BALANCING NEUMANN-NEUMANN PRECONDITIONERS FOR MIXED APPROXIMATIONS OF HETEROGENEOUS PROBLEMS IN LINEAR ELASTICITY

PAULO GOLDFELD *, LUCA F. PAVARINO †, AND OLOF B. WIDLUND ‡

April 8, 2002

Abstract. Balancing Neumann-Neumann methods are extented to mixed formulations of the linear elasticity system with discontinuous coefficients, discretized with mixed finite or spectral elements with discontinuous pressures. These domain decomposition methods implicitly eliminate the degrees of freedom associated with the interior of each subdomain and solve iteratively the resulting saddle point Schur complement using a hybrid preconditioner based on a coarse mixed elasticity problem and local mixed elasticity problems with natural and essential boundary conditions. A polylogarithmic bound in the local number of degrees of freedom is proven for the condition number of the preconditioned operator in the constant coefficient case. Parallel and serial numerical experiments confirm the theoretical results, indicate that they still hold for systems with discontinuous coefficients, and show that our algorithm is scalable, parallel, and robust with respect to material heterogeneities. The results on heterogeneous general problems are also supported in part by our theory.

Key words. domain decomposition methods, linear elasticity, mixed finite elements, spectral elements, preconditioned iterative methods

AMS subject classifications. 65N55, 65N30, 65N35, 65F10, 65Y05

1. Introduction. The purpose of this paper is to introduce and analyze a new domain decomposition method for the symmetric, indefinite linear systems of equations that arise when the equations of linear elasticity for almost incompressible and heterogeneous materials are discretized by mixed finite elements. Many problems in elasticity can successfully be approximated by using displacement variables only but such models suffer increasingly from locking when we approach the incompressible limit. In such situations, the introduction of an additional pressure variable and a mixed finite element method is a well-known remedy.

We will use a balancing Neumann–Neumann domain decomposition method and we note that this is an extension of our recent work on incompressible Stokes's equations [30]; the main results of this paper have also been reported without proofs in a conference paper [17]. The Stokes and elasticity problems in mixed form have much in common but both the algorithm and analysis have to be modified, in particular, when considering large variations in the material properties. The Neumann-Neumann algorithms are iterative substructuring methods and like all of them, our method is based on the implicit elimination of the degrees of freedom associated with the interior of each subdomain. The resulting saddle point Schur complement is solved iteratively

* Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012. E-mail address: paulo.goldfeld@pobox.com, URL: http://www.math.nyu.edu/phd_students/goldfelp. This work was supported by a scholarship of CNPq, of the Ministry for Science and Technology of Brazil, under process 201205/97-1. The work was developed in part at MCS/ANL-DOE, under a Givens Research Associate appointment in Summer 2001.

† Department of Mathematics, Università di Milano, Via Saldini 50, 20133 Milano, ITALY. E-mail address: pavarino@mat.unimi.it, URL: http://mat.unimi.it/~pavarino/. This work was supported in part by the National Science Foundation under Grant NSF-CCR-9732208 and in part by MIUR.

‡ Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012. E-mail address: widlund@cs.nyu.edu. URL: http://cs.nyu.edu/cs/faculty/widlund/index.html. This work was supported in part by the National Science Foundation under Grants qNSF-CCR-9732208, and in part by the U.S. Department of Energy under contracts DE-FC02-01ER25482 and DE-FG02-92ER25127.

using a hybrid preconditioner based on a coarse mixed elasticity problem and local mixed elasticity problems with natural and essential boundary conditions. We will prove that our method is scalable and quasi-optimal in the constant coefficient case, i.e., a polylogarithmic bound in the local number of degrees of freedom holds for the condition number of the preconditioned operator. We will show numerically that these good convergence properties also hold for heterogeneous materials with discontinuous coefficients and that overall our algorithm is very efficient, parallel, and robust. The results on general heterogeneous problems are also supported in part by our theory.

Neumann-Neumann methods were first introduced and analyzed for second order elliptic problems; see Cowsar, Mandel, and Wheeler [11], Dryja and Widlund [12], Mandel [24], Mandel and Brezina [25], and Pavarino [28]. More recently, this family of methods has been extended to plate and shell problems, see Le Tallec, Mandel, and Vidrascu [21], to convection-diffusion problems, see Achdou, Le Tallec, Nataf, and Vidrascu [1] and Alart, Barboteu, Le Tallec, and Vidrascu [3], and to vector field problems, see Toselli [37]. We also note that the connection between Neumann-Neumann and FETI methods has been considered recently by Klawonn and Widlund [20].

There is a considerable literature on domain decomposition methods for incompressible Stokes equations. Iterative substructuring methods have been studied by Ainsworth and Sherwin [2], Bramble and Pasciak [7], Casarin [9], Fischer and Rønquist [15], Le Tallec and Patra [22], Marini and Quarteroni [26], Pasciak [27], Pavarino and Widlund [29], Quarteroni [31], and Rønquist [33]. Overlapping Schwarz methods have been considered by Fischer [13], Fischer, Miller, and Tufo [14], Gervasio [16], Klawonn and Pavarino [18], and Rønquist [34].

For a general introduction to domain decomposition methods, we refer to Quarteroni and Valli [32] and Smith, Bjørstad, and Gropp [35].

The remainder of this paper is organized as follows. We review mixed finite element and spectral element methods in Section 2; we only consider methods with discontinuous pressure spaces. In Section 3, we show how the saddle point problem resulting from a mixed formulation can be reduced to a smaller saddle point problem, which we then solve iteratively. We note that only one pressure degree of freedom per subdomain remains after this reduction. The balancing Neumann–Neumann algorithm is introduced in Section 4, where we also give a detailed description of the global coarse model and the local problems which form the main building blocks of the preconditioner. Additional auxiliary results and our main theorem are given in Section 5. In Section 6, we show how to modify our method to deal with arbitrary constant Lamé parameters in the different subregions. Finally, in Section 7, we discuss the implementation of our methods and two sets of numerical experiments, one parallel and one serial. The first is for quite large lower order finite element problems; these results were obtained on a Beowulf cluster at the Argonne National Laboratory. The second is for spectral elements and were carried out using Matlab. Our experiments uniformly show very good performance.

2. Linear Elasticity and Mixed Discretizations. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain and let Γ_0 be a nonempty subset of its boundary. Let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$.

The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, subject to a surface force of density $\mathbf{g}$, along $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and a body force $\mathbf{f}$:

$$(1) \qquad 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \; dx + \lambda \int_\Omega \text{div}\mathbf{u} \; \text{div}\mathbf{v} \; dx \; = < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, and the inner products are defined as

$$\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}), \; < \mathbf{F}, \mathbf{v} > = \int_\Omega \sum_{i=1}^{3} f_i v_i \; dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i v_i \; ds.$$

In Sections 6 and 7 of this paper, we will consider the case of variable Lamé parameters and show that our algorithm is quite robust. The Lamé parameters can alternatively be expressed in terms of the Poisson ratio ν and Young's modulus E:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

When the material is almost incompressible, the Poisson ratio ν approaches the value $1/2$, i.e., $\lambda/\mu = 2\nu/(1-2\nu)$ approaches infinity. In such cases, finite or spectral element discretizations of this pure displacement formulation will increasingly suffer from locking phenomena and the resulting stiffness matrices become increasingly ill-conditioned. A well-known remedy is based on introducing the new variable $p = -\lambda \text{div}\mathbf{u} \in L^2(\Omega) = U$, that we will call pressure, and replacing the pure displacement problem with a mixed formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(2) \qquad \begin{cases} 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \; dx & - \quad \int_\Omega \text{div}\mathbf{v} \; p \; dx \quad = \quad < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V} \\ -\int_\Omega \text{div}\mathbf{u} \; q \; dx & - \quad 1/\lambda \int_\Omega pq \; dx \quad = \quad 0 \qquad \forall q \in U; \end{cases}$$

see Brezzi and Fortin [8]. In the case of homogeneous Dirichlet boundary conditions for $\mathbf{u}$ on all of $\partial\Omega$, we choose $U = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$, since it can be shown by the divergence theorem that the pressure will have zero mean value.

We could also consider more general saddle point problems with a penalty term: find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(3) \qquad \begin{cases} \mu a(\mathbf{u}, \mathbf{v}) & + \quad b(\mathbf{v}, p) \quad = \quad < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) & - \quad 1/\lambda \; c(p, q) \quad = \quad 0 \qquad \forall q \in U, \end{cases}$$

where $a(\cdot,\cdot)$ is a continuous, coercive bilinear form, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are continuous bilinear forms satisfying some additional hypotheses; see Brezzi and Fortin [8]. In our specific case, we have

$$a(\mathbf{u}, \mathbf{v}) = 2 \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \; dx, \; b(\mathbf{v}, q) = -\int_\Omega \text{div}\mathbf{v} \; q \; dx, \; c(p, q) = \int_\Omega pq \; dx.$$

By letting $\lambda/\mu \to \infty$, we obtain the limiting problem for incompressible linear elasticity or the classical Stokes system for an incompressible fluid; alternatively, the Stokes system is often written using the bilinear form $a(\mathbf{u},\mathbf{v}) = \mu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\ dx$. A penalty term, as in the compressible case, can also originate from stabilization techniques or penalty formulations for Stokes problems.

We will also need to consider problems with natural boundary conditions on all of $\partial\Omega$,

$$2\mu \sum_{j=1}^{d} \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j - p n_i = g_i \ \text{ on } \partial\Omega,\ i = 1,\dots,d \ , \tag{4}$$

derived by using Green's formula. In this case, as for the Laplace operator, the bilinear form $a(\cdot,\cdot)$ has a nontrivial nullspace $\ker(a)$ consisting of the constant velocities in the Stokes case (a d-dimensional nullspace) and of the rigid body motions in the elasticity case (a six-dimensional nullspace in three dimensions and a three-dimensional nullspace in two dimensions). Therefore there is a compatibility condition between the $\mathbf{f}$ and $\mathbf{g}$, namely,

$$\int_\Omega \mathbf{f}\cdot\mathbf{v} dx + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\, ds = 0 \qquad \forall \mathbf{v} \in\ \ker(a).$$

We note that if the boundary conditions are mixed (part essential and part natural), then there is a unique solution without any compatibility conditions.

Using Korn's inequality on the subspace orthogonal to the rigid body motions, we have the following equivalence between the Stokes and mixed elasticity bilinear forms (see, e.g., Klawonn and Widlund [19] for a proof):

LEMMA 2.1. *There exists a constant $c > 0$ such that*

$$c\|\nabla\mathbf{u}\|_{L^2(\Omega)} \le \|\epsilon(\mathbf{u})\|_{L^2(\Omega)} \le \|\nabla\mathbf{u}\|_{L^2(\Omega)}, \qquad \forall \mathbf{u} \in (H^1(\Omega))^d,\ \mathbf{u} \perp ker(a).$$

Here $\|\epsilon(\mathbf{u})\|^2_{L^2(\Omega)} = \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{u}) dx$.

We will consider conforming discretizations of Stokes and mixed elasticity equations using finite as well as spectral elements, all with discontinuous pressures. In fact, our work could easily be extended to the case when the pressure is discontinuous only across the boundaries of the subdomains.

2.1. Finite Element Methods with Discontinuous Pressures. We assume that the domain Ω can be decomposed into N nonoverlapping subdomains Ω_i of characteristic size H forming a hexahedral (quadrilateral) finite element mesh τ_H, which is assumed to be shape regular but not necessarily quasi uniform. This coarse triangulation is further refined into a fine hexahedral (quadrilateral) finite element triangulation τ_h of characteristic size h. Among the many choices of mixed finite elements available for Stokes and mixed elasticity equations, we consider the following:

a) $Q_2(h) - Q_0(h)$ mixed finite elements: the displacement space $\mathbf{V}$ is discretized by continuous, piecewise tri- or bi-quadratic displacements:

$$\mathbf{V}^h = \{\mathbf{v} \in \mathbf{V} : v_k|_T \in Q_2(T)\ \forall T \in \tau_h,\ k = 1,2,\dots,d\},$$

while the pressure space is discretized by discontinuous piecewise constant functions on τ_h :

$$U^h = \{q \in U : q|_T \in Q_0(T)\ \forall T \in \tau_h\}.$$

This method satisfies the uniform inf-sup condition

$$\sup_{\mathbf{v}\in\mathbf{V}^h} \frac{(\mathrm{div}\mathbf{v}, q)}{a(\mathbf{v},\mathbf{v})^{1/2}} \geq \beta_h \|q\|_{L^2} \quad \forall q \in U^h, \tag{5}$$

with $\beta_h \geq c > 0$ independent of h, but it leads to a nonoptimal error estimate; see Brezzi and Fortin [8, chap. VI.4, p. 221].

b) $Q_2(h) - P_1(h)$ mixed finite elements: the displacement space is as before, while the pressure space consists of piecewise linear discontinuous pressures:

$$U^h = \{q \in U : q|_T \in P_1(T)\ \forall T \in \tau_h\}\,.$$

These elements satisfy a uniform inf-sup condition (5) as well. There are also optimal $O(h^2)$ error estimates for both displacements and pressures; see Brezzi and Fortin [8, chap. VI, p. 216].

We note that while finite element methods based on hexahedra and quadrilaterals enjoy popularity, our theory applies equally well to stable mixed methods based on tetrahedra or triangles .

2.2. Spectral Element Methods: $Q_n - Q_{n-2}$. Let Ω_{ref} be the reference cube or square $(-1,1)^d, d = 3, 2$, and let $Q_n(\Omega_{\mathrm{ref}})$ be the set of polynomials on Ω_{ref} of degree n in each variable. We assume that the domain Ω can be decomposed into N nonoverlapping elements Ω_i, each of which is an image $\Omega_i = \phi_i(\Omega_{\mathrm{ref}})$, with ϕ_i an affine mapping. $\mathbf{V}$ is discretized, component by component, by continuous, piecewise tensor product polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\Omega_i} \circ \phi_i \in Q_n(\Omega_{\mathrm{ref}}),\ i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, d\}\,.$$

The pressure space is discretized by piecewise tensor product polynomials of degree $n-2$, which are discontinuous across the boundaries of the elements Ω_i:

$$U^n = \{q \in U : q|_{\Omega_i} \circ \phi_i \in Q_{n-2}(\Omega_{\mathrm{ref}}),\ i = 1, 2, \ldots, N\}\,.$$

These spectral elements are implemented using Gauss-Lobatto-Legendre (GLL(n)) quadrature, which also allows for the construction of a very convenient nodal tensor-product basis for $\mathbf{V}^n$. Denote by $\{\xi_i\}_{i=0}^n$ the set of GLL(n) points of $[-1, 1]$, and by σ_i the quadrature weight associated with ξ_i. Let $l_i(x)$ be the Lagrange interpolating polynomial of degree n that vanishes at all the GLL(n) nodes except at ξ_i, where it equals 1. Each element of $Q_n(\Omega_{\mathrm{ref}})$ is expanded in the GLL(n) basis, and each L^2- inner product of two scalar components u and v is replaced, in the three-dimensional case, by

$$(u, v)_{n,\Omega} = \sum_{s=1}^{N} \sum_{i,j,k=0}^{n} (u \circ \phi_s)(\xi_i, \xi_j, \xi_k)(v \circ \phi_s)(\xi_i, \xi_j, \xi_k)|J_s|\sigma_i\sigma_j\sigma_k\,,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. The mass matrix based on these basis elements and GLL(n) quadrature are diagonal. Similarly, a very convenient basis for U^n consists of the tensor-product Lagrangian nodal basis functions associated with the internal GLL(n) nodes, i.e., the endpoints -1 and $+1$ are excluded. We will call these the *pressure GLL(n) nodes.*

The $Q_n - Q_{n-2}$ method satisfies a nonuniform inf-sup condition

$$\sup_{\mathbf{v}\in\mathbf{V}^n} \frac{(\mathrm{div}\mathbf{v}, q)}{a(\mathbf{v},\mathbf{v})^{1/2}} \geq \beta_n \|q\|_{L^2} \quad \forall q \in U^n\,, \tag{6}$$

where $\beta_n = Cn^{-(d-1)/2}$, $d = 2, 3$, and the constant C is independent of n and q; see Maday, Meiron, Patera, and Rønquist [23] and Stenberg and Suri [36]. However, numerical experiments, reported in [23], have also shown that for practical values of n, e.g., $n \leq 16$, the inf-sup constant β_n of the $Q_n - Q_{n-2}$ method decays much slower than what might be expected from the theoretical bound.

