and interconnecting (FETI) methods were first proposed by Farhat and Roux [?] for elliptic partial differential equations. In this method, the spatial domain is decomposed into non-overlapping subdomains, and the interior subdomain variables are eliminated to form a Schur problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, and a symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using a preconditioned conjugate gradient (PCG) method. This method has been shown to be numerically scalable for second order elliptic problems if a Dirichlet preconditioner is used. For fourth-order problems, a two-level FETI method was developed by Farhat and Mandel [?]. The main idea in this variant is that an extra set of Lagrange multipliers are used to enforce the continuity at the subdomain corners in every step of the PCG algorithm. A similar idea was used by Farhat, Lesoinne, and Pierson [?] to introduce the Dual-Primal FETI (FETI-DP) methods in which the continuity of the primal solution is enforced directly at the corners, i.e., the values of the degrees of freedom at the vertices of the subdomains remain the same. In [?], the FETI-DP methods were further refined to solve three-dimensional problems by introducing Lagrange multipliers to enforce continuity constraints for the averages of the solution on interface edges or faces. This set of Lagrange multipliers, together with the corner variables, form the coarse problem of this FETI-DP method. This richer, primal problem is necessary to obtain satisfactory convergence rates in three dimensions. Mandel and Tezaur [?] proved that the condition number of a FETI-DP algorithm grows at most as $C(1+log(H/h))^2$ for two-dimensional second order and fourth order positive definite elliptic equations;

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here H is the subdomain diameter and h is the element size. Klawonn, Widlund and Dryja [?] proposed new preconditioners and proved that the condition numbers can be bounded from above by $C(1+log(H/h))^2$ for three-dimensional problems; these bounds are also independent of possible jumps of the coefficients of the elliptic problem.

In [?], we developed a dual-primal FETI method for the two-dimensional incompressible Stokes problem and proved that the condition number is bounded from above by $C(1+log(H/h))^2$. In this paper, we will extend this algorithm to solving three-dimensional incompressible Stokes problem, give the same condition number bound, and prove the inf-sup stability of the coarse level saddle point problem, which appeared as an assumption in [?]. We also extend our algorithm to solving linearized non-symmetric Navier-Stokes equations.

2. FETI-DP algorithm for Stokes problem in three dimensions. We are solving the following Stokes problem on a three-dimensional, bounded, polyhedral domain Ω,

$$\left\{ \begin{array}{rcll} -\Delta \mathbf{u} + \nabla p & = & \mathbf{f}, & \text{in } \Omega \\ -\nabla \cdot \mathbf{u} & = & 0, & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{g}, & \text{on } \partial\Omega \ , \end{array} \right. \tag{1}$$

where the boundary velocity $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$.

The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω^i of characteristic size H. $\Gamma = \overline{(\cup \partial\Omega^i) \backslash \partial\Omega}$ is the subdomain interface and $\Gamma^{ij} = \overline{\partial\Omega^i \cap \partial\Omega^j}$ is the interface of two neighboring subdomains Ω^i and Ω^j.

We first consider incompressible Stokes problems on two subdomains with a common face Γ^{ij},

$$\left\{ \begin{array}{rcll} -\Delta \mathbf{u}^i + \nabla p^i & = & \mathbf{f}^i, & \text{in } \Omega^i \\ -\nabla \cdot \mathbf{u}^i & = & 0, & \text{in } \Omega^i \\ \mathbf{u}^i & = & \mathbf{g}^i, & \text{on } \partial\Omega \cap \partial\Omega^i \\ \frac{\partial \mathbf{u}^i}{\partial \mathbf{n}^i} - p^i \mathbf{n}^i & = & \lambda^i, & \text{on } \Gamma^{ij} \ , \end{array} \right.$$

$$\left\{ \begin{array}{rll} -\Delta \mathbf{u}^j + \nabla p^j & = \mathbf{f}^j, & \text{in } \Omega^j \\ -\nabla \cdot \mathbf{u}^j & = 0, & \text{in } \Omega^j \\ \mathbf{u}^j & = \mathbf{g}^j, & \text{on } \partial\Omega \cap \partial\Omega^j \\ \frac{\partial \mathbf{u}^j}{\partial \mathbf{n}^j} - p^j \mathbf{n}^j & = \lambda^j, & \text{on } \Gamma^{ij} \ , \end{array} \right.$$

where $\lambda^i + \lambda^j = 0$. We now form subdomain discrete problems by using an inf-sup stable mixed finite element method on each subdomain. We denote the discrete finite element space for the pressures, with zero average on the subdomain Ω^i, by Π_I^i. $\Pi_I = \prod_{i=1}^N \Pi_I^i$ is the corresponding product space. The space of constant pressure on each subdomain is denoted by Π_0. We denote the discrete finite element space for the velocity components on Ω^i by $\mathbf{W}^h(\Omega^i)$, which is decomposed as $\mathbf{W}^h(\Omega^i) = \mathbf{W}_I^i \oplus \mathbf{W}_\Gamma^i$. $\mathbf{W}_I^i$ is the interior velocity part, which equals zero on $\Gamma \cap \partial\Omega^i$, and $\mathbf{W}_\Gamma^i$ is the subdomain boundary velocity part. $\mathbf{W}_I = \prod_{i=1}^N \mathbf{W}_I^i$ and $\mathbf{W}_\Gamma = \prod_{i=1}^N \mathbf{W}_\Gamma^i$ are the corresponding product spaces. We note that a function in the space $\mathbf{W}_\Gamma$ are not required to be continuous across the interface Γ. We define a subspace $\widetilde{\mathbf{W}}_\Gamma$ of $\mathbf{W}_\Gamma$, which is given by

