Overlapping Schwarz Algorithms using Discontinuous Iterates for Poisson's Equation

by

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DEDICATION

For my father and mother, who always taught me never to give up.

Also to my wife Young-Hee and my son Alex, whose love and support helped me make it through the most difficult periods.

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PREFACE

In a paper by Cai, Casarin, Elliott, and Widlund [5], three algorithms were presented based on the overlapping Schwarz methods for Helmholtz's equation. The algorithms are called Algorithm 1, Algorithm 2, and Algorithm 3 in increasing order of sophistication. Algorithm 3, which is a new type of overlapping Schwarz methods, converges the fastest to the solution of the equation but its analysis and implementation is complicated by the fact that jumps are allowed in the iterates across the domain interfaces. This new algorithm was inspired by the thesis of Després [10] and it can be considered as an overlapping version of Després' method.

In this dissertation, we focus on Algorithm 3 which is also called the Discontinuous Overlapping Schwarz Method (OSM-D) in Casarin and Widlund [7]. The discontinuity of the iterates of Algorithm 3 (OSM-D), which is the fundamental distinction from the classical algorithms, is implemented by allowing multiple values on the artificial interfaces. To handle this important property, we use a saddle-point approach. We study the basic properties and formulation of Algorithm 3 (OSM-D) and develop a convergence theory for simple problems as well as more complicated problems using a formulation based on Lagrange multipliers. Several numerical results including some for multi-level variants of Algorithm 3 (OSM-D) are also presented and analyzed.

ABSTRACT

A new type of overlapping Schwarz methods, the overlapping Schwarz algorithms using discontinuous iterates is constructed from the classical overlapping Schwarz algorithm. It allows for discontinuities at each artificial interface. The new algorithm, for Poisson's equation, can be considered as an overlapping version of Lions' Robin iteration method for which little is known concerning the convergence. Since overlap improves the performance of the classical algorithms considerably, the existence of a uniform convergence factor is the fundamental question for our new algorithm.

The first part of this thesis concerns the formulation of the new algorithm. A variational formulation of the new algorithm is derived from the classical algorithms. The discontinuity of the iterates of the new algorithm is the fundamental distinction from the classical algorithms. To analyze this important property, we use a *saddle-point approach*. We show that the new algorithm can be interpreted as a *block Gauss-Seidel* method with dual and primal variables.

The second part of the thesis deals with algebraic properties of the new algorithm. We prove that the fractional steps of the new algorithm are nonsymmetric. The algebraic systems of the primal variables can be reduced to those of the dual variables. We analyze the structure of the dual formulation algebraically and analyze its numerical behavior.

The remaining part of the thesis concerns convergence theory and numerical results for the new algorithm. We first extend the classical convergence theory, without using Lagrange multipliers, in some limited cases. A new theory using Lagrange multiplier is then introduced and we find conditions for the existence of

uniform convergence factors of the dual variables, which implies convergence of the primal variables, in the two overlapping subdomain case with any Robin boundary condition. Our condition shows a relation between the given conditions and the artificial interface condition. The numerical results for the general case with cross points are also presented. They indicate possible extensions of our results to to this more general case.

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Chapter 1

Introduction

1.1 An Overview

The numerical solution of partial differential equations often leads to quite large, sparse linear systems. *Domain decomposition methods* are general flexible iterative methods for solving such problems. Domain decomposition algorithms are divided into two classes, those that use *overlapping domains*, which are often referred to as *Schwarz methods*, and those that use *nonoverlapping domains*, which are called *iterative substructuring methods*. The discrete approximation to a partial differential equation is obtained iteratively by solving problems associated with each subdomain and passing information between neighbors. Numerous domain decomposition methods have been designed, studied, and implemented in last two decades. Two books by Smith, Bjørstad, and Gropp [43] and Quarteroni and Valli [40] have appeared recently, and almost yearly international conferences are being held; see [16, 8, 9, 17, 22, 39, 23, 18, 4, 28, 24, 46].

The first domain decomposition method is the alternating method of H. A. Schwarz [42]. At the core of that work is a proof that the iterative method converges

in the maximum norm at a *geometric* rate. In his work in the 1980's [25, 26], P. L. Lions analyzed the convergence of Schwarz methods using two different methods, a *maximum principle* and *Hilbert spaces*. He also established the convergence of certain nonoverlapping domain decomposition methods by using energy estimate; see [27]. Since he used the Robin boundary condition on the artificial interfaces, we will call this algorithm the Robin iteration method; see [40]. In contrast with other domain decomposition algorithms, we have little information about the rate of convergence of the Robin iteration method. B. Després applied this idea to a more complicated example, Helmholtz's equation in his thesis [10].

In 1998, Cai, Casarin, Elliott, and Widlund presented a new family of overlapping Schwarz methods which uses discontinuous iterates. The idea of the new algorithm was inspired by Després' thesis and can be considered as the overlapping version of Després' algorithm. This new algorithm is quite effective for solving Helmholtz's equation but its analysis and implementation is complicated by the fact that it allows the discontinuity across the artificial interfaces.

In this thesis, we concentrate on **overlapping Schwarz algorithms using discontinuous iterates for Poisson's equation**, which can be considered as an overlapping version of Lions' Robin iteration method. Here, the discontinuity of the iterates of the new algorithm, which is the fundamental distinction from the classical algorithms, is implemented by allowing multiple values on the artificial interfaces. To analyze and implement this new algorithm, we need a redesign the data structures and to define new concepts and notations.

The study of new algorithm has posed many interesting questions related to how far and in what sense the classical theory of domain decomposition methods can be

extended to this new setting, such as to what extent there may be a counterpart of the geometric convergence factor of the classical Schwarz methods and of the energy estimate of the Robin iteration methods. It is also interesting to see what new phenomena occur and what new insight this might lend to the classical algorithms.

The first questions are related to how to extend the relevant notations of the classical algorithm to a general setting and how to formulate the new algorithm carefully to understand the differences between the new and classical algorithms. One focus of this thesis is the extension and interpretation of a new algorithm inside the classical theory. In Chapter 2 concerns the formulation of the new algorithm in terms of the classical algorithms. The notations in [5] are revised and extended to a new setting for Poisson's equation. The new algorithm is derived from two intermediate algorithms with the new concepts and notations related to the *discontinuity of the iterates*, the fundamental distinction from the classical algorithms. To analyze this important property, we use a *saddle-point approach* which is also used for formulating the *Finite Element Tearing and Interconnecting (FETI)* method by Farhat and Roux [13]. We show that Algorithm 3 (OSM-D) can be interpreted as a *Block Gauss-Seidel* method with dual and primal variables; a new dual variable can be computed from given primal variables and the dual variables will then be used to compute a new primal variable.

In Chapter 3, we extend the results of the classical convergence theory without using Lagrange multiplier in some limited cases. In a special geometry in the two overlapping subdomain case, we show that the rates of convergence on the two nonoverlapping parts is better than that of the Robin iteration methods. Inspired by Nataf [33] and [34], we apply his idea to Poisson problem on several overlap-

ping (finite) strips and we extend this results to a general quadrilateral which is conformally equivalent to a rectangular domain.

In Chapter 4, we analyze the algebraic structure of the new algorithm. In classical theory, each fractional step is symmetric with respect to the L^2 inner product. However, the fractional steps of the new algorithm are nonsymmetric in general. Therefore, it is impossible to apply the *classical conjugate gradient method*. The algebraic convergence theory of overlapping Schwarz methods in [3] is applied.

The algebraic systems of the primal variables can be reduced to those of the dual variables. In the two overlapping subdomain case, the dual system results in a *block* 2-*cycle* matrix. We analyze the structure of the subblock matrices algebraically and check their numerical behavior in Chapter 5.

A convergence theory using the Lagrange multipliers is introduced in Chapter 6. First, we prove the convergence of the new algorithm in a special geometry. We find a condition for the existence of a uniform convergence factor for the dual variables, which implies convergence of the primal variables, in the two overlapping subdomain case with any Robin boundary condition. This result shows a relation between the artificial interface condition and the other conditions which are related to the geometry and the overlap. This new idea is extended to the general two overlapping subdomain case with any Robin boundary condition. In this general cases, we also find a condition for the existence of a uniform convergence factor.

Several numerical examples and results are presented in Chapter 7. The numerical results for the general case with cross points indicate possible extensions of this new approach to this more general case.

We will next provide some mathematical background and establish some notations which will be needed throughout.

1.2 Sobolev Spaces

In this section, we recall some basic results on Sobolev spaces for our work. For a description of the general spaces and their properties, see [1, 30].

Let Ω be a bounded Lipschitz region in $\mathtt{R}^d$. The space $L^2(\Omega)$ is defined as the closure of $C^\infty(\Omega)$ in the norm,

$$||u||_{L^2(\Omega)} = (\int_\Omega |u|^2 dx)^{1/2} < \infty.$$

The H^1-seminorm and norm of $u \in H^1(\Omega)$ are, respectively,

$$\begin{aligned} |u|^2_{H^1(\Omega)} &= \int_\Omega |\nabla u|^2 \, dx; \\ ||u||^2_{H^1(\Omega)} &= |u|^2_{H^1(\Omega)} + ||u||^2_{L^2(\Omega)}. \end{aligned}$$

To define traces of Sobolev spaces on boundaries, we also need the fractional order Sobolev spaces,

$$H^\sigma(\Omega) \quad (0 < \sigma < 1)$$

defined by the completion of $C^\infty(\Omega)$ in the following norm,

$$||u||_{H^\sigma(\Omega)} = (||u||_{L^2(\Omega} + |u|_{H^\sigma(\Omega)})^{1/2},$$

where

$$|u|_{H^\sigma(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2\sigma}} \, dx \, dy.$$

A more detail introduction to the important tools used in domain decomposition theory can be found in [41, 6, 44].

1.2.1 Trace Theorems

For a continuous function u on $\overline{\Omega}$, the trace of u can be simply defined by restricting u to $\partial\Omega$. The trace theorems extend this definition to more general functions; see [1] for the general theory.

Theorem 1.1 *If Ω is a Lipschitz domain and $u \in H^s(\Omega)$, $1/2 < s \leq 1$, then,*

$$\gamma_0 u = u \mid_{\partial\Omega} \in H^{s-1/2}(\partial\Omega).$$

Moreover, the restriction operator from $H^s(\Omega)$ to $H^{s-1/2}(\partial\Omega)$ is onto and continuous,

$$||\gamma_0 u||_{H^{s-1/2}(\partial\Omega)} \leq C(s,\Omega)||u||_{H^s(\Omega)},$$

where $C(s,\Omega)$ is a constant that depends only on s and Ω.

In Chapter 6, we will use a variant of this result.

Theorem 1.2 *If Ω is a Lipschitz domain, then*

$$|\gamma_0 u|^2_{H^{1/2}(\partial\Omega)} \;\leq\; C|u|^2_{H^1(\Omega)},$$

and

$$||\gamma_0 u||^2_{L^2(\partial\Omega)} \leq C_T(|u|^2_{H^1(\Omega)} + ||u||^2_{L^2(\Omega)}).$$

1.2.2 Poincaré and Friedrichs Inequalities

The Poincaré and Friedrichs inequalities are important to establish the equivalence of certain norms. Their proofs are using Rellich's theorem, i.e., that the unit ball of $H^m(\Omega)$ is relatively compact in $H^{m-1}(\Omega)$. These results are used in the proof of the existence of weak solutions of elliptic boundary value problems; see [29].

In particular, when considering domain decomposition methods, we are interested in formulations of these inequalities which specify the dependence of the constants on the domain Ω; see [35] for elementary proofs. Let $\Omega \in \mathcal{R}^d$ and let H_Ω be the diameter of Ω.

Theorem 1.3 (Poincaré's Inequality) *There exists a constant $C(\Omega)$ that depends on Ω but is invariant under dilation of Ω, such that*

$$||u||^2_{L^2(\Omega)} \leq C(\Omega) H^2_\Omega (|u|^2_{H^1(\Omega)} + \frac{1}{H^{d+2}_\Omega} |\int_\Omega u dx|^2), \quad \forall u \in H^1(\Omega).$$

When we study elliptic problem with Dirichlet boundary conditions on parts of boundary $\Lambda \subset \partial\Omega$, we need to consider a Sobolev space $H^1_\Lambda(\Omega) = \{u \in H^1(\Omega) | u_{|_\Lambda} = 0\}$. The Poincaré'-Friedrichs' Inequality gives an equivalence of norms on this space. The idea of its proof can be found in [35] and we can also find a proof in [41].

Theorem 1.4 (Poincaré-Friedrichs' Inequality) *Let $\Lambda \subset \partial\Omega$ with positive measure. Then,*

$$||u||^2_{L^2(\Omega)} \leq C(\Omega, \Lambda) H^2_\Omega (|u|^2_{H^1(\Omega)} + \frac{1}{H_\Omega} (\int_\Lambda \gamma_0 u \, d\sigma)^2), \quad \forall u \in H^1(\Omega),$$

where $C(\Omega, \Lambda)$ is a constant that is invariant under dilation of Ω and Λ.

In Chapter 6, we will use the classical Friedrichs' Inequality,

Theorem 1.5 (Friedrichs' Inequality) *If $\Lambda \subset \partial\Omega$ with positive measure. Then,*

$$||u||^2_{L^2(\Omega)} \leq C_F (|u|^2_{H^1(\Omega)} + \int_\Lambda |u|^2 \, d\sigma), \quad \forall u \in H^1(\Omega).$$

where C_F depends on Ω and Λ.

Corollary 1.1 *If $\Lambda \subset \partial\Omega$ with positive measure. Then, the H^1-seminorm is an equivalent norm on $H^1_\Lambda(\Omega)$, i.e.,*

$$||u||_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)}, \quad \forall u \in H^1_\Lambda(\Omega).$$

1.3 Symmetric Positive Definite Matrices

The following properties of symmetric positive definite matrices will be used in Chapter 5; see [20, 21] for a general theory.

Theorem 1.6 *The product of two symmetric positive definite matrices A and B is a diagonalizable matrix, whose eigenvalues are all real and positive.*

Theorem 1.7 *If a matrix A is symmetric positive definite and a matrix B is Hermitian, then there exists a nonsingular square matrix C such that C^*BC is diagonal and $C^*AC = I$.*

From these theorems, we can prove the following two theorems.

Theorem 1.8 *Let A and B be Hermitian matrices, and suppose that A is positive definite. Then $A - B$ is positive definite if and only if all eigenvalues of BA^{-1} are less than 1.*

Proof By Theorem 1.7, we can find a nonsingular square matrix $\tilde{C}$ whose inverse is C in Theorem 1.7 such that $A = \tilde{C}I\tilde{C}^*$ and $B = \tilde{C}D\tilde{C}^*$ where $D = diag(d_1, d_2, \cdots, d_n)$ is diagonal. Then $A - B$ is positive definite if and only if $\tilde{C}(I - D)\tilde{C}^* > 0$, which is the case if and only if $d_i < 1$ for $i = 1, 2, \cdots$. Since $BA^{-1} = \tilde{C}D\tilde{C}^*\tilde{C}^{*-1}\tilde{C}^{-1} = \tilde{C}D\tilde{C}^{-1}$, the eigenvalues of BA^{-1} are less than 1.

Theorem 1.9 *Let A and B be Hermitian square matrices, and suppose that A is positive definite. Then $A + B$ is positive definite if and only if all eigenvalue of BA^{-1} are greater than -1.*

Proof Similar to that of Theorem 1.8.

1.4 M-matrices and H-matrices

The following properties of M-matrices and H-matrices will be used in Chapter 4; see [19, 20, 36] for a general theory.

A order relation in the algebra of all real matrices is defined by elementwise inequalities; For $A = (a_{\alpha\beta})$ and $B = (b_{\alpha\beta})$,

$$A > B \quad \Leftrightarrow \quad a_{\alpha\beta} > b_{\alpha\beta}, \, \forall\alpha, \, \forall\beta,$$
$$A \geq B \quad \Leftrightarrow \quad a_{\alpha\beta} \geq b_{\alpha\beta}, \, \forall\alpha, \, \forall\beta.$$

Definition 1.1 *An N-by-N matrix $A = (a_{\alpha\beta})$ is an M-matrix if*

$$a_{\alpha\alpha} > 0 \quad 1 \leq \alpha \leq N, \qquad a_{\alpha\beta} \leq 0 \quad \forall\alpha \neq \beta, \tag{1.1}$$

and

$$A \text{ is regular (} A^{-1} \text{ exists) and } \quad A^{-1} \geq 0, \tag{1.2}$$

where 0 is zero matrix.

Definition 1.2 *An N-by-N matrix $A = (a_{\alpha\beta})$ is an H-matrix if its comparison matrix $B = (b_{\alpha\beta}) =< A >$ is an M-matrix, where*

$$b_{\alpha\alpha} = |a_{\alpha\alpha}|, \quad b_{\alpha\beta} = -|a_{\alpha\beta}| \leq 0 \quad \forall\alpha \neq \beta. \tag{1.3}$$

Definition 1.3 *An N-by-N matrix $A = (a_{\alpha\beta})$ is strictly diagonally dominant if*

$$|a_{\alpha\alpha}| > \sum_{\beta \neq \alpha} |a_{\alpha\beta}| \quad 1 \leq \alpha \leq N, \tag{1.4}$$

and weakly diagonally dominant if

$$|a_{\alpha\alpha}| \geq \sum_{\beta \neq \alpha} |a_{\alpha\beta}| \quad 1 \leq \alpha \leq N. \tag{1.5}$$

Theorem 1.10 *Let the N-by-N matrix A be strictly diagonally dominant and let the sign conditions (1.1) be satisfied. Then A is an M-matrix.*

We also have the following theorem,

Theorem 1.11 *(a) A strictly diagonally dominant matrix is an H-matrix.*

(b) A positive definite matrix satisfying the condition in (1.1) is an M-matrix.

(c) An Hermitian M-matrix is positive definite.

1.5 Strengthened Cauchy-Schwarz Inequalities

The following Strengthened Cauchy-Schwarz (or Cauchy-Buniakowskii-Schwarz (C.B.S.)) inequality will be used to prove the convergence of a general domain case in section 6.2.5 ; see [11] for details.

Theorem 1.12 *Given a finite-dimensional Hilbert space H, an inner product $(\cdot, \cdot)$ on H and two subspaces H_1, H_2 of H such that*

$$H_1 \cap H_2 = \{0\},$$

then there exists

$$\gamma = \gamma(H_1, H_2) \in (0, 1),$$

such that for all $h_1 \in H_1$ and $h_2 \in H_2$ the following strengthened C.B.S.-inequality holds:

$$|(h_1, h_2)| \leq \gamma \|h_1\| \|h_2\| \tag{1.6}$$

where the norm is induced by the inner product

$$\|h\| = \sqrt{(h, h)}.$$

Corollary 1.2 *Let M be a symmetric positive definite matrix, and let U and V be disjoint vector subspaces of the space on which M operates. Then there exists a $\gamma \in [0, 1)$ such that*

$$(u^t M v)^2 \leq \gamma^2 u^t M u v^t M v \quad \forall u \in U,\, v \in V. \tag{1.7}$$

Chapter 2

Discontinuous Overlapping Schwarz Methods for Poisson's Equation

2.1 Overlapping Schwarz Algorithms for solving Helmholtz's Equation

In a paper by Xiao-Chuan Cai, Mario A. Casarin, Jr, Frank W. Elliott, Jr, and Olof B. Widlund [5], three domain decomposition methods were introduced for the solution of Helmholtz's equation. They are based on the overlapping Schwarz method and are called Algorithm 1, Algorithm 2, and Algorithm 3 in increasing order of sophistication. Algorithm 1 is the classical overlapping multiplicative Schwarz algorithm which uses Dirichlet boundary conditions at each artificial interface and it is known to be successful for solving Poisson's problem. However, Algorithm 1 is unsuccessful in solving Helmholtz's problem. Algorithm 2 improves on Algorithm 1 by using approximate Sommerfeld boundary conditions at each artificial interface while maintaining continuity of the iterates. Algorithm 3 is constructed from Algorithm 2 and allows discontinuities at each artificial interface.

Algorithm 3 is a new family of domain decomposition methods which uses discontinuous iterates. The idea of Algorithm 3 was inspired by the thesis of Després [10].

The basic domain decomposition algorithm considered by Després is defined as follows. The model problem of a Helmholtz's equation with Sommerfeld boundary condition is given by

$$\begin{aligned} -\Delta u - (k(x))^2 u &= f \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where k is a real parameter. The given region Ω is divided into two **nonoverlapping** subregions Ω_1 and Ω_2, and the iteration is advanced by simultaneously solving

$$\begin{aligned} -\Delta u_j^{n+1} - k^2 u_j^{n+1} &= f \quad \text{in} \quad \Omega_j, \\ \frac{\partial u_j^{n+1}}{\partial n_j} - iku_j^{n+1} &= -\frac{\partial u_{out}^n}{\partial n_{out}} - iku_{out}^n \quad \text{on} \quad \Gamma, \\ \frac{\partial u_j^{n+1}}{\partial n_j} - iku_j^{n+1} &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

in the two subregions. Here Γ is the artificial interface which is the intersection of $\partial\Omega_1$ and $\partial\Omega_2$. We also define n_j as the outward normal of Ω_j with $n_{out} = -n_j$ and where u_{out}^n is the previous approximate solution outside Ω_j.

Algorithm 3 is an overlapping version of Després' method and quite effective in different contexts for solving Helmholtz's equation. For experimental results; see Casarin, and Widlund [7], [5].

2.2 Continuous and Discontinuous Overlapping Schwarz Methods

In this thesis, we will study the convergence of Algorithm 3 for solving Poisson's equation. The difference between Algorithm 2 and Algorithm 3 is whether the continuity of the iterates is maintained or not. There are two basic alternatives:

1. the new values replace the old ones on each artificial interface.
2. the new and the old values are kept.

Algorithm 2 results from the first choice and Algorithm 3 from the second. In [7], Algorithm 3 is called the **Discontinuous Overlapping Schwarz Method (OSM-D)**. So, it hereafter will be called Algorithm 3 (OSM-D). Also Algorithm 2 will be called the **Continuous Overlapping Schwarz Method (OSM-C)**.

2.3 Variational Formulation of Algorithm 3 (OSM-D)

Algorithm 3 (OSM-D) is designed to take advantage of discontinuities and is derived from the associated continuous Algorithm 2 (OSM-C). To understand Algorithm 3 (OSM-D), we first review the variational form of Poisson's equation with Robin boundary conditions and also derive the classical algorithm and Algorithm 2 (OSM-C).

We consider the following problem :

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u + \alpha \frac{\partial u}{\partial n} &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.3}$$

where Ω is a bounded region which is decomposed into several overlapping subre-

gions. Using Green's formula, we convert (2.3) into the following variational form: Find $u \in H^1(\Omega)$ such that,

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v + \frac{1}{\alpha}\int_{\partial\Omega} u\, v \\ &= \int_\Omega f\, v + \frac{1}{\alpha}\int_{\partial\Omega} g\, v \qquad \forall v \in H^1(\Omega). \end{aligned} \tag{2.4}$$

Restricting u and v to elements of an appropriate space V^h of P^1 finite element function, the bilinear form of (2.4) can be written in matrix form as

$$u^t A v = a(u,v). \tag{2.5}$$

2.3.1 Classical Algorithm for solving Poisson's equation

The one-level basic overlapping multiplicative Schwarz method is constructed from a collection of overlapping subregions $\{\Omega_j\}$, which covers the given region Ω, and their boundaries $\partial\Omega_j$. The boundary $\partial\Omega_j$ consists of Θ_j, which is the part of the given boundary $\partial\Omega$, and the artificial interface Γ_j which is a common part of $\partial\Omega_j$ and other neighboring subregions. The classical multiplicative Schwarz method, known here as Algorithm 1, imposes a Dirichlet condition on the artificial interface Γ_j,

$$u_j^{n+1} = (u_j^c)^n \quad \text{on} \quad \Gamma_j.$$

We split the bilinear form a in (2.4) into the local bilinear forms a_j and a_j^c defined by,

$$\begin{aligned} a(u,v) &= a_j(u,v) + a_j^c(u,v), \\ a_j(u,v) &= \int_{\Omega_j} \nabla u \cdot \nabla v + \frac{1}{\alpha}\int_{\Theta_j} u\, v \\ a_j^c(u,v) &= \int_{\Omega_j^c} \nabla u \cdot \nabla v + \frac{1}{\alpha}\int_{\Theta_j^c} u\, v. \end{aligned} \tag{2.6}$$

The associated splitting matrices are derived simply from (2.6),

$$A = A_j + A_j^c, \quad u^t A_j v = a_j(u,v), \quad u^t A_j^c v = a_j^c(u,v). \tag{2.7}$$

Let R_j be the rectangular restriction matrix, modified from Smith, Bjørstad, and Gropp [43], that returns the vector of values defined in the interior of Ω_j and the part of the given boudary $\partial\Omega \cap \partial\Omega_j$, i.e.,

$$u_{\Omega_j \cup \partial\Omega_j} = R_j u = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} u_{(\Omega_j \cup \Theta_j)} \\ u_{(\Omega_j^c \setminus \Theta_j)} \end{pmatrix}. \tag{2.8}$$

The local matrices, B_j, B_j^c are represented in terms of the splitting matrices A_j, A_j^c and the restriction matrices, R_j,R_j^c.

$$B_j = R_j A_j (R_j)^t, \quad B_j^c = R_j^c A_j^c (R_j^c)^t.$$

The j-th fractional step of the classical multiplicative Schwarz method can be written as

$$u_{(n+j/p)} = u_{(n+(j-1)/p)} + A_j^+ (b - A u_{(n+(j-1)/p)}), \tag{2.9}$$

with $A_j^+ = (R_j)^t (B_j)^{-1} (R_j)$. In the classical method, the matrix A_j^+ restricts the residual to $\Omega_j \cup \Theta_j$, solves the problem on the subdomain for a correction, and then extends it by zero onto the entire domain Ω.

2.3.2 Algorithm 2 (OSM-C) for solving Poisson's equation

In Algorithm 2 (OSM-C), we use a Robin boundary condition, a more general boundary condition, on Γ_j given by

$$u_j^{n+1} + \tilde{\alpha} \frac{\partial u_j^{n+1}}{\partial n_j} = (u_j^c)^n - \tilde{\alpha} \frac{\partial (u_j^c)^n}{\partial n_j^c} \quad \text{on} \quad \Gamma_j. \tag{2.10}$$

To construct a local problem on each subdomain Ω_j from (2.10), we split the bilinear form a in (2.4) into local bilinear forms a_j and a_j^c defined by,

$$\begin{aligned} a(u,v) &= a_j(u,v) + a_j^c(u,v), \\ a_j(u,v) &= \int_{\Omega_j} \nabla u \cdot \nabla v + \frac{1}{\alpha}\int_{\Theta_j} u\, v + \frac{1}{\tilde{\alpha}}\int_{\Gamma_j} u\, v \\ a_j^c(u,v) &= \int_{\Omega_j^c} \nabla u \cdot \nabla v + \frac{1}{\alpha}\int_{\Theta_j^c} u\, v - \frac{1}{\tilde{\alpha}}\int_{\Gamma_j^c} u\, v, \end{aligned} \tag{2.11}$$

where $\tilde{\alpha}$ is a constant in a Robin boundary condition on the artificial interface, and which is not necessarily identical to α. The associated splitting matrices are derived simply from (2.11),

$$A = A_j + A_j^c, \quad u^t A_j v = a_j(u,v), \quad u^t A_j^c v = a_j^c(u,v). \tag{2.12}$$

To define the local matrices in terms of matrices of smaller size, we introduce some additional notations modified from Smith, Bjørstad, and Gropp [43]. Let R_j be the rectangular restriction matrix that returns the vector of values defined in the interior and the boundary of Ω_j, i.e.,

$$u_{\Omega_j \cup \partial\Omega_j} = R_j u = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} u_{(\Omega_j \cup \partial\Omega_j)} \\ u_{(\Omega_j^c \setminus \Gamma_j)} \end{pmatrix}. \tag{2.13}$$

The local matrices, B_j, B_j^c are represented in terms of the splitting matrices A_j, A_j^c and the restriction matrices, R_j,R_j^c.

$$B_j = R_j A_j (R_j)^t, \quad B_j^c = R_j^c A_j^c (R_j^c)^t.$$

The j-th fractional step of the multiplicative Schwarz method can be written as

$$u_{(n+j/p)} = u_{(n+(j-1)/p)} + (R_j)^t (B_j)^{-1} (R_j)(b - A u_{(n+(j-1)/p)}). \tag{2.14}$$

A local solver is defined by $A_j^+ = (R_j)^t (B_j)^{-1} (R_j)$.

We now define a square projection matrix P_j with zero entries except for the diagonal elements corresponding to the indices of $\Omega_j \cup \partial\Omega_j$. The projection matrix P_j is written in terms of the restriction matrix R_j as $P_j = (R_j)^t R_j$. With this notation, the j-th fractional step of Algorithm 2 (OSM-C) can be rewritten as

$$u_{(n+j/p)} = P_j^c u_{(n+(j-1)/p)} + P_j A_j^+ (b - A_j^c u_{(n+(j-1)/p)}). \tag{2.15}$$

In Algorithm 2 (OSM-C), the matrix A_j^+ restricts the residual to $\Omega_j \cup \partial\Omega_j$, solves the problem on the subdomain for a correction, and then extends it by zero onto the entire domain Ω. Thus, in this j-th fractional step $x_{(n+j/p)}$ is updated only on $\Omega_j \cup \partial\Omega_j$. To maintain continuity, we overwrite the old values by the new values on the artificial interfaces in Algorithm 2 (OSM-C). Therefore, Algorithm 2 (OSM-C) is easy to implement because it does not require multiple values on the artificial interfaces.

We note that Algorithm 2 (OSM-C) is constructed from the local bilinear forms (2.11). Considering these forms, we can rewrite equation (2.4) using local bilinear forms as

$$\begin{aligned} &\int_{\Omega_j} \nabla u^{new} \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta_j} u^{new}\, v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_j} u^{new}\, v + \int_{\Omega_j^c} \nabla u^{old} \cdot \nabla v \qquad (2.16) \\ &+ \frac{1}{\alpha} \int_{\Theta_j^c} u^{old}\, v - \frac{1}{\tilde{\alpha}} \int_{\Gamma_j} u^{old}\, v = \int_{\Omega} f\, v + \frac{1}{\alpha} \int_{\partial\Omega} g\, v \qquad \forall v \in H^1(\Omega). \end{aligned}$$

On Γ_j the artificial interface of j-th step, the old values are overwritten by the new values and are lost.