An alternative, with a uniform bound on the inf-sup constant, is provided by the $Q_n - P_{n-1}$ method; see Bernardi and Maday [6]. However, this pressure space is less convenient than Q_{n-2} as far as implementation is concerned.

2.3. The Discrete System. Let $\widetilde{\mathbf{V}}$ and $\widetilde{U}$ be the discrete displacement and pressure spaces. In the finite element case, we write $\widetilde{\mathbf{V}} \times \widetilde{U} = \mathbf{V}^h \times U^h$, while in the spectral element case we have $\widetilde{\mathbf{V}} \times \widetilde{U} = \mathbf{V}^n \times U^n$. The discrete system obtained from (3) using finite or spectral elements is: find $\mathbf{u} \in \widetilde{\mathbf{V}}$ and $p \in \widetilde{U}$ such that

$$\left\{ \begin{array}{lllll} \mu a(\mathbf{u}, \mathbf{v}) & + & b(\mathbf{v}, p) & = & \mathbf{F}(\mathbf{v}) \quad \forall \mathbf{v} \in \widetilde{\mathbf{V}} \\ b(\mathbf{u}, q) & - & 1/\lambda \; c(p, q) & = & 0 \quad \forall q \in \widetilde{U}, \end{array} \right. \tag{7}$$

where we denote with the same letters the bilinear forms obtained using the appropriate quadrature rule described above.

On the *benign subspace*

$$(\widetilde{\mathbf{V}} \times \widetilde{U})_B = \{(\mathbf{u}, p) \in \widetilde{\mathbf{V}} \times \widetilde{U} : b(\mathbf{u}, q) - 1/\lambda \; c(p, q) = 0 \; \forall q \in \widetilde{U}\},$$

problem (7) is equivalent to the positive definite problem: find $(\mathbf{u}, p) \in (\widetilde{\mathbf{V}} \times \widetilde{U})_B$ such that

$$\mu a(\mathbf{u}, \mathbf{v}) + 1/\lambda \; c(p, q) = \mathbf{F}(\mathbf{v}) \quad \forall (\mathbf{v}, q) \in (\widetilde{\mathbf{V}} \times \widetilde{U})_B \, .$$

In matrix form, equations (7) have the form

$$K \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mu A & B^T \\ B & -1/\lambda \; C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}. \tag{8}$$

We will need the following two results, which give an explicit formula for the solution of a saddle point problem with a penalty term and a stability result for its solution. The first is proven by an explicit computation.

Lemma 2.2. *Let A and C be positive definite matrices and, if $\lambda = \infty$, let B have full row rank. Then,*

$$\begin{bmatrix} \mu A & B^T \\ B & -1/\lambda \; C \end{bmatrix}^{-1} = \begin{bmatrix} 1/\mu (A^{-1} - A^{-1} B^T S^{-1} B A^{-1}) & A^{-1} B^T S^{-1} \\ S^{-1} B A^{-1} & -\mu S^{-1} \end{bmatrix}, \tag{9}$$

where $S = BA^{-1}B^T + \mu/\lambda \; C$.

Lemma 2.3. *Consider the discrete saddle point problem*

$$\begin{bmatrix} \mu A & B^T \\ B & -1/\lambda \; C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix},$$

where A and C are positive definite and, if $\lambda = \infty$, B has full row rank. Let $\beta \geq 0$ be the best inf-sup constant such that

$$p^T B A^{-1} B^T p \geq \beta^2 p^T C p \quad \forall p.$$

Then,

$$i)\ \|\mathbf{u}\|_A \leq 1/\mu\|\mathbf{f}\|_{A^{-1}} + \frac{1}{\sqrt{\beta^2+\mu/\lambda}}\ \|g\|_{C^{-1}},$$

$$ii)\ \|p\|_C \leq \frac{1}{\sqrt{\beta^2+\mu/\lambda}}\ \|\mathbf{f}\|_{A^{-1}} + \frac{\mu}{\beta^2+\mu/\lambda}\ \|g\|_{C^{-1}}.$$

Proof. By the explicit formula (9) for the inverse of an invertible saddle point problem, we have

$$\mathbf{u} = 1/\mu\ (A^{-1} - A^{-1}B^TS^{-1}BA^{-1})\mathbf{f} + A^{-1}B^TS^{-1}g,$$
$$p = S^{-1}BA^{-1}\mathbf{f} - \mu S^{-1}g,$$

and from the inf-sup condition, we have

$$S = BA^{-1}B^T + \mu/\lambda\ C \geq (\beta^2 + \mu/\lambda)\ C.$$

(Here and in the following an inequality between matrices means an inequality between the associated quadratic forms). We note that $\|\mathbf{u}\|_A = \|A^{1/2}\mathbf{u}\|_{l^2}$ and $\|p\|_C = \|C^{1/2}p\|_{l^2}$; moreover $\|\mathbf{f}\|_{A^{-1}} = \|A^{-1/2}\mathbf{f}\|_{l^2}$ and $\|g\|_{C^{-1}} = \|C^{-1/2}g\|_{l^2}$ are the matrix representation of the dual norms of $\mathbf{f}$ and g, respectively. Indeed,

$$\sup_{\mathbf{v}} \frac{(\mathbf{f}^T\mathbf{v})^2}{\mathbf{v}^TA\mathbf{v}} = \sup_{\mathbf{w}} \frac{(\mathbf{f}^TA^{-1/2}\mathbf{w})^2}{\mathbf{w}^T\mathbf{w}} = \frac{(\mathbf{f}^TA^{-1/2}A^{-1/2}\mathbf{f})^2}{\mathbf{f}^TA^{-1}\mathbf{f}} = \mathbf{f}^TA^{-1}\mathbf{f},$$

and similarly for $\|g\|_{C^{-1}}$.

i) The $A-$norm of the displacement component is estimated by

(10) $$\|A^{1/2}\mathbf{u}\|_{l^2} \leq 1/\mu\|(I - A^{-1/2}B^TS^{-1}BA^{-1/2})A^{-1/2}\mathbf{f}\|_{l^2} + \|A^{-1/2}B^TS^{-1}g\|_{l^2}.$$

The first term in (10) is bounded by $1/\mu\|A^{-1/2}\mathbf{f}\|_{l^2}$ because from $S^{-1} \leq (BA^{-1}B^T)^{-1}$ it follows that

$$0 \leq A^{-1/2}B^TS^{-1}BA^{-1/2} \leq A^{-1/2}B^T(BA^{-1}B^T)^{-1}BA^{-1/2} \leq I,$$

since the next to last expression is an orthogonal projection. The square of the second term in (10) is estimated similarly by

$$\|A^{-1/2}B^TS^{-1}g\|_{l^2}^2 = g^TS^{-1}BA^{-1}B^TS^{-1}g \leq g^TS^{-1}g$$
$$\leq \frac{1}{\beta^2+\mu/\lambda}\ g^TC^{-1}g = \frac{1}{\beta^2+\mu/\lambda}\ \|g\|_{C^{-1}}^2,$$

and therefore i) follows.

ii) The $C-$norm of the pressure component is estimated by

(11) $$\|C^{1/2}p\|_{l^2} \leq \|C^{1/2}S^{-1}BA^{-1}\mathbf{f}\|_{l^2} + \mu\|C^{1/2}S^{-1}g\|_{l^2}.$$

The first term on the right in (11) is bounded by $\frac{1}{\sqrt{\beta^2+\mu/\lambda}}\|A^{-1/2}\mathbf{f}\|_{l^2}$ because

$$\|C^{1/2}S^{-1}BA^{-1}\mathbf{f}\|_{l^2}^2 = \mathbf{f}^TA^{-1}B^TS^{-1}CS^{-1}BA^{-1}\mathbf{f}$$
$$\leq \frac{1}{\beta^2+\mu/\lambda}\mathbf{f}^TA^{-1}B^TS^{-1}BA^{-1}\mathbf{f}$$
$$\leq \frac{1}{\beta^2+\mu/\lambda}\mathbf{f}^TA^{-1}B^T(BA^{-1}B^T)^{-1}BA^{-1}\mathbf{f} \leq \frac{1}{\beta^2+\mu/\lambda}\mathbf{f}^TA^{-1}\mathbf{f};$$

we again use that the matrix $A^{-1/2}B^T(BA^{-1}B^T)^{-1}BA^{-1/2}$ is an orthogonal projection. The square of the second term on the right in (11) is estimated by

$$\mu^2 \, \|C^{1/2}S^{-1}g\|_{l^2}^2 = \mu^2 \; g^T S^{-1} C S^{-1} g \le \frac{\mu^2}{\beta^2 + \mu/\lambda} g^T S^{-1} g \le \left(\frac{\mu}{\beta^2 + \mu/\lambda}\right)^2 g^T C^{-1} g,$$

and therefore ii) follows. □

3. Substructuring for Saddle Point Problems. The domain Ω is decomposed into open, nonoverlapping hexahedral (quadrilateral) subdomains Ω_i and the interface Γ, i.e.,

$$\Omega = \cup_{i=1}^N \Omega_i \cup \Gamma.$$

Here $\Gamma = \left(\bigcup_{i=1}^N \partial\Omega_i\right) \setminus \partial\Omega$. Each Ω_i typically consists of one, or a few, spectral elements of degree n or of many finite elements. We note that some versions of our algorithm can be extended to a more general choice of subdomain shapes. We note, in particular, that each Ω_i can be a union of a finite number of shape regular, coarse elements. At present, our technical tools do not allow us to give a full theory for cases when the intersection between the boundaries of the subdomains fail to be smooth.

We denote by Γ_h and $\partial\Omega_h$ the set of nodes belonging to the interface Γ and $\partial\Omega$, respectively. The starting point of our algorithm is the implicit elimination of the interior degrees of freedom, i.e., the interior displacement component and what we will call the interior pressure component which has zero average over the individual subdomains. This process, also known as static condensation, is carried out by solving decoupled local saddle point problems on each subdomain Ω_i with Dirichlet boundary conditions for the displacements given on $\partial\Omega_i$. We then obtain a saddle point Schur complement problem for the interface displacements and a constant pressure in each subdomain. This reduced problem will be solved by a preconditioned Krylov space iteration, normally the preconditioned conjugate gradient method.

For simplicity, we will use the same letters to denote both functions and their associated vector representations; the same convention will also be used for linear operators and their associated matrix forms.

3.1. Substructuring in Matrix Form. In order to eliminate the interior degrees of freedom, we reorder the vector of unknowns as

$$(12) \qquad \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \quad \begin{array}{l} \text{interior displacements} \\ \text{interior pressures with zero average} \\ \text{interface displacements} \\ \text{constant pressures in each } \Omega_i. \end{array}$$

Then, after using the same permutation, the discrete system matrix can be written as

$$\begin{bmatrix} K_{II} & K_{\Gamma I}^T \\ & \\ K_{\Gamma I} & K_{\Gamma\Gamma} \end{bmatrix} = \left[\begin{array}{cc|cc} \mu A_{II} & B_{II}^T & \mu A_{\Gamma I}^T & 0 \\ B_{II} & -1/\lambda\, C_{II} & B_{I\Gamma} & 0 \\ \hline \mu A_{\Gamma I} & B_{I\Gamma}^T & \mu A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & B_0 & -1/\lambda\, C_0 \end{array}\right],$$

where the zero blocks are due to the interior displacements having zero flux across the subdomain boundaries and the interior pressure having a zero average.

Eliminating the interior unknowns $\mathbf{u}_I$ and p_I by static condensation, we obtain the saddle point Schur complement system

$$(13) \qquad S_{\mu,\lambda} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix},$$

where

$$(14) \qquad \begin{aligned} S_{\mu,\lambda} &= K_{\Gamma\Gamma} - K_{\Gamma I} K_{II}^{-1} K_{\Gamma I}^T = \\ &= \begin{bmatrix} \mu A_{\Gamma\Gamma} & B_0^T \\ B_0 & -1/\lambda\ C_0 \end{bmatrix} \\ &\quad - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{\Gamma I}^T & 0 \\ B_{I\Gamma} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mu S_{\Gamma,\mu,\lambda} & B_0^T \\ B_0 & -1/\lambda\ C_0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\Gamma \\ 0 \end{bmatrix} - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda\ C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix}.$$

By using a second permutation that reorders the interior displacements and pressures subdomain by subdomain, we find that K_{II}^{-1} represents the solution of N decoupled saddle point problems, one for each subdomain and all uniquely solvable, with Dirichlet data given on $\partial\Omega_i$:

$$K_{II}^{-1} = \begin{bmatrix} {K_{II}^{(1)}}^{-1} & & 0 \\ & \ddots & \\ 0 & & {K_{II}^{(N)}}^{-1} \end{bmatrix}.$$

This matrix is associated with the discrete extension operator $\mathcal{SH}_{\mu,\lambda}$ described in the next subsection.

The Schur complement $S_{\mu,\lambda}$ does not need to be explicitly assembled since only its action $S_{\mu,\lambda}v$ on a vector v is needed in a Krylov iteration. This operation essentially only requires the action of K_{II}^{-1} on a vector, i.e., the solution of N decoupled saddle point problems. In other words, $S_{\mu,\lambda}v$ is computed by subassembling the actions of the subdomain Schur complements $S_{\mu,\lambda}^{(i)}$ defined for Ω_i by

$$(15) \qquad \begin{aligned} S_{\mu,\lambda}^{(i)} &= K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} {K_{\Gamma I}^{(i)}}^T \\ &= \begin{bmatrix} \mu A_{\Gamma\Gamma}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda\ C_0^{(i)} \end{bmatrix} \\ &\quad - \begin{bmatrix} \mu A_{\Gamma I}^{(i)} & {B_{I\Gamma}^{(i)}}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu A_{II}^{(i)} & {B_{II}^{(i)}}^T \\ B_{II}^{(i)} & -1/\lambda\ C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \mu {A_{\Gamma I}^{(i)}}^T & 0 \\ B_{I\Gamma}^{(i)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mu S_{\Gamma,\mu,\lambda}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda\ C_0^{(i)} \end{bmatrix}. \end{aligned}$$

Once $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$ is known, $\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix}$ can be found by back-substitution:

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix} = \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda\, C_{II} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix} - \begin{bmatrix} \mu A_{\Gamma I}^T & 0 \\ B_{I\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \right).$$

3.2. Substructuring in Variational Form. The substructuring procedure described in the previous section is associated with the space decomposition

$$\widetilde{\mathbf{V}} \times \widetilde{U} = \oplus_{i=1}^{N} \mathbf{V}_i \times U_i \oplus \mathbf{V}_\Gamma \times U_0,$$

where the interior spaces are defined by

$$\mathbf{V}_i = \widetilde{\mathbf{V}} \cap H_0^1(\Omega_i) \qquad\qquad U_i = \widetilde{U} \cap L_0^2(\Omega_i),$$

and the spaces of interface displacements and coarse pressures, constant in each subdomain, are defined by

$$\mathbf{V}_\Gamma = \mathcal{SH}_{\mu,\lambda}(\widetilde{\mathbf{V}}) = \{\mathbf{v} \in \widetilde{\mathbf{V}} : \mathbf{v}_{|\Omega_i} = \mathcal{SH}_{\mu,\lambda}(\mathbf{v}_{|\partial\Omega_i}),\ i = 1, \cdots, N\},$$

$$U_0 = \{q \in \widetilde{U} : q_{|\Omega_i} = \text{ constant},\ i = 1, \cdots, N\}.$$

Here $\mathcal{SH}_{\mu,\lambda} : \widetilde{\mathbf{V}}_{|\Gamma} \longrightarrow \widetilde{\mathbf{V}}$, is the displacement component of the discrete saddle point harmonic extension operator that maps an interface displacement $\mathbf{u}_\Gamma \in \widetilde{\mathbf{V}}_{|\Gamma}$ onto the solution $\begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix}$ of the following homogeneous saddle point problem, which can be defined on each subdomain separately: find $\tilde{\mathbf{u}} \in \widetilde{\mathbf{V}}$ and $\tilde{p} \in \widetilde{\mathbf{U}}$ such that on each Ω_i,

$$(16) \qquad \begin{cases} \mu a_i(\tilde{\mathbf{u}}, \mathbf{v}) & + & b_i(\mathbf{v}, \tilde{p}) & = & 0 & \forall \mathbf{v} \in \mathbf{V}_i \\ b_i(\tilde{\mathbf{u}}, q) & - & 1/\lambda\ c_i(\tilde{p}, q) & = & 0 & \forall q \in U_i \\ \tilde{\mathbf{u}} = \mathbf{u}_\Gamma & \text{on} & \partial\Omega_i. \end{cases}$$

The following comparison of the energy of the discrete saddle point harmonic extension operator and the discrete harmonic extensions $\mathcal{H}$ of each displacement component separately is a generalization of the analogous comparison in the Stokes case (see [7], [22], [9, 10], [2]).