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

where the primal subspace $\mathbf{W}_\Pi$ is spanned by the subdomain vertex nodal finite element basis functions $\theta_{\mathcal{V}^{il}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ associated with all the edges and faces of the interface Γ. $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ equal 1 at each node of the edge $\mathcal{E}^{ik}$ and the face $\mathcal{F}^{ij}$, respectively, and vanish at all other nodes of Γ. $\mathbf{W}_\Delta$ is the dual part of the velocity space, and it is the direct sum of local subspaces $\mathbf{W}_\Delta^i$, which are defined by

$$\mathbf{W}_\Delta^i := \{\mathbf{w} \in \mathbf{W}_\Gamma^i : \mathbf{w}(\mathcal{V}^{il}) = 0, \bar{\mathbf{w}}_{\mathcal{E}^{ik}} = 0, \bar{\mathbf{w}}_{\mathcal{F}^{ij}} = 0, \ \forall \mathcal{V}^{il}, \mathcal{E}^{ik}, \mathcal{F}^{ij} \subset \partial\Omega^i\}, \tag{2}$$

with $\bar{\mathbf{w}}_{\mathcal{E}^{ik}}$ and $\bar{\mathbf{w}}_{\mathcal{F}^{ij}}$ defined by

$$\bar{\mathbf{w}}_{\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} \mathbf{w} d\mathbf{x}}{\int_{\mathcal{E}^{ik}} d\mathbf{x}}, \text{ and } \bar{\mathbf{w}}_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} \mathbf{w} d\mathbf{x}}{\int_{\mathcal{F}^{ij}} d\mathbf{x}}.$$

Using these notations, we can decompose the discrete velocity and pressure space of the original problem (1) as follows:

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

$$\Pi = \Pi_I \oplus \Pi_0.$$

If we further introduce a Lagrange multiplier space Λ to enforce the continuity of the velocities across the subdomain interfaces, we then have the following discrete problem: find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta, \lambda) \in (\mathbf{W}_I, \Pi_I, \mathbf{W}_\Pi, \Pi_0, \mathbf{W}_\Delta, \Lambda)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} & 0 \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T & 0 \\ 0 & 0 & B_{0\Pi} & 0 & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} & B_\Delta^T \\ 0 & 0 & 0 & 0 & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ 0 \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}, \tag{3}$$

where the matrix B_Δ is constructed from $\{0, 1, -1\}$ such that the values of $\mathbf{u}_\Delta$ coincide across subdomain interface Γ when $B_\Delta \mathbf{u}_\Delta = 0$. In this article, we will exclusively work with fully redundant sets of Lagrange multipliers, i.e., all possible constraints are used for each node on Γ. The matrix B_Δ^T then has a null space and to assure uniqueness it is appropriate to restrict the choice of Lagrange multipliers to $range(B_\Delta)$, i.e., for any $\lambda \in \Lambda$, there is a $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, such that $\lambda = B_\Delta \mathbf{w}_\Delta$. We also note that we are not requiring the pressure to be continuous across the subdomain interfaces in our algorithm.

By defining a Schur complement operator $\tilde{S}$ by

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{u}_\Delta \end{pmatrix}, \tag{4}$$

solving linear system (3) is reduced to solving the following linear system

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix}. \tag{5}$$

By using an additional Schur complement procedure, the problem is finally reduced to solving the following linear system with the Lagrange multipliers λ as variables:

$$B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^*. \tag{6}$$

Our preconditioner is $D B_\Delta S_\Delta B_\Delta^T D$, with S_Δ defined by

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Delta \mathbf{u}_\Delta \end{pmatrix}. \tag{7}$$

D is a diagonal scaling matrix. Each of its entries corresponds to a Lagrange multiplier and is given by $\mu^\dagger(x)$ at the interface point x. $\mu^\dagger(x)$ is the pseudoinverse of the counting functions $\mu(x)$ on the interface Γ: at any node $x \in \Gamma$, $\mu(x)$ equals the number of subdomains shared by that node, and $\mu^\dagger(x) = 1/\mu(x)$.

We have now formed the preconditioned linear system

$$D B_\Delta S_\Delta B_\Delta^T D B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = D B_\Delta S_\Delta B_\Delta^T D B_\Delta \tilde{S}^{-1} f_\Delta^* , \tag{8}$$

which defines our FETI-DP algorithm for solving the incompressible Stokes problem (1).

When we use a Krylov subspace iterative method to solve equation (8), both S_Δ and $\tilde{S}^{-1}$ are always applied to vectors in the space $B_\Delta^T D B_\Delta \mathbf{W}_\Delta$. Therefore, we just need to prove that both S_Δ and $\tilde{S}^{-1}$ are symmetric positive definite on the space $B_\Delta^T D B_\Delta \mathbf{W}_\Delta$, in order to establish that a conjugate gradient method can be used to solve the linear system (8).

The following two lemmas can be found in Klawonn et al [**?**].

LEMMA 1. *The operator $B_\Delta^T D B_\Delta$ preserves the jumps in the sense that for any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$,*

$$B_\Delta B_\Delta^T D B_\Delta \mathbf{w}_\Delta(x) = B_\Delta \mathbf{w}_\Delta(x).$$

LEMMA 2. *For any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, $B_\Delta^T D B_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Delta$.*

LEMMA 3. *$\tilde{S}$ is symmetric, positive definite on the space $\mathbf{W}_\Delta$.*

Proof: It is easy to see, from its definition (4), that $\tilde{S}$ is symmetric. We next just need to show that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \mathbf{W}_\Delta$. For any given function $\mathbf{u}_\Delta \in \mathbf{W}_\Delta$, we can always find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0)$ such that the equation (4) is satisfied. Therefore,

$$\begin{aligned}
&(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) = \mathbf{u}_\Delta^T \tilde{S} \mathbf{u}_\Delta \\
= &\begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} \\
= &\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix} \\
&+2 \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} B_{II} & B_{I\Pi} & B_{I\Delta} \\ 0 & B_{0\Pi} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix} + \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_I \\ p_0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix},$$

where the last equality results from $B_{II}\mathbf{u}_I + B_{I\Pi}\mathbf{u}_\Pi + B_{I\Delta}\mathbf{u}_\Delta = 0$ and $B_{0\Pi}\mathbf{u}_\Pi = 0$, because the vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta)$ satisfies equation (4) . Since the matrix

$$\begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}$$

is just a symmetric, positive definite discretion of a direct sum of three Laplace operators, we find that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \mathbf{W}_\Delta$.