2.3.3 Multiple values on the interface

With multiple values on the artificial interface, the approximate solution $u^{n+j/p}$ in the j-th fractional step is updated only on $\Omega_j \cup \partial\Omega_j$ and defined by,

$$u^{n+j/p} = \begin{cases} u^{new} & \text{on} \quad \Omega_j \cup \partial\Omega_j \\ u^{old} & \text{on} \quad \Omega_j^c \end{cases}$$

From (2.4) and (2.16), the residual corresponding to $u^{n+i/p}$ given by,

$$a(u^{n+j/p}, v) - \int_\Omega f\, v - \frac{1}{\alpha}\int_{\partial\Omega} g\, v = \frac{1}{\tilde{\alpha}}\int_{\Gamma_i} (u^{old} - u^{new})\, v = \frac{1}{\tilde{\alpha}}\int_{\Gamma_i} [u]\, v, \qquad (2.17)$$

where $[\cdot]$ means the jump across the interface. Let U be exact solution. The error is defined as $e^{n+j/p} = u^{n+j/p} - U$ and we get the following equation,

$$a(e^{n+j/p}, v) = a(u^{n+j/p}, v) - a(U, v) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_i} [u]\, v.$$

If $[u] = 0$ then $e = 0$. So continuity implies convergence.

With multiple values on the artificial interfaces, we can compute the residual from only the jumps on the artificial interfaces. Therefore, the original problem can be reduced to a problem of small size on the artificial interfaces. From this observation, we see a similarity to algorithms such as FETI. Since the residual has nonzero values in Ω_j^c without multiple values on the artificial interfaces, Algorithm 2 (OSM-C) cannot be formulated only in terms of the values on the artificial interfaces. We now introduce Algorithm 3 (OSM-D) which allows multiple values on the artificial interfaces.

2.3.4 Atomic Subdomains

The Algorithm 3 (OSM-D) can be derived from Algorithm 2 (OSM-C) by allowing discontinuities across the artificial interfaces Γ_j. Since discontinuities are allowed,

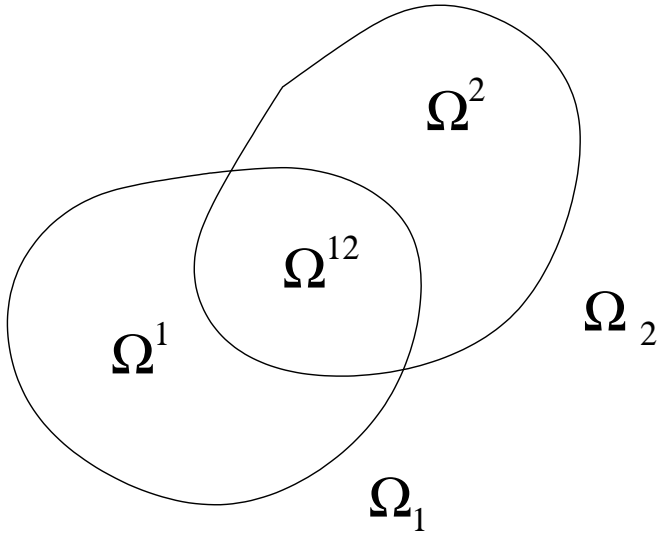

Figure 2.1: Three atomic subdomains and two overlapping subdomains

the variational form in (2.4) must be rewritten in terms of broken forms defined on a collection of certain subsets of the Ω_j's, and the data structure should be redesigned and a proper function space, with multiple values on the interfaces $\{\Gamma_j\}$, needs to be constructed. We need to define a good subset structure which maintains the independent structure for each member of a collection of the subsets with multiple values on each interface.

Since each subdomain is open and every pair of neighboring subregions has a nonempty intersection, we can construct a family of disjoint open sets such that each member is a proper subset of one or more subdomains. We choose the collection of the largest open sets satisfying this condition. We call the elements of such a family of subsets the **atomic subdomains** $\{\Omega^q\}$.

We will use the following notations. We hereafter use the superscript notation for the quantities related to the atomic subdomains and subscripts for those of the

original subdomains. Quantities with tilde are defined on a region including the artificial interfaces where we have multiple values.

We first consider a product space $\tilde{V}$ associated with $\{\Omega^q\}$. Since $\tilde{V}$ is defined independently on each Ω^q, the total function spaces $\tilde{V}$ can be identified with the direct product of all continuous function spaces V^q defined on the Ω^q,

$$\tilde{V} = \bigoplus_{q=1}^{n_a} V^q, \tag{2.18}$$

where n_a is the **number of atomic subdomains**.

$\tilde{u} \in \tilde{V}$ can be interpreted as $(u^1, \cdots, u^q, \cdots, u^{n_a})$ with $u^q \in V^q$. Let V be the continuous function space on Ω. Since V is a proper subspace of $\tilde{V}$, we also define a function $\tilde{\mathcal{R}}$ from the continuous function space V to the discontinuous function space $\tilde{V}$ as a direct product of the $\tilde{\mathcal{R}}^q$, the restricted embeddings onto atomic subdomain Ω^q, defined by

$$\tilde{\mathcal{R}} = \bigoplus_{q=1}^{n_a} \tilde{\mathcal{R}}^q, \tag{2.19}$$

2.3.5 Continuous and Discontinuous Artificial Interfaces

In each fractional step, the discontinuity of the values on each artificial interface depends on its geometric relation to the subdomain of the fractional step. We consider a simple example. In Algorithm 3 (OSM-D), we use the three function spaces of the three atomic subdomains in the two overlapping subdomain case. In the first fractional step, we make the iterates continuous across Γ_2 with no constraints across Γ_1. In the second fractional step, we make the iterates continuous across Γ_1 with no constraints across Γ_2. Thus we need a product space of three

independent atomic subspaces for Algorithm 3 (OSM-D),

$$\tilde{V} = V^1 \bigoplus V^{12} \bigoplus V^2.$$

In fact, we use two proper subsets of $\tilde{V}$ in Algorithm 3 (OSM-D), namely

$$\tilde{V_1} = V_1 \bigoplus V^2 \quad \tilde{V_2} = V_2 \bigoplus V^1.$$

Therefore, we have the following relations,

$$V \subset \tilde{V_1} \subset \tilde{V}, \quad V \subset \tilde{V_2} \subset \tilde{V}.$$

The two fractional steps of Algorithm 3 (OSM-D), in the case of two overlapping subdomains, have two different kinds of artificial interfaces, those that are hereafter called the **continuous artificial interfaces**, which are located inside the subdomain and have the same values on the corresponding boundaries of the atomic subdomains in the subdomain, and those that are hereafter called the **discontinuous artificial interfaces**, which are also formed by the part of the boundaries of the subdomain of the fractional step.

In a general partitioning with cross points, the discontinuity exists on the part of the artificial interface which is not included in the subdomain where the solution is updated. For the sake of convenience, we hereafter define the concept of discontinuous artificial interfaces and continuous artificial interfaces on the closure of the subdomain of the fractional step. Therefore, in each fractional step, we always have a set of discontinuous artificial interfaces and another set of continuous artificial interfaces.

2.3.6 Algorithm 3 (OSM-D) for solving Poisson's equation

The multiple values on the artificial interfaces are an essential part of Algorithm 3 (OSM-D), but the multiple value formulation results in complicated data structure for Algorithm 3 (OSM-D). We now construct the local matrices of Algorithm 3 (OSM-D) in terms of the discontinuous artificial interfaces and the continuous artificial interfaces discussed in previous section.

From the formulas (2.10) and (2.11), we can construct an overall structure of the local matrices for all atomic subdomains as follows. The discontinuous artificial interfaces in the j-th fractional step for Ω_j can be defined as the faces of contact to Ω_j^c. For each atomic subregion Ω^q, the continuous artificial interfaces in the j-th fractional step for Ω_j can be defined as the part of the boundary of the atomic subdomains in Ω_j which have faces of contact to certain Ω_i with $i \neq j$ but is not part of $\partial\Omega_j$.

We therefore define a form $\tilde{a}_{\Omega^q}$ as in (2.4) with $\tilde{u} \in \tilde{V}$ and $v \in V$ and given by,

$$\tilde{a}_{\Omega^q}(\tilde{\mathcal{R}}^q(\tilde{u}), v) = \int_{\Omega^q} \nabla u^q \cdot \nabla v + \frac{1}{\tilde{\alpha}} \int_{\Theta^q} u^q\, v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_+^q} u^q\, v - \frac{1}{\tilde{\alpha}} \int_{\Gamma_-^q} u^q\, v, \qquad (2.20)$$

where

$$\begin{aligned} \Gamma_+^q &= \partial\Omega^q \cap \{\cup_k \{\partial\Omega_k^c | \Omega^q \subseteq \Omega_k\}\}, \qquad (2.21) \\ \Gamma_-^q &= \partial\Omega^q \cap \{\cup_k \{\partial\Omega_k | \Omega^q \subseteq \Omega_k^c\}\}, \\ \Theta^q &= \partial\Omega^q \cap \partial\Omega \quad k = 1, \cdots, n^s, q = 1, \cdots, n_a, \end{aligned}$$

where n^s is the **number of subdomains** and $\tilde{\alpha}$ can differ from α on the artificial interfaces. The bilinear form (2.11) can be written on each subregion as

$$\tilde{a}_{\Omega_j}(\tilde{u}, v) = \sum_{\Omega^q \subseteq \Omega_j} \tilde{a}_{\Omega^q}(u^q, v), \quad \tilde{a}_{\Omega_j^c}(\tilde{u}, v) = \sum_{\Omega^q \subseteq \Omega_j^c} \tilde{a}_{\Omega^q}(u^q, v). \qquad (2.22)$$

We consider $\tilde{P}^1$ discontinuous finite element function $\tilde{V}^h$, permitting multiple values on the artificial interfaces. The discretized associated function space of $\tilde{V}$, can be defined by the direct product of the function spaces $(V^q)^h$, which are finite element space which have degrees of freedom associated with the atomic subregion Ω^q, and where

$$\tilde{V}^h = \bigoplus_{q=1}^{n_a} (V^q)^h. \tag{2.23}$$

An element of $\tilde{V}^h$, $\tilde{u}$, can be interpreted as $(u^1, \cdots, u^q, \cdots, u^{n_a})$ with $u^q \in (V^q)^h$. Assuming v to be an element of V^h, the continuous finite element function space, $\tilde{a}_{\Omega^q}$ can be given, in matrix form on each atomic subregion Ω_j, by

$$(u^q)^t \tilde{A}_{\Omega^q} v = \tilde{a}_{\Omega_j}(u^q, v). \tag{2.24}$$

The matrix form $\tilde{A}_j$, defined on $\Omega_j \cup \partial\Omega_j$, a union of atomic subdomains can be redefined using (2.22) as,

$$\tilde{A}_j = \sum_{\Omega^q \subset (\Omega_j \cup \partial\Omega_j)} \tilde{A}_{\Omega^q}. \tag{2.25}$$

Since $\tilde{V}^h$ allows multiple values on the interface, the number of degrees of freedom of Algorithm 3 (OSM-D) is bigger than that of Algorithm 2 (OSM-C) for the same problem and Algorithm 3 (OSM-D) also requires different, more complicated data structures. To formulate Algorithm 3 (OSM-D), we need to introduce additional matrix forms. Let R^q be the rectangular restriction matrix for each atomic subdomain that returns the vector of values defined in the interior and on the boundary of Ω^q, i.e.,

$$u^q = R^q \tilde{u} = \begin{pmatrix} \tilde{I} & \tilde{0} \end{pmatrix} \begin{pmatrix} \tilde{u}_{\Omega^q} \\ \tilde{u}_{(\Omega^q)^c} \end{pmatrix}. \tag{2.26}$$

The total restriction matrix $\tilde{R}$ is

$$\tilde{R} = \begin{pmatrix} (R^1)^t & \cdots & (R^{n_a})^t \end{pmatrix}^t. \tag{2.27}$$

For each atomic subregion Ω^q, we can define a smaller local matrix B^q in terms of $\tilde{A}_{\Omega^q}$ and R^q as,

$$B^q = R^q \tilde{A}_{\Omega^q} (R^q)^t. \tag{2.28}$$

From these local matrices, we can construct a partitioned matrix $\tilde{\Lambda}$ defined by,

$$\tilde{\Lambda} = diag \begin{pmatrix} B^1 & \cdots & B^{n_a} \end{pmatrix}.$$

The square projection matrix $\tilde{P}_j$ is defined as the square matrix which has zero entries, except for the diagonal entries corresponding to all indices corresponding to $\cup\Omega^q$ such that $\Omega^q \subset (\Omega_j \cup \partial\Omega_j)$. The following two equations relate the A_j of Algorithm 2 and the $\tilde{A}_j$ of Algorithm 3 (OSM-D),

$$A_j = (\tilde{R})^t \tilde{P}_j \tilde{A}_j \tilde{R}, \quad A_j^c = (\tilde{R})^t (\tilde{I} - \tilde{P}_j) \tilde{A}_j \tilde{R}.$$

The j-th fractional step of Algorithm 3 (OSM-D) is built from that of Algorithm 2 and is defined by,

$$\tilde{u}_{(n+j/p)} = \tilde{P}_j^c \tilde{u}_{(n+(j-1)/p)} + \tilde{P}_j \tilde{R} A_j^+ (b - \tilde{R}^t \tilde{P}_j^c \tilde{\Lambda} \tilde{u}_{(n+(j-1)/p)}). \tag{2.29}$$

The residual of each fractional step is computed from the contribution from each atomic subdomain using $\tilde{A}$ which is a discontinuous version of A in (2.5). In this case, each atomic subregion contributes separately to $\tilde{A}$ in the matrix computations. Computational results with Algorithm 3 (OSM-D), which will be described in Chapter 7, show that Algorithm 3 converges under a variety of conditions.

2.4 A Saddle-Point Approach

There are many papers regarding the Finite Element Tearing and Interconnecting (FETI) method including Charbel Farhat and François-Xavier Roux [12] and [13]. The FETI method is a special iterative substructuring method where Lagrange multipliers are used to enforce the continuity conditions across the artificial interface. Since we allow a discontinuity across the interface between neighboring atomic subregions, we will try to study the convergence of Algorithm 3 (OSM-D) using a Lagrange multiplier formulation.

The variational problem (2.4) with two overlapping subdomains is equivalent to the following: Given f and g, find the function u that is a stationary point of the functional

$$J(v) = \frac{1}{2}a(v,v) - (v,f) - \frac{1}{\alpha}(v,g)_{\partial\Omega}, \quad \text{with} \tag{2.30}$$
$$(v,f) = \int_\Omega f \cdot v \quad (v,g)_{\partial\Omega} = \int_{\partial\Omega} g \cdot v.$$

We will use the following notations. Let Ω_1 and Ω_2 be two overlapping subdomains which are embedded in Ω such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. We then have three atomic subregions Ω^1, which is the nonoverlapping part of Ω_1, Ω^2, which is the nonoverlapping part of Ω_2, and Ω^{12}, which is the overlapping part of Ω_1 and Ω_2. Let $\Gamma_1 = \overline{\Omega}^{12} \cap \overline{\Omega}_2$ and $\Gamma_2 = \overline{\Omega}^{12} \cap \overline{\Omega}_1$ and let $\Theta_i = \overline{\Omega}_i \cup \partial\Omega$ for any Ω_i contiguous to $\partial\Omega$. The outward normal of Ω_i is n_i, $i = 1, 2$.

Solving the above problem is equivalent to finding the three functions u^1, u^2, and u^{12} that are stationary points of the functionals:

$$J^1(v^1) \;=\; \frac{1}{2}a(v^1,v^1)_{\Omega^1} - (v^1,f)_{\Omega^1} - \frac{1}{\alpha}(v^1,g)_{\partial\Omega^1}, \tag{2.31}$$

$$
\begin{aligned}
J^2(v^2) &= \frac{1}{2}a(v^2,v^2)_{\Omega^2} - (v^2,f)_{\Omega^2} - \frac{1}{\alpha}(v^2,g)_{\partial\Omega^2},\\
J^{12}(v^{12}) &= \frac{1}{2}a(v^{12},v^{12})_{\Omega^{12}} - (v^{12},f)_{\Omega^{12}} - \frac{1}{\alpha}(v^{12},g)_{\partial\Omega^{12}},
\end{aligned}
$$

where

$$
\begin{aligned}
a(u^1,v^1)_{\Omega^1} &= \int_{\Omega^1} \nabla u^1\cdot\nabla v^1 + \frac{1}{\alpha}\int_{\Theta^1} u^1\,v^1 - \frac{1}{\tilde{\alpha}}\int_{\Gamma_1} u^1\,v^1,\\
a(u^2,v^2)_{\Omega^2} &= \int_{\Omega^2} \nabla u^2\cdot\nabla v^2 + \frac{1}{\alpha}\int_{\Theta^2} u^2\,v^2 - \frac{1}{\tilde{\alpha}}\int_{\Gamma_2} u^2\,v^2,\\
a(u^{12},v^{12})_{\Omega^{12}} &= \int_{\Omega^{12}} \nabla u^{12}\cdot\nabla v^{12} + \frac{1}{\alpha}\int_{\Theta^{12}} u^{12}\,v^{12} + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1\cup\Gamma_2} u^{12}\,v^{12},\\
(v^1,f)_{\Omega^1} &= \int_{\Omega^1} f\,v^1, \qquad (v^1,g)_{\partial\Omega^1} = \int_{\Theta^1} g\,v^1,\\
(v^2,f)_{\Omega^2} &= \int_{\Omega^2} f\,v^2, \qquad (v^2,g)_{\partial\Omega^2} = \int_{\Theta^2} g\,v^2,\\
(v^{12},f)_{\Omega^{12}} &= \int_{\Omega^{12}} f\,v^{12}, \quad (v^{12},g)_{\partial\Omega^{12}} = \int_{\Theta^{12}} g\,v^{12},
\end{aligned}
$$

that satisfy the continuity conditions across the two interfaces,

$$
u^1 = u^{12} \quad \text{on} \quad \Gamma_1 \qquad u^{12} = u^2 \quad \text{on} \quad \Gamma_2. \tag{2.32}
$$

Solving the variational problems in (2.31) with the continuity conditions (2.32) is equivalent to finding the saddle point of the Lagrangian

$$
\begin{aligned}
& J^*(v^1,v^2,v^{12},\mu_1,\mu_2)\\
= \;& J^1(v^1) + J^2(v^2) + J^{12}(v^{12}) + (v^{12}-v^1,\mu_1) + (v^{12}-v^2,\mu_2),\\
& (v^{12}-v^1,\mu_1) = \int_{\Gamma_1}\mu_1(v^{12}-v^1), \quad (v^{12}-v^2,\mu_2) = \int_{\Gamma_2}\mu_2(v^{12}-v^2).
\end{aligned}
$$

This means finding functions u^1, u^2, and u^{12} and Lagrange multiplier λ_1 and λ_2 that satisfy

$$
J^*(u^1,u^2,u^{12},\mu_1,\mu_2) \le J^*(u^1,u^2,u^{12},\lambda_1,\lambda_2) \le J^*(v^1,v^2,v^{12},\lambda_1,\lambda_2), \tag{2.33}
$$

for any admissible v^1, v^2, v^{12}, μ_1, and μ_2.

2.5 An Algebraic Formulation of Algorithm 3 (OSM-D)

2.5.1 Notations in the Two Overlapping Subdomain Case

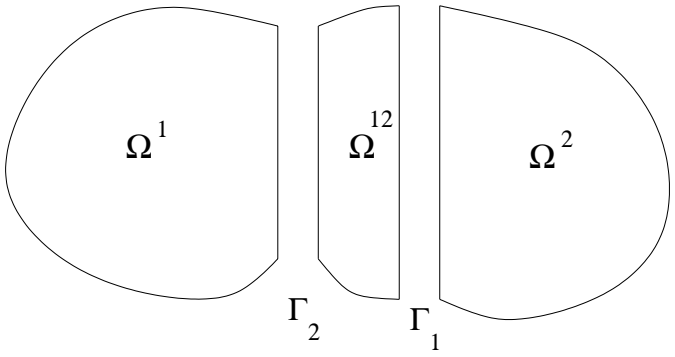

Figure 2.2: An illustration of the two overlapping subdomain with two artificial interfaces which is allowed to have discontinuity (multiple values)

To construct an algebraic formulation of Algorithm 3 (OSM-D) for the two overlapping case, we will use the following notations. Within the three atomic subdomains Ω^1, Ω^{12}, and Ω^2, we denote the number of interior and boundary nodal unknowns except the unknowns on the two artificial interfaces by n_a^1, n_a^{12}, and n_a^2 respectively and the number of the two artificial interfaces Γ_1 and Γ_2 nodal unknowns by n_1^I and n_2^I. We also define the number of interior and boundary nodal unknowns except the unknowns on the artificial interfaces of two subdomain $\Omega_1 = \Omega^1 \cup \Omega^{12} \cup \Gamma_2$, and $\Omega_2 = \Omega^{12} \cup \Omega^2 \cup \Gamma_1$ by n_1^s and n_2^s respectively. With a certain numbering, we have the following two connectivity matrices for the two Lagrange multipliers on the two artificial interfaces,

$$I_1 = \begin{pmatrix} 0_1 & I_2^{12} \end{pmatrix}, \quad I_2 = \begin{pmatrix} I_1^{12} & 0_2 \end{pmatrix},$$
$$I_2^c = \begin{pmatrix} 0^1 & I_2^{12} \end{pmatrix}, \quad I_1^c = \begin{pmatrix} I_1^{12} & 0^2 \end{pmatrix},$$

where 0^i is an $n_i^I \times n_a^i$, $i = 1, 2$, and 0_i is an $n_i^I \times n_i^s$, $i = 1, 2$, zero matrix and I_i^{12} is the $n_i^I \times n_i^I$, $i = 1, 2$, identity matrix between the atomic subdomains Ω^{12} and Ω^i, $i = 1, 2$, respectively.

To formulate a two overlapping subdomain problem, we need to define the following submatrices as well as matrix forms A^1, A^2, and A^{12} and the local matrices B^1, B^2, and B^{12} of section 2.3.6,

$$\begin{aligned}
&\tilde{\Lambda} = diag\begin{pmatrix} B^1 & B^{12} & B^2 \end{pmatrix}, \\
&\tilde{B}_1 = diag\begin{pmatrix} B^1 & B^{12} \end{pmatrix}, \quad \tilde{B}_2 = diag\begin{pmatrix} B^{12} & B^2 \end{pmatrix}, \\
&\tilde{f}_1 = \begin{pmatrix} f^1 & f^{12} \end{pmatrix}^t, \quad \tilde{f}_2 = \begin{pmatrix} f^{12} & f^2 \end{pmatrix}^t, \\
&B_1^c = B^2, \quad B_2^c = B^1, \quad f_1^c = f^2, \quad f_2^c = f^1,
\end{aligned}$$

with

$$\tilde{f} = \begin{pmatrix} f^1 & f^{12} & f^2 \end{pmatrix}^t, \quad f = \tilde{R}^t \tilde{f}, \tag{2.34}$$

and the following vector notations for the solution,

$$\tilde{u}_1 = \begin{pmatrix} u^1 & u^{12} \end{pmatrix}, \quad \tilde{u}_2 = \begin{pmatrix} u^{12} & u^2 \end{pmatrix}, \quad u_1^c = u^2, \quad u_2^c = u^1. \tag{2.35}$$

The two fractional steps of Algorithm 3 (OSM-D) can be written as two problems with two positive definite matrices B_1 and B_2,

$$B_1 = \tilde{R}^t \tilde{B}_1, \quad B_2 = \tilde{R}^t \tilde{B}_2, \quad f_1 = \tilde{R}^t \tilde{f}_1, \quad f_2 = \tilde{R}^t \tilde{f}_2, \quad u_1 = \tilde{R}^t \tilde{u_1}, \quad u_2 = \tilde{R}^t \tilde{u_2}, \tag{2.36}$$

The first problem is related to the first fractional step and is the following,

$$B_1 u_1 = f_1 + (I_1)^T \lambda_2, \quad B_1^c u_1^c = f_1^c - (I_1^c)^T \lambda_2, \quad I_1 u_1 = I_1^c u_1^c, \tag{2.37}$$

the other is,

$$B_2 u_2 = f_2 + (I_2)^T \lambda_1, \quad B_2^c u_2^c = f_2^c - (I_1^c)^T \lambda_1, \quad I_2 u_2 = I_2^c u_2^c. \tag{2.38}$$

We note that the notations and ideas of this section can be generalized to the case of general overlapping subdomains with regions with cross points.

2.5.2 Algebraic System for the Two Overlapping Subdomain Case

Assuming that u^1, u^2, and u^{12} are elements in the appropriate spaces, the problems in section 2.4 results in a discrete problem and the following algebraic system:

$$\begin{aligned} B^1 u^1 &= f^1 - (I^1)^T \lambda_1, \quad B^2 u^2 = f^2 - (I^2)^T \lambda_2, \\ B^{12} u^{12} &= f^{12} + (I_1^{12})^T \lambda_1 + (I_2^{12})^T \lambda_2 \\ I^1 u^1 &= I_1^{12} u^{12}, \quad I^2 u^2 = I_2^{12} u^{12} \end{aligned} \tag{2.39}$$

or

$$\begin{pmatrix} B^1 & 0 & 0 & (I^1)^T & 0 \\ 0 & B^{12} & 0 & -(I_1^{12})^T & -(I_2^{12})^T \\ 0 & 0 & B^2 & 0 & (I^2)^T \\ I^1 & -I_1^{12} & 0 & 0 & 0 \\ 0 & -I_2^{12} & I^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^{12} \\ u^2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} f^1 \\ f^{12} \\ f^2 \\ 0 \\ 0 \end{pmatrix}. \tag{2.40}$$

In this setting, the Lagrange multipliers λ_1 and λ_2 measure the error between the exact solution,

$$U = (U^1, U^{12}, U^2), \quad \text{with} \quad B^1 U^1 = f^1, \quad B^2 U^2 = f^2, \quad B^{12} U = f^{12}$$

and the approximate solution,

$$\tilde{u} = (u^1, u^{12}, u^2).$$

Given the continuity of u^1 and u^{12} in the first fractional step, we have

$$\begin{aligned} B_1 u_1 &= R_1^t (B^1 u^1 + B^{12} u^{12}) \\ &= R_1^t (f^1 + f^{12} + (I_2^{12})^T \lambda_2) = f_1 + (I_2^{12})^T \lambda_2. \end{aligned}$$

Given the continuity of u^2 and u^{12} in the second fractional step, we have

$$\begin{aligned} B_2 u_2 &= R_2^t (B^2 u^2 + B^{12} u^{12}) \\ &= R_2^t (f^2 + f^{12} + (I_1^{12})^T \lambda_1) = f_2 + (I_1^{12})^T \lambda_1. \end{aligned}$$

Finally, if we have continuity across both interfaces,

$$\begin{aligned} Au &= R^t (B^1 u^1 + B^{12} u^{12} + B^2 u^2) \\ &= R^t (f^1 - (I^1)^T \lambda_1 + f^{12} + (I_1^{12})^T \lambda_1 + (I_2^{12})^T \lambda_2 + f^2 - (I^2)^T \lambda_2) \\ &= R^t (f^1 + f^{12} + f^2) = f. \end{aligned}$$

We note that the notations and ideas of this section can be generalized to the case of general overlapping subdomains for regions with cross points.

2.5.3 Algorithm 3 (OSM-D) as a Block Gauss-Seidel method

In this section, we will study Algorithm 3 (OSM-D) for two overlapping subdomains as a Block Gauss-Seidel method. For given u_n^1, u_n^{12}, and u_n^2, a new value λ_2^n can be obtained from the following equation,

$$\lambda_2^n = I^2 (f^2 - B^2 u_n^2) = I_1^c (f^2 - B^2 u_n^2).$$

Given λ_2^n, the first fractional step can be written as in section 2.5.2,

$$\begin{aligned} u_1^{n+1/2} &= (B_1)^{-1} (f_1 + (I_1)^T \lambda_2^n)) \\ &= (B_1)^{-1} (f_1 + (I_1)^T I^2 (f^2 - B^2 u_n^2)) \\ &= (B_1)^{-1} (f_1 + (I_1)^T I_1^c (f^2 - B^2 u_n^2)). \end{aligned}$$

The last part of the previous equations matches Algorithm 3 (OSM-D) which allows new and old data on the artificial interface between Ω_2 and $\Omega_2^c = \Omega^1$. Since the Lagrange multiplier λ_2^n is obtained from the continuous fractional solution u_n in Ω_2, the old and new values of u_n on the artificial interface of Ω_2 are equal and we can use either of them. Since we will update the fractional solution $u_{n+1/2}$ in Ω_1, it is reasonable to use the outside interface values which keep their value in the next fractional step and affects the new fractional solution. From the previous step, we have the fractional solution $u^1_{n+1/2}$ and $u^{12}_{n+1/2}$. Also $\lambda_1^{n+1/2}$ can be obtained from the equation,

$$(I^1)^T \lambda_1 = f^1 - B^1 u^1, \tag{2.41}$$

which can be used to rewrite the next equation for $\lambda_1^{n+1/2}$ and $u^1_{n+1/2}$,

$$\lambda_1^{n+1/2} = I^1(f^1 - B^1 u^1_{n+1/2}) = I_2^c(f^1 - B^1 u^1_{n+1/2}).$$

The second fractional step begins with $\lambda_1^{n+1/2}$ and we get u^{12}_{n+1} and u^2_{n+1} using the same process,

$$\begin{aligned} u_2^{n+1} &= (B_2)^{-1}(f_2 + (I_2)^T \lambda_1^{n+1/2}) \\ &= (B_2)^{-1}(f_2 + (I_2)^T I^1(f^1 - B^1 u^1_{n+1/2})) \\ &= (B_2)^{-1}(f_2 + (I_2)^T I_2^c(f^1 - B^1 u^1_{n+1/2})). \end{aligned}$$

We need the last equation for the same reason as explained in the paragraph below the equation for $u_1^{n+1/2}$. If we add the following relation,

$$u^1_{n+1} = u^1_{n+1/2}, \tag{2.42}$$

and we then obtain u^{n+1}. Finally, we need to get λ_2^{n+1} used in the next iteration step for $u^{n+3/2}$,

$$\lambda_2^{n+1} = I^2(f^2 - B^2 u_{n+1}^2) = I_1^c(f^2 - B^2 u_{n+1}^2).$$

The two vectors λ_1 and λ_2 play a main role in communicating data between the two fractional steps and updating the data. Therefore, the convergence of Algorithm 3 (OSM-D) is strongly related to that of the Lagrange multipliers λ_1 and λ_2. It happens when the two vectors λ_1 and λ_2 reach the exact Lagrange multiplier values of the exact solution. We will therefore study the behavior of the Lagrange multipliers which is related to the data on the artificial interfaces.

The notation and idea of this section can be generalized to the case of the several overlapping subdomains. For an algebraic formulation as a block Gauss-Seidel method, Lagrange multipliers are essential. There is no difference between Algorithm 2 (OSM-C) and Algorithm 3 (OSM-D) for overlapping subdomains without cross points. However, with cross point, Algorithm 2 (OSM-C) does not maintain the old value on the boundary of the complementary subdomain which is essential to produce suitable Lagrange multipliers. Therefore, Algorithm 2 (OSM-C), cannot in the general case be formulated using the methods of this section.