LEMMA 3.1. *Given $\mathbf{u}_\Gamma \in \widetilde{\mathbf{V}}_{|\Gamma}$, let $\mathcal{H}(\mathbf{u}_\Gamma)$ be its componentwise discrete harmonic extension and let $\mathcal{SH}_{\mu,\lambda}(\mathbf{u}_\Gamma) = \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix}$ be its discrete saddle point harmonic extension. Then, $\forall \mathbf{u}_\Gamma \in \mathbf{V}_\Gamma$ such that $\tilde{\mathbf{u}} \perp ker(a_i)$,*

$$\left(1 + \frac{\sqrt{d}}{\sqrt{\beta^2 + \mu/\lambda}}\right)^{-2} \left\| \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix} \right\|_{\mu,\lambda,i}^2 \leq \mu \|\nabla \mathcal{H} \mathbf{u}_\Gamma\|_{L^2(\Omega_i)}^2 \leq C \left\| \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix} \right\|_{\mu,\lambda,i}^2,$$

where β is the inf-sup constant of the chosen mixed finite element spaces $\tilde{\mathbf{V}}_i \times \tilde{U}_i$ and the μ, λ-norm is defined by

$$\left\| \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix} \right\|_{\mu,\lambda,i}^2 = \mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + 1/\lambda\ c_i(\tilde{p}, \tilde{p}).$$

Proof. The right inequality is an easy consequence of the minimal property of the discrete harmonic extension and the lower bound of Lemma 2.1.

In order to prove the left inequality, we choose $\mathbf{v} = \tilde{\mathbf{u}} - \mathcal{H}\mathbf{u}_\Gamma$ in (16) and obtain

$$\mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + b_i(\tilde{\mathbf{u}}, \tilde{p}) = \mu a_i(\tilde{\mathbf{u}}, \mathcal{H}\mathbf{u}_\Gamma) + b_i(\mathcal{H}\mathbf{u}_\Gamma, \tilde{p}).$$

Therefore, since $(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{u})_{L^2(\Omega_i)} \leq d\ (\epsilon(\mathbf{u}) : \epsilon(\mathbf{u}))_{L^2(\Omega_i)} = d/2\ a_i(\mathbf{u}, \mathbf{u})$, $(d = 2, 3)$,

$$(17)\quad \mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + 1/\lambda\ c_i(\tilde{p}, \tilde{p}) \leq \big(\mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}})^{1/2} + \sqrt{d/2}\ \|\tilde{p}\|_{L^2(\Omega_i)}\big) a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)^{1/2}.$$

We will now estimate $\|\tilde{p}\|_{L^2(\Omega_i)}$ by applying Lemma 2.3 ii) to the saddle point problem with homogeneous boundary conditions satisfied by $(\tilde{\mathbf{u}} - \mathcal{H}\mathbf{u}_\Gamma, \tilde{p})$. From (16), we find that on each Ω_i

$$(18)\quad \begin{cases} \mu a_i(\tilde{\mathbf{u}} - \mathcal{H}\mathbf{u}_\Gamma, \mathbf{v}) & + & b_i(\mathbf{v}, \tilde{p}) & = & -\mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_i \\ b_i(\tilde{\mathbf{u}} - \mathcal{H}\mathbf{u}_\Gamma, q) & - & 1/\lambda\ c_i(\tilde{p}, q) & = & -b_i(\mathcal{H}\mathbf{u}_\Gamma, q) & \forall q \in U_i. \end{cases}$$

Then Lemma 2.3 ii) yields

$$\begin{aligned} \|\tilde{p}\|_{L^2(\Omega_i)} &\leq \frac{1}{\sqrt{\beta^2 + \mu/\lambda}} \sup_{\mathbf{v} \in \mathbf{V}_i} \frac{\mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathbf{v})}{a_i(\mathbf{v}, \mathbf{v})^{1/2}} + \frac{\mu}{\beta^2 + \mu/\lambda} \sup_{q \in U_i} \frac{b_i(\mathcal{H}\mathbf{u}_\Gamma, q)}{\|q\|_{L^2(\Omega_i)}} \\ &\leq \left(\frac{1}{\sqrt{\beta^2 + \mu/\lambda}} + \frac{\sqrt{d/2}}{\beta^2 + \mu/\lambda}\right) \mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)^{1/2}. \end{aligned}$$

Therefore, it follows from (17) that

$$\begin{aligned} & \mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + 1/\lambda\ c_i(\tilde{p}, \tilde{p}) \\ \leq\ & \mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}})^{1/2} a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma)^{1/2} + \left(\frac{\sqrt{d/2}}{\sqrt{\beta^2 + \mu/\lambda}} + \frac{d/2}{\beta^2 + \mu/\lambda}\right) \mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \\ \leq\ & 1/2\ \mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \left(1/2 + \frac{\sqrt{d/2}}{\sqrt{\beta^2 + \mu/\lambda}} + \frac{d/2}{\beta^2 + \mu/\lambda}\right) \mu a_i(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma). \end{aligned}$$

By using the upper bound of Lemma 2.1, we then obtain

$$\mu a_i(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + 1/\lambda\ c_i(\tilde{p}, \tilde{p}) \leq \left(1 + \frac{\sqrt{2d}}{\sqrt{\beta^2 + \mu/\lambda}} + \frac{d}{\beta^2 + \mu/\lambda}\right) \mu \|\nabla \mathcal{H}\mathbf{u}_\Gamma\|^2_{L^2(\Omega_i)}.$$

The lower bound of the lemma now follows by an elementary inequality. □

If we define an interface inner product by

$$s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = a(\mathcal{SH}_{\mu,\lambda}(\mathbf{u}_\Gamma), \mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma)) = \mathbf{u}_\Gamma^T S_{\Gamma,\mu,\lambda} \mathbf{v}_\Gamma,$$

and by $b_0(\mathbf{u}_\Gamma, p_0)$ and $c_0(p_0, q_0)$ the restrictions of the other bilinear forms to the saddle point harmonic extensions and the coarse piecewise constant pressures, then the variational formulation of the saddle point Schur complement problem (13) can be given by: find $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma$ and $p_0 \in U_0$ such that,

$$(19)\quad \begin{cases} \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) & + & b_0(\mathbf{v}_\Gamma, p_0) & = & \widetilde{\mathbf{F}}(\mathbf{v}_\Gamma) & \forall \mathbf{v}_\Gamma \in \mathbf{V}_\Gamma \\ b_0(\mathbf{u}_\Gamma, q_0) & - & 1/\lambda\ c_0(p_0, q_0) & = & 0 & \forall q_0 \in U_0. \end{cases}$$

On the benign subspace $(\mathbf{V}_\Gamma \times U_0)_B$ defined by

$$\begin{aligned}(\mathbf{V}_\Gamma \times U_0)_B &= \{(\mathbf{u}_\Gamma, p_0) \in \mathbf{V}_\Gamma \times U_0 : B_0 \mathbf{u}_\Gamma - 1/\lambda\ C_0 p_0 = 0\} \\ &= \{(\mathbf{u}_\Gamma, p_0) \in \mathbf{V}_\Gamma \times U_0 : b_0(\mathbf{u}_\Gamma, q_0) - 1/\lambda\ c_0(p_0, q_0) = 0, \forall q_0 \in U_0\},\end{aligned}$$

problem (19) is equivalent to the positive definite problem: find $(\mathbf{u}_\Gamma, p_0) \in (\mathbf{V}_\Gamma \times U_0)_B$ such that

$$\mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) + 1/\lambda\ c_o(p_0, q_0) = \widetilde{\mathbf{F}}(\mathbf{v}_\Gamma) \quad \forall (\mathbf{v}_\Gamma, q_0) \in (\mathbf{V}_\Gamma \times U_0)_B. \tag{20}$$

4. Balancing Neumann–Neumann Preconditioners. We will solve the saddle point Schur complement problem

$$S_{\mu,\lambda} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mu S_{\Gamma,\mu,\lambda} & B_0^T \\ B_0 & -1/\lambda\ C_0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix} \tag{21}$$

by a preconditioned Krylov space method such as GMRES or PCG. The latter can be applied to this indefinite problem because we will start and keep the iterates in the subspace of benign functions.

The matrix form of the preconditioner is

$$Q_{\mu,\lambda} = Q_H + (I - Q_H S_{\mu,\lambda}) \sum_{i=1}^{N} Q_i (I - S_{\mu,\lambda} Q_H),$$

where the coarse operator Q_H and local operators Q_i are defined below. The preconditioned operator – the Schwarz operator – is then

$$T_{\mu,\lambda} = Q_{\mu,\lambda} S_{\mu,\lambda} = T_0 + (I - T_0) \sum_{i=1}^{N} T_i (I - T_0),$$

where $T_0 = Q_H S_{\mu,\lambda}$ and $T_i = Q_i S_{\mu,\lambda}$. The operators Q_H, Q_i, T_0, and T_i also depend on μ and λ but we leave them without subscripts in order to keep the notation simpler. We note that $Q_{\mu,\lambda}$ can also be written as a three-step preconditioner as in [30]. For simplicity, we will use the same symbol (for example $\mathbf{v}_\Gamma$) for both the interface vector and the function of $\mathbf{V}_\Gamma$ obtained by extension inside each subdomain using the discrete saddle point harmonic extension operator $\mathcal{SH}_{\mu,\lambda}$. In addition, we will avoid writing explicitly finite and spectral element interpolants; therefore, when writing a product of functions (e.g., $\delta_i \mathbf{v}_\Gamma$) we mean the finite or spectral element function with nodal values equal to the product of those of the two functions.

This balancing Neumann-Neumann preconditioner $T_{\mu,\lambda}$ is associated with further decomposing the interface space $\mathbf{V}_\Gamma \times U_0$ as

$$\mathbf{V}_\Gamma \times U_0 = \mathbf{V}_0 \times U_0 + \sum_{i=1}^{N} \mathbf{V}_{\Gamma,i} \times U_{0,i}.$$

Here, the coarse displacement space $\mathbf{V}_0$ is defined in terms of special functions $\delta_i^\dagger$, introduced below, and it is given by either one of the three following choices:

$$\begin{aligned}\mathbf{V}_0^0 &= \left\{\mathbf{v} \in \mathbf{V}_\Gamma : \mathbf{v} \in \left(\mathit{span}\left\{\delta_i^\dagger\right\}\ \textit{multiplied by the functions of}\ \ker(a)\right)\right\}, \\ \mathbf{V}_0^1 &= \mathbf{V}_0^0 + \mathit{span}\{\textit{normal direction quadratic face (edge) bubble functions}\}, \\ \mathbf{V}_0^2 &= \mathbf{V}_0^0 + \left\{\textit{tri- (or bi-)linear coarse piecewise } Q_1^H \textit{ functions}\right\},\end{aligned}$$

while the local spaces are defined by:

$$\mathbf{V}_{\Gamma,i} = \{\mathbf{v} \in \mathbf{V}_\Gamma : \ \ \mathbf{v}(\mathbf{x}) = 0 \ \ \forall x \in \Gamma_h \setminus \partial\Omega_{i,h}\}, \qquad U_{0,i} = span\{1\}.$$

We could also consider richer coarse spaces obtained, e.g., by adding to $\mathbf{V}_0^0$ all the functions of $\mathbf{V}_\Gamma$ that are piecewise quadratic polynomials on Γ, as we did in our study [30] of the Stokes case.

We now describe the coarse and local problems in more detail.

The coarse problem. Given a residual vector r, the coarse term $Q_H r$ is the solution of a coarse, global saddle point problem with a few displacement degrees of freedom and one constant pressure per subdomain Ω_i:

$$Q_H = R_H^T S_{0,\mu,\lambda}^{-1} R_H,$$

where

$$R_H = \begin{bmatrix} L_0^T & 0 \\ 0 & I \end{bmatrix},$$

and

$$S_{0,\mu,\lambda} = R_H S_{\mu,\lambda} R_H^T = \begin{bmatrix} \mu L_0^T S_{\Gamma,\mu,\lambda} L_0 & L_0^T B_0^T \\ B_0 L_0 & -1/\lambda \, C_0 \end{bmatrix}. \tag{22}$$

We will use the notation $\widetilde{S}_{0,\mu,\lambda} = L_0^T S_{\Gamma,\mu,\lambda} L_0$ for the leading block of $S_{0,\mu,\lambda}$. The columns of the matrix L_0 span the coarse space $\mathbf{V}_0$ and in order to define them, we need to define the Neumann-Neumann counting functions $\delta_i \in \mathbf{V}_\Gamma$ associated with each subdomain Ω_i and their pseudo inverses $\delta_i^\dagger$:

- δ_i is zero at all nodes of $\Gamma_h \setminus \partial\Omega_{i,h}$ while its value at any node on $\partial\Omega_i$ equals the number of subdomains shared by that node;

- the pseudo inverse $\delta_i^\dagger$ is the function $1/\delta_i(x)$ for all nodes where $\delta_i(x) \neq 0$, and it vanishes at all other points of $\Gamma_h \cup \partial\Omega_h$.

Then the columns of L_0 are defined by one of the following three choices:

$\mathbf{V}_0^0$: the inverse counting functions $\delta_i^\dagger$ multiplied by the functions of $\ker(a)$;

$\mathbf{V}_0^1$: as in $\mathbf{V}_0^0$ with the addition of the quadratic coarse face (edge) bubble functions for the normal direction;

$\mathbf{V}_0^2$: as in $\mathbf{V}_0^0$ with the addition of the continuous piecewise tri- or bi-linear functions on the coarse mesh τ_H.

The first choice corresponds to the standard choice for second order scalar elliptic problems and it provides a quite minimal coarse displacement space. It turns out to be far from uniformly inf-sup stable and it therefore leads to a nonscalable algorithm in the incompressible case. However, in the compressible case where λ/μ is bounded, it still leads to a scalable algorithm; see our main theorem and the numerical results. The first and second choices are enrichments of the first that turn out to be inf-sup stable uniformly in N and μ/λ.

In order to avoid linearly dependent $\delta_i^\dagger$ functions, and hence a singular coarse space problem, we might have to drop all of the components of these functions for one subdomain, depending on the coarse triangulation.

In variational terms, the coarse problem is defined as follows: given $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \in \mathbf{V}_\Gamma \times U_0$, define $\begin{bmatrix} \mathbf{w}_\Gamma \\ q_0 \end{bmatrix} = T_0 \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \in \mathbf{V}_0 \times U_0$ as the solution of the coarse saddle point problem

$$\left\{\begin{array}{rcll} \mu s_{\mu,\lambda}(\mathbf{w}_\Gamma,\mathbf{v}) \ + \ b_0(\mathbf{v},q_0) & = & \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma,\mathbf{v}) \ + \ b_0(\mathbf{v},p_0) & \forall \mathbf{v}\in \mathbf{V}_0 \\ b_0(\mathbf{w}_\Gamma,q) \ - \ 1/\lambda\ c_0(q_0,q) & = & b_0(\mathbf{u}_\Gamma,q) \ - \ 1/\lambda\ c_0(p_0,q) & \forall q\in U_0, \end{array}\right.$$

or,

$$\left\{\begin{array}{rcll} \mu s_{\mu,\lambda}(\mathbf{w}_\Gamma-\mathbf{u}_\Gamma,\mathbf{v}) \ + \ b_0(\mathbf{v},q_0-p_0) & = & 0 & \forall \mathbf{v}\in \mathbf{V}_0 \\ b_0(\mathbf{w}_\Gamma-\mathbf{u}_\Gamma,q) \ - \ 1/\lambda\ c_0(q_0-p_0,q) & = & 0 & \forall q\in U_0. \end{array}\right. \tag{23}$$

It follows immediately that T_0 is a projection, i.e., $T_0^2 = T_0$. Moreover, from the second equation of (23), we see that $\begin{bmatrix} \mathbf{w}_\Gamma-\mathbf{u}_\Gamma \\ q_0-p_0 \end{bmatrix}$ is balanced, i.e., $Range(I-T_0)\subset (\mathbf{V}_\Gamma,U_0)_B$.