□

In order to prove that S_Δ is symmetric, positive definite on the space $B_\Delta^T D B_\Delta \mathbf{W}_\Delta$, we will make an assumption on the meshes next to each edge. Let n_s be a node on the edge $\mathcal{E}^{ik}$ and φ_s its nodal basis function. Let A_s^{ij} be the area of the intersection of the support of φ_s with the face $\mathcal{F}^{ij}$, and l_s^{ik} be the length of its intersection with the edge $\mathcal{E}^{ik}$. We assume that A_s^{ij}/l_s^{ik} is independent of s, i.e., there is a constant $c^{ij,ik}$ such that $A_s^{ij}/l_s^{ik} = c^{ik,ij}, \forall s$. With this assumption, we can prove that

LEMMA 4. *S_Δ is symmetric, positive definite on the space $B_\Delta^T D B_\Delta \mathbf{W}_\Delta$.*

Proof: We first need to show that S_Δ is well defined on the space $B_\Delta^T D B_\Delta \mathbf{W}_\Delta$. We see, from its definition in equation (7), that to apply S_Δ to a vector $B_\Delta^T D B_\Delta \mathbf{w}_\Delta$, with $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, we need to solve subdomain incompressible Stokes problems with Dirichlet boundary data given by $B_\Delta^T D B_\Delta \mathbf{w}_\Delta$. For these subdomain Dirichlet problems to be well posed, the boundary data $B_\Delta^T D B_\Delta \mathbf{w}_\Delta$ has to satisfy the compatibility condition, $\int_{\partial\Omega^i} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n} = 0$, for each subdomain Ω^i. We have

$$\begin{aligned} &\int_{\partial\Omega^i} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n} \\ &= \sum_{\mathcal{F}^{ij}\in\partial\Omega^i} \left(\int_{\mathcal{F}^{ij}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n}_{\mathcal{F}^{ij}} + \sum_{\delta_k^{ij}\in\mathcal{F}^{ij}} \int_{\delta_k^{ij}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n}_{\mathcal{F}^{ij}}\right) \\ &= \sum_{\mathcal{F}^{ij}\in\partial\Omega^i} \left(\int_{\mathcal{F}^{ij}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n}_{\mathcal{F}^{ij}} + \sum_{\delta_k^{ij}\in\mathcal{F}^{ij}} c^{ij,ik} \int_{\mathcal{E}^{ik}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n}_{\mathcal{F}^{ij}}\right) \end{aligned}$$

where δ_k^{ij} is the strip of finite element mesh on the face $\mathcal{F}^{ij}$ next to the edge $\mathcal{E}^{ik}$. Since $B_\Delta^T D B_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Delta$, we have $\int_{\mathcal{F}^{ij}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i = 0$ and $\int_{\mathcal{E}^{ik}} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i = 0$, $\forall \mathcal{F}^{ij}, \mathcal{E}^{ik} \subset \partial\Omega^i$, therefore

$$\int_{\partial\Omega^i} \left(B_\Delta^T D B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n} = 0.$$

By arguments similar to those in the proof of Lemma 3, we find that S_Δ is symmetric positive definite on the space $\mathbf{W}_\Delta$.

□

When we use a preconditioned conjugate gradient method, or GMRES, to solve the linear equation (8), we need to apply both S_Δ and $\tilde{S}^{-1}$ to a vector in each iteration step. Multiplying S_Δ and a vector requires solving subdomain incompressible Stokes problems with Dirichlet boundary conditions, and multiplying $\tilde{S}^{-1}$ by a vector requires solving a coarse level saddle point problem, as well as subdomain problems. In [?], we made an assumption about the inf-sup stability condition of this coarse level problem. In the next section, we will give a proof of this inf-sup stability condition in the three-dimensional case. This proof is also valid for the two-dimensional case.

3. Inf-sup stability of the coarse saddle point problem. We know, from the definition (4), that to find a vector $\mathbf{u}_\Delta = \tilde{S}^{-1}\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, for any given $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, requires solving the following linear system

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T & A_{\Pi I}^T & 0 \\ A_{\Delta I} & A_{\Delta\Delta} & B_{I\Delta}^T & A_{\Pi\Delta}^T & 0 \\ B_{II} & B_{I\Delta} & 0 & B_{I\Pi} & 0 \\ A_{\Pi I} & A_{\Pi\Delta} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T \\ 0 & 0 & 0 & B_{0\Pi} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

In our FETI-DP algorithm, we solve this linear system by a Schur complement procedure. We first solve a coarse level saddle point problem

$$\begin{pmatrix} S_\Pi & B_{0\Pi}^T \\ B_{0\Pi} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Pi^* \\ 0 \end{pmatrix} ; \tag{10}$$

here S_Π will be defined in (12). We then solve the independent subdomain incompressible Stokes problems

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{I\Delta}^T \\ B_{II} & B_{I\Delta} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{I\Pi} \end{pmatrix} \mathbf{u}_\Pi. \tag{11}$$