2.6 Derivation of the fractional steps

We want to show that Algorithm 3 (OSM-D) in (2.29) is identical to the process given by the Block Gauss-Seidel algorithm in the previous sections. The fractional steps of Algorithm 3 (OSM-D) for two overlapping subdomains are written

according to equation (2.29) as,

$$\tilde{u}_{n+1/2} = \tilde{P}_1^c \tilde{u}_n + \tilde{P}_1 \tilde{R} A_1^+ (b - \tilde{R}^t \tilde{P}_1^c \tilde{\Lambda} \tilde{u}_n)$$
$$\tilde{u}_{n+1} = \tilde{P}_2^c \tilde{u}_{n+1/2} + \tilde{P}_2 \tilde{R} A_2^+ (b - \tilde{R}^t \tilde{P}_2^c \tilde{\Lambda} \tilde{u}_{n+1/2}).$$

We will also check the structure of the Lagrange multipliers and the fractional steps. Since we have used matrices which have smaller dimension than the matrices in (2.29), we need the following identities

$$(I_1)^T I_2^c = R_1 \tilde{R}^t \tilde{P}_1^c, \quad (I_2)^T I_1^c = R_2 \tilde{R}^t \tilde{P}_2^c. \tag{2.43}$$

From these identities, we have the following relation,

$$(B_1)^{-1} = \tilde{P}_1 \tilde{R} A_1^+ \tilde{R}^t \tilde{P}_1, \quad (B_2)^{-1} = \tilde{P}_2 \tilde{R} A_2^+ \tilde{R}^t \tilde{P}_2.$$

We write the fractional solution $u_1^{n+1/2}$ for each subdomain as,

$$\begin{aligned}
u_1^{n+1/2} &= (B_1)^{-1}(f_1 + (I_1)^T \lambda_2^n) \\
&= (B_1)^{-1}(f_1 + (I_1)^T I^2 (f^2 - B^2 u_n^2)) \\
&= (B_1)^{-1}(f_1 + R_1 \tilde{R}^t \tilde{P}_1^c (f^2 - B^2 u_n^2)) \\
&= (B_1)^{-1}(R_1 \tilde{R}^t \tilde{P}_1 \tilde{f} + R_1 \tilde{R}^t \tilde{P}_1^c \tilde{f} - R_1 \tilde{R}^t \tilde{P}_1^c \tilde{\Lambda} u_n) \\
&= (B_1)^{-1} R_1 (\tilde{R}^t \tilde{f} - \tilde{R}^t \tilde{P}_1^c \tilde{\Lambda} u_n) \\
&= (B_1)^{-1} R_1 (b - \tilde{R}^t \tilde{P}_1^c \tilde{\Lambda} \tilde{u}_n),
\end{aligned}$$

and u_2^{n+1} is also obtained similarly,

$$u_2^{n+1} = (B_2)^{-1} R_2 (b - \tilde{R}^t \tilde{P}_2^c \tilde{\Lambda} \tilde{u}_{n+1/2}). \tag{2.44}$$

We can now see the relation between the discontinuous and continuous function spaces in Algorithm 3 (OSM-D). The operator $\tilde{R}$ and the transpose $\tilde{R}^t$ communicate the data between the discontinuous and continuous spaces with the projection

operators $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_1^c$, and $\tilde{P}_2^c$. In each fractional step, we keep the old values outside the subregion where the solution is updated by new values. We can therefore see that the algorithm (2.29) and algorithm of the previous sections are identical.

Chapter 3

Convergence Theory for Overlapping Strips and General Quadrilaterals

3.1 Introduction

In this chapter, we develop a theory of the convergence of Algorithm 3 (OSM-D) for several overlapping strips without cross points. In section 3.2, we review the convergence in the nonoverlapping subdomain case. In section 3.3, we study conditions for convergence of Algorithm 3 in the case of two overlapping general subdomains. In section 3.4, we show the convergence of Algorithm 3 (OSM-D) for two overlapping rectangular subdomains. In section 3.5, we show that the convergence is geometric for the case of several strips. The basic idea of that section is inspired by Nataf [33] and [34]. In the final section 3.6, we extend the results to a general quadrilateral which is conformally equivalent to a rectangular domain.

3.1.1 Basic idea and Notation

We begin this chapter by introducing the basic idea. We consider the Poisson problem with Robin boundary condition as in Chapter 1 and we will also use the definitions and notations therein. The main idea in this chapter is the **energy estimate** used in [27]. In nonoverlapping subdomain case, P. L. Lions has proven the convergence of the Robin iteration method using such an energy estimate. We will study the extension of this methods for the overlapping subdomain cases.

Let Ω be a bounded open set in $\mathcal{R}^2$. We also assume that the various boundaries including the original boundary and artificial interfaces are smooth enough to define an outward normal n for the different subdomains. We consider a harmonic function, i.e., a function v such that

$$-\Delta v = -(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) = 0. \tag{3.1}$$

Multiplying by v and using Green's identity and the identity

$$AB = \frac{1}{4\gamma}[(A+\gamma B)^2 - (A-\gamma B)^2],$$

we have,

$$\begin{aligned} |v|^2_{H^1(\Omega)} &= \int\int_\Omega (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 d\Omega = \int_{\partial\Omega} v\frac{\partial v}{\partial n} dS \qquad (3.2)\\ &= \frac{1}{4\tilde{\alpha}}\int_{\partial\Omega}(v + \tilde{\alpha}\frac{\partial v}{\partial n})^2 dS - \frac{1}{4\tilde{\alpha}}\int_{\partial\Omega}(v - \tilde{\alpha}\frac{\partial v}{\partial n})^2 dS. \end{aligned}$$

Since we use the concept of subdomains and atomic subdomains in this chapter, we need to use the separate notations for the two different kinds of domains. Let $u_i^{n+j/p}$ be the j-th approximation of u in Ω_i (a subdomain) and $(u^i)^{n+j/p}$ be the j-th approximation of u in Ω^i (an atomic subdomain) at step n. With the error

$e_i^{n+j/p} = u_i - u_i^{n+j/p}$, $e_i^{n+j/p}$ is the j-th error of u in Ω_i and $(e^i)^{n+j/p}$ the j-th error of u in Ω^i at step n. The outward normal of Ω_i is n_i and of Ω^i is n^i. Now we define the following,

$$
\begin{aligned}
A_j^{n+j/p} &= \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} (e_j^{n+j/p} + \alpha \frac{\partial e_j^{n+j/p}}{\partial n_j})^2 dS, \\
(A^j)^{n+j/p} &= \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} ((e^j)^{n+j/p} + \alpha \frac{\partial (e^j)^{n+j/p}}{\partial n^j})^2 dS, \\
B_j^{n+j/p} &= \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} (e_j^{n+j/p} - \alpha \frac{\partial e_j^{n+j/p}}{\partial n_j})^2 dS, \\
(B^j)^{n+j/p} &= \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} ((e^j)^{n+j/p} - \alpha \frac{\partial (e^j)^{n+j/p}}{\partial n^j})^2 dS, \\
E_j^{n+j/p} &= \int\int_{\Omega_i} (\frac{\partial e_j^{n+j/p}}{\partial x})^2 + (\frac{\partial e_j^{n+j/p}}{\partial y})^2 d\Omega, \\
(E^j)^{n+j/p} &= \int\int_{\Omega^j} (\frac{\partial (e^j)^{n+j/p}}{\partial x})^2 + (\frac{\partial (e^j)^{n+j/p}}{\partial y})^2 d\Omega.
\end{aligned}
\tag{3.3}
$$

3.2 Convergence on Two Nonoverlapping General Subdomains

Let u_i^n be an approximation of u in Ω_i at step n and let u_i^{n+1} be the solution of

$$
\begin{aligned}
&-\Delta u_1^{n+1/2} = f_1 \quad \text{in} \quad \Omega_i \\
&u_1^{n+1/2} + \tilde{\alpha} \frac{\partial u_1^{n+1/2}}{\partial n_1} = u_2^n - \tilde{\alpha} \frac{\partial u_2^n}{\partial n_2} \quad \text{on} \quad \Gamma_1 \\
&u_1^{n+1/2} = g_1 \quad \text{on} \quad \Theta_1.
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&-\Delta u_2^{n+1} = f_2 \quad \text{in} \quad \Omega_i \\
&u_2^{n+1} + \tilde{\alpha} \frac{\partial u_2^{n+1}}{\partial n_2} = u_1^{n+1/2} - \tilde{\alpha} \frac{\partial u_1^{n+1/2}}{\partial n_1} \quad \text{on} \quad \Gamma_2 \\
&u_2^{n+1} = g_2 \quad \text{on} \quad \Theta_2.
\end{aligned}
$$

With the error $e_i^n = u_i - u_i^n$, we can take $f_i = 0$ and $g_i = 0$. From (3.2), we have,

$$E_1^{n+1} + B_1^{n+1} = A_1^{n+1}, \quad E_2^{n+1} + B_2^{n+1} = A_2^{n+1}. \tag{3.5}$$

From (3.4), we have,

$$A_1^{n+1} = B_2^n, \quad A_2^{n+1} = B_1^{n+1}. \tag{3.6}$$

By summing over $n = 1, 2$ in (3.5) with (3.6), we have,

$$E_1^{n+1} + E_2^{n+1} + B_2^{n+1} = B_2^n. \tag{3.7}$$

By summing over $n = 0, \cdots, M-1$, we have,

$$\sum_{n=1}^{M} (E_1^n + E_2^n) + B_2^M = B^0.$$

By summing over M, we finally have,

$$\sum_{n=1}^{\infty} (E_1^n + E_2^n) = B^0 < \infty.$$

Therefore, we find,

$$\lim_{n\to\infty} (E_1^n + E_2^n) = 0.$$

Since $\{E_i^n\}$ is a sequence of H^1 seminorms and the boundary condition is zero, we can use Friedrichs' inequality and finally have,

$$\lim_{n\to\infty} ||e_i^n||_{H^1} = 0, \quad i = 1, 2.$$

Remark: The convergence for general nonoverlapping domain case has been proved by P.L. Lions; see [27].

3.3 A Condition for Convergence on Two Overlapping General Subdomains

Let u_i^{n+1} be the solution of

$$
\begin{aligned}
&(1)\quad \text{given} && (u^1)^0 \in H^1(\Omega^1), \quad (u^{12})^0 \in H^1(\Omega^{12}), \quad (u^2)^0 \in H^1(\Omega^2) \\
&(2)\quad \forall n \geq 0, && -\Delta u_1^{n+1/2} = f_1 \quad \text{in} \quad \Omega_1 \\
& && u_1^{n+1/2} + \tilde{\alpha}\frac{\partial u_1^{n+1/2}}{\partial n_1} = (u^2)^n - \tilde{\alpha}\frac{\partial (u^2)^n}{\partial n^2} \quad \text{on} \quad \Gamma_1 \\
& && u_1^{n+1/2} + \alpha\frac{\partial u_1^{n+1/2}}{\partial n_1} = g_1 \quad \text{on} \quad \Theta^1 \cup \Theta^{12} \\
&(3) && -\Delta u_2^{n+1} = f_2 \quad \text{in} \quad \Omega_2 \\
& && u_2^{n+1} + \tilde{\alpha}\frac{\partial u_2^{n+1}}{\partial n_2} = (u^1)^{n+1/2} - \tilde{\alpha}\frac{\partial (u^1)^{n+1/2}}{\partial n^1} \quad \text{on} \quad \Gamma_2 \\
& && u_2^{n+1} + \alpha\frac{\partial u_2^{n+1}}{\partial n_2} = g_2 \quad \text{on} \quad \Theta^{12} \cup \Theta^2.
\end{aligned}
$$

With the error $e_i^{n+i/2} = u_i - u_i^{n+i/2}$ (or $(e^i)^{n+i/2} = u^i - (u^i)^{n+i/2}$), we can take $f_i = 0$ and $g_i = 0$. From (3.2), we have,

$$
\begin{aligned}
&A_1^{n+1/2} > B_1^{n+1/2}, \quad B_2^{n+1/2} > A_2^{n+1/2} \\
&A_1^{n+1} > B_1^{n+1}, \quad B_2^{n+1} > A_2^{n+1}, \\
&(E^{n+1/2})^1 < E_1^{n+1/2}, \quad (E^{n+1})^2 < E_2^{n+1}.
\end{aligned}
$$

For Algorithm 3 (OSM-D), we have,

$$
A_2^{n+1/2} = A_2^{n+1}, \quad A_1^{n+1} = A_1^{n+3/2}.
$$

If the following relations are true,

$$
A_1^{n+1/2} > B_2^{n+1/2}, \quad A_2^{n+1} > B_1^{n+1}, \tag{3.8}
$$

then, we have,

$$A_1^{n+1/2} > A_2^{n+1/2}, \quad A_2^{n+1} > A_1^{n+1},$$

and we obtain,

$$A_1^{n+1/2} > A_1^{n+3/2}. \tag{3.9}$$

This is the fundamental idea. In contrast with the nonoverlapping case, which allows comparison of the quantities on the same artificial interfaces, we need to compare certain quantities on two different artificial interfaces in the two overlapping subdomain case. Therefore the relation between the two artificial interfaces is required in the study of the convergence of Algorithm 3 (OSM-D).

3.4 Convergence on a Rectangular Domain

In this section, we will develop a convergence proof of Algorithm 3 (OSM-D) for a rectangular domain. For the first fractional step, we have

$$\Delta e^{n+1/2} = e_{xx}^{n+1/2} + e_{yy}^{n+1/2} = 0 \quad \text{in} \quad D_1, \tag{3.10}$$

where D_1 is the rectangle $\{(x,y)|0 < x < 1, 0 < y < l_{\Gamma_1}\}$ whose east side $\{(x,y)|x = 1, 0 \le y \le l_{\Gamma_1}\}$, west side $\{(x,y)|x = 0, 0 \le y \le l_{\Gamma_1}\}$, and south side $\{(x,y)|0 \le x \le 1, y = 0\}$ have homogeneous boundary condition. The north side $\{(x,y)|0 \le x \le 1, y = l_{\Gamma_1}\}$ has the following nonhomogeneous Robin boundary condition for a given e^n,

$$e^{n+1/2} + \tilde{\alpha}\frac{\partial e^{n+1/2}}{\partial y} = e^n + \tilde{\alpha}\frac{\partial e^n}{\partial y}. \tag{3.11}$$

The second fractional step can be written as follows,

$$\Delta e^{n+1} = e_{xx}^{n+1} + e_{yy}^{n+1} = 0 \quad \text{in} \quad D_2, \tag{3.12}$$

where D_2 is the rectangle $\{(x,y)|0 < x < 1, l_{\Gamma_2} < y < 1\}$ whose east side $\{(x,y)|x = 1, l_{\Gamma_2} \leq y \leq 1\}$, west side $\{(x,y)|x = 0, l_{\Gamma_2} \leq y \leq 1\}$, and north side $\{(x,y)|0 \leq x \leq 1, y = 1\}$ have homogeneous boundary condition. The south side $\{(x,y)|0 \leq x \leq 1, y = l_{\Gamma_2}\}$ has the following nonhomogeneous Robin boundary condition for a given $e^{n+1/2}$,

$$e^{n+1} - \tilde{\alpha}\frac{\partial e^{n+1}}{\partial y} = e^{n+1/2} - \tilde{\alpha}\frac{\partial e^{n+1/2}}{\partial y}. \tag{3.13}$$

3.4.1 A Basic Computation

In this section, we will carry out the basic computation to support the results of the following sections. We will consider the following quantity,

$$G(y) = \frac{1}{\tilde{\alpha}}\int_0^1 |u + \tilde{\alpha}u_y|^2 dx. \tag{3.14}$$

We have the following expression for its first derivative,

$$\begin{aligned} H(y) = G'(y) &= 2\int_0^1 (\frac{1}{\tilde{\alpha}} u\, u_y + |u_y|^2 + u\, u_{yy} + \tilde{\alpha}\, u_y\, u_{yy})\, dx \\ &= 2\,(\frac{1}{\tilde{\alpha}} A_0 + B_0 + \tilde{\alpha}\, C_0). \end{aligned}$$

We also need the first derivative of H at y,

$$\begin{aligned} H'(y) &= 2\int_0^1 (\frac{1}{\tilde{\alpha}}(|u_y|^2 + u\, u_{yy}) + (3\, u_y\, u_{yy} + u\, u_{yyy}) + \tilde{\alpha}\,(|u_{yy}|^2 + u_y\, u_{yyy}))\, dx \\ &= 2\,\frac{1}{\tilde{\alpha}}(A_1 + A_2) + 2\,(3B_1 + B_2) + 2\,\tilde{\alpha}\,(C_1 + C_2). \end{aligned}$$

$H(y) \geq 0$ since

- $A_0 = \int_0^1 u\, u_y dx \geq 0$ since $A_0' = 2\,(\int_0^1 |u_y|^2 + |u_x|^2 dx)$, with $A_0(0) = 0$.
- $B_0 = \int_0^1 (|u_y|^2 + u\, u_{yy}) dx \geq 0$ since $\int_0^1 (|u_y|^2 + u\, u_{yy}) dx = \int_0^1 (|u_y|^2 + |u_x|^2) dx$.

• $C_0 = \int_0^1 u_y\, u_{yy} dx = -\int_0^1 u_y\, u_{xx} dx \geq 0$ since $C_0' = \int_0^1 |u_{yy}|^2 + |u_{xy}|^2\, dx$, with $C_0(0) = 0$.

$H'(y) \geq 0$ since

• $A_1 = \int_0^1 |u_y|^2 dx \geq 0$.

• $A_2 = \int_0^1 u\, u_{yy} dx \geq 0$ since $\int_0^1 u\, u_{yy} dx = -\int_0^1 u\, u_{xx} dx = \int_0^1 |u_x|^2 dx \geq 0$.

• $B_1 = \int_0^1 u_y\, u_{yy} dx = C_0 \geq 0$.

• $B_2 = \int_0^1 u\, u_{yyy} dx \geq 0$ since $B_2'(y) = \int_0^1 (|u_{xy}|^2 + |u_{xx}|^2)\, dx$ with $B_2(0) = 0$.

• $C_1 = \int_0^1 |u_{yy}|^2 dx \geq 0$.

• $C_2 = \int_0^1 u_y\, u_{yyy} dx \geq 0$ since $\int_0^1 u_y\, u_{yyy} dx = \int_0^1 |u_{xy}|^2\, dx \geq 0$.

Therefore, $H'(y) \geq 0$ is true. The function $H(y)$ is also a strictly monotonically increasing function.

3.4.2 The difference between $A_1^{n+i/2}$ and $A_2^{n+i/2}$ as a function of the size of the overlap

Without loss of generality, we will compare $A_1^{n+1/2}$ and $A_2^{n+1/2}$ in the first fractional step,

$$G^{n+1/2}(l_{\Gamma_1}) = A_1^{n+1/2} = \frac{1}{4\tilde{\alpha}} \int_{\Gamma_1} (e^{n+1/2} + \tilde{\alpha} \frac{\partial e^{n+1/2}}{\partial n})^2 dS$$
$$G^{n+1/2}(l_{\Gamma_2}) = A_2^{n+1/2} = \frac{1}{4\tilde{\alpha}} \int_{\Gamma_2} (e^{n+1/2} + \tilde{\alpha} \frac{\partial e^{n+1/2}}{\partial n})^2 dS.$$

From the mean value theorem for $G^{n+1/2}$, we have

$$\begin{aligned} G^{n+1/2}(l_{\Gamma_1}) - G^{n+1/2}(l_{\Gamma_2}) &= (G^{n+1/2})'(z)(l_{\Gamma_1} - l_{\Gamma_2}) \\ &= H^{n+1/2}(z)(l_{\Gamma_1} - l_{\Gamma_2}), \quad l_{\Gamma_2} < z < l_{\Gamma_1}. \end{aligned}$$

From the results of subsection 3.4.1, we have,

$$H^{n+1/2}(l_{\Gamma_2}) < H^{n+1/2}(z) < H^{n+1/2}(l_{\Gamma_1}). \tag{3.15}$$

Therefore, we have,

$$H^{n+1/2}(l_{\Gamma_2})(l_{\Gamma_1} - l_{\Gamma_2}) < G^{n+1/2}(l_{\Gamma_1}) - G^{n+1/2}(l_{\Gamma_2}) < H^{n+1/2}(l_{\Gamma_1})(l_{\Gamma_1} - l_{\Gamma_2}).$$

From this, we have,

$$\begin{aligned}
\frac{G^{n+1/2}(l_{\Gamma_1})}{G^{n+1/2}(l_{\Gamma_2})} &> (1 + \frac{(G^{n+1/2})'(l_{\Gamma_2})}{G^{n+1/2}(l_{\Gamma_2})}(l_{\Gamma_1} - l_{\Gamma_2})), \\
\frac{G^{n+1/2}(l_{\Gamma_2})}{G^{n+1/2}(l_{\Gamma_1})} &> (1 - \frac{(G^{n+1/2})'(l_{\Gamma_1})}{G^{n+1/2}(l_{\Gamma_1})}(l_{\Gamma_1} - l_{\Gamma_2})).
\end{aligned}$$

Let $l = l_{\Gamma_1} - l_{\Gamma_2}$ be the size of the overlap and let $D_1^\beta = (\log(G^\beta)'(l_{\Gamma_1})$ and $D_2^\beta = (\log(G^\beta)'(l_{\Gamma_2})$, respectively. Since $D_1^\beta > 0$ and $D_2^\beta > 0$, we have,

$$(1 - D_1^{n+1/2}\, l)G^{n+1/2}(l_{\Gamma_1}) < G^{n+1/2}(l_{\Gamma_2}) < (1 + D_2^{n+1/2}\, l)^{-1}G^{n+1/2}(l_{\Gamma_1}). \quad (3.16)$$

For the second fractional step, we have a similar formula,

$$(1 - D_2^{n+1}\, l)G^{n+1}(l_{\Gamma_2}) < G^{n+1}(l_{\Gamma_1}) < (1 + D_1^{n+1}\, l)^{-1}G^{n+1}(l_{\Gamma_2}). \quad (3.17)$$

3.4.3 Convergence rate for a single fractional step

In this section, we compare the two quantities $A_1^{n+1/2}$ and $A_1^{(n+1)+1/2}$. We have the following relations for Algorithm 3 (OSM-D),

$$\begin{aligned}
A_2^{n+1/2} - A_2^{n+1} &= A_2^{n+1/2} - B_2^{n+1/2} = (E^{n+1/2})^1 > 0, \quad (3.18) \\
A_1^{n+1} - A_1^{(n+1)+1/2} &= A_1^{n+1} - B_1^{n+1} = (E^{n+1})^2 > 0.
\end{aligned}$$

From the results above, we have,

$$\begin{aligned}
A_1^{n+1/2} &> (1 + D_2^{n+1/2}\, l)A_2^{n+1/2} \\
&> (1 + D_2^{n+1/2}\, l)(1 + D_1^{n+1}\, l)A_1^{n+1} + (1 + D_2^{n+1/2}\, l)(E^{n+1/2})^1 \\
&> (1 + D_2^{n+1/2}\, l)(1 + D_1^{n+1}\, l)A_1^{(n+1)+1/2} \\
&+ (1 + D_2^{n+1/2}\, l)(1 + D_1^{n+1}\, l)(E^{n+1})^2 + (1 + D_2^{n+1/2}\, l)(E^{n+1/2})^1,
\end{aligned}$$

and

$$\begin{aligned} A_2^{n+1} &> (1+D_1^{n+1}\,l)(1+D_2^{(n+1)+1/2}\,l)A_2^{(n+1)+1} \\ &+ (1+D_1^{n+1}\,l)(1+D_2^{(n+1)+1/2}\,l)(E^{(n+1)+1/2})^1 + (1+D_1^{n+1}\,l)(E^{n+1})^2, \end{aligned}$$

with

$$(1+D_2^{n+1/2}\,l)^{-1}(1+D_1^{n+1}\,l)^{-1} < 1,$$

$$(1+D_1^{n+1}\,l)^{-1}(1+D_2^{(n+1)+1/2}\,l)^{-1} < 1.$$

The convergence rate factors thus depend on the overlapping size.

3.4.4 The convergence of Algorithm 3 (OSM-D)

If we apply the idea in the previous section for $n = 0$, we have,

$$\begin{aligned} A_1^{1/2} &> (1+D_2^{n+1/2}\,l)(1+D_1^{n+1}\,l)A_1^{(n+1)+1/2} \\ &+ \sum_{i=0}^{i=n}(\prod_{j=0}^{j=i}(1+D_2^{j+1/2}\,l)(1+D_1^{j+1}\,l))(E^{i+1})^2 \\ &+ \sum_{i=0}^{i=n}(\prod_{j=0}^{j=i-1}(1+D_2^{j+1/2}\,l)(1+D_1^{j+1}\,l))(1+D_2^{i+1/2}\,l)(E^{i+1/2})^1, \end{aligned}$$

and

$$\begin{aligned} A_2^1 &> (1+D_1^{n+1}\,l)(1+D_2^{(n+1)+1/2}\,l)A_2^{(n+1)+1} \\ &+ \sum_{i=0}^{i=n}(\prod_{j=0}^{j=i}(1+D_1^{j+1}\,l)(1+D_2^{(j+1)+1/2}\,l))(E^{(i+1)+1/2})^1 \\ &+ \sum_{i=0}^{i=n}(\prod_{j=0}^{j=i-1}(1+D_1^{j+1}\,l)(1+D_2^{(j+1)+1/2}\,l))(1+D_1^{i+1}\,l)(E^{i+1})^2. \end{aligned}$$

Finally we can let n go to infinity and we have,

$$\begin{aligned} A_1^{1/2} &> \sum_{i=0}^{i=\infty}(\prod_{j=0}^{j=i}(1+D_2^{j+1/2}\,l)(1+D_1^{j+1}\,l))(E^{i+1})^2 \\ &+ \sum_{i=0}^{i=\infty}(\prod_{j=0}^{j=i-1}(1+D_2^{j+1/2}\,l)(1+D_1^{j+1}\,l))(1+D_2^{i+1/2}\,l)(E^{i+1/2})^1, \end{aligned}$$

and

$$\begin{aligned} A_2^1 &> \sum_{i=0}^{i=\infty}(\prod_{j=0}^{j=i}(1+D_1^{j+1}\,l)(1+D_2^{(j+1)+1/2}\,l))(E^{(i+1)+1/2})^1 \\ &+ \sum_{i=0}^{i=\infty}(\prod_{j=0}^{j=i-1}(1+D_1^{j+1}\,l)(1+D_2^{(j+1)+1/2}\,l))(1+D_1^{i+1}\,l)(E^{i+1})^2. \end{aligned}$$

This means that

$$(\prod_{j=0}^{j=n}(1+D_2^{j+1/2}\,l)(1+D_1^{j+1}\,l))(E^{n+1})^2 \to 0 \quad \text{as} \quad n\to\infty,$$

and that

$$(\prod_{j=0}^{j=n}(1+D_1^{j+1}\,l)(1+D_2^{(j+1)+1/2}\,l))(E^{(n+1)+1/2})^1 \to 0 \quad \text{as} \quad n\to\infty.$$

and that

$$(E^{n+1})^2 \to 0, \quad (E^{(n+1)+1/2})^1 \to 0 \quad \text{as} \quad n\to\infty. \tag{3.19}$$

Since $\{(E^n)^1\}$ and $\{(E^{n+1/2})^2\}$ are sequences of H_1 seminorms over Ω^1 and Ω^2 respectively and the original boundary condition is zero, we can use Friedrichs' inequality and we finally have,

$$\lim_{n\to\infty}||e^n||_{H^1(\Omega^1)} = 0, \quad \lim_{n\to\infty}||e^{n+1/2}||_{H^1(\Omega^2)} = 0. \tag{3.20}$$

Compared with the nonoverlapping case, we just have shown that we have a better rate of convergence on the two parts that are covered only once since the weighted values converge to zero and the weights are strictly greater than 1. We will extend the geometric convergence result for Algorithm 3 (OSM-D) on two overlapping strip subdomains in the next section. Using conformal mapping and the results in this section, we can extend the result on the convergence of Algorithm 3 (OSM-D) to a more general domain, see section 3.6.