We prove in the next lemma that the coarse space correction is independent of p_0 and we can therefore drop the terms involving p_0 in equation (23). This lemma provides the displacement and pressure components of the coarse operator $Q_H S_{\mu,\lambda}$ in matrix form. The formulas follow by using Lemma 2.2 for the coarse matrix (22).

LEMMA 4.1. *Let* $\widetilde{S}_{0,\mu,\lambda} = L_0^T S_{\Gamma,\mu,\lambda} L_0$ *and let* $\widetilde{S}_{\mu,\lambda} = B_0 L_0 \widetilde{S}_{0,\mu,\lambda}^{-1} L_0^T B_0^T + \mu/\lambda\ C_0$. *Then,* $\begin{bmatrix} \mathbf{v}_\Gamma \\ q_0 \end{bmatrix} = (I - Q_H S_{\mu,\lambda}) \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$ *depends only on the displacement component* $\mathbf{u}_\Gamma$ *and equals*

$$\begin{array}{rcl} \mathbf{v}_\Gamma & = & (I-T_0^u)\mathbf{u}_\Gamma, \\ q_0 & = & -T_0^p \mathbf{u}_\Gamma, \end{array}$$

where

$$\begin{array}{rcl} T_0^u & = & L_0\widetilde{S}_{0,\mu,\lambda}^{-1}L_0^T S_{\Gamma,\mu,\lambda} - L_0\widetilde{S}_{0,\mu,\lambda}^{-1}L_0^T B_0^T \widetilde{S}_{\mu,\lambda}^{-1} B_0 L_0 \widetilde{S}_{0,\mu,\lambda}^{-1} L_0^T S_{\Gamma,\mu,\lambda} \\ & & + L_0\widetilde{S}_{0,\mu,\lambda}^{-1}L_0^T B_0^T \widetilde{S}_{\mu,\lambda}^{-1} B_0, \\ T_0^p & = & \mu \widetilde{S}_{\mu,\lambda}^{-1} B_0 L_0 \widetilde{S}_{0,\mu,\lambda}^{-1} L_0^T S_{\Gamma,\mu,\lambda} - \mu\ \widetilde{S}_{\mu,\lambda}^{-1} B_0. \end{array}$$

We also note that equation (23) implies that $\langle T_0\mathbf{u},(I-T_0)\,\mathbf{v}\rangle_{S_{\mu,\lambda}} = 0$ for all $\mathbf{u},\mathbf{v}$. This, together with the fact that T_0 is a projection, implies that T_0 is symmetric with respect to the bilinear form $\langle\cdot,\cdot\rangle_{S_{\mu,\lambda}}$, defined by

$$\left\langle \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ q_0 \end{bmatrix} \right\rangle_{S_{\mu,\lambda}} = \left\langle S_{\mu,\lambda} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ q_0 \end{bmatrix} \right\rangle.$$

Local problems. Each local operator Q_i is based on the solution of a local saddle point problem on Ω_i with a natural boundary condition on $\partial\Omega_i\setminus\Gamma_0$. This local problem is singular for any subdomain Ω_i the boundary of which does not intersect the Dirichlet boundary Γ_0; all the rigid body motions are in the nullspace. Such a subregion is called a *floating* subdomain. To avoid possible complications with singular problems, we modify the local saddle point problems on the floating subdomains by adding ϵ times the displacement mass matrix to the local stiffness matrix $K^{(i)}$. We could also make these solutions unique by requiring that each displacement component is orthogonal to the nullspace of $a(\cdot,\cdot)$; the right hand sides will always be compatible. The matrix form of Q_i is

$$Q_i = R_i^T \begin{bmatrix} D_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mu S^{(i)}_{\Gamma,\mu,\lambda,\epsilon} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda\ C_0^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} D_i^{-1} & 0 \\ 0 & I \end{bmatrix} R_i \tag{24}$$

Here R_i are $0, 1$ restriction matrices mapping $\mathbf{V}_\Gamma \times U_0$ into $\mathbf{V}_{\Gamma,i} \times U_{0,i}$ and D_i are diagonal matrices representing multiplication by the counting functions δ_i. Moreover,

$$S^{(i)}_{\mu,\lambda,\epsilon} = \begin{bmatrix} \mu S^{(i)}_{\Gamma,\mu,\lambda,\epsilon} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda\ C_0^{(i)} \end{bmatrix}$$

is the local saddle point Schur complement, associated with the subdomain Ω_i, of the regularized local stiffness matrix

$$K^{(i)}_\epsilon = \begin{bmatrix} \mu A^{(i)}_{II,\epsilon} & {B^{(i)}_{II}}^T & \mu {A^{(i)}_{\Gamma I,\epsilon}}^T & 0 \\ B^{(i)}_{II} & -1/\lambda\ C^{(i)}_{II} & B^{(i)}_{I\Gamma} & 0 \\ \mu A^{(i)}_{\Gamma I,\epsilon} & {B^{(i)}_{I\Gamma}}^T & \mu A^{(i)}_{\Gamma\Gamma,\epsilon} & {B^{(i)}_0}^T \\ 0 & 0 & B^{(i)}_0 & -1/\lambda\ C^{(i)}_0 \end{bmatrix},$$

where

$$A^{(i)}_\epsilon = A^{(i)} + \epsilon M^{(i)}.$$

Here $M^{(i)}$ is the local displacement mass matrix.

The local operators Q_i will only be applied to residuals of benign displacement fields and thus the second residual component will vanish. We have also shown, in Lemma 4.1, that the pressure components obtained in this step of the preconditioner plays no further role when we next apply the operator $(I - T_0)$. Therefore, the identity block I in the scaling matrix in (24) can equally well be replaced by zero.

In preparation for writing the local problems in variational form, we define the operator $\tilde{T}_i : \mathbf{V}_\Gamma \times U_0 \to \mathbf{V}_{\Gamma,i} \times U_{0,i}$ as $\tilde{T}_i = R_i T_i$. The local problems are now defined in variational terms: for $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$, $\tilde{T}_i \mathbf{w} = \begin{bmatrix} \tilde{T}^u_i \mathbf{w} \\ \tilde{T}^p_i \mathbf{w} \end{bmatrix} \in \mathbf{V}_{\Gamma,i} \times U_{0,i}$ is the solution of a local saddle point problem with natural boundary conditions given by $\forall \mathbf{v}_i \in \mathbf{V}_{\Gamma,i}, \forall q_i \in U_{0,i}$,

$$\begin{cases} \mu s_{\mu,\lambda,\epsilon,i}(\delta_i \tilde{T}^u_i \mathbf{u}_\Gamma, \delta_i \mathbf{v}_i) + b_{0,i}(\delta_i \mathbf{v}_i, \tilde{T}^p_i \mathbf{u}_\Gamma) = \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_i) + b_0(\mathbf{v}_i, p_0) \\ b_{0,i}(\delta_i \tilde{T}^u_i \mathbf{u}_\Gamma, q_i) - 1/\lambda\ c_{0,i}(\tilde{T}^p_i \mathbf{u}_\Gamma, q_i) = b_0(\mathbf{u}_\Gamma, q_i) - 1/\lambda\ c_0(p_0, q_i). \end{cases} \tag{25}$$

In the formula above,

$$\begin{aligned} s_{\mu,\lambda,\epsilon,i}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) &= a_{\epsilon,i}(\mathcal{SH}_{\mu,\lambda,\epsilon,i}(\mathbf{u}_\Gamma), \mathcal{SH}_{\mu,\lambda,\epsilon,i}(\mathbf{v}_\Gamma)), \\ a_{\epsilon,i}(\mathbf{u}, \mathbf{v}) &= a_i(\mathbf{u}, \mathbf{v}) + \epsilon \int_{\Omega_i} \mathbf{u} \cdot \mathbf{v} dx, \end{aligned} \tag{26}$$

and $\mathcal{SH}_{\mu,\lambda,\epsilon,i}$ is the displacement component of the discrete saddle point harmonic extension operator defined in terms of the regularized $a_{\epsilon,i}(\cdot,\cdot)$ displacement bilinear form instead of the standard $a_i(\cdot,\cdot)$ form.

We note that, since we need to apply local saddle point solvers only to elements $\mathbf{w} \in Range(I - T_0) \subset (\mathbf{V}_\Gamma \times U_0)_B$, the right-hand side of the second equation in (25) equals zero.

Defining the space $(\mathbf{V}_{\Gamma,i} \times U_{0,i})_B$ by

$$(\mathbf{V}_{\Gamma,i} \times U_{0,i})_B = \left\{ \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] \in \mathbf{V}_{\Gamma,i} \times U_{0,i} \;\middle|\; b_{0,i}(\delta_i \mathbf{u}, q) = \frac{1}{\lambda} c_{0,i}(p,q) \quad \forall q \in U_{0,i} \right\},$$

and assuming $\mathbf{w} \in (\mathbf{V}_\Gamma \times U_0)_B$, we can restate the definition of $\tilde{T}_i$ as follows: $\tilde{T}_i \mathbf{w} \in (\mathbf{V}_{\Gamma,i} \times U_{0,i})_B$ and satisfies, $\forall \left[\begin{array}{c} \mathbf{v}_i \\ q_i \end{array} \right] \in (\mathbf{V}_{\Gamma,i} \times U_{0,i})_B$,

$$\mu s_{\mu,\lambda,\epsilon,i}(\delta_i \tilde{T}_i^u \mathbf{w}, \delta_i \mathbf{v}_i) + 1/\lambda \; c_{0,i}(\tilde{T}_i^p \mathbf{w}, q_i) = \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_i) + b_0(\mathbf{v}_i, p_0), \tag{27}$$

or,

$$\left\langle \left[\begin{array}{c} \delta_i \tilde{T}_i^u \mathbf{w} \\ \tilde{T}_i^p \mathbf{w} \end{array} \right], \left[\begin{array}{c} \delta_i \mathbf{v}_i \\ q_i \end{array} \right] \right\rangle_{\mu,\lambda,\epsilon,i} = \left\langle \mathbf{w}, \left[\begin{array}{c} \mathbf{v}_i \\ q_i \end{array} \right] \right\rangle_{S_{\mu,\lambda}}. \tag{28}$$

Here the inner-product $\langle \cdot, \cdot \rangle_{\mu,\lambda,\epsilon,i}$ is defined by the left-hand side of (27).

5. Analysis of the Method.

5.1. Auxiliary Results.

We will work with the $\mu, \lambda-$inner product

$$\left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{\mu,\lambda} = \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) + 1/\lambda \; c_0(p_0, q_0).$$

On the benign subspace $(\mathbf{V}_\Gamma \times U_0)_B$, this inner product coincides with the bilinear form defined by $S_{\mu,\lambda}$, i.e.,

$$\left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{\mu,\lambda} = \left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{S_{\mu,\lambda}} \forall \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \in (\mathbf{V}_\Gamma \times U_0)_B,$$

since $B_0 \mathbf{u}_\Gamma - 1/\lambda \; C_0 p_0 = B_0 \mathbf{v}_\Gamma - 1/\lambda \; C_0 q_0 = 0$.

Lemma 5.1. *Let* $\left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right] \in Range(I - T_0)$ *and let* $\left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right]$ *be arbitrary. Then,*

$$\left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{S_{\mu,\lambda}} = \left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], (I - T_0) \left[\begin{array}{c} \mathbf{v}_\Gamma \\ \star \end{array} \right] \right\rangle_{\mu,\lambda},$$

where $\star$ is an arbitrary piecewise constant pressure vector.

Proof. By the symmetry of $(I - T_0)$ with respect to the $S_{\mu,\lambda}$ bilinear form and the fact that $(I - T_0)$ is a projection, we have

$$\left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{S_{\mu,\lambda}} = \left\langle (I - T_0) \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{S_{\mu,\lambda}}$$

$$= \left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], (I - T_0) \left[\begin{array}{c} \mathbf{v}_\Gamma \\ q_0 \end{array} \right] \right\rangle_{S_{\mu,\lambda}} = \left\langle \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right], (I - T_0) \left[\begin{array}{c} \mathbf{v}_\Gamma \\ \star \end{array} \right] \right\rangle_{S_{\mu,\lambda}},$$

where we can replace q_0 with an arbitrary piecewise constant pressure vector denoted by $\star$ because, by Lemma 4.1, the result of the action of $(I - T_0)$ on a vector does not depend on its pressure component. Since now both arguments are benign, we can switch to the $\mu, \lambda-$inner product and the lemma follows. □

In the proof of our main result, we need a bound on the norm to the coarse correction operator. We note that this operator has norm 1 when restricted to the space of benign functions but that it is applied to more general functions in our algorithm.

The analog of the following lemma has been proven for Stokes' equations in [30, Lemma 5.2]. Here, we will show that the incompressible case gives the worst bound and that we therefore have a bound which is uniform in μ and λ.

LEMMA 5.2. *The coarse space $\mathbf{V}_0^1 \times U_0$ satisfies the inf-sup condition*

$$\sup_{\mathbf{v}_\Gamma \in \mathbf{V}_0^1} \frac{(\mathrm{div}\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma), q_0)^2}{a(\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma), \mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma))} \geq \beta_0^2 \|q_0\|_{L^2}^2 \qquad \forall q_0 \in U_0,$$

with

$$\beta_0^2 = \begin{cases} \frac{C}{1+\log(H/h)} & \text{for finite elements} \\ \frac{C}{1+\log n} & \text{for spectral elements,} \end{cases}$$

where the constant C is independent of the Lamé parameters.

Proof. We first note that the numerator of the expression in the lemma is independent of μ and λ since, by the divergence theorem applied on each subdomain, we have

$$\int_{\Omega_i} \mathrm{div}\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma) q_0 dx = \int_{\partial\Omega_i} \mathbf{v}_\Gamma \cdot \mathbf{n}\, q_0 ds.$$

We will now show that the denominator increases with decreasing values of μ/λ. We do so by considering $S_{\Gamma,\mu,\lambda}$. We have, by (14),

$$\mu S_{\Gamma,\mu,\lambda} = \mu A_{\Gamma\Gamma} - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{\Gamma I}^T \\ B_{I\Gamma} \end{bmatrix},$$

where $A_{\Gamma\Gamma}, A_{II}, C_{II}$ are positive definite matrices. A direct computation shows that

$$S_{\Gamma,\mu,\lambda} = A_{\Gamma\Gamma} - \begin{bmatrix} A_{\Gamma I} & B_{I\Gamma}^T \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & -\mu/\lambda\, C_{II} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Gamma I}^T \\ B_{I\Gamma} \end{bmatrix}.$$

Since the Lamé parameters only enter in one of the matrices, we only have to consider that matrix. Factoring it, we find,

$$\begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & -\mu/\lambda\, C_{II} \end{bmatrix} = \begin{bmatrix} I & 0 \\ B_{II}A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & 0 \\ 0 & -S_{II} \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1}B_{II}^T \\ 0 & I \end{bmatrix},$$

where $S_{II} = B_{II}A_{II}^{-1}B_{II}^T + \mu/\lambda\, C_{II}$. Again only one of the matrices depend on the Lamé parameters. It is now easy to show that the denominator is at its maximum in the incompressible limit. We can replace the denominator by that for the Stokes' case, for which the result of the lemma already has been established in [30, Lemma 5.2]. This follows by noticing that

$$a(\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma), \mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma)) \leq a(\mathcal{H}(\mathbf{v}_\Gamma), \mathcal{H}(\mathbf{v}_\Gamma)) \leq \|\nabla \mathcal{H}\mathbf{v}_\Gamma\|_{L^2}^2.$$

□

Our numerical results, reported in Section 7, indicate that a uniform inf-sup condition does not hold for the first coarse space $\mathbf{V}_0^0 \times U_0$. The results for the third coarse space $\mathbf{V}_0^2 \times U_0$ are quite satisfactory although we do not have a full theory. We note that the $Q_1 - Q_0$ elements by themselves are not inf-sup stable but that we are using a richer velocity space which also includes the $\delta_i^\dagger$ functions times basis elements for the space of rigid body motions. We also work in the somewhat different context of saddle point harmonic extensions of traces on Γ.