In equation (10), S_Π is defined by:

$$S_\Pi = A_{\Pi\Pi} - \begin{pmatrix} A_{\Pi I} & A_{\Pi\Delta} & B_{I\Pi}^T \end{pmatrix} \begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{I\Delta}^T \\ B_{II} & B_{I\Delta} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{I\Pi} \end{pmatrix}. \tag{12}$$

S_Π corresponds to a discrete Stokes harmonic extension operator $\mathcal{SH}_\Pi : \mathbf{W}_\Pi \to \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$, defined as follows: for any given primal velocity $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$, find $\mathcal{SH}_\Pi \mathbf{u}_\Pi \in \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ and $p_I \in \prod_{i=1}^N \Pi_I^i$ such that on each subdomain $\Omega^i, i = 1, ..., N$,

$$\begin{cases} a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathbf{v}^i) + b(\mathbf{v}^i, p_I^i) & = & 0, & \forall \mathbf{v}^i \in \mathbf{W}^h(\Omega^i) \\ b(\mathcal{SH}_\Pi \mathbf{u}_\Pi, q_I^i) & = & 0, & \forall q_I^i \in \Pi^i \\ \mathcal{SH}_\Pi \mathbf{u}_\Pi & = & \mathbf{u}_\Pi. & \end{cases} \tag{13}$$

If we define an inner product $s_\Pi(.,.)$, corresponding to the Schur operator S_Π, on the space $\mathbf{W}_\Pi$, as

$$s_\Pi(\mathbf{u}_\Pi, \mathbf{u}_\Pi) = \mathbf{u}_\Pi^T S_\Pi \mathbf{u}_\Pi = a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathcal{SH}_\Pi \mathbf{u}_\Pi), \quad \forall \mathbf{u}_\Pi \in \mathbf{W}_\Pi, \tag{14}$$

then the matrix form of the coarse problem (10) can be written in the following variational form: find $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in \Pi_0$ such that,

$$\begin{cases} s_\Pi(\mathbf{u}_\Pi, \mathbf{v}_\Pi) + b(\mathbf{v}_\Pi, p_0) & = & < \mathbf{f}_\Pi, \mathbf{v}_\Pi >, \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi \\ b(\mathbf{u}_\Pi, q_0) & = & 0, \forall q_0 \in \Pi_0. \end{cases} \tag{15}$$

We now give an inf-sup stability estimate for this coarse problem. We first introduce an inner product $s_\Gamma(.,.)$ which is defined by

$$s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma) = a(\mathcal{SH}_\Gamma \mathbf{w}_\Gamma, \mathcal{SH}_\Gamma \mathbf{w}_\Gamma), \quad \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma, \tag{16}$$

where $\mathcal{SH}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$, is the standard Stokes harmonic extension. We have the following lemma, from Pavarino and Widlund [?],

LEMMA 5. *The saddle point problem*

$$\left\{ \begin{array}{lcl} s_\Gamma(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) + b(\mathbf{v}_\Gamma, p_0) & = & < \mathbf{f}_\Gamma, \mathbf{v}_\Gamma >, \forall \mathbf{v}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \\ b(\mathbf{u}_\Gamma, q_0) & = & 0, \forall q_0 \in \Pi_0, \end{array} \right. \tag{17}$$

is inf-sup stable, i.e., there is a constant β_Γ such that

$$\sup_{\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma} \frac{b(\mathbf{w}_\Gamma, q_0)^2}{s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma)} \geq \beta_\Gamma^2 ||q_0||_{L^2}^2, \quad \forall q_0 \in \Pi_0. \tag{18}$$

We also need the following lemmas. Lemma 6 can be found in Klawonn et al [?] and Lemma 7 in Bramble and Pasciak [?].

LEMMA 6. *Define an interpolation operator $I_{\Gamma\Pi} : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Pi$, such that for all $\mathbf{w}_\Gamma(\mathbf{x}) \in \widetilde{\mathbf{W}}_\Gamma$,*

$$I_{\Gamma\Pi}\mathbf{w}_\Gamma(\mathbf{x}) = \sum_{\mathcal{V}^{il}} \mathbf{w}_\Gamma(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}(x) + \sum_{\mathcal{E}^{ik}} \bar{\mathbf{w}}_{\mathcal{E}^{ik}}\theta_{\mathcal{E}^{ik}}(x) + \sum_{\mathcal{F}^{ij}} \bar{\mathbf{w}}_{\mathcal{F}^{ij}}\theta_{\mathcal{F}^{ij}}(x). \tag{19}$$

We then have

$$|I_{\Gamma\Pi}\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2 \leq C(1 + log(H/h))|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2, \forall \mathbf{w}_\Gamma(\mathbf{x}) \in \widetilde{\mathbf{W}}_\Gamma, \tag{20}$$

where C is a constant independent of H and h.

LEMMA 7. *There exist positive constants C_1 and C_2, such that*

$$C_1\beta^2 s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma) \leq |\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2 \leq C_2 s_\Gamma(\mathbf{w}_\Gamma, \mathbf{w}_\Gamma), \forall \mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma, \tag{21}$$

and

$$C_1\beta^2 s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi) \leq |\mathbf{w}_\Pi|_{H^{1/2}(\Pi)}^2 \leq C_2 s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi), \forall \mathbf{w}_\Pi \in \mathbf{W}_\Pi, \tag{22}$$

where β is the inf-sup stability constant of the subdomain Stokes problem, and the inner products $s_\Gamma(.,.)$ and $s_\Pi(.,.)$ are defined in (16) and (14), respectively.

We now prove the inf-sup stability for the coarse saddle point problem (15).

THEOREM 1.

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathbf{w}_\Pi, q_0)^2}{s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi)} \geq C\beta^2\beta_\Gamma^2(1 + log(H/h))^{-1}||q_0||_{L^2}^2, \quad \forall q_0 \in \Pi_0, \tag{23}$$

where β is the inf-sup stability constant of the subdomain Stokes problem solver, and β_Γ is the inf-sup stability constant of Lemma 5.