3.5 Geometric Convergence on Several Strips

We will now show the geometric convergence of Algorithm 3 (OSM-D) for several overlapping strips following F. Nataf [33] and [34]. The author of these papers uses open boundary condition for the convection-diffusion equation with an approximate factorization of the convection-diffusion operator and proves the convergence on several overlapping strips for convection-diffusion equations. Since we use a Robin boundary condition, we can find certain similarity between these two different cases. To apply the idea of [33] and [34], we will consider the following factorization of the operator $-\Delta$ in the two-dimensional case,

$$-\Delta e = -(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x})(\frac{\partial}{\partial y} - i\frac{\partial}{\partial x})e. \tag{3.21}$$

In this section, we assume the domain Ω is a unit square. We also assume zero Dirichlet boundary condition on the two boundary segments $\{(x,y)|0 \leq x \leq 1, y = 0\} \cup \{(x,y)|0 \leq x \leq 1, y = 1\}$ and zero Robin boundary condition with $\alpha = \tilde{\alpha}$ on the other two boundary segments $\{(x,y)|0 \leq y \leq 1, x = 0\} \cup \{(x,y)|0 \leq y \leq 1, x = 1\}$. We have the following simple boundary conditions on the two different sides of the artificial interfaces Γ_j,

$$\begin{aligned} e + \tilde{\alpha}e_y &= \tilde{\alpha}(e_y + \frac{e}{\tilde{\alpha}}), \quad \text{on the top side,} \\ e - \tilde{\alpha}e_y &= -\tilde{\alpha}(e_y - \frac{e}{\tilde{\alpha}}) \quad \text{on the bottom side.} \end{aligned}$$

So we have the following equation for the factorization of the operator,

$$-\Delta e = -(\frac{\partial}{\partial y} + \frac{1}{\tilde{\alpha}})(\frac{\partial}{\partial y} - \frac{1}{\tilde{\alpha}})(e) - (\frac{1}{\tilde{\alpha}^2} + \frac{\partial^2}{\partial x^2})(e). \tag{3.22}$$

We multiply (3.22) by $e_y - \frac{e}{\tilde{\alpha}}$ and integrate over the unit square. Since the following relation holds,

$$\frac{\partial(|v|^2)}{\partial y} = 2\frac{\partial v}{\partial y}\, v, \tag{3.23}$$

we have the following equation,

$$\begin{aligned} - & \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx + \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx \\ - & \int_0^1\int_0^1 \frac{1}{\tilde{\alpha}}|e_y - \frac{e}{\tilde{\alpha}}|^2\, dydx - \int_0^1\int_0^1 (\frac{e}{\tilde{\alpha}^2} + e_{xx})\,(e_y - \frac{e}{\tilde{\alpha}})\, dydx = 0. \end{aligned} \tag{3.24}$$

Similarly, multiplying (3.22) by $-(e_y + \frac{e}{\tilde{\alpha}})$ gives the following equation,

$$\begin{aligned} + & \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx - \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx \\ - & \int_0^1\int_0^1 \frac{1}{\tilde{\alpha}}|e_y + \frac{e}{\tilde{\alpha}}|^2\, dydx + \int_0^1\int_0^1 (\frac{e}{\tilde{\alpha}^2} + e_{xx})\,(e_y + \frac{e}{\tilde{\alpha}})\, dydx = 0. \end{aligned} \tag{3.25}$$

We add the two equation in (3.24) and (3.25) and apply integration by part to the last term, and obtain,

$$\begin{aligned} - & \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx + \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx \\ + & \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx - \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx \\ - & \frac{2}{\tilde{\alpha}}\int_0^1\int_0^1 (|e_y|^2\, dydx - \frac{2}{\tilde{\alpha}}\int_0^1\int_0^1 |e_x|^2)\, dydx = 0. \end{aligned}$$

Finally, we have the following relation,

$$\begin{aligned} & \frac{2}{\tilde{\alpha}}\int_0^1\int_0^1 |e_y|^2\, dydx + \frac{2}{\tilde{\alpha}}\int_0^1\int_0^1 |e_x|^2\, dydx \\ + & \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx + \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx \\ = & \frac{1}{2}\int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=0}\, dx + \frac{1}{2}\int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=1}\, dx. \end{aligned} \tag{3.26}$$

3.5.1 Nonoverlapping Strips

In this case, we can define the subdomains by,

$$\Omega(m) = \{(x,y)|0 \le x \le 1, l_{\Gamma_2(m)} \le y \le l_{\Gamma_1(m)}\} \quad m = 1, \cdots, N,$$

$$\text{with} \quad l_{\Gamma_1(m)} = l_{\Gamma_2(m+1)} \quad l_{\Gamma_2(1)} = 0 \quad \text{and} \quad l_{\Gamma_1(N)} = 1.$$

We can apply the same computation for each of the subdomains and have the following equation,

$$\begin{aligned} & \frac{2}{\tilde{\alpha}} \int_0^1 \int_{l_{\Gamma_2(m)}}^{l_{\Gamma_1(m)}} |e_y|^2 dydx + \frac{2}{\tilde{\alpha}} \int_0^1 \int_{l_{\Gamma_2(m)}}^{l_{\Gamma_1(m)}} |e_x|^2 dydx \\ + & \frac{1}{2} \int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=l_{\Gamma_1(m)}} dx + \frac{1}{2} \int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=l_{\Gamma_2(m)}} dx \\ = & \frac{1}{2} \int_0^1 |e_y - \frac{e}{\tilde{\alpha}}|^2|_{y=l_{\Gamma_2(m)}} dx + \frac{1}{2} \int_0^1 |e_y + \frac{e}{\tilde{\alpha}}|^2|_{y=l_{\Gamma_1(m)}} dx. \end{aligned} \tag{3.27}$$

To show the convergence of Algorithm 3 (OSM-D), we define the following,

$$\begin{aligned} & A_m^n(y) = \frac{1}{2} \int_0^1 |e_y^n + \frac{e^n}{\tilde{\alpha}}|^2 dx, \quad B_m^n(y) = \frac{1}{2} \int_0^1 |e_y^n - \frac{e^n}{\tilde{\alpha}}|^2 dx, \\ & E_m^n = \frac{2}{\tilde{\alpha}} \int_0^1 \int_{l_{\Gamma_2(m)}}^{l_{\Gamma_1(m)}} |e_y^n|^2 dydx + \frac{2}{\tilde{\alpha}} \int_0^1 \int_{l_{\Gamma_2(m)}}^{l_{\Gamma_1(m)}} |e_x^n|^2 dydx. \end{aligned} \tag{3.28}$$

We have the following relation from the boundary condition of Algorithm 3 (OSM-D),

$$A_m^n(l_{\Gamma_1(m)}) = A_{m+1}^{n-1}(l_{\Gamma_1(m)}) = A_{m+1}^{n-1}(l_{\Gamma_2(m+1)}), \quad m \le N-1$$

$$B_m^n(l_{\Gamma_2(m)}) = B_{m-1}^n(l_{\Gamma_2(m)}) = B_{m-1}^n(l_{\Gamma_1(m-1)}), \quad 2 \le m$$

$$A_N^n(l_{\Gamma_1(N)}) = 0, \quad \text{and} \quad B_1^n(l_{\Gamma_2(1)}) = 0.$$

Equation (3.27) can be written as,

$$\begin{aligned} E_m^n + B_m^n(l_{\Gamma_1(m)}) + A_m^n(l_{\Gamma_2(m)}) &= B_m^n(l_{\Gamma_2(m)}) + A_m^n(l_{\Gamma_1(m)}) \\ &= B_{m-1}^n(l_{\Gamma_1(m-1)}) + A_{m+1}^{n-1}(l_{\Gamma_2(m+1)}). \end{aligned}$$

Summing over m, we obtain,

$$\sum_m (E_m^n + B_m^n(l_{\Gamma_1(m)}) + A_m^n(l_{\Gamma_2(m)})) = \sum_m (B_{m-1}^n(l_{\Gamma_1(m-1)}) + A_{m+1}^{n-1}(l_{\Gamma_2(m+1)})).$$

Summing this equation over n, we have the following,

$$\begin{aligned} & \sum_n (\sum_m E_m^n) + \sum_n (\sum_m B_m^n(l_{\Gamma_1(m)})) + \sum_n (\sum_m A_m^n(l_{\Gamma_2(m)})) \\ = \; & \sum_n (\sum_m B_{m-1}^n(l_{\Gamma_1(m-1)})) + \sum_n (\sum_m A_{m+1}^{n-1}(l_{\Gamma_2(m+1)})). \end{aligned}$$

By cancellation, we finally have the following,

$$\sum_n (\sum_m E_m^n) = \sum_m A_{m+1}^0(l_{\Gamma_2(m+1)})). \tag{3.29}$$

Therefore

$$\lim_{n\to\infty} |e^n|_{H^1} = \lim_{n\to\infty} (\sum_m E_m^n) = 0. \tag{3.30}$$

Since the original boundary condition is zero, we can use Friedrichs' inequality and finally have,

$$\lim_{n\to\infty} ||e^n||_{H^1} = 0. \tag{3.31}$$

3.5.2 Several Overlapping Strips

In this case, we define the subdomains by a uniform subdomain size L and a uniform overlap of δ between neighboring subdomains,

$$\Omega_m = \{(x,y) | 0 \le x \le 1, l_{\Gamma_2(m)} \le y \le l_{\Gamma_1(m)}\} \quad m = 1, \cdots, N,$$

$$\text{with} \quad L = l_{\Gamma_1(m)} - l_{\Gamma_2(m)} \quad \delta = l_{\Gamma_1(m)} - l_{\Gamma_2(m+1)} \quad , \quad l_{\Gamma_2(1)} = 0 \quad \text{and} \quad l_{\Gamma_1(N)} = 1.$$

We define the function $G(y)$ as follows,

$$G(y) = \int_0^1 |e|^2 \, dx. \tag{3.32}$$

The function $G(y)$ has first and second derivatives given by

$$G'(y) = 2\int_0^1 e\, e_y\, dx \quad \text{and} \quad G''(y) = 2\int_0^1 (e\, e_{yy} + |e_y|^2)\, dx. \tag{3.33}$$

By integration by part, we find

$$\frac{1}{2}G'' = \int_0^1 (e\, e_{yy} + |e_y|^2)\, dx = \int_0^1 (|e_x|^2 + |e_y|^2)\, dx \geq 0.$$

We introduce H(y) as the solution of the following ordinary differential equation,

$$H'' = 0 \quad H(0) = G(0) \quad H(L) = G(L). \tag{3.34}$$

We have,

$$H(y) = G(0)(1 - \frac{1}{L}y) + \frac{G(L)}{L}y. \tag{3.35}$$

We also have the following inequality,

$$(H'' - G'') = -G'' \leq 0. \tag{3.36}$$

From this equation, we conclude that $G(y) \leq H(y)$ and we have the following inequality,

$$G(y) \leq G(0)(1 - \frac{1}{L}y) + G(L)\frac{y}{L}. \tag{3.37}$$

We will use the same notations and boundary conditions as in (3.27) and (3.28) in this section. We also have the following,

$$\begin{aligned} A_m^n(l_{\Gamma_1(m)}) &= A_{m+1}^{n-1}(l_{\Gamma_1(m)}), \quad m \leq N-1 \\ B_m^n(l_{\Gamma_2(m)}) &= B_{m-1}^n(l_{\Gamma_2(m)}), \quad 2 \leq m \\ A_N^n(l_{\Gamma_1(N)}) &= 0, \quad \text{and} \quad B_1^n(l_{\Gamma_2(1)}) = 0. \end{aligned} \tag{3.38}$$

We now define the following function in each subdomain Ω_m,

$$W_m^n(x,y) = e_y + \frac{e^n}{\tilde{\alpha}} \quad Z_m^n(x,y) = e_y - \frac{e^n}{\tilde{\alpha}}$$

Since $W_m^n(x,y)$ and $Z_m^n(x,y)$ also are harmonic with zero value on the two boundary segments $\{(x,y)|0 \le x \le 1, y = 0\} \cup \{(x,y)|0 \le x \le 1, y = 1\}$, we can apply the inequality in (3.37) and we have the following relations,

$$A_m^n(l_{\Gamma_1(m)}) \le (1 - \frac{\delta}{L})A_m^n(l_{\Gamma_2(m+1)}) + (\frac{\delta}{L})A_m^n(l_{\Gamma_1(m+1)}) \tag{3.39}$$
$$B_m^n(l_{\Gamma_2(m)}) \le (\frac{\delta}{L})B_m^n(l_{\Gamma_2(m-1)}) + (1 - \frac{\delta}{L})B_m^n(l_{\Gamma_1(m-1)}).$$

Now we want to show the following from the previous results for $0 \le j \le N-2$,

$$\begin{aligned} & \sum_{m=1}^{N} E_m^n + \sum_{i=0}^{j} (\frac{\delta}{L})^i B_{N-i}^n(l_{\Gamma_1(N-i)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \qquad (3.40) \\ \le\ & (\frac{\delta}{L})^{j+1} \sum_{m=2}^{N-1-j} B_m^n(l_{\Gamma_2(m)}) + (-\frac{\delta}{L})^{j+1} \sum_{m=1}^{N-1-j} B_{m-1}^n(l_{\Gamma_1(m)}) \\ +\ & (1 - \frac{\delta}{L}) \sum_{i=1}^{j+1} (\sum_{m=1}^{N} A_m^{n-i}(l_{\Gamma_2(m)}) (\frac{\delta}{L})^{i-1}) + (\frac{\delta}{L})^{j+1} \sum_{m=j+2}^{N-1} A_{m+1}^{n-2-j}(l_{\Gamma_1(m)}). \end{aligned}$$

We will use mathematical induction. We will begin with the case of $j = 0$.

3.5.3 Geometric Convergence

(I) j=0:

We sum (3.27) over m and use the last relation of (3.38) to obtain,

$$\begin{aligned} & \sum_{m=1}^{N} E_m^n + \sum_{m=1}^{N} B_m^n(l_{\Gamma_1(m)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \\ =\ & \sum_{m=1}^{N} B_m^n(l_{\Gamma_2(m)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_1(m)}) \\ =\ & \sum_{m=2}^{N} B_m^n(l_{\Gamma_2(m)}) + \sum_{m=1}^{N-1} A_m^n(l_{\Gamma_1(m)}). \end{aligned}$$

From the relations in (3.38), we have,

$$\begin{aligned} & \sum_{m=1}^{N} E_m^n + \sum_{m=1}^{N} B_m^n(l_{\Gamma_1(m)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \\ =\ & \sum_{m=2}^{N} B_{m-1}^n(l_{\Gamma_2(m)}) + \sum_{m=1}^{N-1} A_{m+1}^{n-1}(l_{\Gamma_1(m)}). \end{aligned}$$

Applying relation (3.39), we have,

$$\begin{aligned}
&\sum_{m=1}^{N} E_m^n + \sum_{m=1}^{N} B_m^n(l_{\Gamma_1(m)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \\
\leq\ & \sum_{m=2}^{N} (\frac{\delta}{L}) B_{m-1}^n(l_{\Gamma_2(m-1)}) + \sum_{m=2}^{N} (1-\frac{\delta}{L}) B_{m-1}^n(l_{\Gamma_1(m-1)}) \\
+\ & \sum_{m=1}^{N-1} (1-\frac{\delta}{L}) A_{m+1}^{n-1}(l_{\Gamma_2(m+1)}) + \sum_{m=1}^{N-1} (\frac{\delta}{L}) A_{m+1}^{n-1}(l_{\Gamma_1(m+1)}).
\end{aligned}$$

From the last relation of (3.38) and simplification, we obtain,

$$\begin{aligned}
&\sum_{m=1}^{N} E_m^n + B_N^n(l_{\Gamma_1(N)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \\
\leq\ & (\frac{\delta}{L}) \sum_{m=2}^{N-1} B_m^n(l_{\Gamma_2(m)}) + (-\frac{\delta}{L}) \sum_{m=1}^{N-1} B_m^n(l_{\Gamma_1(m)}) \\
+\ & (1-\frac{\delta}{L}) \sum_{m=1}^{N} A_m^{n-1}(l_{\Gamma_2(m)}) + (\frac{\delta}{L}) \sum_{m=2}^{N-1} A_m^{n-1}(l_{\Gamma_1(m)}).
\end{aligned}$$

From the relation (3.38), we have,

$$\begin{aligned}
&\sum_{m=1}^{N} E_m^n + B_N^n(l_{\Gamma_1(N)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \\
\leq\ & (\frac{\delta}{L}) \sum_{m=2}^{N-1} B_m^n(l_{\Gamma_2(m)}) + (-\frac{\delta}{L}) \sum_{m=1}^{N-1} B_{m-1}^n(l_{\Gamma_1(m)}) \\
+\ & (1-\frac{\delta}{L}) \sum_{m=1}^{N} A_m^{n-1}(l_{\Gamma_2(m)}) + (\frac{\delta}{L}) \sum_{m=2}^{N-1} A_{m+1}^{n-2}(l_{\Gamma_1(m)}).
\end{aligned}$$

(II) j to j+1:

We will now show the general step in the mathematical induction. We suppose the inequality holds for some j; we want to show it also holds with $j+1$. From the relation in (3.39) and the assumption follows,

$$\sum_{m=1}^{N} E_m^n + \sum_{i=0}^{j} (\frac{\delta}{L})^i B_{N-i}^n(l_{\Gamma_1(N-i)}) + \sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) \tag{3.41}$$

$$
\begin{aligned}
&\leq (\frac{\delta}{L})^{j+1}\sum_{m=2}^{N-1-j}((\frac{\delta}{L})B^n_{m-1}(l_{\Gamma_2(m-1)})+(1-\frac{\delta}{L})B^n_{m-1}(l_{\Gamma_1(m-1)})))\\
&+ (-\frac{\delta}{L})^{j+1}\sum_{m=1}^{N-1-j}B^n_{m-1}(l_{\Gamma_1(m)})+(1-\frac{\delta}{L})\sum_{i=1}^{j+1}(\sum_{m=1}^{N}A^{n-i}_m(l_{\Gamma_2(m)})\,(\frac{\delta}{L})^{i-1})\\
&+ (\frac{\delta}{L})^{j+1}\sum_{m=j+2}^{N-1}((1-\frac{\delta}{L})A^{n-2-j}_{m+1}(l_{\Gamma_2(m+1)})+(\frac{\delta}{L})A^{n-2-j}_{m+1}(l_{\Gamma_1(m+1)})).
\end{aligned}
$$

After simplification, we have,

$$
\begin{aligned}
&\sum_{m=1}^{N}E^n_m+\sum_{i=0}^{j}(\frac{\delta}{L})^iB^n_{N-i}(l_{\Gamma_1(N-i)})+\sum_{m=1}^{N}A^n_m(l_{\Gamma_2(m)})\\
&\leq (\frac{\delta}{L})^{j+2}\sum_{m=1}^{N-2-j}B^n_m(l_{\Gamma_2(m)})+(-\frac{\delta}{L})^{j+2}\sum_{m=1}^{N-2-j}((\frac{\delta}{L})B^n_m(l_{\Gamma_1(m)})\\
&+ (-\frac{\delta}{L})^{j+1}B^n_{N-1-j}(l_{\Gamma_1(N-1-j)})+(1-\frac{\delta}{L})\sum_{i=1}^{j+2}(\sum_{m=1}^{N}A^{n-i}_m(l_{\Gamma_2(m)})\,(\frac{\delta}{L})^{i-1})\\
&+ (\frac{\delta}{L})^{j+2}\sum_{m=j+3}^{N-1}A^{n-2-j}_m(l_{\Gamma_1(m)}).
\end{aligned}
$$

From (3.38), we have,

$$
\begin{aligned}
&\sum_{m=1}^{N}E^n_m+\sum_{i=0}^{j+1}(\frac{\delta}{L})^iB^n_{N-i}(l_{\Gamma_1(N-i)})+\sum_{m=1}^{N}A^n_m(l_{\Gamma_2(m)})\\
&\leq (\frac{\delta}{L})^{j+2}\sum_{m=1}^{N-2-j}B^n_m(l_{\Gamma_2(m)})+(-\frac{\delta}{L})^{j+2}\sum_{m=1}^{N-2-j}((\frac{\delta}{L})B^n_m(l_{\Gamma_1(m)})\\
&+ (1-\frac{\delta}{L})\sum_{i=1}^{j+2}(\sum_{m=1}^{N}A^{n-i}_m(l_{\Gamma_2(m)})\,(\frac{\delta}{L})^{i-1})+(\frac{\delta}{L})^{j+2}\sum_{m=j+3}^{N-1}A^{n-3-j}_{m+1}(l_{\Gamma_1(m)}).
\end{aligned}
$$

So we have proved the inequality in (3.40) for $j+1$.

(III) Geometric convergence:

We have the following inequality from (3.40) with $j = N-2$ and the relation (3.38),

$$
\sum_{m=1}^{N}E^n_m+\sum_{i=0}^{N-1}(\frac{\delta}{L})^iB^n_{N-i}(l_{\Gamma_1(N-i)})+\sum_{m=1}^{N}A^n_m(l_{\Gamma_2(m)}) \tag{3.42}
$$

$$\leq \quad (1-\frac{\delta}{L})\sum_{i=1}^{N-1}(\sum_{m=1}^{N} A_m^{n-i}(l_{\Gamma_2(m)})\,(\frac{\delta}{L})^{i-1}).$$

We define the following,

$$C^n = sup_{0\leq i\leq N-2}\sum_{m=1}^{N} A_m^{n-i}(l_{\Gamma_2(m)}). \tag{3.43}$$

We prove that

(i) $C^n \leq C^{n-1}$**:** From (3.42),

$$\begin{aligned}\sum_{m=1}^{N} A_m^n(l_{\Gamma_2(m)}) &\leq (1-\frac{\delta}{L})\sum_{i=1}^{N-1} C^{n-1}\,(\frac{\delta}{L})^{i-1}) \leq (1-(\frac{\delta}{L})^{N-1})C^{n-1}\\ &\leq C^{n-1} \quad \text{for} \quad 1\leq i\leq N-1.\end{aligned}$$

The following relation is obvious from the definition of C^n,

$$\sum_{m=1}^{N} A_m^{n-i}(l_{\Gamma_2(m)}) \leq C^{n-1} \quad \text{for} \quad 1\leq i\leq N-1. \tag{3.44}$$

(ii) $C^{n-1+\alpha(N-1)} \leq (1-(\frac{\delta}{L})^{N-1})^{\alpha}C^{n-1}$**:** From equation (3.42),

$$\begin{aligned}\sum_{m=1}^{N} A_m^{n+j}(l_{\Gamma_2(m)}) &\leq (1-(\frac{\delta}{L})^{N-1})C^{n-1-j}\\ &\leq (1-(\frac{\delta}{L})^{N-1})C^{n-1} \quad \text{for} \quad j\geq 0.\end{aligned}$$

From which, we have,

$$C^{n+N-2} \leq (1-(\frac{\delta}{L})^{N-1})C^{n-1} \quad \text{for} \quad n\geq N-1. \tag{3.45}$$

Finally, we have,

$$\begin{aligned}C^{n-1+\alpha(N-1)} &\leq (1-(\frac{\delta}{L})^{N-1})C^{n-1+(\alpha-1)(N-1)}\\ &\leq (1-(\frac{\delta}{L})^{N-1})^{\alpha}C^{n-1} \quad \text{for} \quad n\geq N-1.\end{aligned}$$

So we have the relation,

$$C^{n+N-1} \leq (1 - (\frac{\delta}{L})^{N-1})C^n \quad \text{for} \quad n \geq N-1. \tag{3.46}$$

From this inequality, we can see that Algorithm 3 (OSM-D) with several overlapping strip subdomains converges geometrically in a certain sense.

3.6 Convergence using a Conformal Mapping

3.6.1 Quadrilaterals and their Conformal Module

In this section, we consider a more general domain which is the image under a conformal mapping of a certain rectangular domain. We consider the conformal mapping for a bounded domain with a Jordan curve Γ as its boundary which is a union of 4 analytic arcs Γ_j $(j = 1, 2, 3, 4)$ and such that the conformal mapping is continuous over the closed domain $\overline{\Omega'} = \Omega' \cup \Gamma'$. Here Ω' is a Jordan domain in the complex (or two-dimensional Euclidean) w-plane ($w = x' + iy'$or (x', y')), and we consider a system consisting of Ω' and four distinct points w_1 $((x'_1, y'_1))$, w_2 $((x'_2, y'_2))$, w_3 $((x'_3, y'_3))$, w_4 $((x'_4, y'_4))$ in a counter-clockwise order on its boundary $\partial\Omega' = \Gamma'$. Such a domain is said to be a quadrilateral Q' (see [37, 38, 15]) and is denoted by

$$Q' := \{\Omega' | w_1(x'_1, y'_1), w_2(x'_2, y'_2), w_3(x'_3, y'_3), w_4(x'_4, y'_4)\}. \tag{3.47}$$

The *conformal module* $m(Q')$ of Q' is defined as follows. Let $\Omega = R_h$ denote a rectangle of the form

$$\Omega = R_h := \{(x, y) | 0 < x < a, 0 < y < b\} \quad h = a/b, \tag{3.48}$$

in the z-plane ($z = x + iy$). Then $m(Q')$ is the unique value of h for which Q' is conformally equivalent to the rectangular quadrilateral

$$\begin{aligned} Q_h \quad := \quad & \{\Omega = R_h | z_1 = 0 (x_1 = 0, y_1 = 0), z_2 = 1 (x_2 = 1, y_2 = 0), \\ & z_3 = 1 + ih (x_3 = 1, y_3 = h), z_4 = ih (x_4 = 0, y_4 = h)\}. \end{aligned}$$

This means, for $h = m(Q')$, there exists a unique conformal map

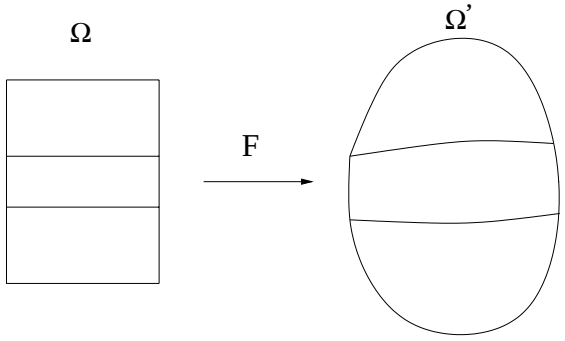

Figure 3.1: An example of the conformal mapping between a rectangular domain and a general quadrilateral

$$F : \Omega \to \Omega' \quad \text{or} \quad F(x, y) = (x'(x, y), y'(x, y)), \tag{3.49}$$

which takes the four vertices of Ω, $z_j, j = 1, 2, 3, 4$, respectively, onto the four corner points of Ω', i.e.,

$$F(z_1) = w_1, F(z_2) = w_2, F(z_3) = w_3, F(z_4) = w_4$$

Figure 3.1 shows an example of such an F. Since F is a conformal mapping, we can define F^{-1} as another conformal mapping defined by,

$$F^{-1}(w_1) = z_1, F^{-1}(w_2) = z_2, F^{-1}(w_3) = z_3, F^{-1}(w_4) = z_4$$

3.6.2 Conformal Mapping

To establish the convergence of Algorithm 3 (OSM-D) on a general quadrilateral, we need to consider several quantities on such a region. Since the results have been established for a rectangular domain, we use the conformal mapping in our analysis. The difference between two quantities corresponding to the two domains should result from the conformal mappings F and F^{-1}. We can define the conformal mapping F and F^{-1} as,

$$F(x,y) = (x'(x,y), y'(x,y)) \quad F^{-1}(x',y') = F^{-1}(x(x',y'), y(x',y')). \tag{3.50}$$

Also we define the error vector $e(x,y)$ on the rectangular domain and $e'(x',y')$ on the general quadrilateral can be written as follows,

$$\begin{aligned} e(x,y) &= e'(x'(x,y), y'(x,y)) \\ e'(x',y') &= e(x(x',y'), y(x',y')). \end{aligned}$$

The normal derivative on the general quadrilateral

The normal derivative is given by

$$\frac{\partial (e')^{\beta}}{\partial n'} = \nabla (e')^{\beta} \cdot n' \tag{3.51}$$

To know the relation between the normal derivative on the general quadrilateral, $\frac{\partial (e')^{\beta}}{\partial n'}$, and the corresponding normal derivative on the rectangular domain $\frac{\partial e^{\beta}}{\partial n}$, we need to develop a relation ∇e and $\nabla (e')$. From a routine calculation, we have

$$(\nabla (e'))^t = ((e')_{x'}, (e')_{y'}) = (e_x, e_y) \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix} = (\nabla e)^t \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix}.$$

Since F is a conformal mapping, it preserve angles. Therefore the normal direction vectors on the artificial interfaces on the rectangular domain map on to the normal

direction vectors on the artificial interfaces on the general quadrilateral. Thus, we have,

$$n' = (\frac{\partial x'}{\partial n}, \frac{\partial y'}{\partial n})/\sqrt{|\frac{\partial x'}{\partial n}|^2 + |\frac{\partial y'}{\partial n}|^2}. \tag{3.52}$$

From the Cauchy-Riemann equations, we have

$$|\frac{\partial x'}{\partial n}|^2 + |\frac{\partial y'}{\partial n}|^2 = |F'(z)|^2, \tag{3.53}$$

and therefore,

$$n' = (\frac{\partial x'}{\partial n}, \frac{\partial y'}{\partial n})/|F'(z)|. \tag{3.54}$$

From this result, we have,

$$\frac{\partial (e')^\beta}{\partial n'} = \frac{\partial e^\beta}{\partial n}/|F'(z)|. \tag{3.55}$$

3.6.3 The three factors of the $(A')^\beta_\alpha$ on the artificial interface Γ'_β

Since we consider the conformal mapping F between the rectangular domain and the general quadrilateral, there exists a C^1 mapping (path) between the artificial interface on the rectangular domain and that of the general quadrilateral. Let this C^1 mapping (path) between the standard interface of the rectangular domain Γ_β and that of the general quadrilateral Γ'_β be given by,

$$\sigma : \Gamma_\beta = \{(x, y) | 0 \leq x \leq 1, y = y_0\} \rightarrow \Gamma'_\beta \subset R^2. \tag{3.56}$$

Since the conformal mapping F is defined on the rectangular domain, we can consider $e'(x', y')$ on the artificial interface as a function of one variable t. Let $F'(\Gamma_\beta)$ be Γ'_β. Therefore, the artificial interface of the general quadrilateral Γ'_β can

be considered as the path over a certain interval $[0,1]$ and we need to review the definition of path integrals. We have,

$$\int_\sigma f \quad d\sigma = \int_a^b f(\sigma(t))|\sigma'(t)|dt \tag{3.57}$$

From the definition of the complex derivative, we have the following relation,

$$|\sigma'(t)| = |F'_\beta(z)|, \tag{3.58}$$

on the artificial interfaces. From the result above, we have,

$$\begin{aligned}
&\int_{\Gamma'_\beta} |u'|^2 dS' = \int_{\Gamma_\beta} |u|^2 |F'_\beta(z)| dS, \qquad (3.59)\\
&\int_{\Gamma'_\beta} |\frac{\partial e'}{\partial n'}|^2 dS' = \int_{\Gamma_\beta} |\frac{\partial e^\beta}{\partial n}/|F'_\beta(z)||^2 |F'_\beta(z)| dS = \int_{\Gamma_\beta} |\frac{\partial e^\beta}{\partial n}|^2/|F'_\beta(z)| dS,\\
&\int_{\Gamma'_\beta} e' \frac{\partial e'}{\partial n'} dS' = \int_{\Gamma_\beta} e \frac{\partial e^\beta}{\partial n}/|F'_\beta(z)||F'_\beta(z)| dS = \int_{\Gamma_\beta} e \frac{\partial e^\beta}{\partial n} dS.
\end{aligned}$$

The third quantity is preserved under the conformal mapping and equals the Dirichlet integral given in (3.2); we have just shown that the Dirichlet integral is invariant under the conformal mapping.

$(A')^\beta_\alpha$ on the artificial interfaces

From the previous results, we have

$$\begin{aligned}
& \frac{1}{4\tilde{\alpha}} \int_{\Gamma'_\beta} (e' + \tilde{\alpha} \frac{\partial e'}{\partial n'})^2 dS' \\
= \;& \frac{1}{4\tilde{\alpha}} \int_{\Gamma'_\beta} |\frac{\partial e'}{\partial n'}|^2 dS' + \frac{1}{2} \int_{\Gamma'_\beta} e' \frac{\partial e'}{\partial n'} dS' + \frac{\tilde{\alpha}}{4} \int_{\Gamma'_\beta} |\frac{\partial e'}{\partial n'}|^2 dS' \\
= \;& \frac{1}{4\tilde{\alpha}} \int_{\Gamma_\beta} |u|^2 |F'(z)| dS + \frac{1}{2} \int_{\Gamma_\beta} e \frac{\partial e^\beta}{\partial n} + \frac{\tilde{\alpha}}{4} \int_{\Gamma_\beta} |\frac{\partial e^\beta}{\partial n}|^2/|F'(z)| dS
\end{aligned}$$

From the property of conformal mappings, we have

$$\exists \gamma_\beta, \quad \gamma_\beta \geq 1, \quad \frac{1}{\gamma_\beta} \leq |F'_\beta(z)| \leq \gamma_\beta \quad \frac{1}{\gamma_\beta} \leq \frac{1}{|F'_\beta(z)|} \leq \gamma_\beta. \tag{3.60}$$

From this, we have,

$$\frac{1}{\gamma_\beta} A_\alpha^\beta \leq (A')_\alpha^\beta \leq \gamma_\beta A_\alpha^\beta. \tag{3.61}$$

In the case of two overlapping subdomains, we have two artificial interface, Γ_1' and Γ_2'. Therefore, we can find a uniform constant,

$$\gamma = \max\{\gamma_1, \gamma_2\}, \tag{3.62}$$

with the following property,

$$\frac{1}{\gamma} A_\alpha^1 \leq (A')_\alpha^1 \leq \gamma A_\alpha^1 \quad \frac{1}{\gamma} A_\alpha^2 \leq (A')_\alpha^1 \leq \gamma A_\alpha^2. \tag{3.63}$$

3.6.4 Convergence and Geometric Convergence on overlapping Quadrilaterals

From (3.19), we have the following results for the standard rectangular domain,

$$(E^{n+1})^2 \to 0 \quad (E^{(n+1)+1/2})^1 \to 0 \quad \text{as} \quad n \to \infty.$$

From the result of the previous section,

$$(E^{n+1})^2 = ((E')^{n+1})^2, \quad (E^{(n+1)+1/2})^1 = ((E')^{(n+1)+1/2})^1,$$

and we have,

$$((E')^{n+1})^2 \to 0 \quad ((E')^{(n+1)+1/2})^1 \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, we have the same result as in section 3.4.