LEMMA 5.3. *The coarse correction operator $(I - T_0)$ satisfies the stability estimate*

$$\left\| \begin{bmatrix} (I - T_0^u)\mathbf{u}_\Gamma \\ -T_0^p \mathbf{u}_\Gamma \end{bmatrix} \right\|_{\mu,\lambda}^2 \leq \mu\rho^2 \|\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}^2,$$

where

$$\rho^2 = 2\left(2 + \frac{\sqrt{d/2}}{\sqrt{\beta_0^2 + \mu/\lambda}}\right)^2,$$

and β_0 is the inf-sup constant of the coarse space.

We note that $\rho^2 \leq C(1 + \lambda/\mu)$ whatever the value of β_0. We will establish such a bound by a direct argument in the general case discussed in Section 6.

Proof. We apply the stability estimates of Lemma 2.3. Setting $p_0 = 0$, the coarse problem (23) can be rewritten as

$$\begin{cases} \mu s_{\mu,\lambda}(T_0^u \mathbf{u}_\Gamma, \mathbf{v}) & + & b_0(\mathbf{v}, T_0^p \mathbf{u}_\Gamma) & = & \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_0 \\ b_0(T_0^u \mathbf{u}_\Gamma, q) & - & 1/\lambda \; c_0(T_0^p \mathbf{u}_\Gamma, q) & = & b_0(\mathbf{u}_\Gamma, q) & \forall q \in U_0. \end{cases}$$

We recall that the matrix form of the coarse operator is

$$T_0 = Q_H S_{\mu,\lambda} = R_H^T S_{0,\mu,\lambda}^{-1} R_H S_{\mu,\lambda},$$

where $S_{0,\mu,\lambda}$ is given by (22). Let $\begin{bmatrix} \tilde{\mathbf{u}}_0 \\ \tilde{p}_0 \end{bmatrix}$ be the solution of $S_{0,\mu,\lambda} \begin{bmatrix} \tilde{\mathbf{u}}_0 \\ \tilde{p}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ g_0 \end{bmatrix}$, where the right hand side is given by

$$\begin{aligned} \mathbf{f}_0^T \tilde{\mathbf{v}} &= \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, L_0 \tilde{\mathbf{v}}) = \mu s_{\mu,\lambda}(\mathbf{u}_\Gamma, \mathbf{v}) \\ g_0^t \tilde{q} &= b_0(\mathbf{u}_\Gamma, \tilde{q}). \end{aligned}$$

Then, $T_0^u \mathbf{u}_\Gamma = L_0 \tilde{\mathbf{u}}_0$, $T_0^p \mathbf{u}_\Gamma = \tilde{p}_0$, and

$$\begin{aligned} \|\tilde{\mathbf{u}}_0\|_{\widetilde{S}_{0,\mu,\lambda}}^2 &= \tilde{\mathbf{u}}_0^T \widetilde{S}_{0,\mu,\lambda} \tilde{\mathbf{u}}_0 = \tilde{\mathbf{u}}_0^T L_0^T S_{\Gamma,\mu,\lambda} L_0 \tilde{\mathbf{u}}_0 = \|L_0 \tilde{\mathbf{u}}_0\|_{S_{\Gamma,\mu,\lambda}}^2 = \|T_0^u \mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}^2, \\ \|\tilde{p}_0\|_{C_0}^2 &= \|T_0^p \mathbf{u}_\Gamma\|_{C_0}^2. \end{aligned}$$

Since,

$$\begin{aligned} \|\mathbf{f}_0\|_{\widetilde{S}_{0,\mu,\lambda}^{-1}}^2 &= \sup_{\tilde{\mathbf{v}}} \frac{(\mathbf{f}_0^T \tilde{\mathbf{v}})^2}{\tilde{\mathbf{v}}^T \widetilde{S}_{0,\mu,\lambda} \tilde{\mathbf{v}}} = \sup_{\tilde{\mathbf{v}}} \frac{\mu^2 s_{\mu,\lambda}(\mathbf{u}_\Gamma, L_0 \tilde{\mathbf{v}})^2}{\tilde{\mathbf{v}}^T \widetilde{S}_{0,\mu,\lambda} \tilde{\mathbf{v}}} \leq \mu^2 \|\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}^2, \\ \|g_0\|_{C_0^{-1}}^2 &= \sup_{\tilde{q}} \frac{(g_0^T \tilde{q})^2}{\tilde{q}^T C_0 \tilde{q}} = \sup_{\tilde{q}} \frac{b_0(\mathbf{u}_\Gamma, \tilde{q})^2}{\tilde{q}^T C_0 \tilde{q}} \leq \|\mathrm{div} \mathcal{SH}_{\mu,\lambda}(\mathbf{u}_\Gamma)\|_{L^2}^2 \leq d/2 \; \|\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}^2. \end{aligned}$$

We can then apply the estimates of Lemma 2.3 and obtain

$$\begin{aligned} \|T_0^u \mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}} &\leq \left(1 + \frac{\sqrt{d/2}}{\sqrt{\beta_0^2 + \mu/\lambda}}\right) \|\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}, \\ \|T_0^p \mathbf{u}_\Gamma\|_{C_0} &\leq \mu \left(\frac{1}{\sqrt{\beta_0^2 + \mu/\lambda}} + \frac{\sqrt{d/2}}{\beta_0^2 + \mu/\lambda} \right) \|\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}. \end{aligned}$$

From the definition of the $\mu, \lambda-$norm

$$\left\| \begin{bmatrix} (I - T_0^u)\mathbf{u}_\Gamma \\ -T_0^p \mathbf{u}_\Gamma \end{bmatrix} \right\|_{\mu,\lambda}^2 = \mu \|(I - T_0^u)\mathbf{u}_\Gamma\|_{S_{\Gamma,\mu,\lambda}}^2 + 1/\lambda \|T_0^p \mathbf{u}_\Gamma\|_{C_0}^2.$$

The lemma now follows by some elementary estimates. □

5.2. Main Result. We are now ready to formulate our main theorem.

THEOREM 5.4. *On the benign subspace $(\mathbf{V}_\Gamma \times U_0)_B$ the balancing Neumann-Neumann operator $T_{\mu,\lambda}$ is symmetric, positive definite with respect to the $\mu, \lambda-$inner product, and*

$$cond(T_{\mu,\lambda}) \leq C \left(2 + \frac{\sqrt{d/2}}{\sqrt{\beta_0^2 + \mu/\lambda}}\right) \left(1 + \frac{\sqrt{d}}{\sqrt{\beta^2 + \mu/\lambda}}\right)^2 \alpha,$$

where

$$\alpha = \begin{cases} (1 + \log(H/h))^2 & \text{for finite elements} \\ (1 + \log n)^2 & \text{for spectral elements,} \end{cases}$$

and β_0 and β are the inf-sup constants of the coarse problem and the original discrete saddle point problem, respectively. The constant C in the bound is uniform in the parameter ϵ used in the regularization of the local Neumann problems.

Proof. Let $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$ be benign. Then, $T_0\mathbf{w}$ as well as $(I - T_0)\mathbf{w}$ are benign and we can use either $\langle \cdot, \cdot \rangle_{\mu,\lambda}$ or $\langle \cdot, \cdot \rangle_{S\mu,\lambda}$ in our formulas. Since T_0 is a $\langle \cdot, \cdot \rangle_{\mu,\lambda}$-orthogonal projection on the benign subspace, we find that

$$\begin{aligned} &\langle T_{\mu,\lambda}\mathbf{w}, \mathbf{w} \rangle_{\mu,\lambda} \\ = \;& \langle T_0 \mathbf{w}, \mathbf{w} \rangle_{\mu,\lambda} + \left\langle (I - T_0) \sum_i T_i (I - T_0)\mathbf{w}, \mathbf{w} \right\rangle_{\mu,\lambda} \\ = \;& \|T_0 \mathbf{w}\|_{\mu,\lambda}^2 + \left\langle \sum_i T_i (I - T_0)\mathbf{w}, (I - T_0)\mathbf{w} \right\rangle_{S_{\mu,\lambda}} \qquad (29) \\ = \;& \|\mathbf{w}\|_{\mu,\lambda}^2 - \|(I - T_0)\mathbf{w}\|_{\mu,\lambda}^2 + \left\langle \sum_i T_i (I - T_0)\mathbf{w}, (I - T_0)\mathbf{w} \right\rangle_{S_{\mu,\lambda}}. \end{aligned}$$

Our goal is to find lower and upper bounds for this expression in terms of $\|\mathbf{w}\|_{\mu,\lambda}^2$.

Lower bound: Define $\tilde{\mathbf{w}} = \begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_0 \end{bmatrix} = (I - T_0)\mathbf{w}$. Since the pseudo inverses $\delta_i^\dagger$ of the counting functions define a partition of unity, we have $\tilde{\mathbf{u}}_\Gamma = \sum_i \tilde{\mathbf{u}}_i$ with

$\tilde{\mathbf{u}}_i = \delta_i^\dagger \tilde{\mathbf{u}}_\Gamma \in \mathbf{V}_{\Gamma,i}$. Let $\tilde{q}_i$ be such that $\begin{bmatrix} \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \in (\mathbf{V}_{\Gamma,i} \times U_{0,i})_B$. From the definition of the local problems (27), we have

$$\begin{aligned} \mu s_{\mu,\lambda}(\tilde{\mathbf{u}}_\Gamma, \tilde{\mathbf{u}}_\Gamma) &= \sum_i \mu s_{\mu,\lambda}(\tilde{\mathbf{u}}_\Gamma, \tilde{\mathbf{u}}_i) \\ = \quad & \sum_i \mu s_{\mu,\lambda,\epsilon,i}(\delta_i \tilde{T}_i^u \tilde{\mathbf{w}}, \delta_i \tilde{\mathbf{u}}_i) + \sum_i 1/\lambda \; c_{0,i}(\tilde{T}_i^p \tilde{\mathbf{w}}, \tilde{q}_i) - \sum_i b_0(\tilde{\mathbf{u}}_i, \tilde{p}_0). \end{aligned}$$

But $\sum_i b_0(\tilde{\mathbf{u}}_i, \tilde{p}_0) = b_0(\tilde{\mathbf{u}}_\Gamma, \tilde{p}_0) = 1/\lambda \; c_0(\tilde{p}_0, \tilde{p}_0)$ because $\begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_0 \end{bmatrix}$ is benign. Then,

$$\begin{aligned} \|\tilde{\mathbf{w}}\|^2_{\mu,\lambda} &= \mu s_{\mu,\lambda}(\tilde{\mathbf{u}}_\Gamma, \tilde{\mathbf{u}}_\Gamma) + 1/\lambda \; c_0(\tilde{p}_0, \tilde{p}_0) \\ &= \sum_{i=1}^N \left[\mu s_{\mu,\lambda,\epsilon,i}(\delta_i \tilde{T}_i^u \tilde{\mathbf{w}}, \delta_i \tilde{\mathbf{u}}_i) + 1/\lambda \; c_{0,i}(\tilde{T}_i^p \tilde{\mathbf{w}}, \tilde{q}_i) \right] \\ &= \sum_{i=1}^N \left\langle \begin{bmatrix} \delta_i \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \delta_i \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right\rangle_{\mu,\lambda,\epsilon,i} \\ &\le \left(\sum_{i=1}^N \left\| \begin{bmatrix} \delta_i \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i} \right)^{1/2} \left(\sum_i \left\| \begin{bmatrix} \delta_i \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i} \right)^{1/2}. \end{aligned} \tag{30}$$

We note that $\delta_i \tilde{\mathbf{u}}_i = \delta_i \delta_i^\dagger \tilde{\mathbf{u}}_\Gamma = \tilde{\mathbf{u}}_\Gamma|_{\partial\Omega_i}$. From the definition of $(\mathbf{V}_{\Gamma,i} \times U_{0,i})_B$, we have that $b_{0,i}(\delta_i \tilde{\mathbf{u}}_i, r_i) = 1/\lambda \; c_{0,i}(\tilde{q}_i, r_i) \;\; \forall r_i \in U_{0,i}$. Summing over i and recalling that $\begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_0 \end{bmatrix}$ is benign, we conclude that $\tilde{q}_i = \tilde{p}_0|_{\Omega_i}$. The square of the second factor in (30) can then be estimated as in [30]:

$$\begin{aligned} \sum_{i=1}^N \left\| \begin{bmatrix} \delta_i \tilde{\mathbf{u}}_i \\ \tilde{q}_i \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i} &= \left\| \begin{bmatrix} \tilde{\mathbf{u}}_\Gamma \\ \tilde{p}_0 \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon} = \mu s_{\mu,\lambda,\epsilon}(\tilde{\mathbf{u}}_\Gamma, \tilde{\mathbf{u}}_\Gamma) + 1/\lambda \; c_0(\tilde{p}_0, \tilde{p}_0) \\ \le \quad & \left(1 + \frac{\epsilon}{\eta}\right) \mu s_{\mu,\lambda}(\tilde{\mathbf{u}}_\Gamma, \tilde{\mathbf{u}}_\Gamma) + 1/\lambda \; c_0(\tilde{p}_0, \tilde{p}_0) \le \left(1 + \frac{\epsilon}{\eta}\right) \|\tilde{\mathbf{w}}\|^2_{\mu,\lambda}, \end{aligned} \tag{31}$$

where

$$\eta = \inf_{\mathbf{v}_\Gamma} \frac{a(\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma), \mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma))}{\|\mathcal{SH}_{\mu,\lambda}(\mathbf{v}_\Gamma)\|^2_{L^2(\Omega)}} > 0.$$

The square of the first factor in (30) is estimated by using the definition (28) of the local problems:

$$\begin{aligned} \sum_i \left\| \begin{bmatrix} \delta_i \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i} &= \sum_i \left\langle \begin{bmatrix} \delta_i \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \delta_i \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\rangle_{\mu,\lambda,\epsilon,i} \\ = \quad & \sum_i \left\langle \tilde{\mathbf{w}}, \begin{bmatrix} \tilde{T}_i^u \tilde{\mathbf{w}} \\ \tilde{T}_i^p \tilde{\mathbf{w}} \end{bmatrix} \right\rangle_{S_{\mu,\lambda}} = \left\langle (I - T_0)\mathbf{w}, \sum_i T_i (I - T_0)\mathbf{w} \right\rangle_{S_{\mu,\lambda}}. \end{aligned} \tag{32}$$

Putting (30), (31), and (32) together, we obtain

$$\|(I - T_0)\mathbf{w}\|^2_{\mu,\lambda} \le \left(1 + \frac{\epsilon}{\eta}\right) \left\langle \sum_i T_i (I - T_0)\mathbf{w}, (I - T_0)\mathbf{w} \right\rangle_{S_{\mu,\lambda}}. \tag{33}$$

Finally, from (29) and (33),

$$\begin{aligned}\langle T_{\mu,\lambda}\mathbf{w},\mathbf{w}\rangle_{\mu,\lambda} &\geq \|\mathbf{w}\|_{\mu,\lambda}^2 - \frac{\epsilon}{\eta}\left\langle \sum_i T_i(I-T_0)\mathbf{w},(I-T_0)\mathbf{w}\right\rangle_{S_{\mu,\lambda}} \\ &\geq \|\mathbf{w}\|_{\mu,\lambda}^2 - \frac{\epsilon}{\eta}\langle T_{\mu,\lambda}\mathbf{w},\mathbf{w}\rangle_{\mu,\lambda}.\end{aligned}$$

Therefore,

$$\langle T_{\mu,\lambda}\mathbf{w},\mathbf{w}\rangle_{\mu,\lambda} \geq \left(1+\frac{\epsilon}{\eta}\right)^{-1}\|\mathbf{w}\|_{\mu,\lambda}^2.$$

Upper bound: We recall that T_0 restricted to the benign subspace is an orthogonal projection with respect to $\langle\cdot,\cdot\rangle_{\mu,\lambda}$. Therefore, the only term we have to control in (29) is $\langle\sum_i T_i(I-T_0)\mathbf{w},(I-T_0)\mathbf{w}\rangle_{S_{\mu,\lambda}}$. This expression will be bounded from above in terms of the square of the norm of $\mathbf{w}$. Since the norm of $(I-T_0)\mathbf{w}$ is less than or equal to that of $\mathbf{w}$, we will assume, henceforth, that $\mathbf{w}\in Range(I-T_0)$.