Proof: Given the inf-sup stability estimate in Lemma 5, we know, from Fortin [?], that there exit an interpolant $\Pi_\Gamma : H^{1/2}(\Gamma) \to \widetilde{\mathbf{W}}_\Gamma$, satisfying

$$\left\{ \begin{array}{l} b(\Pi_\Gamma \mathbf{w} - \mathbf{w}, q_0) = 0, \forall q_0 \in \Pi_0 \\ s_\Gamma(\Pi_\Gamma \mathbf{w}, \Pi_\Gamma \mathbf{w}) \leq \frac{C}{\beta_\Gamma^2}|\mathbf{w}|_{H^{1/2}(\Gamma)}^2. \end{array} \right. \tag{24}$$

In order to prove (23), we just need to show that there exists an operator $\Pi_\Pi : H^{1/2}(\Gamma) \to \mathbf{W}_\Pi$, such that

$$\left\{ \begin{array}{c} b(\Pi_\Pi \mathbf{w} - \mathbf{w}, q_0) = 0, \forall q_0 \in \Pi_0 \\ s_\Pi(\Pi_\Pi \mathbf{w}, \Pi_\Pi \mathbf{w}) \leq C\frac{1}{\beta^2\beta_\Gamma^2}(1 + log(H/h))|\mathbf{w}|_{H^{1/2}(\Gamma)}^2. \end{array} \right. \tag{25}$$

By defining $\Pi_\Pi = I_{\Gamma\Pi} \circ \Pi_\Gamma : H^{1/2}(\Gamma) \to \mathbf{W}_\Pi$, we have

$$\begin{array}{rcl} b(\Pi_\Pi \mathbf{w} - \mathbf{w}, q_0) & = & b(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \mathbf{w}, q_0) \\ & = & b(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \Pi_\Gamma \mathbf{w}, q_0) + b(\Pi_\Gamma \mathbf{w} - \mathbf{w}, q_0) \\ & = & \sum_i q_0^i \int_{\Omega^i} \operatorname{div}(I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w}) - \Pi_\Gamma \mathbf{w}) \\ & = & \sum_i q_0^i \int_{\partial\Omega^i} (I_{\Gamma\Pi} \mathbf{w}_\Gamma - \mathbf{w}_\Gamma) \cdot \mathbf{n} \\ & = & -\sum_i q_0^i \int_{\partial\Omega^i} \mathbf{w}_\Delta \cdot \mathbf{n} \\ & = & 0 \, . \end{array}$$

At the same time, by using Lemma 6, Lemma 7, and equation (24), we have

$$\begin{array}{rcl} s_\Pi(\Pi_\Pi \mathbf{w}, \Pi_\Pi \mathbf{w}) & = & a(\mathcal{SH}(\Pi_\Pi \mathbf{w}), \mathcal{SH}(\Pi_\Pi \mathbf{w})) \\ & \leq & C \frac{1}{\beta^2} |\Pi_\Pi \mathbf{w}|^2_{H^{1/2}(\Gamma)} \\ & = & C \frac{1}{\beta^2} |I_{\Gamma\Pi}(\Pi_\Gamma \mathbf{w})|^2_{H^{1/2}(\Gamma)} \\ & \leq & C \frac{1}{\beta^2} (1 + log(H/h)) |\Pi_\Gamma \mathbf{w}|^2_{H^{1/2}(\Gamma)} \\ & \leq & C \frac{1}{\beta^2} (1 + log(H/h)) s_\Gamma(\Pi_\Gamma \mathbf{w}, \Pi_\Gamma \mathbf{w}) \\ & \leq & C \frac{1}{\beta^2 \beta_\Gamma^2} (1 + log(H/h)) |\mathbf{w}|^2_{H^{1/2}(\Gamma)} . \end{array}$$

Therefore, equation (25) holds and (23) is proved.

□

4. Condition number estimate in the three-dimensional case. In this section, we give a proof of the condition number bound of the preconditioned linear system (8).

LEMMA 8. $|\mathbf{w}_\Delta|_{\tilde{S}} \leq |\mathbf{w}_\Delta|_{S_\Delta}, \forall \mathbf{w}_\Delta \in \mathbf{W}_\Delta$.

Proof: We know, from the definitions of $\tilde{S}$ and S_Δ, that

$$\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta = \min_{\mathbf{v}_I} \min_{\mathbf{v}_\Pi} \max_{p_I} \{ \mathbf{v}^T K \mathbf{v} \mid \mathbf{v}_\Delta = \mathbf{w}_\Delta \text{ and } B_{0\Pi} \mathbf{v}_\Pi = 0 \} \, , \tag{26}$$

and

$$\mathbf{w}_\Delta^T S_\Delta \mathbf{w}_\Delta = \min_{\mathbf{v}_I} \max_{p_I} \{ \mathbf{v}^T K \mathbf{v} \mid \mathbf{v}_\Delta = \mathbf{w}_\Delta \text{ and } \mathbf{v}_\Pi = 0 \} \, , \tag{27}$$

for any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, where K and $\mathbf{v}$ denote

$$K = \begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}, \mathbf{v} = \begin{pmatrix} \mathbf{v}_I \\ p_I \\ \mathbf{v}_\Pi \\ \mathbf{v}_\Delta \end{pmatrix} .$$

Since $\mathbf{v}_\Pi = 0$ satisfies the constraint $B_{0\Pi} \mathbf{v}_\Pi = 0$ in equation (26), it is easy to see that $\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta \leq \mathbf{w}_\Delta^T S_\Delta \mathbf{w}_\Delta$ for any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$.