From the section 3.5, we have geometric convergence of the following quantity,

$$C^n = sup_{0 \leq i \leq N-2} \sum_{m=1}^{N} A_m^{n-i}(l_{\Gamma_2(m)}), \tag{3.64}$$

with the following relation

$$C^{n+N-1} \leq (1 - (\frac{\delta}{L})^{N-1})C^n \quad \text{for} \quad n \geq N-1, \tag{3.65}$$

with N is the number of the overlapping subdomains.

We have established the following relation in the previous section,

$$\frac{1}{\gamma_\beta} A_\alpha^\beta \leq (A')_\alpha^\beta \leq \gamma_\beta A_\alpha^\beta. \tag{3.66}$$

Since we have N overlapping subdomains in the general cases, we have

$$\gamma = \max\{\gamma_1, \cdots, \gamma_N\}, \tag{3.67}$$

with

$$\frac{1}{\gamma} A_\alpha^k \leq (A')_\alpha^k \leq \gamma A_\alpha^k \quad \forall k = 1, \cdots, N. \tag{3.68}$$

Because of the following relation,

$$A_m^{n-i}(l_{\Gamma_2(m)}) = G_m^{n-i}(l_{\Gamma_2(m)}), \tag{3.69}$$

we have

$$\frac{1}{\gamma} C^\beta \leq (C')^\beta \leq \gamma C^\beta. \tag{3.70}$$

Therefore,

$$\begin{aligned}
(C')^{n+N-1} &\leq \gamma C^{n+N-1} \\
&\leq \gamma(1 - (\frac{\delta}{L})^{N-1})C^n \quad \text{for} \quad n \geq N-1 \\
&\leq (\gamma)^2(1 - (\frac{\delta}{L})^{N-1})(C')^n \quad \text{for} \quad n \geq N-1.
\end{aligned}$$

From this inequality, we see that Algorithm 3 (OSM-D) converges geometrically in the general overlapping quadrilaterals if

$$(\gamma)^2(1 - (\frac{\delta}{L})^{N-1}) < 1. \tag{3.71}$$

We have $\gamma \geq 1$ and the value depends on the behavior of the conformal mapping F on the artificial interfaces Γ_β. With a certain overlapping size δ, the value $(1-(\frac{\delta}{L})^{N-1}) < 1$. Also a larger overlap will make the value $(1-(\frac{\delta}{L})^{N-1})$ smaller and the value of $(\gamma)^2(1-(\frac{\delta}{L})^{N-1})$ decreases. Therefore, given a value of γ, there exist a certain value δ_0 which makes,

$$(\gamma)^2(1-(\frac{\tilde{\delta}}{L})^{N-1}) < 1, \quad \delta_0 \leq \tilde{\delta} \leq L, \tag{3.72}$$

hold. Therefore, this result on the geometric convergence of Algorithm 3 (OSM-D) depends on the conformal mapping between a rectangular domain and the general domain as well as on the overlapping size. With sufficient overlapping size, Algorithm 3 (OSM-D) has geometric convergence in the general quadrilateral and multiple strip case.

Chapter 4

Algebraic Properties of Algorithm 3 (OSM-D)

4.1 Nonsymmetry of Algorithm 3 (OSM-D)

4.1.1 The symmetry of the fractional steps of Algorithm 2 (OSM-C)

To establish the symmetry of the symmetric multiplicative Schwarz variant of Algorithm 2, it is enough to check each step. From formula (2.15), we see that the mapping is affine because of the constant term. So it is sufficient to check the linear part of (2.15) which can be computed by subtracting the value of the fractional step corresponding to a zero input. We get,

$$M_j = P_j^c - P_j A_j^+ A_j^c. \tag{4.1}$$

The symmetry of this problem is defined with respect to the A-norm: $(u, v)_A = u^t A v$. What we want to check is the following property,

$$u^t A M_j v = (M_j u)^t A v.$$

We can rewrite (4.1) as,

$$\begin{aligned}
M_j &= I - P_j - P_j A_j^+ A + P_j A_j^+ A_j, \text{ since } \quad A_j^+ = (R_j)^t (B_j)^{-1} (R_j), \qquad (4.2)\\
&= I - P_j - P_j (R_j)^t (B_j)^{-1} (R_j) A + P_j (R_j)^t (B_j)^{-1} (R_j) A_j,\\
&\quad \text{with} \quad P_j = (R_j)^t R_j,\\
&= I - P_j - ((R_j)^t R_j)(R_j)^t (B_j)^{-1} (R_j) A + ((R_j)^t R_j)(R_j)^t (B_j)^{-1} (R_j) A_j,\\
&\quad \text{because of} \quad (R_j)^t R_j (R_j)^t = (R_j)^t,\\
&= I - P_j - (R_j)^t (B_j)^{-1} (R_j) A + (R_j)^t (B_j)^{-1} (R_j) A_j,\\
&\quad \text{from the two identities} \quad A_j = A_j (R_j)^t (R_j) \quad \text{and} \quad B_j = R_j A_j (R_j)^t,\\
&= I - P_j - (R_j)^t (B_j)^{-1} (R_j) A + (R_j)^t (B_j)^{-1} (B_j)(R_j),\\
&\quad \text{and since the second and fourth terms are identically the same,}\\
&= I - (R_j)^t (B_j)^{-1} (R_j) A.
\end{aligned}$$

Since

$$M_j = I - (R_j)^t (B_j)^{-1} (R_j) A, \qquad (4.3)$$

the relation $AM_j = (M_j)^t A$ is clear.

4.1.2 The nonsymmetry of the fractional steps of Algorithm 3 (OSM-D)

The same idea can be applied to (2.29) to provide a formula for Algorithm 3 (OSM-D),

$$\tilde{M}_j = \tilde{P}_j^c - \tilde{P}_j \tilde{R} A_j^+ \tilde{R}^t \tilde{P}_j^c \tilde{\Lambda}. \qquad (4.4)$$

Equation in (4.4) can be rewritten as,

$$\tilde{M}_j = \tilde{I} - \tilde{P}_j - \tilde{P}_j \tilde{R} A_j^+ \tilde{R}^t \tilde{\Lambda} + \tilde{P}_j \tilde{R} A_j^+ \tilde{R}^t \tilde{P}_j \tilde{\Lambda}. \qquad (4.5)$$

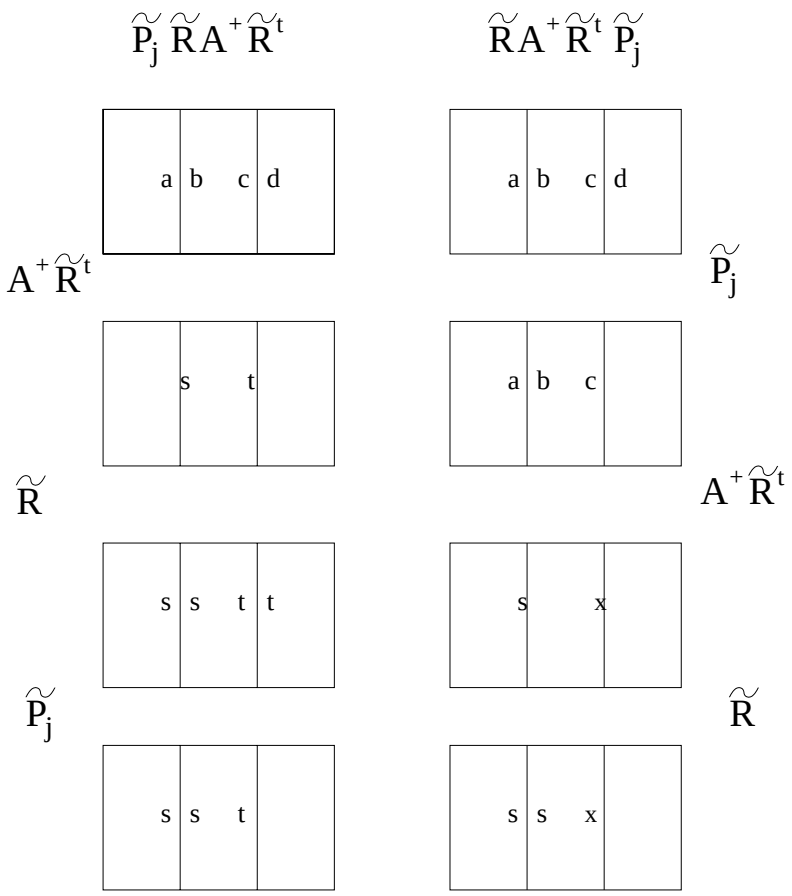

Figure 4.1: An illustration of nonsymmetry of Algorithm 3 (OSM-D)

We can first try the same computation as (4.2) to check for symmetry of $\tilde{\Lambda}\tilde{M}_j$. It is clear that $\tilde{I}$ and $\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t\tilde{P}_j\tilde{\Lambda}$ satisfy the symmetry relation. Since $\tilde{\Lambda}$ is defined as a partitioned matrix and the following equation holds,

$$\begin{pmatrix} \tilde{I} & \tilde{0} \\ \tilde{0} & \tilde{0} \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{0} \\ \tilde{0} & \tilde{B} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{0} \\ \tilde{0} & \tilde{B} \end{pmatrix} \begin{pmatrix} \tilde{I} & \tilde{0} \\ \tilde{0} & \tilde{0} \end{pmatrix}, \tag{4.6}$$

it is true that $\tilde{\Lambda}\tilde{P}_j = \tilde{P}_j\tilde{\Lambda}$. Since we have checked the symmetry of the three terms, it is now only necessary to check the symmetry condition for $\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t\tilde{\Lambda}$. The

symmetry of $\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t\tilde{\Lambda}$ would mean that

$$\tilde{\Lambda}\tilde{R}A_j^+\tilde{R}^t\tilde{P}_j\tilde{\Lambda} = \tilde{\Lambda}\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t\tilde{\Lambda}. \tag{4.7}$$

Comparing $\tilde{R}A_j^+\tilde{R}^t\tilde{P}_j$ and $\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t$, we realize that these cannot be same because of the discontinuous artificial boundary of the subdomain. Since we allow a discontinuity on the artificial interfaces in Algorithm 3 (OSM-D), we have multiple values which contribute to the matrix computation in every cycle. The nonsymmetry of Algorithm 3 (OSM-D) arises from the handling of the data on the artificial interfaces. We recall that since we consider overlapping subregions and two neighboring subregions have a common atomic subregion, each subregion Ω_j has two kinds of interfaces, the continuous artificial interfaces and the discontinuous artificial interfaces, in the j-th fractional step which updates the data on that subregion.

We now consider only the discontinuous artificial interfaces which is Γ_j, the artificial interface of the subregion Ω_j. Since Algorithm 3 (OSM-D) allows different values on the discontinuous artificial interfaces, we can again divide the discontinuous artificial interfaces into two kinds, which is the interface which is a subset of $\Omega_j \cup \partial\Omega$ and that which is a subset of the complement of Ω_j. We call the first **inside interfaces** and the others **outside interfaces**. For $\tilde{R}A_j^+\tilde{R}^t\tilde{P}_j$, the $\tilde{P}_j$ takes only data on the inside boundary, $\tilde{R}^t$ uses only this data, A_j^+ restricts, solves, and extends, and $\tilde{R}$ copies the result on both inside and the outside interfaces. However in $\tilde{P}_j\tilde{R}A_j^+\tilde{R}^t$, $\tilde{R}^t$ combines the data on both interfaces, A_j^+ restricts, solves, and extends, $\tilde{R}$ copies the result on both interfaces, and $\tilde{P}_j$ takes only the result of the inside interfaces. So the two results are not same. The linear part of Algorithm 3 (OSM-D) is not symmetric in general.

4.2 The Matrix of Algorithm 3 (OSM-D) of Unit Square

In this section, we will consider the structure of $\tilde{\Lambda}$. We consider the general matrix of Algorithm 3 on a unit square and $n \times n$ overlapping subdomains. With a uniform size on each overlapping subdomain, we have several different types of atomic subregions; for example, in the case of the nine overlapping subdomains with different α and $\tilde{\alpha}$, we have 25 atomic subregions of 16 different types.

A partitioned matrix $\tilde{\Lambda}$ is defined by,

$$\tilde{\Lambda} = diag\left(\begin{array}{ccc} \tilde{B}_1(\alpha,\tilde{\alpha}) & \cdots & \tilde{B}_n(\alpha,\tilde{\alpha}) \end{array} \right),$$

where n = the number of atomic subdomains with

$$B_j(\alpha,\tilde{\alpha}) = \begin{pmatrix} M(\alpha,\tilde{\alpha}) & K(\alpha,\tilde{\alpha}) & 0 & & \cdots & 0 \\ K(\alpha,\tilde{\alpha}) & L(\alpha,\tilde{\alpha}) & K(\alpha,\tilde{\alpha}) & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & K(\alpha,\tilde{\alpha}) & L(\alpha,\tilde{\alpha}) & K(\alpha,\tilde{\alpha}) \\ 0 & \cdots & & 0 & K(\alpha,\tilde{\alpha}) & M(\alpha,\tilde{\alpha}) \end{pmatrix},$$

and with

$$M(\alpha,\tilde{\alpha}) = \begin{pmatrix} M_c(\alpha,\tilde{\alpha}) & M_o(\alpha,\tilde{\alpha}) & 0 & & \cdots & 0 \\ M_o(\alpha,\tilde{\alpha}) & M_d(\alpha,\tilde{\alpha}) & M_o(\alpha,\tilde{\alpha}) & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & M_o(\alpha,\tilde{\alpha}) & M_d(\alpha,\tilde{\alpha}) & M_o(\alpha,\tilde{\alpha}) \\ 0 & \cdots & & 0 & M_o(\alpha,\tilde{\alpha}) & M_c(\alpha,\tilde{\alpha}) \end{pmatrix},$$

$$L(\alpha,\tilde{\alpha}) = \begin{pmatrix} M_d(\alpha,\tilde{\alpha}) & -1 & 0 & & & \cdots & 0 \\ -1 & 4 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & -1 & 4 & -1 \\ 0 & \cdots & & & 0 & -1 & M_d(\alpha,\tilde{\alpha}) \end{pmatrix},$$

and

$$K(\alpha,\tilde{\alpha}) = \begin{pmatrix} M_o(\alpha,\tilde{\alpha}) & & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & & M_o(\alpha,\tilde{\alpha}) \end{pmatrix}. \tag{4.8}$$

Here,

$$M_c(\alpha,\tilde{\alpha}) = \begin{cases} 1 \\ 1+\frac{1}{\alpha}\frac{2}{3}h \\ 1+\frac{1}{\tilde{\alpha}}\frac{2}{3}h \\ 1-\frac{1}{\tilde{\alpha}}\frac{2}{3}h \\ 1+\frac{1}{\alpha}\frac{1}{3}h+\frac{1}{\tilde{\alpha}}\frac{1}{3}h \\ 1+\frac{1}{\alpha}\frac{1}{3}h-\frac{1}{\tilde{\alpha}}\frac{1}{3}h \end{cases} \qquad M_o(\alpha,\tilde{\alpha}) = \begin{cases} -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h \\ -\frac{1}{2}+\frac{1}{\tilde{\alpha}}\frac{1}{6}h \\ -\frac{1}{2}-\frac{1}{\tilde{\alpha}}\frac{1}{6}h \end{cases}$$

$$M_d(\alpha,\tilde{\alpha}) = \begin{cases} 2+\frac{1}{\alpha}\frac{2}{3}h \\ 2+\frac{1}{\tilde{\alpha}}\frac{2}{3}h \\ 2-\frac{1}{\tilde{\alpha}}\frac{2}{3}h \end{cases}$$

The different combinations of subblock matrices produce different partitioned matrices for the different atomic subregions.

4.3 Splitting Theory of Algorithm 3 (OSM-D)

4.3.1 Basic Theory of Splitting

There are many papers on splittings of matrices, including [36, 3, 47]. Let us consider the iterative solution of the linear equation system

$$Ax = b, \tag{4.9}$$

where the square matrix A is nonsingular with real vectors x, b. Many iterative methods for the linear system considered can be formulated by means of a suitable splitting of the matrix A,

$$A = M - N \quad \text{where} \quad M \quad \text{is nonsingular}, \tag{4.10}$$

and the iterative solution x^n is generated, as follows,

$$Mx^{n+1} = Nx^n + b, \quad n = 1, \cdots, \tag{4.11}$$

or equivalently

$$x^{n+1} = M^{-1}Nx^n + M^{-1}b, \quad n = 1, \cdots, \tag{4.12}$$

with a given initial vector x^0.

Theorem 4.1 *(The Fundamental Theorem of Linear Iterative Methods) Let H be an $N \times N$ matrix and assume that the equation $x = Hx + d$ has a unique solution x^*. Then the iterates (4.12) with $H = M^{-1}N$ converge to x^* for any x^0 if and only if $\rho(H) < 1$ where $\rho(H)$ is the spectral radius of H.*

Proof See Theorem 7.1.1. in [36].

For any matrix $A = (a_{ij}) \in \mathcal{R}^{n\times n}$, we define the matrix $|A| = (|a_{ij}|)$ and its comparison matrix $< A >= (\alpha_{ij})$ by

$$\alpha_{ii} = |a_{ii}|, \quad \alpha_{ij} = -|a_{ij}|, \, i \neq j.$$

Definition 4.1 *For a nonsingular matrix A the decomposition $A = M - N$ is called:*

(a) a regular splitting of A if $M^{-1} \geq 0$ (nonnegative) and $N \geq 0$;

(b) a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$;

(c) a weak regular splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ or $NM^{-1} \geq 0$;

(d) a P-regular splitting of A if $M^T + N \succ 0$ (positive definite);

(e) a strong P-regular splitting of $A \succ 0$ if $N \succeq 0$ (semi positive definite);

(f) a M-splitting of A if M is an M-matrix and $N \geq 0$;

(g) a H-splitting of A if $< M > - |N|$ is an M-matrix;

(h) a H-compatible splitting of A if $< A > = < M > - |N|$;

We have the following theorem regarding the uniqueness of the splitting of a linear iteration matrix.

Theorem 4.2 *Let A and T be square matrices such that A and $I - T$ are nonsingular. Then, there exists a unique pair of matrices B, C, such that B is nonsingular, $T = B^{-1}C$, and $A = B - C$. The matrices are $B = A(I - T)^{-1}$ and $C = B - A$.*

Proof See [3].

Theorem 4.3 *For a nonsingular matrix A,*

(a) any regular splitting of A is a nonnegative splitting of A;

(b) any nonnegative splitting of A is a weak regular splitting of A;

(c) any strong P-regular splitting of $A \succ 0$ is a P-regular splitting of A;

(d) any M-splitting of A is a regular splitting of A;

(e) any M-splitting of M-matrix A is an H-splitting of A and also as H-compatible splitting of A;

(f) any H-compatible splitting of H-matrix A is an H-splitting of A.

Proof Easy or see [14] for (d), (e) and (f).

Theorem 4.4 *Let $A = M - N$ be a splitting.*

(a) If the splitting is regular or weak regular, then

$$\rho(M^{-1}N) < 1 \quad \textit{if and only if} \quad A^{-1} \geq 0;$$

(b) If the splitting of the symmetric matrix A is P-regular, then

$$\rho(M^{-1}N) < 1 \quad \textit{if and only if} \quad A \succ 0;$$

(c) If the splitting is a M-splitting, then

$$\rho(M^{-1}N) < 1 \quad \textit{if and only if } A \textit{ is an } M\textit{-matrix;}$$

(d) If the splitting is a H-splitting, then

$$\rho(M^{-1}N) \leq \rho(< M >^{-1} |N|) < 1.$$

Proof See [36, 14].

We need to define the following two norms for the nonnegative or positive definite matrices.

Definition 4.2 *(a) $||\cdot||_A$ (A-norm): For a positive definite matrix $A \succ 0$,*

$$||B||_A = ||A^{1/2}BA^{-1/2}||$$

(b) $||\cdot||_w$ (Weighted max norm): The weighted max norm for the vector y is defined as,

$$||y||_w = \max_{j=1,\cdots,n} |\frac{y_j}{w_j}| \quad w > 0,$$

and the weighted max norm for the matrix B is given by

$$||B||_w = \sup_{||x||_w=1} ||Bx||_w.$$

We can find more detailed explanation and several theorems on the A-norm and the weighted max norm in [45]. We will introduce the following basic properties of the two norms.

Theorem 4.5 *(a) Let $A \succ 0$ and H be a real square matrix. Then $A \succ H^T AH$ if and only if $||H||_A < 1$.*

(b) Let B be a real square matrix and w be a real vector with $w > 0$, and let $\gamma > 0$ be such that

$$|B|w \leq \gamma w.$$

Then, $||B||_w \leq \gamma$. The inequality is strict if

$$|B|w < \gamma w.$$

Proof See [45]. From the definition above, we have the following comparison theorem.

Theorem 4.6 *Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings.*

(a) If the two splittings are regular splittings with $A^{-1} \geq 0$ and $M_1^{-1} \leq M_2^{-1}$, then

$$\rho(M_1^{-1}N) \leq \rho(M_2^{-1}N) < 1.$$

The inequality is strict if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$.

(b) If the two splittings are nonnegative splittings with $A^{-1} \geq 0$ and $M_1^{-1} \leq M_2^{-1}$, then

$$\rho(M_1^{-1}N) \leq \rho(M_2^{-1}N) < 1.$$

The inequality is strict if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$.

(c) If the two splittings are weak splittings with $A^{-1} \geq 0$ *and* $M_1^{-1} \leq M_2^{-1}$*, then*

$$||M_1^{-1}N||_w \leq ||M_2^{-1}N||_w < 1.$$

The inequality is strict if $A^{-1} > 0$ *and* $M_1^{-1} > M_2^{-1}$.

(d) If the two splittings are strong P-regular splittings with $A \succ 0$ *and* $0 \preceq N_1 \preceq N_2$*, then*

$$||M_1^{-1}N||_A = \rho(M_1^{-1}N) \leq \rho(M_2^{-1}N) = ||M_2^{-1}N||_A < 1.$$

The inequality is strict if $0 \preceq N_1 \prec N_2$.

Proof See [3, 47].

4.3.2 Splitting theory for Multiplicative Schwarz Methods

To define a splitting of the multiplicative Schwarz methods, we need to define the following concepts of multisplittings which has been discussed in several recent papers including [32].

Definition 4.3 *A multisplitting of a square matrix* A *is a collection of triples of matrices* $(M_k, N_k, E_k), k = 1, \cdots, n$*, satisfying:*

(a) $A = M_k - N_k, k = 1, \cdots, n$,

(b) M_k *is nonsingular for* $k = 1, \cdots, n$,

(c) $E_k, k = 1, \cdots, n$*, are diagonal matrices with nonnegative entries which satisfy,*

$$\sum_{k=1}^{n} E_k \geq I.$$

The iteration to solve $Ax = b$ *is defined by*

$$x^{i+1} = \sum_{k=1}^{n} E_k (M_k^{-1} N_k x^i + M^{-1} b).$$

Using the matrices

$$H = \sum_{k=1}^{n} E_k M_k^{-1} N_k, \quad G = \sum_{k=1}^{n} E_k M^{-1},$$

we can express, if G is nonsingular, the multisplitting $(M_k, N_k, E_k)_{k=1}^m$ as a (single) splitting $(G^{-1}, G^{-1}H)$ of A.

In this section, we will review part of the convergence theory in [3]. The classical multiplicative Schwarz is defined as a stationary iteration with given initial data x^0,

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \cdots,$$

where

$$T = (I - P_p)(I - P_{p-1}) \cdots (I - P) = \Pi_{i=p}^{1} (I - P_i)$$

and c is a certain vector. We also define,

$$P_i = R_i^T (R_i A R_i^T)^{-1} R_i A.$$

Therefore to define the classical multiplicative Schwarz methods in terms of a multisplitting, we need the following expression of each single splitting $A = M_i - N_i$,

$$E_i M_i^{-1} = R_i^T (R_i A R_i^T)^{-1} R_i$$

where

$$E_i = R_i^T R_i.$$

The general restriction matrix R_i for the i-th subdomain is defined as the matrix whose rows are rows of the identity matrix satisfying the following relation,

$$R_i = [I_i|0]\pi_i,$$

with I_i the identity matrix for a smaller real space and π_i a permutation matrix on that space.

We also have the following expression for the diagonal matrix E_i,

$$E_i = \pi_i^T \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} \pi_i.$$

The given matrix A can be partitioned with A_i a $n_i \times n_i$ principal submatrix corresponding the i-th subdomain of A:

$$\pi_i A \pi_i^T = \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix},$$

where A_i^c is the complementary principal submatrix of A_i.

Let A be a symmetric positive definite matrix (or a general matrix which has a positive definite principal submatrix). For each $i = 1, \cdots, p$, we construct matrices M_i as follows,

$$M_i = \pi_i^T \begin{pmatrix} A_i & 0 \\ 0 & A_i^c \end{pmatrix} \pi_i.$$

We then have,

$$\begin{aligned} E_i M_i^{-1} &= R_i^T R_i \pi_i^T \begin{pmatrix} A_i^{-1} & 0 \\ 0 & (A_i^c)^{-1} \end{pmatrix} \pi_i = R_i^T [I_i|0] \pi_i \pi_i^T \begin{pmatrix} A_i^{-1} & 0 \\ 0 & (A_i^c)^{-1} \end{pmatrix} \pi_i \\ &= R_i^T [I_i|0] \begin{pmatrix} A_i^{-1} & 0 \\ 0 & (A_i^c)^{-1} \end{pmatrix} \pi_i = R_i^T [A_i|0] \pi_i = R_i^T A_i [I_i|0] \pi_i \\ &= R_i^T A_i R_i. \end{aligned}$$

If we replace A_i^c with any nonsingular matrix C (or a more general matrix which is singular but for which we can define a generalized inverse), we obtain

$$\tilde{M}_i = \pi_i^T \begin{pmatrix} A_i & 0 \\ 0 & C \end{pmatrix} \pi_i.$$

We then have the same process since

$$\begin{aligned} E_i \tilde{M}_i^{-1} &= R_i^T R_i \pi_i^T \begin{pmatrix} A_i^{-1} & 0 \\ 0 & C^+ \end{pmatrix} \pi_i = R_i^T [I_i|0] \pi_i \pi_i^T \begin{pmatrix} A_i^{-1} & 0 \\ 0 & C^+ \end{pmatrix} \pi_i \\ &= R_i^T [I_i|0] \begin{pmatrix} A_i^{-1} & 0 \\ 0 & C^+ \end{pmatrix} \pi_i = R_i^T [A_i|0] \pi_i = R_i^T A_i [I_i|0] \pi_i \\ &= R_i^T A_i R_i. \end{aligned}$$

Let the real vectors x and y be defined by

$$y = (I - E_i M_i^{-1} A)x.$$

with

$$x = \pi_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \pi_i \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We then have that

$$y_2 = x_2.$$

We have the following process for the general partitioned matrix with an invertible A_i,

$$\begin{aligned} & \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix} \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} \\ = \;& \pi_i A \pi_i^T \pi_i y = \pi_i A y = \pi_i A (I - E_i M_i^{-1} A) x \\ = \;& \pi_i A (I - \pi_i^T \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} \pi_i \pi_i^T \begin{pmatrix} A_i^{-1} & 0 \\ 0 & (A_i^c)^{-1} \end{pmatrix} \pi_i \pi_i^T \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix} \pi_i) \pi_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ = \;& \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix} (I - \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_i^{-1} & 0 \\ 0 & (A_i^c)^{-1} \end{pmatrix} \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix}(I - \begin{pmatrix} A_i^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} A_i & K_i \\ K_i^T & A_i^c \end{pmatrix}(I - \begin{pmatrix} I_i & A_i^{-1}K_i \\ 0 & 0 \end{pmatrix})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & -K_i^T A_i^{-1} K_i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -K_i^T A_i^{-1} K_i x_2 \end{pmatrix}.
\end{aligned}
$$

Therefore, we have,

$$A_i y_1 = -K_i x_2.$$

If we replace A_i^c with any nonsingular matrix $\tilde{A}_i^c$ (or a more general matrix which is singular but for which we can define a generalized inverse) such that

$$\pi_i \tilde{A} \pi_i^T = \begin{pmatrix} \tilde{A}_i & \tilde{K}_i \\ \tilde{K}_i^T & \tilde{A}_i^c \end{pmatrix},$$

then

$$\begin{pmatrix} \tilde{A}_i & \tilde{K}_i \\ \tilde{K}_i^T & \tilde{A}_i^c \end{pmatrix}\begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde{K}_i^T \tilde{A}_i^{-1} \tilde{K}_i x_2 \end{pmatrix}.$$

Therefore, we have,

$$\tilde{A}_i y_1 = -\tilde{K}_i x_2.$$

From this result, we have the following,

Theorem 4.7 *Let A be a symmetric positive definite matrix. Let x, y be such that,*

$$y = (I - E_i M_i^{-1} A)x.$$

Then the following identity holds:

$$||y||_A^2 - ||x||_A^2 = -(y - x)^T E_i A E_i (y - x) \leq 0.$$

Proof See [3].

From the previous computation, we can revise the theorem above to obtain,

Theorem 4.8 *Let $\tilde{A}$ be a general matrix which has a nonsingular principal submatrix $\tilde{A}_i$ such that*

$$\tilde{A}_i = E_i \tilde{A} E_i.$$

Let x, y satisfy,

$$y = (I - E_i \tilde{M}_i^{-1} A)x.$$

Then the following identity holds:

$$y^T \tilde{A} y - x^T \tilde{A} x = -(y - x)^T E_i \tilde{A} E_i (y - x).$$

Theorem 4.9 *Let A be a symmetric positive definite matrix. Then the multiplicative Schwarz iteration converges to the solution of $Ax = b$ for any choice of the initial guess x^0. In fact, we have*

$$\rho(T) \leq ||T||_A < 1.$$

There exists a unique splitting $A = B - C$ such that $T = B^{-1}C$, and this splitting is P-regular.