Let p_{T_i} be the piecewise constant pressure so that $\begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}$ is benign; we remark that both $T_i^u\mathbf{w}$ and p_{T_i} are supported in Ω_i and the subdomains adjacent to it. By Lemma 5.1,

$$\begin{aligned}&\left\langle \sum_i T_i(I-T_0)\mathbf{w},(I-T_0)\mathbf{w}\right\rangle_{S_{\mu,\lambda}} \\ &= \left\langle \mathbf{w},(I-T_0)\sum_i T_i\mathbf{w}\right\rangle_{S_{\mu,\lambda}} = \left\langle \mathbf{w},(I-T_0)\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ T_i^p\mathbf{w}\end{bmatrix}\right\rangle_{\mu,\lambda} \\ &= \left\langle \mathbf{w},\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}\right\rangle_{S_{\mu,\lambda}} = \left\langle \mathbf{w},\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}\right\rangle_{\mu,\lambda} \qquad (34) \\ &\leq \|\mathbf{w}\|_{\mu,\lambda}\left\|\sum_i \begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}\right\|_{\mu,\lambda}\end{aligned}$$

and we are left with bounding the second factor from above. By a standard coloring argument, it suffices to bound the $\mu,\lambda-$norm of just one term, $\begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}$, of the sum. By the comparison of the energy of the discrete saddle point and harmonic extensions in Lemma 3.1, we have

$$\left\|\begin{bmatrix} T_i^u\mathbf{w} \\ p_{T_i}\end{bmatrix}\right\|_{\mu,\lambda}^2 \leq \left(1+\frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}}\right)^2 \mu\|\nabla\mathcal{H}(T_i^u\mathbf{w})\|_{L^2(\Omega)}^2. \qquad (35)$$

We then apply to each scalar component of $\mathcal{H}(T_i^u\mathbf{w})$ the decomposition lemma for the scalar Neumann-Neumann algorithm (see Dryja and Widlund [12, lemma 4] for finite elements and Pavarino [28, lemma 6.2] for spectral elements) and obtain

$$\mu\|\nabla\mathcal{H}(T_i^u\mathbf{w})\|_{L^2(\Omega)}^2 \leq C\alpha\mu\Big(\|\nabla\mathcal{H}_{\epsilon,i}(\delta_i T_i^u\mathbf{w})\|_{L^2(\Omega_i)}^2 + \epsilon\|\mathcal{H}_{\epsilon,i}(\delta_i T_i^u\mathbf{w})\|_{L^2(\Omega_i)}^2\Big),$$

where

$$\alpha = \begin{cases} (1+\log(H/h))^2 & \text{for finite elements} \\ (1+\log n)^2 & \text{for spectral elements.} \end{cases}$$

By using on each subdomain a variant of Lemma 3.1 for the regularized $\epsilon-$forms, we can return to the discrete saddle point harmonic extension:

$$\begin{aligned} & \mu\Big(\|\nabla \mathcal{H}_{\epsilon,i}(\delta_i T_i^u \mathbf{w})\|^2_{L^2(\Omega_i)} + \epsilon \|\mathcal{H}_{\epsilon,i}(\delta_i T_i^u \mathbf{w})\|^2_{L^2(\Omega_i)} \Big) \\ \leq \quad & C\mu\Big(a_i(\mathcal{S}\mathcal{H}_{\epsilon,i}(\delta_i T_i^u \mathbf{w}), \mathcal{S}\mathcal{H}_{\epsilon,i}(\delta_i T_i^u \mathbf{w})) + \epsilon \|\mathcal{S}\mathcal{H}_{\epsilon,i}(\delta_i T_i^u \mathbf{w}))\|^2_{L^2(\Omega_i)} \Big). \end{aligned}$$

Hence,

$$\mu \|\nabla \mathcal{H}(T_i^u \mathbf{w})\|^2_{L^2(\Omega)} \leq C\alpha \left\| \begin{bmatrix} \delta_i T_i^u \mathbf{w} \\ T_i^p \mathbf{w} \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i}. \tag{36}$$

From (35) and (36), we obtain

$$\begin{aligned} & \left\| \begin{bmatrix} T_i^u \mathbf{w} \\ p_{T_i} \end{bmatrix} \right\|^2_{\mu,\lambda} \leq C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \left\| \begin{bmatrix} \delta_i T_i^u \mathbf{w} \\ T_i^p \mathbf{w} \end{bmatrix} \right\|^2_{\mu,\lambda,\epsilon,i} \\ = \quad & C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \langle \mathbf{w}, T_i \mathbf{w} \rangle_{S_{\mu,\lambda}} \quad \text{(definition of the local problems)} \\ = \quad & C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \langle \mathbf{w}, (I-T_0) T_i \mathbf{w} \rangle_{\mu,\lambda} \quad \text{(Lemma 5.1)} \\ \leq \quad & C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \|\mathbf{w}\|_{\mu,\lambda} \|(I-T_0) T_i \mathbf{w}\|_{\mu,\lambda} \quad \text{(Cauchy-Schwarz).} \end{aligned}$$

Lemma 5.3 now gives

$$\|(I-T_0) T_i \mathbf{w}\|^2_{\mu,\lambda} \leq \mu \rho^2 \|T_i^u \mathbf{w}\|^2_{S_{\Gamma,\mu,\lambda}}.$$

Since $\mu \|T_i^u \mathbf{w}\|^2_{S_{\Gamma,\mu,\lambda}} \leq \left\| \begin{bmatrix} T_i^u \mathbf{w} \\ p_{T_i} \end{bmatrix} \right\|^2_{\mu,\lambda}$, we then have

$$\left\| \begin{bmatrix} T_i^u \mathbf{w} \\ p_{T_i} \end{bmatrix} \right\|^2_{\mu,\lambda} \leq C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \rho \|\mathbf{w}\|_{\mu,\lambda} \left\| \begin{bmatrix} T_i^u \mathbf{w} \\ p_{T_i} \end{bmatrix} \right\|_{\mu,\lambda},$$

i.e.,

$$\left\| \begin{bmatrix} T_i^u \mathbf{w} \\ p_{T_i} \end{bmatrix} \right\|_{\mu,\lambda} \leq C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \rho \|\mathbf{w}\|_{\mu,\lambda},$$

and finally, from (34), we obtain

$$\Big\langle \sum_i T_i (I-T_0)\mathbf{w}, (I-T_0)\mathbf{w} \Big\rangle_{S_{\mu,\lambda}} \leq C\alpha \Big(1 + \frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}} \Big)^2 \rho \|\mathbf{w}\|^2_{\mu,\lambda}.$$

The upper bound is derived from (29) by using the fact that $\|T_0\mathbf{w}\|_{\mu,\lambda} \le \|\mathbf{w}\|_{\mu,\lambda}$ for any benign $\mathbf{w}$:

$$\langle T_{\mu,\lambda}\mathbf{w},\mathbf{w}\rangle_{\mu,\lambda} \le C\alpha\left(1+\frac{\sqrt{d}}{\sqrt{\beta^2+\mu/\lambda}}\right)^2 \rho\,\|\mathbf{w}\|^2_{\mu,\lambda}$$

The result on the condition number of $T_{\mu,\lambda}$ now follows from the upper and lower bounds derived above. □

6. Heterogeneous Materials with Variable Coefficients. Our algorithm can be extended to heterogeneous materials with different Lamé constants λ_i, μ_i in the different subdomains Ω_i:

$$\begin{cases} 2\sum_{i=1}^{N}\int_{\Omega_i} \mu_i\ \epsilon(\mathbf{u}):\epsilon(\mathbf{v})\ dx & - \quad \int_{\Omega} \mathrm{div}\mathbf{v}\ p\ dx & = \ <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v}\in\mathbf{V} \\ -\int_{\Omega}\mathrm{div}\mathbf{u}\ q\ dx & - \quad \sum_{i=1}^{N}\int_{\Omega_i} 1/\lambda_i\ \ pq\ dx & = \quad 0 \qquad \forall q \in U. \end{cases}$$

The global stiffness matrix K is constructed by subassembling the contributions

$$\begin{bmatrix} \mu_i A^{(i)} & {B^{(i)}}^T \\ B^{(i)} & -1/\lambda_i\ C^{(i)} \end{bmatrix}$$

from the individual substructures; cf. the discussion at the end of Subsection 3.1. A saddle point Schur complement matrix can similarly be assembled from the matrices

$$\begin{bmatrix} \mu_i S^{(i)}_{\Gamma,\mu,\lambda} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda_i\ C_0^{(i)} \end{bmatrix},$$

which are obtained by static condensation. The balancing Neumann-Neumann preconditioner $Q_{\mu,\lambda}$ for $S_{\mu,\lambda}$ has the same form as before, but uses modified local and coarse spaces. As in the scalar elliptic case, the jumps in the coefficients μ_i are accounted for by appropriately scaling the special counting functions δ_i and their pseudo inverses $\delta_i^\dagger$. As in [25], we now use the definition

$$\delta_i^\dagger(x) = \frac{\mu_i^\gamma(x)}{\sum_{j\in N_x}\mu_j^\gamma(x)}, \tag{37}$$

where $\gamma \in [1/2,\infty)$ and N_x is the set of indices of all the subdomains that have x on their boundaries. The new δ_i is the pseudo inverse of $\delta_i^\dagger$. As before, both δ_i and $\delta_i^\dagger$ vanish at all interface nodes outside $\partial\Omega_{i,h}$ and are extended inside each subdomain by discrete saddle point harmonic extensions. The pseudo inverses $\delta_i^\dagger$ still form a partition of unity. We have chosen $\gamma = 1$ in our numerical experiments reported in the next section.

The local and coarse problems are then defined formally as before but using the modified functions δ_i and $\delta_i^\dagger$. In particular, the coarse problem is now written as

$$\begin{cases} \sum_{i=1}^{N}\mu_i s_{\mu,\lambda,i}(T_0^u\mathbf{u}_\Gamma,\mathbf{v}) + b_0(\mathbf{v},T_0^p\mathbf{u}_\Gamma) & = \sum_{i=1}^{N}\mu_i s_{\mu,\lambda,i}(\mathbf{u}_\Gamma,\mathbf{v}) \quad \forall \mathbf{v}\in\mathbf{V}_0 \\ b_0(T_0^u\mathbf{u}_\Gamma,q) - \sum_{i=1}^{N} 1/\lambda_i\ c_{0,i}(T_0^p\mathbf{u}_\Gamma,q) & = \quad b_0(\mathbf{u}_\Gamma,q) \qquad \forall q\in U_0. \end{cases} \tag{38}$$

Our balancing Neumann-Neumann preconditioner is therefore well defined also in the case of variable coefficients and our numerical experiments, reported in the next section, indicate that indeed our preconditioner retains its excellent convergence rate also for heterogeneous materials.

Unfortunately, we have not been able to completely extend our analysis to the case of variable coefficients. While it is straightforward to check that all other parts of the proof still works, we have not been able to extend Lemma 5.3 to the general case with variable coefficients. We note that we do not know how to prove the uniform inf-sup stability for the underlying finite element discretization or for the continuous problem for arbitrary heterogeneous coefficients and that is at the heart of our difficulties.

We can nevertheless prove a weaker result by selecting the test functions $\mathbf{v} = T_0^u \mathbf{u}_\Gamma$ and $q = T_0^p \mathbf{u}_\Gamma$ in (38). We can now eliminate $b_0(T_0^u \mathbf{u}_\Gamma, T_0^p \mathbf{u}_\Gamma)$ and obtain,

$$\sum_{i=1}^N \mu_i s_{\mu,\lambda,i}(T_0^u \mathbf{u}_\Gamma, T_0^u \mathbf{u}_\Gamma) + \sum_{i=1}^N \frac{1}{\lambda_i} c_{0,i}(T_0^p \mathbf{u}_\Gamma, T_0^p \mathbf{u}_\Gamma) =$$
$$= \sum_{i=1}^N \mu_i s_{\mu,\lambda,i}(\mathbf{u}_\Gamma, T_0^u \mathbf{u}_\Gamma) - b_0(\mathbf{u}_\Gamma, T_0^p \mathbf{u}_\Gamma).$$

Since $\|\mathrm{div}\mathbf{u}_\Gamma\|^2_{L^2(\Omega_i)} \leq (d/2) s_{\mu,\lambda,i}(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma)$, we find by using elementary inequalities that

$$\sum_{i=1}^N \mu_i s_{\mu,\lambda,i}(T_0^u \mathbf{u}_\Gamma, T_0^u \mathbf{u}_\Gamma) + \sum_{i=1}^N \frac{1}{\lambda_i} c_{0,i}(T_0^p \mathbf{u}_\Gamma, T_0^p \mathbf{u}_\Gamma)$$
$$\leq \Big(1 + (d/2) \max_i (\lambda_i/\mu_i)\Big) \sum_{i=1}^N \mu_i s_{\mu,\lambda,i}(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma).$$

Thus, we have a bound which is satisfactory only if the Poisson ratio ν remains bounded away from $1/2$; cf. the remark before the proof of Lemma 5.3. The weaker bound here reflects the fact that we have not been able to estimate the inf-sup constant of the coarse spaces in the case of greatly varying Lamé parameters. We have computed the norm of $I - T_0$ in a number of cases and always found it to be less than 1.5, which would indicate that Lemma 5.3 might always be valid.

Another partial result can be obtained for the case when the μ_i vary moderately. The proof to Lemma 5.3 can then be modified; the resulting estimate will depend on the ratio of the largest and smallest values of the μ_i.

7. Numerical Experiments and Implementation Details. In this section, we will discuss a few practical aspects of the implementation of the method, before presenting numerical results for both finite and spectral element implementations of our algorithm. The finite element results were obtained in parallel experiments on a Beowulf cluster using the parallel PETSc library; see [4], [5]. The spectral element results were obtained in serial experiments on a Unix workstation using Matlab 5.3.

7.1. Avoiding a special basis for the pressure. In our discussion, we have assumed that the basis functions for the pressure degrees of freedom can be divided into two sets: functions with zero average and functions constant in each subdomain Ω_i; see formula (12). Although our method requires a pressure space that admits such a partition, it still can be implemented using a standard nodal basis for the pressure.