□

We now prove a key estimate,

LEMMA 9. *For all* $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, *we have,*

$$|B_\Delta^T D B_\Delta \mathbf{w}_\Delta|^2_{S_\Delta} \leq C \frac{1}{\beta^2} (1 + \log(H/h))^2 |\mathbf{w}_\Delta|^2_{\tilde{S}},$$

where $C > 0$ *is independent of* h, H.

Proof: We consider an arbitrary $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$. In order to compute its $\tilde{S}-$norm, we determine the element $\mathbf{w} = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Gamma, \mathbf{w}_\Pi \in \mathbf{W}_\Pi$, with the correct minimal property. Then, by the definition of $\tilde{S}$, we know that $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}|_{s_\Gamma}$. We next note that we can subtract any continuous function from $\mathbf{w}_\Delta$ without changing the values of $B_\Delta^T D B_\Delta \mathbf{w}_\Delta$; thus, $B_\Delta^T D B_\Delta \mathbf{w} = B_\Delta^T D B_\Delta \mathbf{w}_\Delta$.

We introduce the notation $(\mathbf{v}^i)_{i=1,\dots,N} := B_\Delta^T D B_\Delta \mathbf{w}$. Then, we have to estimate

$$|B_\Delta^T D B_\Delta \mathbf{w}|_{S_\Delta}^2 = |B_\Delta^T D B_\Delta \mathbf{w}|_{s_\Gamma}^2 = \sum_{i=1}^N |\mathbf{v}^i|_{s_\Gamma^i}^2.$$

We can therefore focus on the estimate of the contribution from a single subdomain Ω^i. By noticing that $\mathbf{v}^i$ vanishes at the subdomain vertices, we can separate the function $\mathbf{v}^i$ by using the cutoff functions $\theta_{\mathcal{F}^{ij}}$ and $\theta_{\mathcal{E}^{ik}}$,

$$\mathbf{v}^i = \sum_{\mathcal{F}^{ij} \subset \partial\Omega^i} I^h(\theta_{\mathcal{F}^{ij}} \mathbf{v}^i) + \sum_{\mathcal{E}^{ik} \subset \partial\Omega^i} I^h(\theta_{\mathcal{E}^{ik}} \mathbf{v}^i).$$

By using a similar procedure as in Lemma 9 of Klawonn et al [?], we can show that

$$|\mathbf{v}^i|_{s_\Gamma^i}^2 \le C(1 + \log(\frac{H}{h}))^2 \left(|\mathbf{w}^i|_{H^{1/2}(\partial\Omega^i)}^2 + \sum_{j \in \mathcal{N}^i} |\mathbf{w}^j|_{H^{1/2}(\partial\Omega^j)}^2 \right) ,$$

where $\mathcal{N}^i$ is the set of indices of all the subdomains which surround the subdomain Ω^i. Then, by using Lemma 7, we have we have

$$|\mathbf{v}^i|_{s_\Gamma^i}^2 \le C \frac{1}{\beta^2} (1 + \log(\frac{H}{h}))^2 \left(|\mathbf{w}^i|_{s_\Gamma^i}^2 + \sum_{j \in \mathcal{N}^i} |\mathbf{w}^j|_{s_\Gamma^j}^2 \right).$$

□

THEOREM 2. *The condition number of the preconditioned linear system (8) is bounded from above by $C\frac{1}{\beta^2}(1 + \log(H/h))^2$, where C is independent of h, H.*

Proof: We will show that

$$\lambda^T M \lambda \le \lambda^T F \lambda \le C \frac{1}{\beta} (1 + log \frac{H}{h})^2 \lambda^T M \lambda \ , \forall \lambda \in \Lambda \ ,$$

where $M^{-1} = D B_\Delta S_\Delta B_\Delta^T D, F = B_\Delta \tilde{S}^{-1} B_\Delta^T$.

Lower bound: From Klawonn et al [?], or Mandel and Tezaur [?], we have

$$\lambda^T F \lambda = \max_{0 \ne \mathbf{v}_\Delta \in \mathbf{W}_\Delta} \frac{|(\lambda, B_\Delta \mathbf{v}_\Delta)|^2}{|\mathbf{v}_\Delta|_{\tilde{S}}^2} .$$

From Lemma 2, we know that $B_\Delta^T D B_\Delta \mathbf{W}_\Delta \subset \mathbf{W}_\Delta$, and from Lemma 1 we know that $B_\Delta B_\Delta^T D B_\Delta \mathbf{w}_\Delta(x) = B_\Delta \mathbf{w}_\Delta(x)$, for any $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$. We also know from Lemma 8 that $|\mathbf{w}_\Delta|_{\tilde{S}} \le |\mathbf{w}_\Delta|_{S_\Delta}$ for all $\mathbf{w}_\Delta \in B_\Delta^T D B_\Delta \mathbf{W}_\Delta$. Therefore, we have

$$\lambda^T F \lambda \ge \max_{0 \ne \mathbf{w}_\Delta \in \mathbf{W}_\Delta} \frac{|(\lambda, B_\Delta B_\Delta^T D B_\Delta \mathbf{w}_\Delta)|^2}{|B_\Delta^T D B_\Delta \mathbf{w}_\Delta|_{\tilde{S}}^2} \ge \max_{0 \ne \mathbf{w}_\Delta \in \mathbf{W}_\Delta} \frac{|(\lambda, B_\Delta \mathbf{w}_\Delta)|^2}{|B_\Delta^T D B_\Delta \mathbf{w}_\Delta|_{S_\Delta}^2} .$$

Since for any $\nu \in \Lambda$ there is a $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$ such that $\nu = B_\Delta \mathbf{w}_\Delta$, we have

$$\lambda^T F \lambda \ge \frac{|(\lambda, \nu)|^2}{|B_\Delta^T D \nu|_{S_\Delta}^2} = \frac{|(\lambda, \nu)|^2}{(M^{-1}\nu, \nu)} .$$

It follows that $\lambda^T M\lambda \leq \lambda^T F\lambda$ by choosing $\nu = M\lambda$.