Proof See [3].

We also have the following theorem.

Theorem 4.10 *Let A be a symmetric positive definite matrix. Let x, y be defined by*

$$y = (I - E_i \tilde{M}_i^{-1} A)x,$$

where $\tilde{M}$ is defined by

$$\tilde{M}_i = \pi_i^T \begin{pmatrix} \tilde{A}_i & 0 \\ 0 & A_i^c \end{pmatrix} \pi_i,$$

and $\tilde{A}_i$ satisfies,

$$A_i = \tilde{A}_i - (\tilde{A}_i - A_i) \quad with \quad \tilde{A}_i^T + \tilde{A}_i - A_i \succ 0.$$

Then the following identity holds:

$$||y||_A^2 - ||x||_A^2 = -(y-x)^T E_i(\tilde{M}_i^T + \tilde{M}_i - A)E_i(y-x) \leq 0.$$

Proof See [3].

We have the following convergence theorem,

Theorem 4.11 *Let A be a symmetric positive definite matrix. Then the multiplicative Schwarz iteration with the iteration matrix,*

$$\tilde{T} = (I - E_p\tilde{M}_p^{-1}A)(I - E_{p-1}\tilde{M}_{p-1}^{-1}A)\cdots(I - E_1\tilde{M}_1^{-1}A),$$

with same condition on $\tilde{M}$ and $\tilde{A}_i$ as in Theorem 4.10, converges to the solution of $Ax = b$ for any choice of the initial guess x^0. In fact, we have

$$\rho(\tilde{T}) \leq ||\tilde{T}||_A < 1.$$

There exists a unique splitting $A = B - C$ such that $\tilde{T} = B^{-1}C$, and this splitting is P-regular.

Proof See [3].

4.3.3 Splitting theory for Algorithm 3 (OSM-D)

We will consider Algorithm 3 (OSM-D) with $\alpha = \tilde{\alpha}$. The original matrix A of Algorithm 3 (OSM-D) is

$$A = \begin{pmatrix} M & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ 0 & K & L & K & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K & 0 \\ 0 & \cdots & & & 0 & K & L & K \\ 0 & \cdots & & & & 0 & K & M \end{pmatrix}, \tag{4.13}$$

where

$$M = \begin{pmatrix} 1+\frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & 0 & & \cdots & 0 \\ -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & 2+\frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & 2+\frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h \\ 0 & \cdots & & 0 & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & 1+\frac{1}{\alpha}\frac{2}{3}h \end{pmatrix},$$

$$L = \begin{pmatrix} 2+\frac{1}{\alpha}\frac{2}{3}h & -1 & 0 & & & \cdots & 0 \\ -1 & 4 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & -1 & 4 & -1 \\ 0 & \cdots & & & 0 & -1 & 2+\frac{1}{\alpha}\frac{2}{3}h \end{pmatrix},$$

$$K = \begin{pmatrix} -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h \end{pmatrix}.$$

From this structure, we have the following properties of A.

- A is strictly diagonally dominant and positive definite.
- If $\beta = \frac{h}{\alpha} \leq 3$, then A satisfies the sign condition and is a M-matrix.
- If $\beta = \frac{h}{\alpha} \geq 3$, A is a H-matrix.

Since the local matrix B_i for the i-th fractional step has the same structure as A with a much smaller size, B_i is not same as A_i. The condition for Theorem 4.11 is given by the following computation,

$$
\begin{aligned}
B_i^T + B_i - A_i &= 2B_i - A_i \\
&= \begin{pmatrix}
M & K & 0 & & & \cdots & 0 \\
K & L & K & 0 & & \cdots & 0 \\
0 & K & L & K & 0 & \cdots & 0 \\
\vdots & & & \ddots & & & \vdots \\
0 & \cdots & & 0 & K & L & K & 0 \\
0 & \cdots & & & 0 & K & L & K \\
0 & \cdots & & & & 0 & K & N
\end{pmatrix},
\end{aligned}
$$

with

$$
N = \begin{pmatrix}
\frac{2h}{3\alpha} & \frac{h}{3\alpha} & 0 & & & \cdots & 0 \\
\frac{h}{3\alpha} & \frac{4h}{3\alpha} & \frac{h}{3\alpha} & 0 & & \cdots & 0 \\
0 & \frac{h}{3\alpha} & \frac{4h}{3\alpha} & \frac{h}{3\alpha} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & & \vdots \\
0 & \cdots & & 0 & \frac{h}{3\alpha} & \frac{4h}{3\alpha} & \frac{h}{3\alpha} \\
0 & \cdots & & & 0 & \frac{h}{3\alpha} & \frac{2h}{3\alpha}
\end{pmatrix}.
$$

If we have the condition $\beta = \frac{h}{\alpha} \geq 3/2$, then

$$B_i^T + B_i - A_i = 2B_i - A_i \succ 0.$$

This is the sufficient condition for the convergence of Algorithm 3 (OSM-D). With this condition Algorithm 3 (OSM-D) converges according to Theorem 4.10.

Theorem 4.12 *Algorithm 3 (OSM-D) converges in two overlapping subdomain with* $\frac{h}{\alpha} \geq 3/2$.

Chapter 5

Lagrange Multiplier Formulation

5.1 Lagrange Multipliers for Two Overlapping Subdomains

We will now consider the relation between the Lagrange multipliers λ_1^n and $\lambda_2^{n+1/2}$ and the fractional solutions $\tilde{u}_1^{n+1/2}$ and $\tilde{u}_2^{n+1}$ for two overlapping subdomains. From the relations between these quantities given in chapter 2, we have the following,

$$\begin{aligned}
\lambda_1^{n+1/2} &= I^1(f^1 - B^1 u^1_{n+1/2}) = I^1(f^1 - B^1 \tilde{R}^1 u_1^{n+1/2}) \\
&= I^1 f^1 - I^1 B^1 \tilde{R}^1 (B_1)^{-1} f_1 - I^1 B^1 \tilde{R}^1 (B_1)^{-1} (I_1)^T \lambda_2^n, \\
\lambda_2^{n+1} &= I^2(f^2 - B^2 u^2_{n+1}) = I^2(f^2 - B^2 \tilde{R}^2 u^{n+1})_2) \\
&= I^2 f^2 - I^2 B^2 \tilde{R}^2 (B_2)^{-1} f_2 - I^2 B^2 \tilde{R}^2 (B_2)^{-1} (I_2)^T \lambda_1^{n+1/2},
\end{aligned}$$

where

$$\tilde{R}^1 = \begin{pmatrix} I^1 & 0 \end{pmatrix}, \quad \tilde{R}^2 = \begin{pmatrix} 0 & I^2 \end{pmatrix}.$$

Eliminating the superscripts of the fractional steps, this can be written as the following system,

$$\begin{pmatrix} \lambda_1^{new} \\ \lambda_2^{new} \end{pmatrix} = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} + \begin{pmatrix} 0 & -\Lambda^1 \\ -\Lambda^2 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{old} \\ \lambda_2^{old} \end{pmatrix}, \tag{5.1}$$

where

$$\begin{aligned} H^1 &= I^1 f^1 - I^1 B^1 \tilde{R}^1 (B_1)^{-1} f_1, \quad H^2 = I^2 f^2 - I^2 B^2 \tilde{R}^2 (B_2)^{-1} f_2 \\ \Lambda^1 &= I^1 B^1 \tilde{R}^1 (B_1)^{-1} (I_1)^T, \quad \Lambda^2 = I^2 B^2 \tilde{R}^2 (B_2)^{-1} (I_2)^T. \end{aligned}$$

The convergence of (5.1) depends on the spectral radius of the following matrix,

$$\Lambda = \begin{pmatrix} 0 & -\Lambda^1 \\ -\Lambda^2 & 0 \end{pmatrix}. \tag{5.2}$$

Let λ be an eigenvalue and $\phi = (\phi_1, \phi_2)^T$ the eigenvector corresponding to λ,

$$\begin{pmatrix} 0 & -\Lambda^1 \\ -\Lambda^2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5.3}$$

We have the following relations,

$$\Lambda^2 \Lambda^1 \phi_2 = \lambda^2 \phi_2, \quad \Lambda^1 \Lambda^2 \phi_1 = \lambda^2 \phi_1.$$

We know the following results from [20]:

1. The spectra $\sigma(\Lambda^1 \Lambda^2)$ and $\sigma(\Lambda^2 \Lambda^1)$ coincide except for a possible zero eigenvalue:

$$\sigma(\Lambda^1 \Lambda^2) \backslash \{0\} = \sigma(\Lambda^2 \Lambda^1) \backslash \{0\}.$$

2. The spectrum $\sigma(J)$ satisfies the following,

$$\sigma(\Lambda) = \pm\sqrt{\sigma(\Lambda^1 \Lambda^2)} \cup \pm\sqrt{\sigma(\Lambda^2 \Lambda^1)}.$$

3. The following identity holds for the spectral radii,

$$\varrho(\Lambda) = \sqrt{\varrho(\Lambda^1 \Lambda^2)} = \sqrt{\varrho(\Lambda^2 \Lambda^1)}.$$

Therefore, we need estimates of the eigenvalues of $\Lambda^1 \Lambda^2$. We will prove that

$$\Lambda^1 = \Lambda^2, \tag{5.4}$$

on a unit square with uniform mesh size and we will obtain,

$$\varrho(\Lambda) = \sqrt{\varrho((\Lambda^1)^2)} = \varrho(\Lambda^1). \tag{5.5}$$

5.2 Proof that $\Lambda^1 = \Lambda^2$

To begin the proof, we need to develop algebraic expressions for Λ^1 and Λ^2. We first study the structure of these operators.

5.2.1 Matrices for two overlapping subdomains

In this section, we will consider the structure of Λ^1 and Λ^2. We assume that $\tilde{\alpha}$, the interior parameter, equals α, the original parameter. Because of the relation $\Lambda^1 = I^1 B^1 \tilde{R}^1 (B_1)^{-1} (I_1)^T$ and $\Lambda^2 = I^2 B^2 \tilde{R}^2 (B_2)^{-1} (I_2)^T$, we first check the structure of $(B_1)^{-1}$, $(B_2)^{-1}$, B^1, and B^2 . Since we consider the two overlapping subdomain case on the unit square, we have the following original matrices B_1 and B_2 which have the two important properties of being block diagonal and positive definite,

$$B_1 = B_2 = \begin{pmatrix} M & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ 0 & K & L & K & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K & 0 \\ 0 & \cdots & & & 0 & K & L & K \\ 0 & \cdots & & & & 0 & K & M \end{pmatrix}, \tag{5.6}$$

where

$$M = \begin{pmatrix} 1 + \frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h & 0 & & \cdots & 0 \\ -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h & 2 + \frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h & 2 + \frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h \\ 0 & \cdots & & 0 & -\frac{1}{2} + \frac{1}{\alpha}\frac{1}{6}h & 1 + \frac{1}{\alpha}\frac{2}{3}h \end{pmatrix},$$

$$L = \begin{pmatrix} 2+\frac{1}{\alpha}\frac{2}{3}h & -1 & 0 & & & \cdots & 0 \\ -1 & 4 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & -1 & 4 & -1 \\ 0 & \cdots & & & 0 & -1 & 2+\frac{1}{\alpha}\frac{2}{3}h \end{pmatrix},$$

and

$$K = \begin{pmatrix} -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h & & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & & -\frac{1}{2}+\frac{1}{\alpha}\frac{1}{6}h \end{pmatrix}.$$

The block matrices B_1 and B_2 have the following form,

$$B_1 = \begin{pmatrix} C & G \\ G^T & D \end{pmatrix}, \quad B_2 = \begin{pmatrix} D & H^T \\ H & C \end{pmatrix}, \tag{5.7}$$

where

$$C = \begin{pmatrix} M & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ 0 & K & L & K & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K & 0 \\ 0 & \cdots & & & 0 & K & L & K \\ 0 & \cdots & & & & 0 & K & L \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ K & \cdots & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & \cdots & K \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \quad D = \begin{pmatrix} L & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K \\ 0 & \cdots & & & 0 & K & M \end{pmatrix},$$

$$B^1 = \begin{pmatrix} M & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K \\ 0 & \cdots & & & 0 & K & m \end{pmatrix}, \quad B^2 = \begin{pmatrix} m & K & 0 & & & \cdots & 0 \\ K & L & K & 0 & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & K & L & K \\ 0 & \cdots & & & 0 & K & M \end{pmatrix},$$

where

$$m = L - M, \tag{5.8}$$

and

$$m = \begin{pmatrix} 1 & -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h & 0 & & \cdots & 0 \\ -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h & 2 - \frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h & 2 - \frac{1}{\alpha}\frac{2}{3}h & -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h \\ 0 & \cdots & & 0 & -\frac{1}{2} - \frac{1}{\alpha}\frac{1}{6}h & 1 \end{pmatrix}.$$

Since the matrix B_1 is positive definite, it is invertible and C and D are also invertible being principal minors of B_1. From the following matrix relation,

$$\begin{pmatrix} I & 0 \\ -C^{-1}G^T & I \end{pmatrix} \begin{pmatrix} C & G \\ G^T & D \end{pmatrix} \begin{pmatrix} I & -C^{-1}G \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & (D - G^T C^{-1} G) \end{pmatrix},$$

we know that the matrix $D - G^T C^{-1} G$ is positive definite and invertible. The matrix $C - GD^{-1}G^T$ is also positive definite and invertible by a similar argument. We now have the following expression for $(B_1)^{-1}$ the inverse of the block matrix B_1,

$$(B_1)^{-1} = \begin{pmatrix} C & G \\ G^T & D \end{pmatrix}^{-1} = \begin{pmatrix} C' & G' \\ (G')^T & D' \end{pmatrix}, \tag{5.9}$$

where

$$\begin{aligned} C' &= (C - GD^{-1}G^T)^{-1} = C^{-1} + (C^{-1}G)D'(G^T C^{-1}), \\ D' &= (D - G^T C^{-1} G)^{-1} = D^{-1} + (D^{-1}G^T)C'(GD^{-1}), \\ G' &= -(C^{-1}G)D', \quad (G')^T = -D'(G^T C^{-1}). \end{aligned}$$

Now we can directly compute the matrix Λ^1 from the previous formulas,

$$\Lambda^1 = I^1 B^1 \tilde{R}^1 (B_1)^{-1} (I_1)^T$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} \left(C - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & M \end{pmatrix} \right) \\
&\cdot \quad \left(\begin{array}{cccc|ccc} I_\gamma & \cdots & 0 & 0 & 0 & \cdot & 0 \\ \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdot & 0 \\ 0 & \cdots & 0 & I_\gamma & 0 & \cdot & 0 \end{array} \right) \left(\begin{array}{ccc} C' & | & G' \\ - & - & - \\ (G')^T & | & D' \end{array} \right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ - \\ 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} \left(C - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & M \end{pmatrix} \right) \begin{pmatrix} C' & | & G' \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ - \\ 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} \left(C - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & M \end{pmatrix} \right) \left(G' \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \right) \\
&= K G'_{l-1,m} + m G'_{l,m}.
\end{aligned}
$$

The same idea can be applied to B^2 and we have the following relation,

$$
\begin{pmatrix} I & 0 \\ -D^{-1}H & I \end{pmatrix} \begin{pmatrix} D & H^T \\ H & C \end{pmatrix} \begin{pmatrix} I & -D^{-1}H^T \\ 0 & I \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & (C - HD^{-1}H^T) \end{pmatrix}.
$$

The inverse of the block matrix B_2 can be expressed as,

$$
(B_2)^{-1} = \begin{pmatrix} D & H^T \\ H & C \end{pmatrix}^{-1} = \begin{pmatrix} D'' & (H')^T \\ H' & C'' \end{pmatrix}. \tag{5.10}
$$

where

$$\begin{aligned} D'' &= (D - H^T C^{-1} H)^{-1} = D^{-1} + (D^{-1}H^T)C''(HD^{-1}), \\ C'' &= (C - HD^{-1}H^T)^{-1} = C^{-1} + (C^{-1}H)D''(H^T C^{-1}), \\ H' &= -C''(D^{-1}H), \quad (H')^T = -(H^T D^{-1})C''. \end{aligned}$$

We also find, from a similar computation,

$$\Lambda^2 = I^2 B^2 \tilde{R}^2 (B_2)^{-1} (I_2)^T = K H'_{2,1} + m H'_{1,1}.$$

In the following sections, we will study the subblock matrices $G'_{l-1,m}$, $G'_{l,m}$, $H'_{2,1}$, and $H'_{1,1}$ and show that we obtain $\Lambda^1 = \Lambda^2$.

5.2.2 The inverse of symmetric block tridiagonal matrices

To analyze the structure of Λ^1 and Λ^2, we need to know explicit formulas for the inverses of B_1 and B_2. Since the matrix $b(B) = B_1 = B_2$ is symmetric and block tridiagonal, we need a formula for symmetric block tridiagonal matrices.

We assume that block matrix $b(A)$ has the following structure,

$$b(A) = \begin{pmatrix} D_1 & -A_2^T & 0 & & & & \cdots & 0 \\ -A_2 & D_2 & -A_3^T & 0 & & & \cdots & 0 \\ 0 & -A_3 & D_3 & -A_4^T & 0 & & \cdots & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & \cdots & & 0 & -A_{n-2} & D_{n-2} & -A_{n-1}^T & 0 \\ 0 & \cdots & & & 0 & -A_{n-1} & D_{n-1} & -A_n^T \\ 0 & \cdots & & & & 0 & -A_n & D_n \end{pmatrix}. \quad (5.11)$$

We define the following,

$$\left\{ \begin{array}{c} \triangle_1 = D_1 \\ \triangle_i = D_i - A_i(\triangle_{i-1})^{-1}(A_i)^T \end{array} \right\}, \left\{ \begin{array}{c} \sum_n = D_n \\ \sum_i = D_i - (A_{i+1})^T(\sum_{i-1})^{-1} A_{i+1} \end{array} \right\}. \quad (5.12)$$

The block matrix $b(A)$ is said to be **proper** if the matrices A_i are nonsingular; in our case the matrix $b(B)$ satisfies this condition except when $\frac{1}{\alpha} = \frac{3}{h}$.

We have the following theorem if $b(A)$ is proper,

Theorem 5.1 *If $b(A)$ is proper, there exist two (nonunique) sequences of matrices of $\{U_i\}$, $\{V_i\}$ such that for $j \leq i$*

$$(b(A)^{-1})_{i,j} = U_i V_j^T, \tag{5.13}$$

with $U_i = A_i^{-T} \triangle_{i-1} \cdots A_2^{-T} \triangle_1$ and $V_j^T = \sum_1^{-1} A_2^T \cdots A_j^T \sum_j^{-1}$. In other words, $b(A)^{-1}$ can be written as,

$$b(A)^{-1} = \begin{pmatrix} U_1V_1^T & U_1V_2^T & U_1V_3^T & \cdots & U_1V_n^T \\ V_2U_1^T & U_2V_2^T & U_2V_3^T & \cdots & U_2V_n^T \\ V_3U_1^T & V_3U_2^T & U_3V_3^T & \cdots & U_3V_n^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_nU_1^T & V_nU_2^T & V_nU_3^T & \cdots & U_nV_n^T \end{pmatrix}. \tag{5.14}$$

Proof See [31].

Therefore, we know that the submatrices of $b(A)^{-1}$ are represented as a finite products of persymmetric and symmetric submatrices.

5.2.3 Symmetry, persymmetry, and centrosymmetry

In this section, we review the property of each subblock and the whole matrices. We recall that the matrix $b(B)$ and its submatrix $b(B)_{i,j}$ are symmetric and persymmetric. The matrix K is said to be persymmetric if it is symmetric about the anti-diagonal, i.e. $k_{i,j} = k_{n-j+1,n-i+1}$ for all i and j; see [19]. This is equivalent to

requiring that $K = EK^TE$ where

$$E = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

Since $b(B)$ is both symmetric and persymmetric, it also satisfies another property; we call it centrosymmetry. The matrix K is said to be centrosymmetric if it is symmetric about its center point, i.e., $k_{j,i} = k_{n-j+1,n-i+1}$ for all i and j. This is equivalent to requiring that $K = EKE$. We can prove the following properties easily.

1) If K is symmetric, so is K^{-1}.

2) If K is persymmetric, so is K^{-1}.

3) If K is centrosymmetric, so is K^{-1}.

4) Even if K and P are both symmetric, KP may not be symmetric.

5) Even if K and P are both persymmetric, KP may not be persymmetric.

6) If K and P are both centrosymmetric, so is KP.

From these properties, we know that $b(B)^{-1}$ is symmetric, persymmetric, and centrosymmetric but $b(B)_{i,j}^{-1}$ might be neither symmetric nor persymmetric. However it is centrosymmetric. The Figure 5.1 shows an example. Combining the symmetry and persymmetry of $b(B)^{-1}$ and the centrosymmetry of each subblock matrix $b(B)_{ij}^{-1}$, we have the following block centrosymmetry of the subblock matrix $b(B)_{ij}^{-1}$,

$$b(B)_{ij}^{-1} = b(B)_{n-i+1\,n-j+1}^{-1}. \tag{5.16}$$

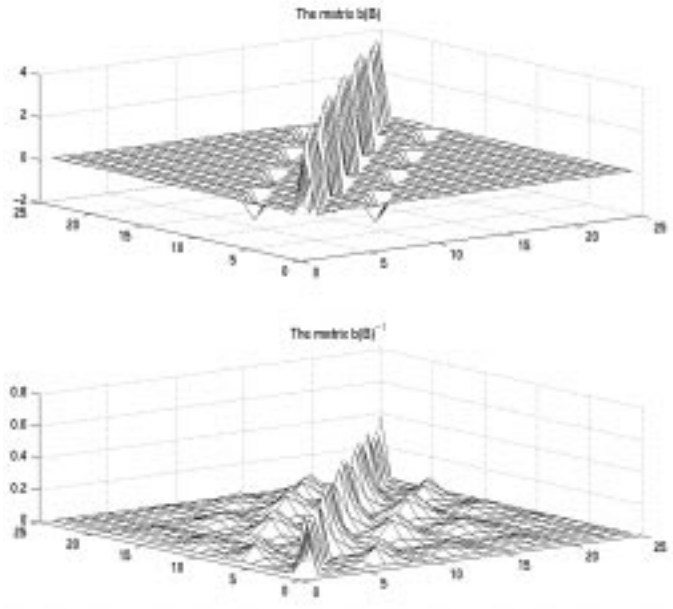

Figure 5.1: An example of the matrices $b(B)$ and $b(B)^{-1}$ with $n = 5$

Theorem 5.2 *The matrix $b(B)^{-1}$ are block centrosymmetric.*

Proof We want to prove that each entry of the two matrices are the same. Let $c(k,l)_{i,j}$ be the ij entry of $b(B)^{-1}_{k,l}$. Since $b(B)^{-1}$ is symmetric, $c(k,l)_{i,j}$ is equal $c(l,k)_{j,i}$ of $b(B)^{-1}_{l,k}$. Now, we exploit the persymmetry of $b(B)^{-1}$ and find that $c(l,k)_{j,i}$ of $b(B)^{-1}_{l,k}$ is equal to $c(n-k+1, n-l+1)_{n-i+1,n-j+1}$ of $b(B)^{-1}_{n-k+1,n-l+1}$. Finally, we apply the centrosymmetry of $b(B)^{-1}_{n-k+1,n-l+1}$ and we have equality between $c(n-k+1, n-l+1)_{n-i+1,n-j+1}$ and $c(n-k+1, n-l+1)_{i,j}$ of $b(B)^{-1}_{n-k+1,n-l+1}$. Therefore we have equality between $c(l,k)_{i,j}$ of $b(B)^{-1}_{l,k}$ and $c(n-k+1, n-l+1)_{i,j}$ of $b(B)^{-1}_{n-k+1,n-l+1}$.

From this property, we have equality of Λ^1 and Λ^2. As the subblock matrices, G'_{l-1m} and H'_{21} are block symmetric about a center block, so are G'_{lm}, and $H'_{1,1}$.

Since we have the relations $\Lambda^1 = KG'_{l-1,m} + mG'_{l,m}$ and $\Lambda^2 = KH'_{2,1} + mH'_{1,1}$, we have the relation $\Lambda^1 = \Lambda^2$.

5.3 Spectral radius of $F = \Lambda$

We now assume that the vector z is the eigenvector of Λ^1 with eigenvalue λ. The absolute value of λ gives us the required information about the convergence of the system (5.1). The difference between the matrices B^1 and C is zero except for the block submatrix $m = L - M$. Since M is a principal submatix of a positive definite matrix B_1, it is positive definite and it can be represented by a Cholesky decomposition,

$$M = ll^T. \tag{5.17}$$

To analyze the absolute value of eigenvalue $|\lambda|$, we will use the Woodbury formula given in [19],

5.3.1 The Woodbury formula

If $A = B + UV^T$, with U and V are $n \times p$ matrices, we have,

$$A^{-1} = B^{-1} - B^{-1}U(I_p + V^T B^{-1} U)^{-1} V^T B^{-1} \quad \text{with } I_p : p \times p \text{ identity matrix.} \tag{5.18}$$

Given the relation between B^1 and C, we have the following relation,

$$C = B^1 + LL^T \quad \text{where} \quad L = \begin{pmatrix} 0 & \cdots & 0 & l^T \end{pmatrix}^T \tag{5.19}$$

So we have,

$$\begin{aligned} C^{-1} &= (B^1)^{-1} - (B^1)^{-1} L (I_p + L^T (B^1)^{-1} L)^{-1} L^T (B^1)^{-1} \\ &= (B^1)^{-1} - (B^1)^{-1} L (I_p + l^T m' l)^{-1} L^T (B^1)^{-1}, \end{aligned}$$

with $m' = S^{-1}(m)$ where $S(m)$ is the Schur complement in B^1 corresponding to eliminating all diagonal block except m.

From this result, we have the following expression for Λ^1,

$$
\begin{aligned}
\Lambda^1 &= I^1 B^1 \tilde{R}^1 (B_1)^{-1} (I_1)^T \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} \left(C - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & M \end{pmatrix} \right) G' \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} B^1 \ (C^{-1}G)(G^T C^{-1} G - D)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} B^1 ((B^1)^{-1} - (B^1)^{-1} L (I_p + l^T m' l)^{-1} L^T (B^1)^{-1}) \\
&\quad G(G^T C^{-1} G - D)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} (I - L(I_p + l^T m' l)^{-1} L^T (B^1)^{-1}) \\
&\quad \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ K & \cdots & 0 \end{pmatrix} (G^T C^{-1} G - D)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cdots & 0 & I_\gamma \end{pmatrix} (I - L(I_p + l^T m' l)^{-1} L^T (B^1)^{-1}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&\quad K \begin{pmatrix} I_\gamma & 0 & \cdots & 0 \end{pmatrix} (G^T C^{-1} G - D)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_\gamma \end{pmatrix} \\
&= (I_p - l(I_p + l^T m' l)^{-1} l^T m') K (G^T C^{-1} G - D)^{-1}_{1,s}
\end{aligned}
$$

$$
\begin{aligned}
&= (I_p - (M^{-1} + m')^{-1}m')K(G^TC^{-1}G - D)^{-1}_{1,s} \\
&= (I_p - ((Mm')^{-1} + I_p)^{-1})K(G^TC^{-1}G - D)^{-1}_{1,s} \\
&\quad (A^{-1} - (A+E)^{-1}) = (A+E)^{-1}EA^{-1} \\
&= (((Mm')^{-1} + I_p)^{-1}(Mm')^{-1})K(G^TC^{-1}G - D)^{-1}_{1,s} \\
&= (I_p + Mm')^{-1}K(G^TC^{-1}G - D)^{-1}_{1,s}
\end{aligned}
$$

where s is the number of row block matrices of $(G^TC^{-1}G - D)^{-1}$.

We define the spectral norm $||\cdot||_2$ as the matrix norm corresponding to the Euclidean vector norm and which has the following properties; see [21],

$$
\begin{aligned}
1.||A||_2 &= \sqrt{\varrho(A^HA)} = \sqrt{\varrho(AA^H)} \quad \text{for all square matrices} \\
2.||A||_2 &= \varrho(A) \quad \text{for all normal matrices.}
\end{aligned}
$$

We know the following relation between the spectral radius and the matrix norm of any matrix A,

$$\varrho(A) \leq ||A||. \tag{5.20}$$

Therefore,

$$
\begin{aligned}
\varrho(\Lambda^1) &\leq ||(\Lambda^1)|| \\
&= ||(I_p + Mm')^{-1}K(G^TC^{-1}G - D)^{-1}_{1,s}|| \\
&= ||(I_p + Mm')^{-1}||||K||||(G^TC^{-1}G - D)^{-1}_{1,s}|| \\
&= ||F_1||||F_2||||F_3||,
\end{aligned}
$$

with

$$F_1 = (I_p + Mm')^{-1}, \quad F_2 = K, \quad F_3 = (G^TC^{-1}G - D)^{-1}_{1,s}. \tag{5.21}$$

5.3.2 Eigenvalues and L^2 norm of $F_1 = ((I_p + Mm')^{-1})$

5.3.2.1 Eigenvalues of $(Mm')^{-1}$

In this section, we consider two special cases.

Case 1: B^1 positive definite

If we assume that B^1 is positive definite, then m' and the other diagonal blocks of the inverse of B^1 are positive definite. From the definition of m', $m' = S^{-1}(m)$, and we have,

$$\begin{aligned} & M - (m')^{-1} = M - S(m) = M - m + KL'K \\ = \; & M - (L - M) + KL'K = 2M - L + KL'K \\ = \; & \begin{pmatrix} \frac{1}{\alpha}\frac{2}{3}h & \frac{1}{\alpha}\frac{1}{3}h & 0 & & \cdots & 0 \\ \frac{1}{\alpha}\frac{1}{3}h & \frac{1}{\alpha}\frac{4}{3}h & \frac{1}{\alpha}\frac{1}{3}h & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & \frac{1}{\alpha}\frac{1}{3}h & \frac{1}{\alpha}\frac{4}{3}h & \frac{1}{\alpha}\frac{1}{3}h \\ 0 & \cdots & & 0 & \frac{1}{\alpha}\frac{1}{3}h & \frac{1}{\alpha}\frac{4}{3}h \end{pmatrix} + KL'K > 0, \end{aligned}$$

with $L' = S^{-1}(L)$ where $S(L)$ is the Schur complement of the positive definite upper left block in B^1 as in (5.9). Since the first matrix is diagonally dominant, it is positive definite and since the second matrix is similar to a subblock of the inverse of a positive definite matrix, it is positive definite. From Theorem 1.8, we know that all eigenvalues of $(Mm')^{-1} = S(m)M^{-1}$ are less than 1.