In our actual implementation, we generate a stiffness matrix $\widetilde{K}$ using a standard nodal basis, instead of the stiffness matrix K of equation (8), and introduce a Lagrange multiplier to enforce that the average of the pressure is zero. Furthermore, we never assemble the entire matrix $\widetilde{K}$, but rather work with the local stiffness matrices $\widetilde{K}^{(i)}$:

$$\widetilde{K}^{(i)} = \begin{bmatrix} \mu A^{(i)} & {\widetilde{B}^{(i)}}^T & 0 \\ \widetilde{B}^{(i)} & -1/\lambda\, \widetilde{C}^{(i)} & w^{(i)} \\ 0 & {w^{(i)}}^T & 0 \end{bmatrix}. \tag{39}$$

Here, the matrices $\widetilde{B}^{(i)}$ and $\widetilde{C}^{(i)}$ differ from $B^{(i)}$ and $C^{(i)}$ of Subsection 3.1, since a standard basis for the pressure is used. The entries of the vector $w^{(i)}$ are the integrals of the pressure basis functions. In each of the local matrices $\widetilde{K}^{(i)}$, we eliminate the interior velocities, all the pressures and the Lagrange multiplier. This corresponds to taking the Schur complement with respect to the (2,2)-block in the following matrix, which is a reordering of (39):

$$\left[\begin{array}{ccc|c} \mu A_{II}^{(i)} & {\widetilde{B}_I^{(i)}}^T & 0 & \mu A_{I\Gamma}^{(i)} \\ \widetilde{B}_I^{(i)} & -1/\lambda\, \widetilde{C}^{(i)} & w^{(i)} & \widetilde{B}_\Gamma^{(i)} \\ 0 & {w^{(i)}}^T & 0 & 0 \\ \hline \mu A_{\Gamma I}^{(i)} & {\widetilde{B}_\Gamma^{(i)}}^T & 0 & \mu A_{\Gamma\Gamma}^{(i)} \end{array}\right].$$

We can show that the result of this static condensation is precisely $\mu S_{\Gamma,\mu,\lambda}^{(i)}$, the $(1,1)$-block of $S_{\mu,\lambda}^{(i)}$, as defined in (15). The remaining blocks of $S_{\mu,\lambda}^{(i)}$, the vector $B_0^{(i)}$ and the scalar $-1/\lambda\; C_0^{(i)}$, are computed using the formula:

$$\begin{bmatrix} \mu A_{\Gamma\Gamma}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & -1/\lambda\; C_0^{(i)} \end{bmatrix} = \begin{bmatrix} I \\ {e^{(i)}}^T \end{bmatrix} \begin{bmatrix} \mu A_{\Gamma\Gamma}^{(i)} & {\widetilde{B}_\Gamma^{(i)}}^T \\ \widetilde{B}_\Gamma^{(i)} & -1/\lambda\; \widetilde{C}^{(i)} \end{bmatrix} \begin{bmatrix} I & e^{(i)} \end{bmatrix}.$$

Here the entries of the vector $e^{(i)}$ are the coefficients that express the constant pressure on subdomain Ω_i in terms of the standard basis functions, i.e.,

$$\sum_{k=1}^{\widetilde{n}_p} e_k^{(i)} \widetilde{\psi}_k = \chi_{\Omega_i},$$

where $\left\{ \widetilde{\psi}_k \right\}_{k=1,\ldots,\widetilde{n}_p}$ is the regular pressure basis and χ_{Ω_i} is the characteristic function of the subdomain Ω_i.

7.2. Solution of the local problems. Our algorithm requires the solution of local problems with essential boundary conditions (when computing the action of $S_{\mu,\lambda}$ on a vector) and natural or mixed boundary conditions (when computing the action of Q_i for floating and non-floating subdomains, respectively). These local problems are of the same nature as the original problem, only much smaller. The problems involving essential boundary conditions must be solved exactly, since their results are used for the evaluation of the residual of the Schur complement problem. Those with natural and mixed boundary conditions could, in principle, be solved approximately. In our implementation, we have chosen to use a direct method (sparse LU factorization) in all cases, which leaves us with the question of how to order equations and unknowns in order to exploit sparsity.

When handling incompressible local problems, the block $\widetilde{C}^{(i)}$ is zero. In this case, the matrices of the problems with an essential boundary condition have the form:

$$\begin{bmatrix} \mu A_{II}^{(i)} & \widetilde{B}_I^{(i)^T} & 0 \\ \widetilde{B}_I^{(i)} & 0 & w^{(i)} \\ 0 & w^{(i)^T} & 0 \end{bmatrix}.$$

Our experience is that a combination of an off-the-shelf reordering algorithm with an off-the-shelf sparse direct solver without pivoting for stability (as the one now provided in PETSc) typically breaks down because of encountering a zero pivot. It is interesting to note that factorization without any reordering also fails since the leading two-by-two block is singular and that therefore the matrix does not admit a LU factorization. One alternative, when using $Q_2 - Q_0$ elements, is simply to interchange the last two rows and columns of the matrix since the resulting matrix can be factored without pivoting. But the sparsity of the matrix is then not explored. Similar devices can be developed for other finite element methods. A better alternative, which we have used in our parallel code, is to reorder for sparsity only the displacement equations and unknowns, corresponding to the block $\mu A_{II}^{(i)}$, in addition to exchanging the last two rows and columns. In the case of $Q_2 - Q_0$ elements, when about 89% of the dofs are displacements (in two dimensions; 96% in three dimensions), this approach yields a substantial gain in performance.

In the case of compressible materials, when $\widetilde{C}^{(i)}$ is positive definite, we have encountered no problem in using the off-the-shelf reordering/factoring approach, which yields much smaller fill-in than when only the displacement variables are reordered as described above.

The local problems with natural or mixed boundary conditions do not require a Lagrange multiplier and there is no need to exchange the two rows and columns. Moreover, artificial numerical compressibility could be introduced in these problems, which would allow us to use an off-the-shelf reordering/factoring approach.

One could also avoid the use of exact solvers, even for the local Dirichlet problems, at the expense of operating on the entire space, including interior displacements and zero-average pressures; see [35, sect. 4.4, p. 141]. This requires two more local solves per iteration and, in addition, we have to expect an increase in the number of iterations. For the incompressible case, this might possibly be advantageous even when using direct solvers, since it would allow us to introduce artificial compressibility (and the use of efficient reordering algorithms) even for the problems with essential boundary conditions.

7.3. Parallel results for elasticity and $Q_2 - Q_0$ mixed finite elements. In this section, we report on some results of parallel numerical experiments on the Beowulf cluster Chiba City at Argonne National Laboratory (with 256 Dual Pentium III processors). The algorithm has been implemented by the first author in C, using the PETSc library. We report on results for compressible/almost-incompressible elasticity only, although similar results have been obtained for incompressible elasticity (and also Stokes and generalized Stokes equations), see [17].

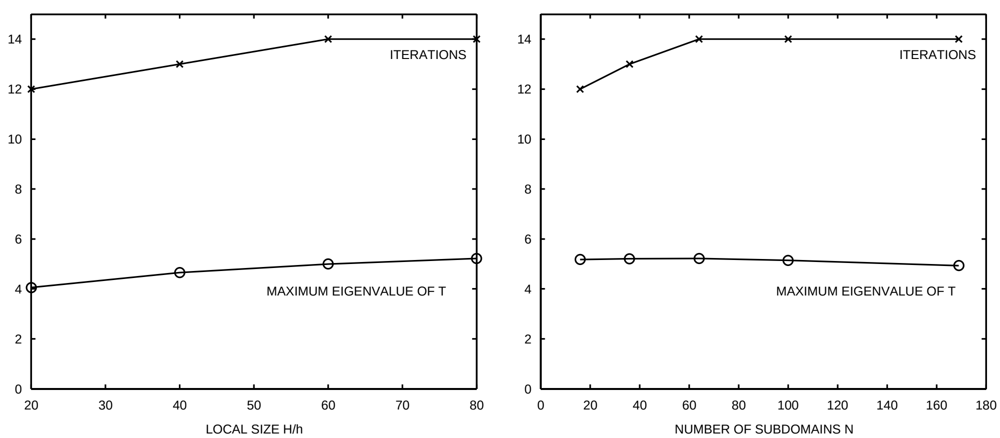
The domain considered is the unit square and the boundary conditions are of Dirichlet type. The Lamé parameters are constant in each subdomain but are discontinuous across the interface. This corresponds to a heterogeneous medium, which is

Table 1

Parallel results for elasticity system (heterogeneous medium) and $Q_2 - Q_0$ finite elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ for the balancing Neumann-Neumann preconditioner with coarse space V_0^2

Fixed number of subdomains $N = 8 \times 8$						
					CPU time (sec.)	
mesh size	local size	# unkn.	iter.	eig max	fact.	total
160 × 160	20 × 20	230,000	12	4.06	1.4	18.0
320 × 320	40 × 40	920,000	13	4.65	18.2	40.9
480 × 480	60 × 60	2,080,000	14	4.99	84.2	126.3
640 × 640	80 × 80	3,690,000	14	5.22	260.8	345.3
Fixed local size 80 × 80 elements (58,242 unknowns)						
					CPU time (sec.)	
mesh size	# subdom.	# unkn.	iter.	eig max	fact.	total
320 × 320	4 × 4	920,000	12	5.18	258.0	321.4
480 × 480	6 × 6	2,080,000	13	5.21	253.7	317.4
640 × 640	8 × 8	3,690,000	14	5.22	260.8	345.3
800 × 800	10 × 10	5,770,000	14	5.14	262.8	356.7
1040 × 1040	13 × 13	9,740,000	14	4.93	261.2	363.9

Fig. 1. *Parallel results for elasticity system (heterogeneous medium) and $Q_2 - Q_0$ finite elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ vs. local size H/h (left) and number of subdomains N (right), from Table 1*



composed of a $\sqrt{N} \times \sqrt{N}$ array of three different materials in the following pattern:

s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a
s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a

,

where

$s =$ steel-like:	$\mu_s = 8.20$	$\lambda_s = 10.00$	$\nu_s = 0.275$
$a =$ aluminum-like:	$\mu_a = 2.60$	$\lambda_a = 5.60$	$\nu_a = 0.341$.
$r =$ rubber-like:	$\mu_r = 0.01$	$\lambda_r = 0.99$	$\nu_r = 0.495$

Note that the material r is almost incompressible, with a Poisson ratio close to 0.5.

The problem is discretized with $Q_2 - Q_0$ finite elements and the saddle point Schur complement (21) is solved iteratively by PCG with our balancing Neumann-Neumann preconditioner and the third coarse space $V_0^2 = \{\text{scaled rigid body motions}\} + Q_1^H$. The initial guess is a random vector modified so that the initial error is in the range of $(I - T_0)$, the right hand side is a random, uniformly distributed vector, and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate.

In the upper half of Table 1, we show results for decreasing mesh sizes for a case of 64 subdomains. The condition number and the iteration count grow weakly as we increase the size of the local problems, as can also be observed in the left part of Figure 1. The last two columns of this table display CPU-time for these runs. The last column gives the total time for the code to run, while the column labeled "fact." gives the time spent on LU factorizations; there are three of them: two local, namely a Dirichlet and a Neumann subdomain-level problem, and one global coarse problem. We note that the cost of the factorizations grows rapidly and dominates the cost of the computation. The lower part of Table 1 shows results for an increasing number of subdomains of fixed size (about 58,000 degrees of freedom). The corresponding graph, on the right in Figure 1, shows an almost horizontal tail, indicating independence of the condition number and the iteration count on the number of subdomains. This is numerical evidence that our main result, Theorem 5.4, remains valid in the case of discontinuous coefficients. The fact that the factorization time remained constant for the entire range of problem sizes tested (from 16 to 169 subdomains) indicates that the cost associated with the factorization of the coarse problem is still tiny compared with that of the local problems. One can expect this scenario to change if the number of subdomains increases significantly.

7.4. Serial results for elasticity and $Q_n - Q_{n-2}$ mixed spectral elements. In this section, we report on serial numerical experiments, carried out in Matlab 5.3 on Unix workstations, for model mixed elasticity problems on the unit square or the unit cube and with homogeneous Dirichlet boundary conditions. The problem was discretized with $Q_n - Q_{n-2}$ spectral elements and the domain Ω divided into $\sqrt{N} \times \sqrt{N}$ square subdomains or $N^{1/3} \times N^{1/3} \times N^{1/3}$ cubic subdomains. After the implicit elimination of the interior unknowns, the saddle point Schur complement system (21) is solved iteratively by PCG with our balancing Neumann-Neumann preconditioner. The initial guess is always zero, the right hand side is random and uniformly distributed, and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate.

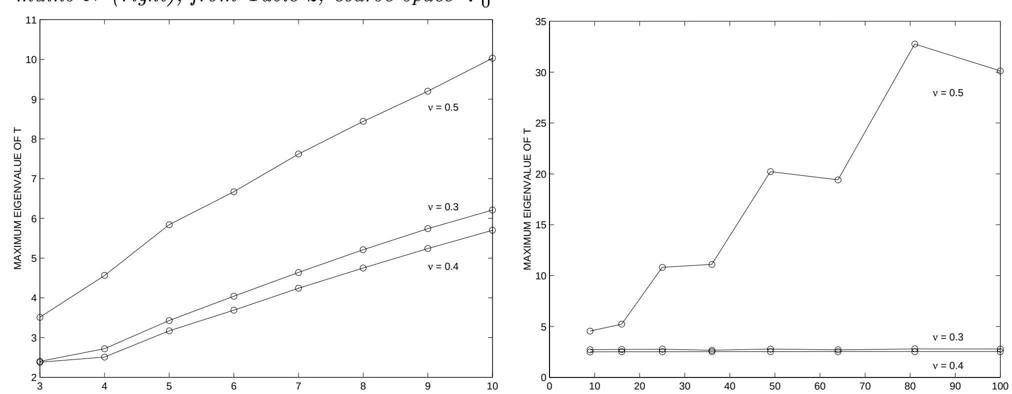
Homogeneous materials. We consider first the case of homogeneous materials with fixed Lamé constants over the whole domain Ω. We report the results in two tables corresponding to two of the coarse spaces introduced in Section 4, $\mathbf{V}_0^0$ in Table 2 and $\mathbf{V}_0^1$ in Table 3. In the upper half of each table the number of subdomains, $N = 3 \times 3$, is fixed, while the spectral degree n is increased from 2 to 10; in the lower half the spectral degree $n = 4$ is fixed and the number of subdomains N is increased from 2×2 to 10×10. Each table reports the PCG iteration counts and, in brackets, the maximum eigenvalue of the preconditioned operator $T_{\mu,\lambda}$; the minimum eigenvalue is

TABLE 2

Serial results for elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ (in brackets) for the balancing Neumann-Neumann preconditioner with coarse space V_0^0

Fixed number of subdomains $N = 3 \times 3$				
spectral degree n	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5$
3	8 (2.40)	8 (2.38)	9 (2.86)	9 (3.51)
4	9 (2.72)	9 (2.51)	10 (3.51)	10 (4.56)
5	10 (3.43)	10 (3.17)	11 (4.18)	11 (5.84)
6	11 (4.04)	11 (3.69)	12 (4.69)	12 (6.67)
7	12 (4.64)	11 (4.24)	13 (5.11)	13 (7.62)
8	12 (5.21)	12 (4.75)	13 (5.66)	14 (8.44)
9	13 (5.74)	12 (5.24)	14 (5.93)	15 (9.20)
10	13 (6.21)	13 (5.70)	15 (6.56)	15 (10.03)
Fixed spectral degree $n = 4$				
# of subdomains N	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5$
3×3	9 (2.72)	9 (2.51)	10 (3.51)	10 (4.56)
4×4	10 (2.75)	10 (2.52)	11 (3.92)	12 (5.23)
5×5	10 (2.77)	10 (2.52)	13 (6.07)	14 (10.82)
6×6	10 (2.67)	10 (2.53)	14 (6.20)	15 (11.11)
7×7	11 (2.79)	10 (2.53)	15 (8.05)	17 (20.22)
8×8	11 (2.71)	10 (2.54)	16 (7.98)	19 (19.42)
9×9	11 (2.80)	10 (2.54)	17 (9.40)	21 (32.76)
10×10	11 (2.79)	10 (2.54)	18 (9.24)	22 (30.12)

FIG. 2. *Serial results for elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ vs. spectral degree n (left) and number of subdomains N (right), from Table 2, coarse space $\mathbf{V}_0^0$*



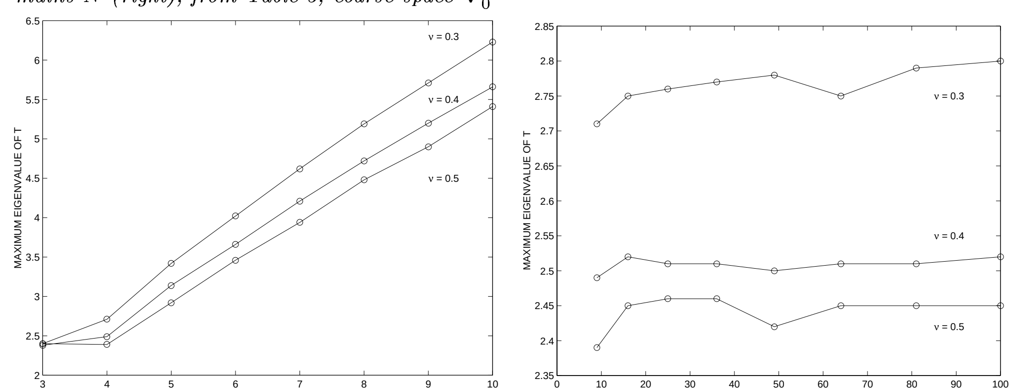
always very close to 1 and therefore not reported. In each table, we consider, in four different columns, the four cases $\nu = 0.3, 0.4, 0.49, 0.5$, ranging from compressible to incompressible materials. The results show, in agreement with the theory, that our balancing Neumann-Neumann algorithm is quasi-optimal, i.e., there is only a weak dependence on the spectral degree n, and scalable, i.e., there is no dependence on