Upper bound: Using Lemma 9, we have

$$\begin{aligned}
\lambda^T F\lambda &= \max_{0\neq \mathbf{v}_\Delta \in \mathbf{W}_\Delta} \frac{(\lambda, B_\Delta v_\Delta)^2}{|\mathbf{v}_\Delta|^2_{\tilde{S}}} \\
&\leq C\frac{1}{\beta}(1+log\frac{H}{h})^2 \max_{0\neq \mathbf{v}_\Delta \in \mathbf{W}_\Delta} \frac{(\lambda, B_\Delta \mathbf{v}_\Delta)^2}{|B_\Delta^T D B_\Delta \mathbf{v}_\Delta|^2_{S_\Delta}} \\
&= C\frac{1}{\beta}(1+log\frac{H}{h})^2 \max_{0\neq \mathbf{v}_\Delta \in \mathbf{W}_\Delta} \frac{(\lambda, B_\Delta \mathbf{v}_\Delta)^2}{(M^{-1}B_\Delta \mathbf{v}_\Delta, B_\Delta \mathbf{v}_\Delta)} \\
&= C\frac{1}{\beta}(1+log\frac{H}{h})^2 \max_{\nu\in\Lambda} \frac{(\lambda,\nu)^2}{(M^{-1}\nu,\nu)} \\
&= C\frac{1}{\beta}(1+log\frac{H}{h})^2 (M\lambda,\lambda)\ .
\end{aligned}$$

□

5. Extension to solving linearized Navier-Stokes equations. When we solve the nonlinear Navier-Stokes equations using a Picard iteration, we need to solve a linearized problem in each iteration step:

$$\left\{ \begin{array}{rcl} -\mu\Delta\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f}, \\ -\nabla\cdot\mathbf{u} &=& 0, \\ \mathbf{u}|_{\partial\Omega} &=& \mathbf{g}, \end{array} \right. \tag{28}$$

where μ is the viscosity, $\nabla\cdot\mathbf{a} = 0$, and $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n} = 0$.

To solve this non-symmetric equation, the bilinear form $\int_{\Omega^i}(\mathbf{a}\cdot\nabla)\mathbf{uv}$, on each subdomain Ω^i, is written as the sum of a skew-symmetric term and an interface term:

$$\left(\frac{1}{2}\int_{\Omega^i}(\mathbf{a}\cdot\nabla)\mathbf{uv} - \frac{1}{2}\int_{\Omega^i}(\mathbf{a}\cdot\nabla)\mathbf{vu}\right) + \frac{1}{2}\int_{\partial\Omega^i}(\mathbf{a}\cdot\mathbf{n})\mathbf{uv}. \tag{29}$$

By doing this, we are identifying the correct bilinear form describing the action of the above non-symmetric operator on any given subdomain Ω^i, and the subdomain incompressible Navier-Stokes problem appears as:

$$\left\{ \begin{array}{rcll} -\mu\Delta\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega^i \\ -\nabla\cdot\mathbf{u} &=& 0, & \text{in } \Omega^i \\ \mathbf{u} &=& \mathbf{g}, & \text{on } \partial\Omega\cap\partial\Omega^i \\ \mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} - \frac{\mathbf{a}\cdot\mathbf{n}}{2}\mathbf{u} &=& \lambda, & \text{on } \Gamma^{ij}\ . \end{array} \right. \tag{30}$$

The idea to write the non-symmetric bilinear form $\int_{\Omega^i}(\mathbf{a}\cdot\nabla)\mathbf{uv}$ as in (29) has been used by Achdou, Le Tallec, Nataf, and Vidrascu [?] to solve advection-diffusion problems.

After discretizing the subdomain problems (30), we can use the same procedure as in section 2 to design a FETI-DP algorithm. The conjugate gradient method cannot be used here to solve the preconditioned linear system, because this problem is no longer symmetric positive definite; instead we use GMRES.

6. Numerical experiments. In [?], we gave some numerical results to demonstrate the scalability of the FETI-DP algorithm for solving two-dimensional incompressible problems. Here we first describe a three-dimensional experiment. We are solving a lid-driven cavity problem in the domain $\Omega = [0,1]\times[0,1]\times[0,1]$, with $\mathbf{f} = \mathbf{0}$, the boundary conditions $\mathbf{g} = (1,0,0)$ on the face $z = 1$ and $\mathbf{g} = \mathbf{0}$ elsewhere on the boundary. We use GMRES to solve the preconditioned linear system (8), as well as the non-preconditioned linear system (6). The initial guess is $\lambda = 0$ and the stopping criterion is $||r_k||_2/||r_0||_2 \leq 10^{-6}$, where r_k is the residual of the Lagrange multiplier equation at the kth iteration. In our experiments here, we are only using the face average constraints in our primal velocity space to make the algorithm simpler.

Figure 1 gives the number of GMRES iterations for different number of subdomains with a fixed subdomain problem size $H/h = 4$, and for different subdomain problem size H/h with $4 \times 4 \times 4$ subdomains. We see, from the left figure, that the convergence of the augmented FETI-DP method, with or without a preconditioner, is independent of the number of subdomains, and that the preconditioned version needs fewer iterations. The right figure shows that the GMRES iteration count increases, in both the preconditioned and the non-preconditioned cases, with a increase of the size of subdomain problem, but that the growth is much slower with the Dirichlet preconditioner than without.

In Figure 2, we are using two-dimensional numerical results to verify that the inf-sup stability condition for the coarse level saddle point problem is consistent with our estimate in Theorem 1. We can see, from the left figure, that β_C has a lower bound which is independent of the number of subdomains, and that $1/{\beta_C}^2$ appears to be a linear function of $log(H/h)$, from the right figure.