We now need to consider the term L' in the next computation. The positive definiteness of the matrix B^1 is important for that of L'. Even though L is positive definite, it does not give the positive definiteness of L'. Also, the positive definiteness of m' is decided by that of the whole matrix B^1 which depends on having a sufficiently small $\frac{1}{\alpha}h$. Even if m is positive definite, it is possible that m' is not.

Case 2 : $L > KL'K$

With this condition, we have

$$\begin{aligned} M + (m')^{-1} &= M + S(m) = M + m - KL'K = M + (L - M) - KL'K \\ &= L - KL'K = S(L) > 0. \end{aligned}$$

From Theorem 1.9, we know that all eigenvalues of $(Mm')^{-1} = S(m)M^{-1}$ are greater than -1. From the result above we know that all eigenvalues of $(Mm')^{-1} = S(m)M^{-1}$ are located in $(-1, 1)$.

5.3.2.2 Eigenvalues of $(I_p + Mm')^{-1}$

In this section, we consider a special case. Here $\sigma(A)$ is the set of eigenvalues of the square matrix A.

Case 1 : B^1 positive definite

If we assume positive definiteness of B^1, we can apply Theorem 1.6 and we have $\sigma((Mm')^{-1}) \subset (0, 1)$.

Then, we have the following,

1. all eigenvalues of Mm' are in $(1, \infty)$.

2. all eigenvalues of $I_p + Mm'$ are $(2, \infty)$.

3. all eigenvalues of $(I_p + Mm')^{-1}$ are $(0, 1/2)$.

So we have $\varrho(I_p + Mm')^{-1} < 1/2$. Even though M and m' are symmetric positive definite matrices, the matrix Mm' might not be symmetric. Therefore we cannot prove a norm bound directly. However, since M is a positive definite matrix, we have the following similarity,

$$Mm' = ll^t m' ll^{-1}. \tag{5.22}$$

Table 5.1: Condition number of l.

	n=5	n=10	n=15	n=20	n=25
$\frac{1}{\alpha}h = 10^{-4}$	1.9662	1.9838	1.9923	1.9957	1.9972
$\frac{1}{\alpha}h = 10^{-3}$	1.9653	1.9829	1.9914	1.9948	1.9963
$\frac{1}{\alpha}h = 10^{-2}$	1.9561	1.9741	1.9826	1.9859	1.9874
$\frac{1}{\alpha}h = 10^{-1}$	1.8720	1.8931	1.9012	1.9043	1.9057
$\frac{1}{\alpha}h = 1$	1.4333	1.4559	1.4604	1.4620	1.4626
$\frac{1}{\alpha}h = 10^{1}$	1.2854	1.3082	1.3131	1.3150	1.3159
$\frac{1}{\alpha}h = 10^{2}$	1.5438	1.6278	1.6483	1.6562	1.6600
$\frac{1}{\alpha}h = 10^{3}$	1.5849	1.6803	1.7036	1.7126	1.7169
$\frac{1}{\alpha}h = 10^{4}$	1.5893	1.6859	1.7095	1.7186	1.7230

Therefore, we have the following,

$$I_p + Mm' = l(I_p + l^t m' l)l^{-1}. \tag{5.23}$$

Finally we have,

$$(I_p + Mm')^{-1} = l(I_p + l^t m' l)^{-1} l^{-1}. \tag{5.24}$$

Since the eigenvalues of two similar matrices are the same and $(I_p + l^t m' l)^{-1}$ is symmetric, we have,

$$\begin{aligned} ||(I_p + Mm')^{-1}||_2 &= ||l(I_p + l^t m' l)^{-1} l^{-1}||_2 \\ &\leq ||l||_2 ||(I_p + l^t m' l)^{-1}||_2 ||l^{-1}||_2 \\ &= ||l||_2 \varrho(I_p + l^t m' l)^{-1}) ||l^{-1}||_2 \\ &= ||l||_2 \varrho((I_p + Mm')^{-1}) ||l^{-1}||_2 \\ &= cond(l)\varrho((I_p + Mm')^{-1}). \end{aligned}$$

Now we want to know the condition number of l and a precise estimate of $\varrho((I_p + Mm')^{-1})$. Table 5.1 is a table of the condition number of l which shows

Table 5.2: Table for $||F||_2$.

$n = 31$	$\|\|F_1\|\|_2$	$\|\|F_2\|\|_2$	$\|\|F_3\|\|_2$	$\|\|F\|\|_2$
$\alpha = 100$, $ovlp = 27$	0.4759	1.000	796.2	0.0171
$\alpha = 100$, $ovlp = 15$	0.4891	1.000	892.4	0.1627
$\alpha = 100$, $ovlp = 3$	0.4890	1.000	1014	0.7116
$\alpha = 10$, $ovlp = 27$	0.4758	1.000	74.99	0.0151
$\alpha = 10$, $ovlp = 15$	0.4889	1.000	87.72	0.1505
$\alpha = 10$, $ovlp = 3$	0.4888	1.000	104.4	0.6674
$\alpha = 1$, $ovlp = 27$	0.4743	1.000	4.779	0.0046
$\alpha = 1$, $ovlp = 15$	0.4872	1.000	7.739	0.0761
$\alpha = 1$, $ovlp = 3$	0.4871	1.000	12.75	0.3873
$\alpha = 0.1$, $ovlp = 27$	0.4605	1.000	0.1445	0.0088
$\alpha = 0.1$, $ovlp = 15$	0.4723	1.000	0.6038	0.1117
$\alpha = 0.1$, $ovlp = 3$	0.4724	1.000	1.942	0.4112
$\alpha = 0.01$, $ovlp = 27$	1.4350	1.000	0.0076	0.0110
$\alpha = 0.01$, $ovlp = 15$	2.7126	1.000	0.0573	0.1553
$\alpha = 0.01$, $ovlp = 3$	2.8609	1.000	0.2326	0.6652

that it is less than 2. Since $\varrho((I_p + Mm')^{-1})$ is strictly less than $1/2$, this shows that $||F_1|| = ||(I_p + Mm')^{-1}||_2 < 1$.

5.3.2.3 Conclusions and numerical results for the general case

We can conclude that the first factor F_1 is composed of a product of a positive definite matrix and a symmetric matrix and therefore, it is similar to a symmetric matrix. Under the condition of positive definiteness of B^1, $||F_1||_2$ is less than 1. From Table 5.2, we can see the 2-norm of F_1 is not so big in numerical results for the general case.

5.3.3 L^2 norm of $F_2 = K$

Consider a positive definite matrix $P = -K$. If $\frac{1}{\alpha}h < 6$, then $I - P \geq 0$. So we have the following,

$$||K||_2 = ||P||_2 \leq ||I|| = 1. \tag{5.25}$$

The 2-norm of K can be greater than 1 under the condition $\frac{1}{\alpha}h \leq 6$ which means a very small α.

5.3.4 Eigenvalues and L^2 norm of $F_3 = (G^T C^{-1} G - D)^{-1}_{1,s}$

5.3.4.1 Eigenvalues of $(G^T C^{-1} G - D)^{-1}_{1,s}$

We know that $(G^T C^{-1} G - D)^{-1}$ is a partitioned block of $B_1^{-1} = b(B)^{-1}$. The subblock matrix $(G^T C^{-1} G - D)^{-1}_{1,s}$ is an off-diagonal and boundary block matrix. Table 5.2 shows an example of the 2-norm of the boundary subblock matrices $b(B)^{-1}_{(n,i)}$ of $b(B)^{-1}$ with $n = 30$. The size of the block matrix $(G^T C^{-1} G - D)^{-1}$ depends on the overlap of the two subdomains. A larger overlap means that the number s is bigger. The block $(G^T C^{-1} G - D)^{-1}_{1,s}$ is the block submatrix $b(B)^{-1}_{n,n-s+1}$ of $b(B)^{-1}$. Therefore, a larger overlap makes $||(G^T C^{-1} G - D)^{-1}_{1,s}||_2$ smaller.

Table 5.2 and other numerical results show that $||(G^T C^{-1} G - D)^{-1}_{1,s}||_2$ is less than 1 for small α. Since the matrix $b(B)$ is a M-matrix, its inverse $b(B)^{-1}$ is a positive definite matrix. With a small $\frac{1}{\alpha}$, the diagonal entries of the matrix $b(B)$ dominate the off-diagonal entries. For a large α, $||F_3||_2$ is relatively large. Also it is generally not symmetric.

Remark In Chapter 8 of [2], there is a formula for the decay of block submatrices of inverses of block tridiagonal positive definite square matrices. Theorem 8.33 in [2] shows that an upper bound of the norm of an off-diagonal subblock

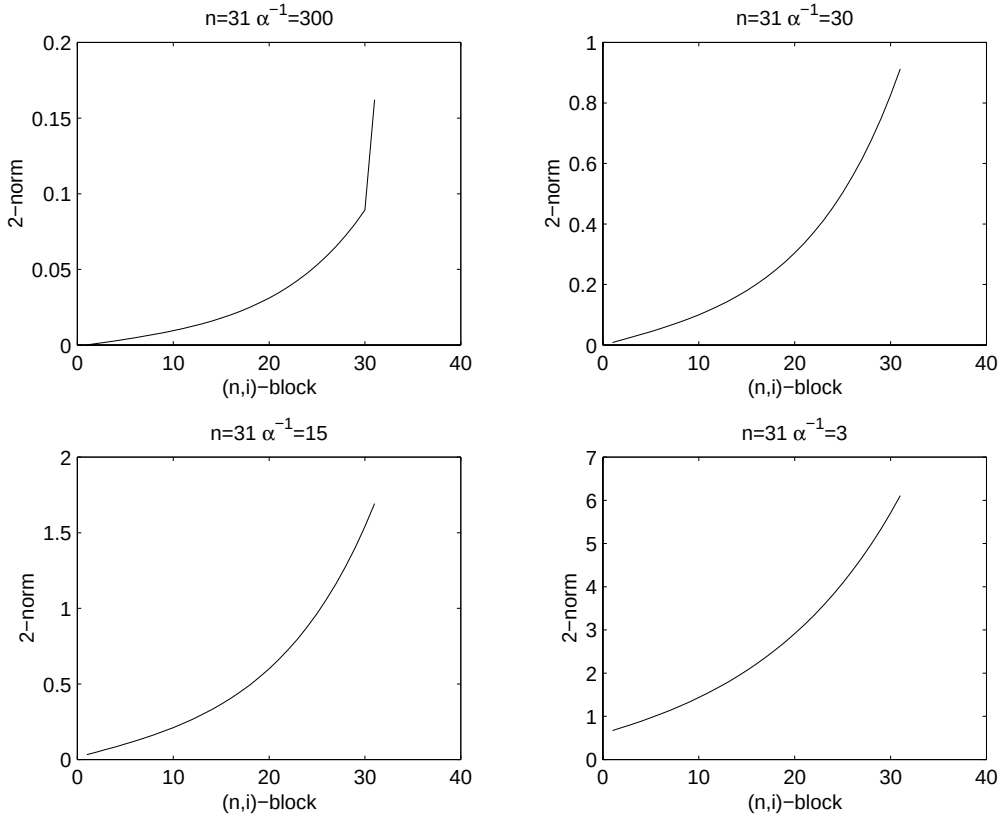

Figure 5.2: An example of $||\cdot||_2$ of boundary sub block matrices $b(B)^{-1}_{n,i}$ of $b(B)^{-1}$ with $h = 1/30$

matrix which is product of two corresponding diagonal sub block and the C.B.S. (Cachy-Schwarz-Bunyakowski) constant (see [2] section 9.1).

5.3.4.2 L^2 norm of $(G^T C^{-1} G - D)^{-1}_{1,s}$

The effect of F_3 looks disastrous for large α. Figure 5.3 shows an example of $log_{10}(\sigma(F_3))$ with $\alpha = 10^i, i = -2, \cdot, 2$ and $n = 31$. In this figure, the eigenvalues which have magnitude larger than 1 are isolated from other eigenvalues which have absolute values less than 1.

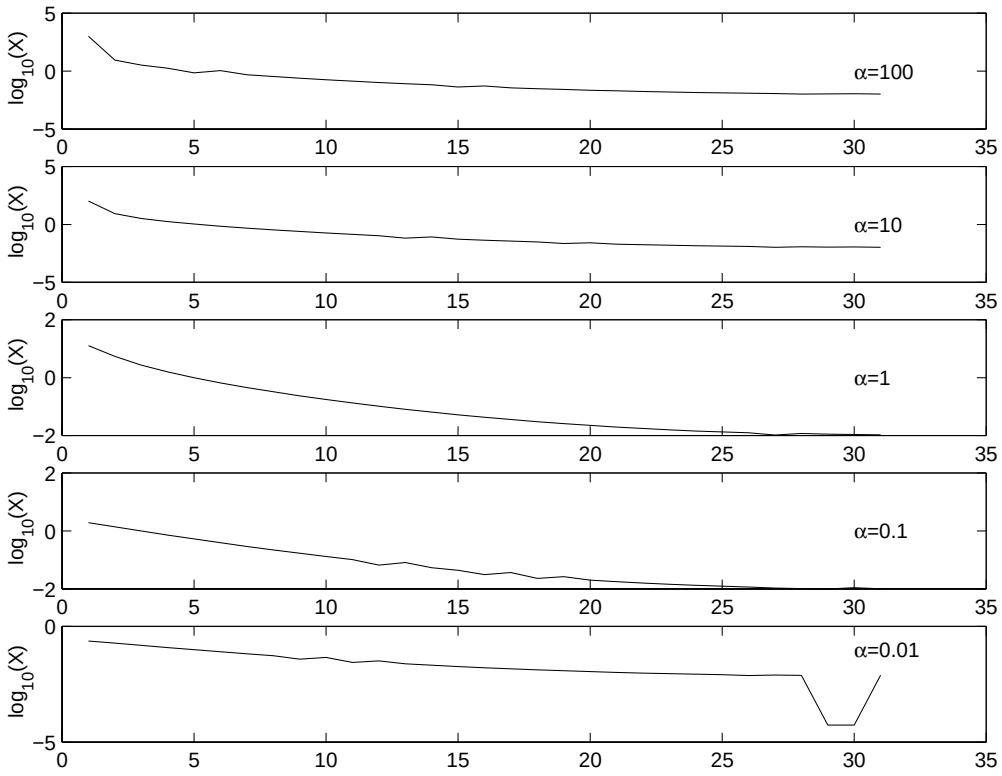

Figure 5.3: An example of the spectrum of $log_{10}(\sigma(F_3))$ with $\alpha = 10^i, i = -2, \cdots, 2$ and $n = 31$.

5.3.5 L^2 norm of $F = \Lambda$

Table 5.2 shows the 2-norm of F and its three factors. The table show that even if one of three is greater than 1, the estimate of the 2-norm of F is still less than 1. With a small α, the 2-norm of F_3 has a good effect and from Table 5.2, the product of the 2-norms of the three factor is less than 1. With a large α, the 2-norm of F_3 is disastrously big but we know from Figure 5.3 that it comes from isolated eigenvalues and that most other eigenvalues are less than 1.

Chapter 6

Analysis based on Lagrange Multipliers

6.1 Basic Concepts and Notations

6.1.1 Splitting of the Error Vector

We define a positive atomic subdomain as an atomic subdomain participating in all fractional steps and whose corresponding local matrix is positive definite. In the two overlapping subdomain case, we have one positive atomic subdomain which is the filled-in region of Figure 6.1.

From the following formula (2.39) for the positive atomic subregion,

$$B^{12}u^{12} = f^{12} + (I_1^{12})^T\lambda_1 + (I_2^{12})^T\lambda_2, \tag{6.1}$$

we obtain the following formula for the error vector in the overlapping atomic subregion,

$$B^{12}e^{12} = (I_1^{12})^T\lambda_1 + (I_2^{12})^T\lambda_2. \tag{6.2}$$

Since B^{12} is invertible, we can decompose e^{12} as,

$$e^{12} = e_1^{12} + e_2^{12},$$

with

$$B^{12}e_2^{12} = (I_2^{12})^T\lambda_2 \quad B^{12}e_1^{12} = (I_1^{12})^T\lambda_1. \tag{6.3}$$

We define a matrix m' as

$$m' = I_2^{12}(B^{12})^{-1}(I_2^{12})^T.$$

Since m' is positive definite, being a principal submatrix of $(B^{12})^{-1}$, we have,

$$\begin{aligned} ||e_2^{12}||^2_{B^{12}} &= (e_2^{12})^T B^{12} e_2^{12} = (e_2^{12})^T (I_2^{12})^T \lambda_2 \\ &= (\lambda_2)^T I_2^{12}(B^{12})^{-1}(I_2^{12})^T\lambda_2 \\ &= (\lambda_2)^T m'\lambda_2 = ||\lambda_2||^2_{m'}, \end{aligned}$$

and

$$||e_1^{12}||^2_{B^{12}} = (e_2^{12})^T B^{12} e_2^{12} = (\lambda_1)^T m'\lambda_1 = ||\lambda_1||^2_{m'}. \tag{6.4}$$

Therefore, a comparison between $||\lambda_i||_{m'}, i = 1, 2$ is reduced to that between $||e_i^{12}||_{B^{12}}, i = 1, 2$.

6.1.2 A Relation between the Split Error Vectors

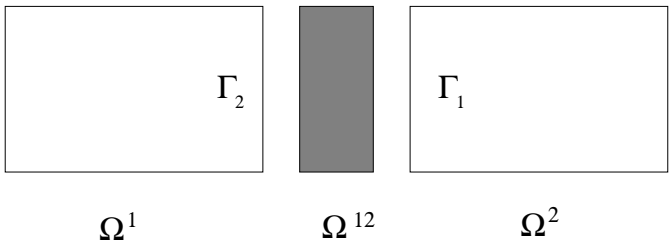

Figure 6.1: The positive atomic subregion of two overlapping subdomains

Let u and v be elements of the appropriate space V^1 of P^1 finite element functions. We then have the following identity from the functionals in (2.31),

$$u^t B^{12} v = \int_{\Omega^{12}} \nabla u \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta^{12}} u\, v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} u\, v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} u\, v.$$

From the relation (6.3), e_1^{12} is the solution with nonzero Robin boundary data on Γ_2 and zero Robin boundary data on Γ_1 and Θ^{12} and e_2^{12} is the solution with nonzero Robin boundary data on Γ_1 and zero Robin boundary data on Γ_2 and Θ^{12}.

Therefore, we have the following,

$$\begin{aligned}
(e_1^{12})^T B^{12} e_1^{12} &= \int_{\Omega^{12}} \nabla e_1^{12} \cdot \nabla e_1^{12} + \frac{1}{\alpha} \int_{\Theta^{12}} |e_1^{12}|^2 + \frac{1}{\tilde{\alpha}} \int_{\Gamma_1 \cup \Gamma_2} |e_1^{12}|^2 \\
&= \frac{1}{\alpha} \int_{\Theta^{12}} e_1^{12} \, (e_1^{12} + \alpha \frac{\partial e_1^{12}}{\partial n}) + \frac{1}{\tilde{\alpha}} \int_{\Gamma_1 \cup \Gamma_2} e_1^{12} \, (e_1^{12} + \tilde{\alpha} \frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}) \\
&= \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e_1^{12} \, (e_1^{12} + \tilde{\alpha} \frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}).
\end{aligned}$$

We also have,

$$(e_2^{12})^T B^{12} e_2^{12} = \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} e_2^{12} \, (e_2^{12} + \tilde{\alpha} \frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}).$$

In addition,

$$(e^{12})^T B^{12} e^{12} = \frac{1}{\tilde{\alpha}} \int_{\Gamma_1 \cup \Gamma_2} e^{12} \, (e^{12} + \tilde{\alpha} \frac{\partial e^{12}}{\partial n_{\Gamma_1}}).$$

Therefore,

$$(e_1^{12})^T B^{12} e_2^{12} = \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} e_1^{12} \, (e_2^{12} + \tilde{\alpha} \frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}),$$

and

$$(e_2^{12})^T B^{12} e_1^{12} = \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e_2^{12} \, (e_1^{12} + \tilde{\alpha} \frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}).$$

Since the following identity

$$(e_1^{12})^T B^{12} e_2^{12} = (e_2^{12})^T B^{12} e_1^{12}$$

holds, we have,

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e_1^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e_2^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}).$$

From Green's identity and the relation between the Robin boundary conditions and $e_i^{12}, i=1,2$, the identity above can be established directly by noticing that

$$\begin{aligned}
\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e_1^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) &= \frac{1}{\tilde{\alpha}}\int_{\Gamma_1\cup\Gamma_2} e_1^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) \\
&= \int_{\Omega^{12}} \nabla e_1^{12}\cdot\nabla e_2^{12} + \frac{1}{\alpha}\int_{\Theta^{12}\cup\Gamma_1\cup\Gamma_2} e_1^{12}\,e_2^{12} \\
&= \frac{1}{\tilde{\alpha}}\int_{\Gamma_1\cup\Gamma_2} e_2^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_2}}) \\
&= \frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e_2^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}).
\end{aligned}$$

Also we have the following,

$$\begin{aligned}
&\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e_1^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}) + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e_2^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}) \\
= &\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} (e_1^{12}+e_2^{12})\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e_1^{12}+\tilde{\alpha}\frac{\partial e_1^{12}}{\partial n_{\Gamma_1}}) \\
= &\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e_1^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e_2^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) \\
= &\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} (e_1^{12}+e_2^{12})\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e_2^{12}+\tilde{\alpha}\frac{\partial e_2^{12}}{\partial n_{\Gamma_2}}) \\
= &\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}).
\end{aligned}$$

The two quantities are then reduced to,

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) = ||e_2^{12}||^2_{B^{12}} + (e_1^{12})^T B^{12} e_2^{12}, \tag{6.5}$$

and

$$\frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e^{12} \, (e^{12} + \tilde{\alpha} \frac{\partial e^{12}}{\partial n_{\Gamma_1}}) = ||e_1^{12}||^2_{B^{12}} + (e_1^{12})^T B^{12} e_2^{12}. \tag{6.6}$$

Therefore, a comparison between the two integrals above gives us a way of comparing $||e_2^{12}||_{B^{12}}$ and $||e_1^{12}||_{B^{12}}$.

6.2 Convergence Theory for the Rectangular Two Subdomain Case

In this section, we will study the convergence for two overlapping rectangular subdomains of a unit square (= Ω). We will study the convergence of the first fractional step with

$$\Omega_1 = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq L\},$$

and we have the same result for the second fractional step.

6.2.1 Basic Properties and Notations for the Boundary Segments

Without loss of generality, we assume that the Robin boundary condition vanishes on the following set,

$$\Theta_1 = \{(x, y) | (x, 0) \cup (x, 1) \cup (0, y), 0 \leq x \leq 1, 0 \leq y \leq L\}.$$

This means that the following conditions hold on the three different boundary segments,

$$\begin{aligned} \phi^S(x) = u - \alpha u_y = 0 \quad \text{on} \quad \Theta_1^S = \{(x, y) | (x, 0), 0 \leq x \leq 1\}, \\ \phi^E(y) = u + \alpha u_x = 0 \quad \text{on} \quad \Theta_1^E = \{(x, y) | (1, y), 0 \leq y \leq L\}, \end{aligned} \tag{6.7}$$

$$\phi^W(y) = u - \alpha u_x = 0 \quad \text{on} \quad \Theta_1^W = \{(x,y)|(0,y), 0 \le y \le L\}.$$

Therefore we have the following equation for any solution u which has sufficient regularity,

$$\begin{aligned}
&\phi^S(x)_x = u_x - \alpha u_{yx} = 0, \quad \phi^S(x)_{xx} = u_{xx} - \alpha u_{yxx} = 0 \quad \text{on} \quad \Theta_1^S, \qquad (6.8)\\
&\phi^E(y)_y = u_y + \alpha u_{xy} = 0, \quad \phi^E(x)_{yy} = u_{yy} + \alpha u_{xyy} = 0 \quad \text{on} \quad \Theta_1^E,\\
&\phi^W(y)_y = u_y - \alpha u_{xy} = 0, \quad \phi^W(x)_{yy} = u_{yy} - \alpha u_{xyy} = 0 \quad \text{on} \quad \Theta_1^W.
\end{aligned}$$

We define the nonoverlapped atomic subdomain Ω^1 in the first fractional step by

$$\Omega^1 = \{(x,y)|0 \le x \le 1, 0 \le y \le l\},$$

We also introduce notations for the two artificial boundaries and the other three boundary segments for Ω^1,

$$\begin{aligned}
\Gamma_1 &= \{(x,y)|(x,L), 0 \le x \le 1\}\\
\Gamma_2 &= \{(x,y)|(x,l), 0 \le x \le 1\}\\
\Theta_S^1 &= \{(x,y)|(x,0), 0 \le x \le 1\}\\
\Theta_W^1 &= \{(x,y)|(0,y), 0 \le y \le l\}\\
\Theta_E^1 &= \{(x,y)|(1,y), 0 \le y \le l\}.
\end{aligned}$$

6.2.2 A Basic Tool for the Computation

Let $H(y)$ be the solution of the following ordinary differential equation with a given differentiable function $G(y)$,

$$H'' = 0 \quad H(0) = G(0) \quad H(L) = G(L).$$

We then have,

$$H(y) = G(0)(1 - \frac{1}{L}y) + \frac{G(L)}{L}y.$$

If we have the following inequality,

$$G'' \geq 0, \tag{6.9}$$

then,

$$(H'' - G'') = -G'' \leq 0.$$

From this equation, we can conclude that $G(y) \leq H(y)$ and that we have the following inequality,

$$G(y) \leq H(y) = G(0)(1 - \frac{1}{L}y) + G(L)\frac{y}{L}. \tag{6.10}$$

6.2.3 Basic Computation for $\alpha \geq 0$

We now define the function $G_1(y)$ as,

$$G_1(y) = \int_0^1 u^2 \, dx. \tag{6.11}$$

The function $G_1(y)$ has the first and second derivatives

$$G_1'(y) = 2\int_0^1 u \, u_y \, dx \quad \text{and} \quad G_1''(y) = 2\int_0^1 (u \, u_{yy} + u_y^2) \, dx.$$

Therefore, by integration by parts and using the relation in (6.7),

$$\begin{aligned} G_1''(y) &= 2\int_0^1 u \, u_{yy} + u_y^2 \, dx = -2\int_0^1 u \, u_{xx} \, dx + 2\int_0^1 u_y^2 \, dx \\ &= -2u \, u_x(1) + 2u \, u_x(0) + 2\int_0^1 u_x^2 \, dx + 2\int_0^1 u_y^2 \, dx \\ (\alpha > 0): &= \frac{2}{\alpha}(u^2(1) + u^2(0)) + 2\int_0^1 u_x^2 \, dx + 2\int_0^1 u_y^2 \, dx \geq 0, \end{aligned}$$

or

$$(\alpha = 0): \quad = \quad 2(\int_0^1 u_x^2 \, dx + \int_0^1 u_y^2 \, dx) \geq 0.$$

We then define H_1 as,

$$H_1(y) = G_1(0)(1 - \frac{1}{L}y) + \frac{G_1(L)}{L}y. \tag{6.12}$$

From the relation in (6.9) and (6.10), we have,

$$G_1(y) \leq H_1(y) = G_1(0)(1 - \frac{1}{L}y) + G_1(L)\frac{y}{L}.$$

We now define another function $G_2(y)$ as,

$$G_2(y) = \int_0^1 u\, u_y\, dx. \tag{6.13}$$

The function $G_2(y)$ has the first and second derivatives

$$G_2'(y) = \int_0^1 u_y^2\, dx + \int_0^1 u\, u_{yy}\, dx,$$

and

$$G_2''(y) = 3\, \int_0^1 u_y\, u_{yy}\, dx + \int_0^1 u\, u_{yyy}\, dx.$$

To check (6.9), we need to consider the following two terms separately,

$$K_1(y) = \int_0^1 u_y\, u_{yy}\, dx, \quad K_2(y) = \int_0^1 u\, u_{yyy}\, dx. \tag{6.14}$$

For the first term,

$$\begin{aligned} K_1'(y) &= \int_0^1 u_{yy}^2\, dx + \int_0^1 u_y\, u_{yyy}\, dx = \int_0^1 u_{yy}^2\, dx - \int_0^1 u_y\, u_{xxy}\, dx \qquad (6.15)\\ &= \int_0^1 u_{yy}^2\, dx + \int_0^1 u_{xy}^2\, dx - u_y u_{xy}(1) + u_y u_{xy}(0)\\ (\alpha > 0): &= \int_0^1 u_{yy}^2\, dx + \int_0^1 u_{xy}^2\, dx + \frac{1}{\alpha}(u_y^2(1) + u_y^2(0)) \geq 0, \end{aligned}$$

or

$$(\alpha = 0): \quad = \int_0^1 u_{yy}^2\, dx + \int_0^1 u_{xy}^2\, dx \geq 0.$$

from the properties in (6.8).

For the second term,

$$\begin{aligned} K_2'(y) &= \int_0^1 u_y\, u_{yyy}\, dx + \int_0^1 u\, u_{yyyy}\, dx \geq -\int_0^1 u\, u_{xxyy}\, dx \quad \text{(by (6.15))} \\ &= \int_0^1 u_x\, u_{xyy}\, dx - u\, u_{xyy}(1) + u\, u_{xyy}(0) \\ &= \int_0^1 u_{yy}^2\, dx - u\, u_{xyy}(1) + u\, u_{xyy}(0) + u_x\, u_{yy}(1) - u_x\, u_{yy}(0) \\ (\alpha > 0): \quad &\geq \frac{1}{\alpha}(u\, u_{yy}(1) + u\, u_{yy}(0)) + u_x\, u_{yy}(1) - u_x\, u_{yy}(0) \\ &= (\frac{1}{\alpha}u + u_x)\, u_{yy}(1) + (\frac{1}{\alpha}u - u_x)\, u_{yy}(0) = 0, \end{aligned}$$

or

$$(\alpha = 0): \quad \geq \int_0^1 u_{yy}^2\, dx \geq 0.$$

from the properties in (6.7) and (6.8).

Let $L(y)$ be given by

$$L(y) = G_2''(y) = K(y) + B(y).$$

From the results above, we have

$$L'(y) \geq 0,$$

and

$$\begin{aligned} L(0) &= 3\int_0^1 u_y\, u_{yy}(0)\, dx + \int_0^1 u\, u_{yyy}(0)\, dx \\ (\alpha > 0): \quad &= -\frac{3}{\alpha}\int_0^1 u\, u_{xx}(0)\, dx - \int_0^1 u\, u_{yxx}(0)\, dx \\ &= -\frac{4}{\alpha}\int_0^1 u\, u_{xx}(0)\, dx = \frac{4}{\alpha}(\int_0^1 u_x^2\, dx - u\, u_x(1) + u\, u_x(0)) \\ &= \frac{4}{\alpha}(\int_0^1 u_x^2\, dx + \alpha\, u_x^2(1) + \alpha\, u_x^2(0)) \geq 0, \end{aligned}$$

or

$$(\alpha = 0): \quad = -3\int_0^1 u_y\, u_{xx}(0)\, dx + \int_0^1 u\, u_{yyy}(0)\, dx = 0,$$

from the properties in (6.7) and (6.8).