Table 3

Serial results for elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ (in brackets) for the balancing Neumann-Neumann preconditioner with coarse space V_0^1

Fixed number of subdomains $N = 3 \times 3$				
spectral degree n	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5$
3	8 (2.40)	8 (2.38)	8 (2.40)	9 (2.40)
4	9 (2.71)	9 (2.49)	9 (2.39)	9 (2.39)
5	10 (3.42)	10 (3.14)	10 (2.94)	10 (2.92)
6	11 (4.02)	11 (3.66)	11 (3.45)	11 (3.46)
7	11 (4.62)	11 (4.21)	12 (3.93)	12 (3.94)
8	12 (5.19)	12 (4.72)	13 (4.45)	12 (4.48)
9	13 (5.71)	12 (5.20)	13 (4.86)	13 (4.90)
10	12 (6.23)	13 (5.66)	14 (5.35)	14 (5.41)
Fixed spectral degree $n = 4$				
# of subdomains N	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5$
9	9 (2.71)	9 (2.49)	9 (2.39)	9 (2.39)
16	10 (2.75)	10 (2.52)	10 (2.45)	10 (2.45)
25	10 (2.76)	10 (2.51)	10 (2.45)	10 (2.46)
36	11 (2.77)	10 (2.51)	10 (2.46)	10 (2.46)
49	10 (2.78)	10 (2.50)	10 (2.42)	10 (2.42)
64	11 (2.75)	10 (2.51)	10 (2.45)	10 (2.45)
81	11 (2.79)	10 (2.51)	10 (2.45)	10 (2.45)
100	11 (2.80)	10 (2.52)	10 (2.45)	10 (2.45)

Fig. 3. *Serial results for elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ vs. spectral degree n (left) and number of subdomains N (right), from Table 3, coarse space $\mathbf{V}_0^1$*



the number of subdomains N, for both coarse spaces. There are exceptions, with the first coarse space $\mathbf{V}_0^0$ in the incompressible and almost incompressible cases; see the last two columns of Table 2, lower part. This is due to the fact that $\mathbf{V}_0^0$ is not uniformly inf-sup stable with respect to H and that therefore β_0 approaches zero with increasing N. In the compressible case, the non-zero factor μ/λ in Theorem 5.4 allows

TABLE 4

Serial results for elasticity system (heterogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ for the balancing Neumann-Neumann preconditioner. Fixed number of subdomains $N = 4 \times 4$ and spectral degree $n = 5$.

$\mathbf{V}_0^0$ coarse space				
exponent t in μ	# iterations	eig max	eig min	estim. cond(K)
-3	11	3.48	1.0012	$2.4{\cdot}10^5$
-2	11	3.53	1.0024	$3.3{\cdot}10^4$
-1	12	3.73	1.0026	$5.3{\cdot}10^3$
0	12	3.47	1.0017	$1.7{\cdot}10^4$
1	12	3.46	1.0017	$8.3{\cdot}10^5$
2	12	3.35	1.0016	$7.2{\cdot}10^7$
3	11	3.47	1.0017	$7.1{\cdot}10^9$
4	10	3.48	1.0003	$7.1{\cdot}10^{11}$
5	10	3.49	1.0001	$7.1{\cdot}10^{13}$
6	9	3.49	1.0000	$7.1{\cdot}10^{15}$
$\mathbf{V}_0^1$ coarse space				
exponent t in μ	# iterations	eig max	eig min	estim. cond(K)
-3	10	2.87	1.0011	$2.4{\cdot}10^5$
-2	10	2.91	1.0018	$3.3{\cdot}10^4$
-1	11	3.30	1.0018	$5.3{\cdot}10^3$
0	12	3.41	1.0010	$1.7{\cdot}10^4$
1	12	3.41	1.0009	$8.3{\cdot}10^5$
2	12	3.33	1.0011	$7.2{\cdot}10^7$
3	11	3.44	1.0016	$7.1{\cdot}10^9$
4	11	3.46	1.0005	$7.1{\cdot}10^{11}$
5	11	3.46	1.0001	$7.1{\cdot}10^{13}$
6	10	3.46	1.0000	$7.1{\cdot}10^{15}$

us to still obtain an upper bound independent of N and therefore scalability, but in the incompressible case μ/λ vanishes and we lose scalability. On the other hand, the use of the inf-sup stable coarse space $\mathbf{V}_0^1$ yields a scalable algorithm independently of the compressibility of the material; see Table 3.

Heterogeneous materials and 3D results. We next consider the case of a heterogeneous material occupying a domain Ω divided into 4×4 subdomains and with the following distribution of Lamé coefficients:

$$\lambda = \begin{array}{|c|c|c|c|}\hline 1 & \infty & 1 & \infty \\ \hline \infty & 1 & 10 & 0.1 \\ \hline 1 & 10 & \infty & 1 \\ \hline 0.1 & \infty & 0.1 & 10 \\ \hline \end{array} \qquad \mu = \begin{array}{|c|c|c|c|}\hline 1 & 10^t & 1 & 10^t/3 \\ \hline 3 \cdot 10^t & 1/2 & 10^t/5 & 2 \\ \hline 1 & 10^t/2 & 1/3 & 10^t \\ \hline 10^t & 2/5 & 5 \cdot 10^t & 3 \\ \hline \end{array}$$

where the exponent t assumes the values $t = -3, -2, \cdots, 5, 6$. We have set $\gamma = 1$ in the definition of the scaled δ_i and $\delta_i^\dagger$ functions; cf. (37). This example does not reflect any physical model, but illustrates the robustness of our algorithm with respect to variations of the Lamé coefficients, that have jumps of many orders of magnitude across subdomain boundaries. We have tried many additional combinations of compressible and incompressible materials and have found our algorithm to be virtually independent of these variations. This is clearly shown in the results of Table 4, where

FIG. 4. *3D results: Distribution of the Lamé coefficients λ (left) and μ (right) on a cubic domain*

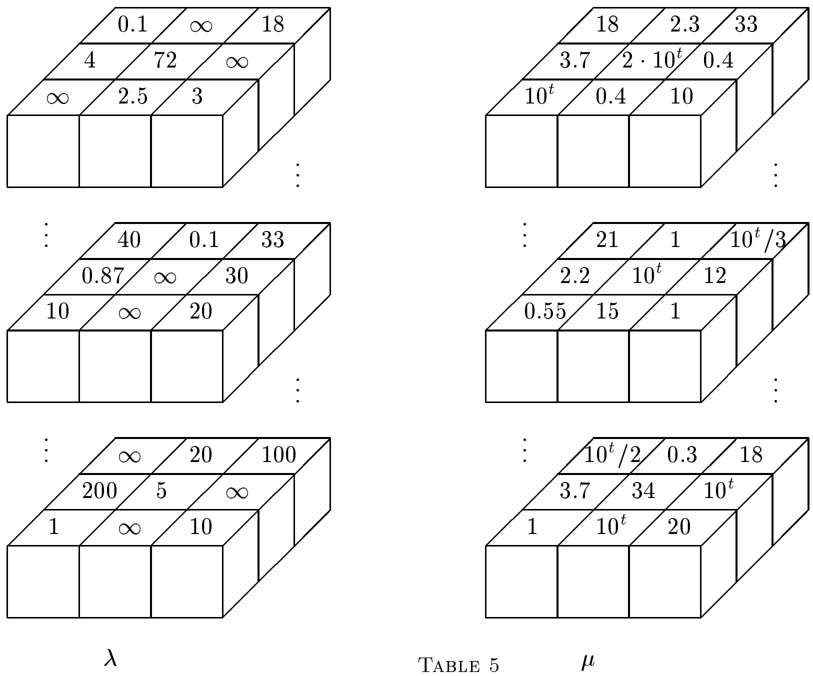


TABLE 5

Serial results in 3D for elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of $T_{\mu,\lambda}$ for the balancing Neumann-Neumann preconditioner Fixed number of subdomains $N = 3 \times 3$ and spectral degree $n = 4$.

	$\mathbf{V}_0^0$ coarse space		
exponent t in μ	# iterations	eig max	eig min
-3	18	6.68	1.0003
-2	17	6.65	1.0017
-1	16	6.43	1.0039
0	15	5.16	1.0049
1	17	6.39	1.0048
2	18	7.10	1.0039
3	19	7.52	1.0029
4	19	7.58	1.0029
5	19	7.59	1.0030
6	19	7.59	1.0030

variations of up to ten orders of magnitude in the Lamé coefficients cause a comparable increase in the condition number of the discrete problem (computed by the Matlab function `condest`) but do not affect the performance of our algorithm or the spectrum of the preconditioned operator.

Similar results have been obtained in three dimensions. Figure 4 shows the distribution of the Lamé coefficients λ (left) and μ (right) on a cubic domain and the exponent t assumes the values $t = -3, -2, \cdots, 5, 6$, as before. The results reported in Table 5 show that also in three dimensions the performance of our algorithm and the spectrum of the preconditioned operator are independent of the jumps in the Lamé coefficients.

REFERENCES

[1] Yves Achdou, Patrick Le Tallec, Frédéric Nataf, and Marina Vidrascu. A domain decomposition preconditioner for an advection-diffusion problem. *Comput. Methods Appl. Mech. Engrg*, 184:145–170, 2000.

[2] Mark Ainsworth and Spencer Sherwin. Domain decomposition preconditioners for p and hp finite element approximation of Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 175(3-4):243–266, 1999.

[3] Pierre Alart, Mikaël Barboteu, Patrick Le Tallec, and Marina Vidrascu. Méthode de Schwarz additive avec solveur grossier pour problèmes non symétriques. *Comp. Rend. Acad. Sci. Paris, serie I*, 331:399–404, 2000.

[4] Satish Balay, Kris Buschelman, William D. Gropp, Dinesh Kaushik, Lois Curfman McInnes, and Barry F. Smith. PETSc home page. http://www.mcs.anl.gov/petsc, 2001.

[5] Satish Balay, William D. Gropp, Lois Curfman McInnes, and Barry F. Smith. PETSc users manual. Technical Report ANL-95/11 - Revision 2.1.1, Argonne National Laboratory, 2001.

[6] Christine Bernardi and Yvon Maday. Uniform inf-sup conditions for the spectral discretization of the Stokes problem. *Math. Models Methods Appl. Sci*, 9(3):395–414, 1999.

[7] James H. Bramble and Joseph E. Pasciak. A domain decomposition technique for Stokes problems. *Appl. Numer. Math.*, 6:251–261, 1989/90.

[8] Franco Brezzi and Michel Fortin. *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, Berlin, 1991.

[9] Mario A. Casarin. *Schwarz Preconditioners for Spectral and Mortar Finite Element Methods with Applications to Incompressible Fluids.* PhD thesis, Dept. of Mathematics, Courant Institute of Mathematical Sciences, New York University, March 1996. Tech. Rep. 717, Department of Computer Science, Courant Institute.

[10] Mario A. Casarin. Schwarz preconditioners for the spectral element discretization of the steady Stokes and Navier-Stokes equations. *Numer. Math.*, 89(2):307–339, August 2001.

[11] Lawrence C. Cowsar, Jan Mandel, and Mary F. Wheeler. Balancing domain decomposition for mixed finite elements. *Math. Comp.*, 64(211):989–1015, 1995.

[12] Maksymilian Dryja and Olof B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, February 1995.

[13] Paul F. Fischer. An overlapping Schwarz method for spectral element solution of the incompressible Navier-Stokes equations. *J. Comp. Phys.*, 133 (1):84–101, 1997.

[14] Paul F. Fischer, Neil I. Miller, and Henry M. Tufo. An overlapping Schwarz method for spectral element simulation of three-dimensional incompressible flows. In P. Bjørstad and M. Luskin, editors, *Parallel Solution of PDE*, number 120 in IMA Volumes in Mathematics and Its Applications, pages 1–30. Springer-Verlag, 2000.

[15] Paul F. Fischer and Einar Rønquist. Spectral element methods for large scale parallel Navier-Stokes calculations. *Comp. Meths. Appl. Mech. Eng.*, 116:69–76, 1994.

[16] Paola Gervasio. *Risoluzione delle equazioni alle derivate parziali con metodi spettrali in regioni partizionate in sottodomini.* PhD thesis, Università di Milano, 1995.

[17] Paulo Goldfeld, Luca F. Pavarino, and Olof B. Widlund. Balancing Neumann-Neumann methods for mixed approximations of linear elasticity. In Luca F. Pavarino and Andrea Toselli, editors, *Recent Developments in Domain Decomposition Methods*, volume 23 of *Lecture Notes in Computational Science and Engineering*, pages 53–76. Springer-Verlag, 2002.

[18] Axel Klawonn and Luca F. Pavarino. Overlapping Schwarz methods for mixed linear elasticity and Stokes problems. *Comp. Meths. Appl. Mech. Eng.*, 165:233–245, 1998.

[19] Axel Klawonn and Olof B. Widlund. A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity. *SIAM J. Sci. Comp.*, 22(4):1199–1219, 2000.

[20] Axel Klawonn and Olof B. Widlund. FETI and Neumann-Neumann iterative substructuring methods: Connections and new results. *Comm. Pure Appl. Math.*, 54(1):57–90, 2001.

[21] Patrick Le Tallec, Jan Mandel, and Marina Vidrascu. A Neumann-Neumann domain decomposition algorithm for solving plate and shell problems. *SIAM J. Numer. Anal.*, 35(2):836–867, 1998.

[22] Patrick Le Tallec and Abani Patra. Non-overlapping domain decomposition methods for adaptive hp approximations of the Stokes problem with discontinuous pressure fields. *Comp.*

Meths. Appl. Mech. Eng., 145:361–379, 1997.

[23] Yvon Maday, Dan Meiron, Anthony Patera, and Einar Rønquist. Analysis of iterative methods for the steady and unsteady Stokes problem: Application to spectral element discretizations. *SIAM J. Sci. Comp.*, 14(2):310–337, 1993.

[24] Jan Mandel. Balancing domain decomposition. *Comm. Numer. Methods Engrg.*, 9(3):233–241, 1993.

[25] Jan Mandel and Marian Brezina. Balancing domain decomposition for problems with large jumps in coefficients. *Math. Comp.*, 65(216):1387–1401, 1996.

[26] Donatella Marini and Alfio Quarteroni. A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math.*, 55(5):575–598, 1989.

[27] Joseph E. Pasciak. Two domain decomposition techniques for Stokes problems. In T. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Domain Decomposition Methods*, pages 419–430, Philadelphia, 1989. SIAM.

[28] Luca F. Pavarino. Neumann-Neumann algorithms for spectral elements in three dimensions. *RAIRO* M^2AN, 31(4):471–493, 1997.

[29] Luca F. Pavarino and Olof B. Widlund. Iterative substructuring methods for spectral element discretizations of elliptic systems. II: Mixed methods for linear elasticity and Stokes flow. *SIAM J. Numer. Anal.*, 37(2):375–402, 1999.

[30] Luca F. Pavarino and Olof B. Widlund. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Comm. Pure Appl. Math.*, 55(3):302–335, 2002.

[31] Alfio Quarteroni. Domain decomposition algorithms for the Stokes equations. In T. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Domain Decomposition Methods*, pages 431–442, Philadelphia, 1989. SIAM.

[32] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.

[33] Einar Rønquist. A domain decomposition solver for the steady Navier-Stokes equations. In A.V. Ilin and L.R. Scott, editors, *Proc. of ICOSAHOM.95, published by the Houston Journal of Mathematics*, pages 469–485, 1996.

[34] Einar Rønquist. Domain decomposition methods for the steady Stokes equations. In C-H. Lai, P. Bjørstad, M. Cross, and O. Widlund, editors, *Proc. of DD11*, pages 326–336. DDM.org, 1999.

[35] Barry F. Smith, Petter Bjørstad, and William D. Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge University Press, 1996.

[36] Rolf Stenberg and Manil Suri. Mixed *hp* finite element methods for problems in elasticity and Stokes flow. *Numer. Math.*, 72(3):367–390, 1996.

[37] Andrea E. Toselli. Neumann–Neumann methods for vector field problems. *ETNA*, 11:1–24, 2000.