We also test the FETI-DP algorithm for solving the linearized non-symmetric Navier-Stokes problem (28) in a two-dimensional domain $\Omega = [0,1] \times [0,1]$, with $\mathbf{f} = 0$, the boundary conditions $\mathbf{g} = (1,0)$ on the upper side $y = 1$ and $\mathbf{g} = \mathbf{0}$ on the three other sides, and the convection coefficient

$$\mathbf{a} = \begin{pmatrix} 2(2y-1)(1-(2x-1)^2) \\ -2(2x-1)(1-(2y-1)^2) \end{pmatrix},$$

cf. Elman, Silvester and Wathen [**?**]. We can see, from the left graph in Figure 3, that the convergence of GMRES method becomes even better for both the preconditioned and the non-preconditioned algorithms when we increase the number of subdomains. The right graph shows that the convergence of the non-preconditioned algorithm depends on linearly on H/h and that the convergence of the preconditioned algorithm appears to depend on H/h in a logarithmic manner.

Figure 4 shows how the convergence of the preconditioned algorithm depends on the Reynolds number. The bigger the Reynolds number, the slower the convergence. These experiments are carried out for the case of 10×10 subdomains and $H/h = 16$ with different Reynolds numbers.

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FIG. 1. *GMRES iterations counts for the 3D Stokes solver vs. number of subdomains for $H/h = 4$ (left) and vs. H/h for $4 \times 4 \times 4$ subdomains (right).*

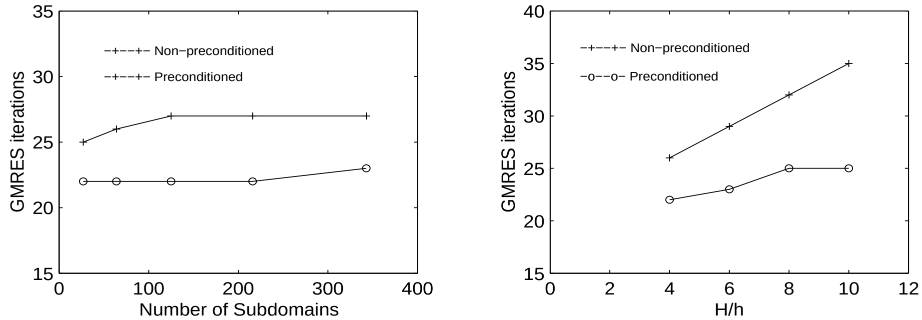

FIG. 2. *Inf-sup stability condition of the coarse level saddle point problem.*

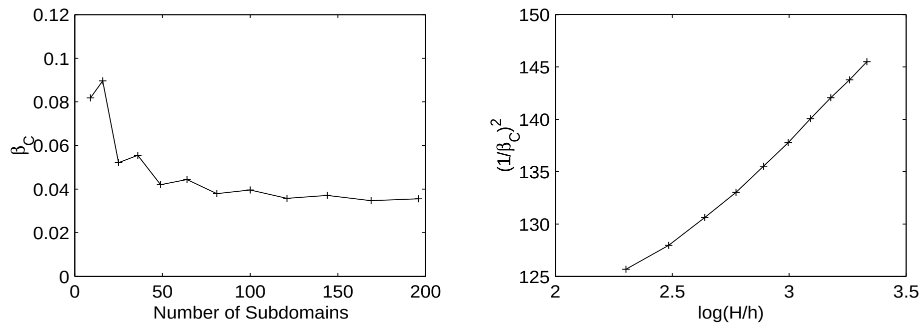

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FIG. 3. *GMRES iterations counts for 2D linearized Navier-Stokes solver vs. number of subdomains for $H/h = 8$ (left) and vs. H/h for 4×4 subdomains (right) with $\mu = 0.01$.*

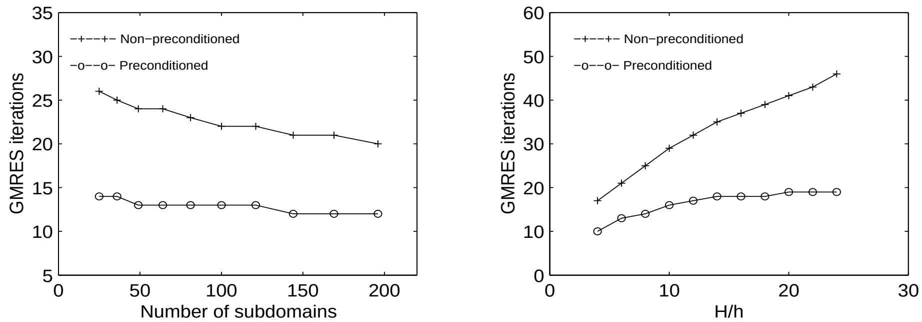
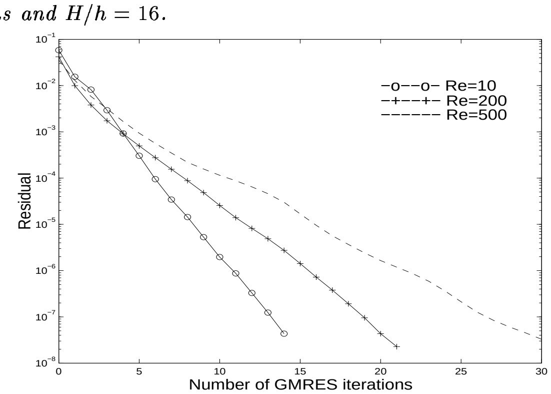


FIG. 4. *Convergence of GMRES method corresponding to different Reynolds number in the case of* 10×10 *subdomains and* $H/h = 16$.



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