From the computational result above, we have

$$G_2''(y) = L(y) \geq 0.$$

Then, define H_2 as,

$$H_2(y) = G_1(0)(1 - \frac{1}{L}y) + \frac{G_1(L)}{L}y. \tag{6.16}$$

From the relation in (6.9) and (6.10), we have,

$$G_2(y) \leq H_2(y) = G_2(0)(1 - \frac{1}{L}y) + G_2(L)\frac{y}{L}. \tag{6.17}$$

Let G_3 be given by,

$$G_3(y) = \frac{1}{\tilde{\alpha}}G_1(y) + G_2(y).$$

From the previous result, we have,

$$G_3'' \geq 0.$$

Then define H_3 as,

$$H_3(y) = G_3(0)(1 - \frac{1}{L}y) + \frac{G_3(L)}{L}y. \tag{6.18}$$

From the relation in (6.9) and (6.10), we have,

$$G_3(y) \leq H_3(y) = G_3(0)(1 - \frac{1}{L}y) + G_3(L)\frac{y}{L}. \tag{6.19}$$

As a last part of this section, we have the following inequality

$$G_2(y) = \int_0^1 u\, u_y\, dx \geq 0, \tag{6.20}$$

with $G_2'(y) \geq 0$ from the previous results and,

$$(\alpha > 0): \quad G_2(0) = \int_0^1 u\, u_y\, dx = \frac{1}{\alpha}\int_0^1 u_y^2\, dx \geq 0, \tag{6.21}$$

or

$$(\alpha = 0): \quad G_2(0) = \int_0^1 u\, u_y\, dx = 0,$$

from the properties in (6.7).

6.2.4 Analysis for the case of $\alpha \geq 0$

The result in this section is important for the analysis of the general Robin boundary condition with two overlapping subdomains as well as for a general number of subdomains with general Robin boundary conditions on the original boundaries and the artificial interfaces.

We begin this section with the following basic property of harmonic functions,

$$\int\int_\Omega e_x^2 + e_y^2 d\Omega = \int_{\partial\Omega} e \frac{\partial e}{\partial n} dS. \tag{6.22}$$

From the relation (6.22), we have the following equation with zero Robin boundary conditions on Θ_S^1, Θ_W^1, and Θ_E^1 with $\alpha > 0$,

$$\begin{aligned} & \int\int_\Omega e_x^2 + e_y^2 d\Omega = \int_{\partial\Omega} e \frac{\partial e}{\partial n} dS = \int_{\Gamma_2} e \frac{\partial e}{\partial n}\, dx + \int_{\Theta_1} e \frac{\partial e}{\partial n}\, dx \\ = & \int_{\Gamma_2} e\, e_y\, dx - \int_{\Theta_S^1} e\, e_y\, dx - \int_{\Theta_W^1} e\, e_x\, dy + \int_{\Theta_E^1} e\, e_x\, dy \\ = & \int_{\Gamma_2} e\, e_y\, dx - \frac{1}{\alpha}\int_{\Theta_S^1} e^2\, dx - \frac{1}{\alpha}\int_{\Theta_W^1} e^2\, dy - \frac{1}{\alpha}\int_{\Theta_E^1} e^2\, dy. \end{aligned}$$

Since the H^1- semi-norm is nonnegative, we have the following inequality,

$$\int_{\Gamma_2} e\, e_y\, dx \geq \frac{1}{\alpha}\int_{\Theta_S^1} e^2\, dx + \frac{1}{\alpha}\int_{\Theta_W^1} e^2\, dy + \frac{1}{\alpha}\int_{\Theta_E^1} e^2\, dy > \frac{1}{\alpha}\int_{\Theta_S^1} e^2\, dx. \tag{6.23}$$

We have the following with the general Robin boundary condition with the notation above,

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12} + \tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) = -\int_{\Gamma_2} e^{12}\, e_y^{12}\, dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2} |e^{12}|^2\, dx, \tag{6.24}$$

and

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}}) = \int_{\Gamma_1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2\,dx. \tag{6.25}$$

From the result of (6.19) and (6.7), we have,

$$\begin{aligned}
&\int_{\Gamma_2} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2}|e^{12}|^2\,dx \\
\leq\ & C_0\,(\int_{\Theta_S^1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Theta_S^1}|e^{12}|^2\,dx) + C_1\,(\int_{\Gamma_1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2\,dx) \\
=\ & C_0\,(\frac{1}{\alpha}+\frac{1}{\tilde{\alpha}})\int_{\Theta_S^1}|e^{12}|^2\,dx + C_1\,(\int_{\Gamma_1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2\,dx)
\end{aligned}$$

with

$$C_0 = (1-\frac{l}{L}),\quad C_1 = \frac{l}{L},\quad C_0+C_1=1,\quad 0<C_0<1,\quad 0<C_1<1.$$

We have the following inequality using (6.23)

$$\begin{aligned}
&-\int_{\Gamma_2} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2}|e^{12}|^2\,dx \qquad (6.26)\\
=\ & \int_{\Gamma_2} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2}|e^{12}|^2\,dx - 2\int_{\Gamma_2} e^{12}\,e_y^{12}\,dx \\
\leq\ & C_1\,(\int_{\Gamma_1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2\,dx) + C_0\,(\frac{1}{\alpha}+\frac{1}{\tilde{\alpha}})\int_{\Theta_S^1}|e^{12}|^2\,dx - 2\int_{\Gamma_2} e^{12}\,e_y^{12}\,dx \\
<\ & C_1\,(\int_{\Gamma_1} e^{12}\,e_y^{12}\,dx + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2\,dx) \quad \text{with} \quad C_0\,(\frac{1}{\alpha}+\frac{1}{\tilde{\alpha}}) \leq \frac{2}{\alpha}.
\end{aligned}$$

For $\alpha = 0$, we have the following results,

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) = -\int_{\Omega^1}\nabla e^1\cdot\nabla e^1 + \frac{1}{\tilde{\alpha}}\int_{\Gamma_2}|e^{12}|^2, \tag{6.27}$$

and

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}}) = \int_{\Omega_1}\nabla e_1\cdot\nabla e_1 + \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2. \tag{6.28}$$

From the result above and (6.20), we have the following results with $\tilde{\alpha}\geq\alpha(\frac{2-C_0}{C_0})$,

$$\begin{aligned}
\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) &< C_1(\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}})) \\
&< (\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}})),
\end{aligned}$$

with $C_1 = \frac{l}{L} < 1$ which is the ratio between the original length and the overlapped length which has a smaller value with more overlap. From (6.5), we have,

$$||e_2^{12}||_{B^{12}}^2 < ||e_1^{12}||_{B^{12}}^2.$$

Finally we have,

$$||\lambda_1||_{m'}^2 < ||\lambda_2||_{m'}^2. \tag{6.29}$$

Therefore the norm of the Lagrange multipliers decreases strictly in Algorithm 3 (OSM-D).

6.2.5 Geometric Convergence of the Lagrange Multipliers

The main results of the inequalities (6.29) is that there exist a uniform factor which is strictly less than 1 and is independent of the error vector. To prove the convergence of Algorithm 3 (OSM-D), we need to compare the two quantities $||e_2^{12}||_{B^{12}}^2$, and $||e_1^{12}||_{B^{12}}^2$. In the subsection 6.2.4, we have studied the relation of two other quantities $||e_2^{12}||_{B^{12}}^2 + (e_1^{12})^T B^{12} e_2^{12}$ and $||e_1^{12}||_{B^{12}}^2 + (e_1^{12})^T B^{12} e_2^{12}$. Therefore, we can simplify the problem as the follows, If we know that

$$A + C < \rho(B + C) \quad \text{with} \quad \rho < 1, \tag{6.30}$$

then is it possible to find a uniform factor $\mu < 1$ such that

$$A < \mu B \quad ?$$

This statement is generally not true.

However, we can prove this in our case from the strengthened C.B.S. inequality in 1.5. Let $V(\Gamma_1)$ be the vector space with an element $e_1 \in V(\Gamma_1)$ which corresponds to the Lagrange multiplier λ_1 on the discontinuous artificial interface Γ_1

such that,

$$B^{12}e_1 = (I_1^{12})^T\lambda_1 \quad \forall e_1 \in V(\Gamma_1).$$

We also define $V(\Gamma_2)$ as

$$B^{12}e_2 = (I_2^{12})^T\lambda_2 \quad \forall e_2 \in V(\Gamma_2).$$

We then have,

$$V(\Gamma_1) \cap V(\Gamma_2) = \{0\}.$$

Therefore, the following strengthened C.B.S.-inequality holds:

$$|(e_1, e_2)_{B^{12}}| \leq \beta ||e_1||_{B^{12}} ||e_2||_{B^{12}}, \quad \forall e_1 \in V(\Gamma_1), \quad \forall e_2 \in V(\Gamma_2) \quad \beta < 1. \tag{6.31}$$

We can identify each term in (6.30) as,

$$A = ||e_2^{12}||^2_{B^{12}}, \quad B = ||e_1^{12}||^2_{B^{12}}, \quad C = (e_1^{12})^T B^{12} e_2^{12}, \quad \rho = C_1 = \frac{y}{L} < 1.$$

From (6.31), we have,

$$|C| \leq \beta\sqrt{AB}, \quad \text{with} \quad \beta < 1. \tag{6.32}$$

The inequality (6.30) can be rewritten as,

$$A + (1 - \rho)C < \rho B.$$

If $C \geq 0$, then we have $A < \rho B$.

If $C < 0$, we have the following from (6.32),

$$0 < -C \leq \beta\sqrt{AB}.$$

We also have,

$$A + (\rho - 1)(-C) < \rho B,$$

which implies

$$\frac{A}{B} + (\rho - 1)\beta\sqrt{\frac{A}{B}} < \rho.$$

Finally, we have

$$0 < \sqrt{\frac{A}{B}} < b,$$

with

$$b = \frac{(1-\rho)\beta + \sqrt{\beta^2(1-\rho)^2 + 4\rho}}{2} < 1.$$

Therefore, we have

$$A < b^2 B.$$

Finally we have the main inequality,

$$||e_2^{12}||^2_{B^{12}} < \mu ||e_1^{12}||^2_{B^{12}} \quad \mu < 1, \tag{6.33}$$

where μ is independent of the error vectors.

6.2.6 Geometric Convergence of Algorithm 3 (OSM-D) on Two Overlapping Subdomains

We have studied the behavior of the error vectors with the general Robin boundary condition on two overlapping subdomains in subsection 6.2.5. From the inequality (6.33), the Lagrange multiplier converges to zero. The meaning of the Lagrange multiplier on Γ_1 is the nonzero Robin boundary condition on the discontinuous artificial interface. Since the other boundary segments have zero Robin conditions, the error vector on the subdomain depends only on the Lagrange multiplier. We have compared two Lagrange multipliers, the Lagrange multipliers on the discontinuous artificial interface and that on the continuous artificial interface. The Lagrange multipliers on the continuous artificial interface will be the Lagrange multipliers

on the discontinuous artificial interface for next step. Therefore, we have actually compared two boundary conditions on the new and old problems and a certain quantity of the new boundary condition is strictly less than that of the old boundary condition. Since the sequence of the quantities of the boundary condition converges to zero. We obtain a zero Robin boundary condition on all boundary segments in the limit. Therefore the error vector converges to zero. Since, in this case, we have a uniform convergence factor, we have geometric convergence of Algorithm 3 (OSM-D) on two overlapping subdomains

6.3 Convergence Theory for the General Two Subdomain Case

The main issue for this case is to find an inequality similar to (6.29) in the general two subdomain case. Since we have following two equalities,

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}}) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_1}|e^{12}|^2 + \int_{\Omega_1}\nabla e^{12}\cdot\nabla e^{12} + \frac{1}{\alpha}\int_{\Theta_1}|e^{12}|^2, \quad (6.34)$$

and

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) = \frac{1}{\tilde{\alpha}}\int_{\Gamma_2}|e^{12}|^2 - (\int_{\Omega^1}\nabla e^{12}\cdot\nabla e^{12} + \frac{1}{\alpha}\int_{\Theta^1}|e^{12}|^2), \quad (6.35)$$

we want find a uniform factor $\rho(\Omega_1,\Omega^1)<1$ such that

$$\frac{1}{\tilde{\alpha}}\int_{\Gamma_2} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_2}}) \;<\; \rho(\Omega_1,\Omega^1)\,(\frac{1}{\tilde{\alpha}}\int_{\Gamma_1} e^{12}\,(e^{12}+\tilde{\alpha}\frac{\partial e^{12}}{\partial n_{\Gamma_1}})). \quad (6.36)$$

From Friedrichs' Inequality, we have

$$\int_{\Omega_1}\nabla e^{12}\cdot\nabla e^{12} + \frac{1}{\alpha}\int_{\Theta_1^1}|e^{12}|^2 \geq (C_F+1)\;F(\alpha)\;\left(|e^{12}|^2_{H^1(\Omega_1)} + ||e^{12}||^2_{L^2(\Omega_1)}\right),$$

with

$$F(\alpha) = \frac{1}{\max(1,\alpha)}.$$

From the trace theorem, we have the following sequence of inequalities,

$$\begin{aligned}
\int_{\Omega_1} \nabla e^{12} \cdot \nabla e^{12} + \frac{1}{\alpha} \int_{\Theta_1^1} e^{12}\, e^{12} &\geq (C_F + 1)\ F(\alpha)\ \left(|e^{12}|^2_{H^1(\Omega_1)} + ||e^{12}||^2_{L^2(\Omega_1)} \right) \\
&> (C_F + 1)\ F(\alpha)\ \left(|e^{12}|^2_{H^1(\Omega^1)} + ||e^{12}||^2_{L^2(\Omega^1)} \right) \\
&\geq (C_F + 1)\ F(\alpha)\ C_T\ ||e^{12}||^2_{L^2(\partial\Omega^1)} \\
&> (C_F + 1)\ F(\alpha)\ C_T\ \int_{\Gamma_2} |e^{12}|^2.
\end{aligned}$$

The two constants C_F and C_T depend on the geometry of the subdomain and the atomic subdomain. With a sufficiently large $\tilde{\alpha}$, we have,

$$\rho(\Omega_1, \Omega^1) = \frac{1}{(C_F + 1)\ F(\alpha)\ C_T\ \tilde{\alpha}} < 1.$$

Finally we have the following results for a sufficiently large $\tilde{\alpha}$,

$$\begin{aligned}
& \rho(\Omega_1, \Omega^1)\ \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e^{12}\, (e^{12} + \tilde{\alpha} \frac{\partial e^{12}}{\partial n_{\Gamma_1}}) - \int_{\Gamma_2} e^{12}\, \frac{\partial e^{12}}{\partial n_{\Gamma_2}} \qquad (6.37) \\
= \ & \rho(\Omega_1, \Omega^1)\ \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e^{12}\, (e^{12} + \tilde{\alpha} \frac{\partial e^{12}}{\partial n_{\Gamma_1}}) + \int_{\Omega^1} \nabla e^{12} \cdot \nabla e^{12} + \frac{1}{\alpha} \int_{\Theta^1} e^{12}\, e^{12} \\
> \ & \rho(\Omega_1, \Omega^1)\ \frac{1}{\tilde{\alpha}} \int_{\Gamma_1} e^{12}\, (e^{12} + \tilde{\alpha} \frac{\partial e^{12}}{\partial n_{\Gamma_1}}) \\
> \ & \rho(\Omega_1, \Omega^1)\ (\int_{\Omega_1} \nabla e^{12} \cdot \nabla e^{12} + \frac{1}{\alpha} \int_{\Theta_1^1} e^{12}\, e^{12}) \\
> \ & \rho(\Omega_1, \Omega^1)\ (C_F + 1)\ F(\alpha)\ C_T\ \int_{\Gamma_2} |e^{12}|^2 \\
= \ & \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} e^{12}\, e^{12}.
\end{aligned}$$

Therefore, (6.36) holds.

Chapter 7

Numerical Results

7.1 Two-level Algorithms

The convergence rate of one-level classical Schwarz algorithms deteriorates rapidly with the number of subdomains. This is due to the fact that in the one-level classical algorithms, information is passed only between neighboring subdomains. To overcome this weakness of the algorithms, we can introduce a coarse space which has a small number of degrees of freedom in each subdomain. The coarse global problem set over the coarse space provides the mechanism for global communication of information between all subdomains in each iteration.

In this section, we will compare three two-level Overlapping Schwarz Algorithms. We will use the notations and definitions of Chapter 2.

7.1.1 Two-level Classical Algorithm

The fractional step of the classical overlapping Schwarz algorithm can be written as,

$$u_{(n+j/p)} = u_{(n+(j-1)/p)} + A_j^+(b - Au_{(n+(j-1)/p)}). \tag{7.1}$$

To facilitate global communication between distant subregions, we can include a coarse grid correction in the classical overlapping Schwarz algorithm. Let A^c be a coarse grid discretization of the form $a(\cdot,\cdot)$ in (2.4). Let R^c be a change of basis map from the coarse grid basis to the fine grid base. Then the coarse grid correction in the fractional step is given by c_n^c which satisfies

$$c_n^c = R^c(A^c)^{-1}(R^c)^t(b - Au_n). \tag{7.2}$$

From the above equation, a complete listing of the classical overlapping Schwarz algorithm is

$$\begin{aligned} r_n &:= b - Au_n \\ c_n^c &:= R^c(A^c)^{-1}(R^c)^t r_n \\ u_n &:= u_n + c_n^c \\ u_{n+1/p} &:= u_n + A_1^+(b - Au_n) \\ &\cdots \\ u_{n+1} &:= u_{(n+(p-1)/p)} + A_p^+(b - Au_{(n+(p-1)/p)}). \end{aligned} \tag{7.3}$$

7.1.2 Two-level Algorithm 2 (OSM-C)

The two-level cycle of Algorithm 2 (OSM-C) has the same form as the classical overlapping Schwarz algorithm. In the j-th fractional step of Algorithm 2 (OSM-C), all nodal values in the closure of Ω_j are changed but all other nodal values are left unchanged. Using the projection P_j, we modify the listing of the classical overlapping Schwarz algorithm and obtain

$$r_n := b - Au_n$$

$$
\begin{aligned}
c_n^c &:= R^c (A^c)^{-1} (R^c)^t r_n \\
u_n &:= u_n + c_n^c \\
u_{(n+1/p)} &:= P_1^c u_n + P_1 A_1^+ (b - A_1^c u_n) \\
&\cdots \\
u_{n+1} &:= P_p^c u_{(n+(p-1)/p)} + P_p A_p^+ (b - A_p^c u_{(n+(p-1)/p)}).
\end{aligned} \tag{7.4}
$$

7.1.3 Two-level Algorithm 3 (OSM-D)

The two-level cycle of Algorithm 3 (OSM-D) will be derived from (7.4). In the previous algorithm, we constructed the coarse grid correction from the residual. To define the coarse grid correction in Algorithm 3 (OSM-D), we need to consider the special aspects of the residual in Algorithm 3 (OSM-D). Since Algorithm 3 (OSM-D) allows multiple values on the artificial interfaces, the residuals of the fractional steps are acquired through their contributions from the atomic subregions and according to (2.17), the residual can also be computed from the jump directly. Using partitioned matrices $\tilde{\Lambda}$ and $\tilde{A}$, we have the listing of two-level Algorithm 3 (OSM-D) given by

$$
\begin{aligned}
r_n &:= b - \tilde{R}^t \tilde{A} \tilde{u}_n \\
c_n^c &:= R^c (A^c)^{-1} (R^c)^t r_n \\
\tilde{u}_n &:= \tilde{u}_n + \tilde{R} c_n^c \\
\tilde{u}_{(n+1/p)} &:= \tilde{P}_1^c \tilde{u}_n + \tilde{P}_1 \tilde{R} A_1^+ (b - \tilde{R}^t \tilde{P}_1^c \tilde{\Lambda} \tilde{u}_n) \\
&\cdots \\
\tilde{u}_{n+1} &:= \tilde{P}_p^c \tilde{u}_{(n+(p-1)/p)} + \tilde{P}_p \tilde{R} A_p^+ (b - \tilde{R}^t \tilde{P}_p^c \tilde{\Lambda} \tilde{u}_{(n+(p-1)/p)}).
\end{aligned} \tag{7.5}
$$

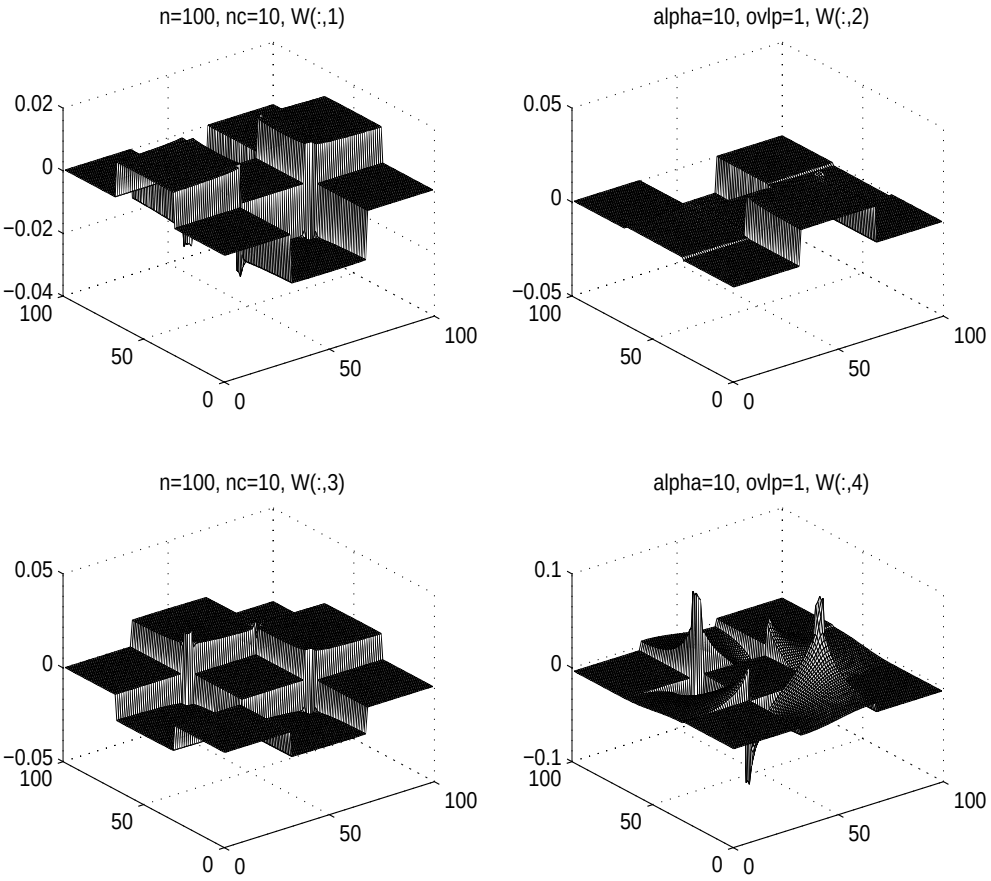

Figure 7.1: An example of the four approximate eigenvectors corresponding to negative real eigenvalues of two level Algorithm 2 (OSM-C) with GMRES on nine overlapping domain with $n = 100$, $nc = 10$, and $ovlp = 1$

7.1.4 Algorithm 2 (OSM-C) and Algorithm 3 (OSM-D) with GMRES

As we can see from the listings, Algorithm 2 (OSM-C) and Algorithm 3 (OSM-D) do not use the conventional residuals in the fractional steps. To apply GMRES to the two algorithms for solving $Ax = b$, we need to change to another related problem. Algorithm 2 (OSM-C) and Algorithm 3 (OSM-D) can be thought of in

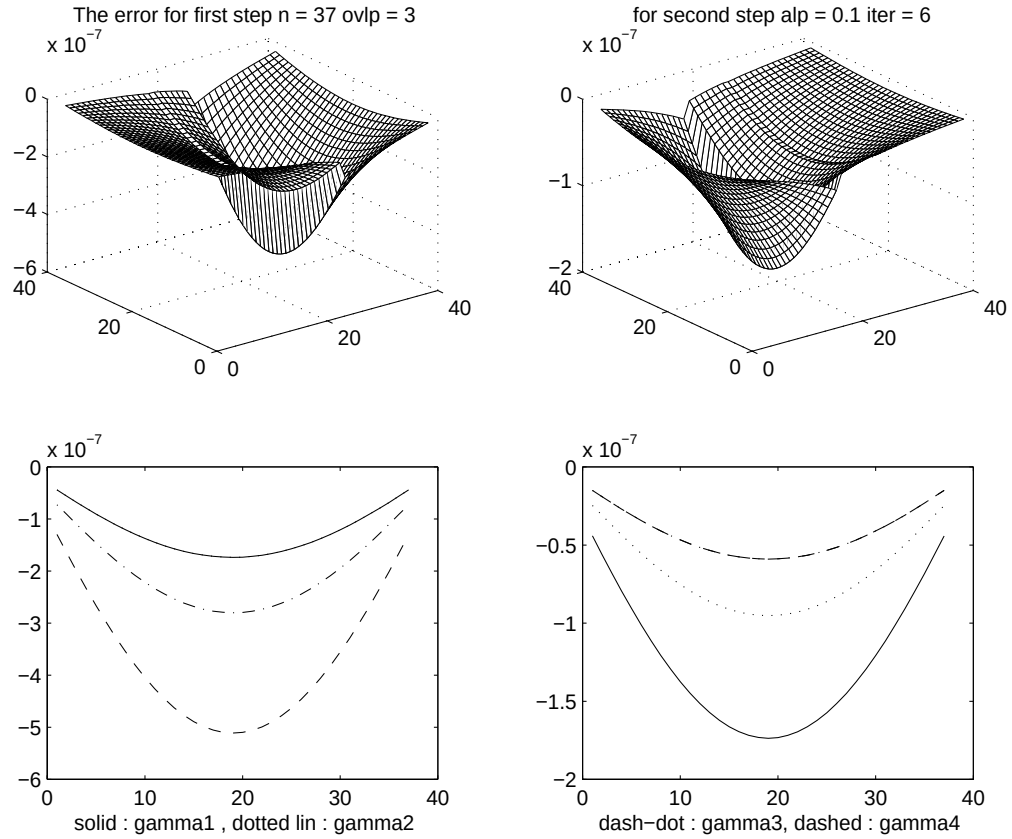

Figure 7.2: An example of error vectors on the artificial interfaces and the atomic subdomains on two overlapping subdomains; $e^{5+1/2}$ and e^6 of Ω^1 on Γ_2 (solid line), of Ω^{12} on Γ_2 (dotted line), of Ω^{12} on Γ_1 (dash-dotted line), of Ω^2 on Γ_1 (dashed line) in 6-th step with $n = 37, ovlp = 3, \alpha = 0.1$.

terms of affine linear maps,

$$x_{n+1} = F(x_n) \quad \text{with} \quad F(x) = Mx + F(0). \tag{7.6}$$

If the problem has a solution, it is the fixed point of

$$(I - M)x = C \quad \text{with} \quad C = F(0). \tag{7.7}$$

We can apply GMRES to equation (7.7).

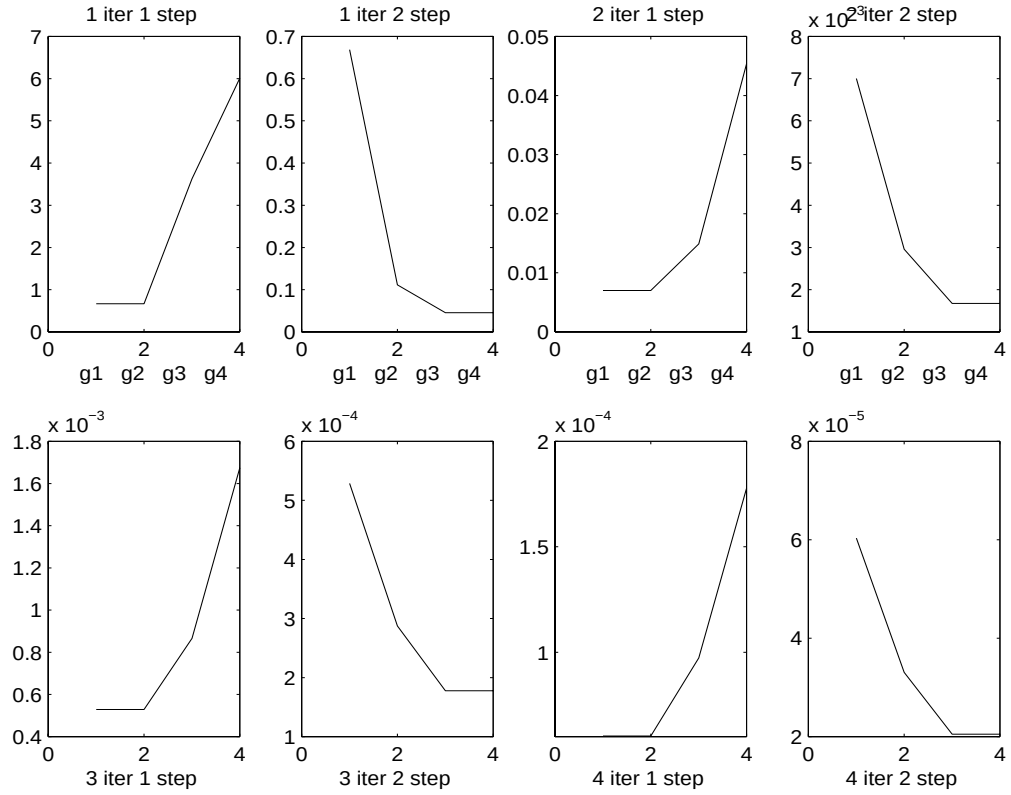

Figure 7.3: An example of l^2 norm of error vectors on the artificial interfaces on two overlapping subdomains of Algorithm 3 (OSM-D); e^{β} of Ω^1 on Γ_2 (1), of Ω^{12} on Γ_2 (2), of Ω^{12} on Γ_1 (3), of Ω^2 on Γ_1 (4) in 1 - 4 iteration with $n = 37, ovlp = 3, \alpha = 0.1$. (Here $g1$ and $g2$ are two artificial interfaces (inside and outside interfaces) for the Γ_2 and $g3$ and $g4$ for Γ_1.)

7.2 Approximate Eigenvalues and Divergence of Algorithm 2 (OSM-C)

We have implemented several cases of Algorithm 2 (OSM-C) with nine overlapping subdomains. For fixed $n = 100$, we used $\alpha = \tilde{\alpha} = 0.01, 0.1, 1$, and 10 for one-level method with GMRES and two level method with a $nc = 10$ coarse grid and GMRES to check the dominating eigenvalues and eigenvectors of the iteration

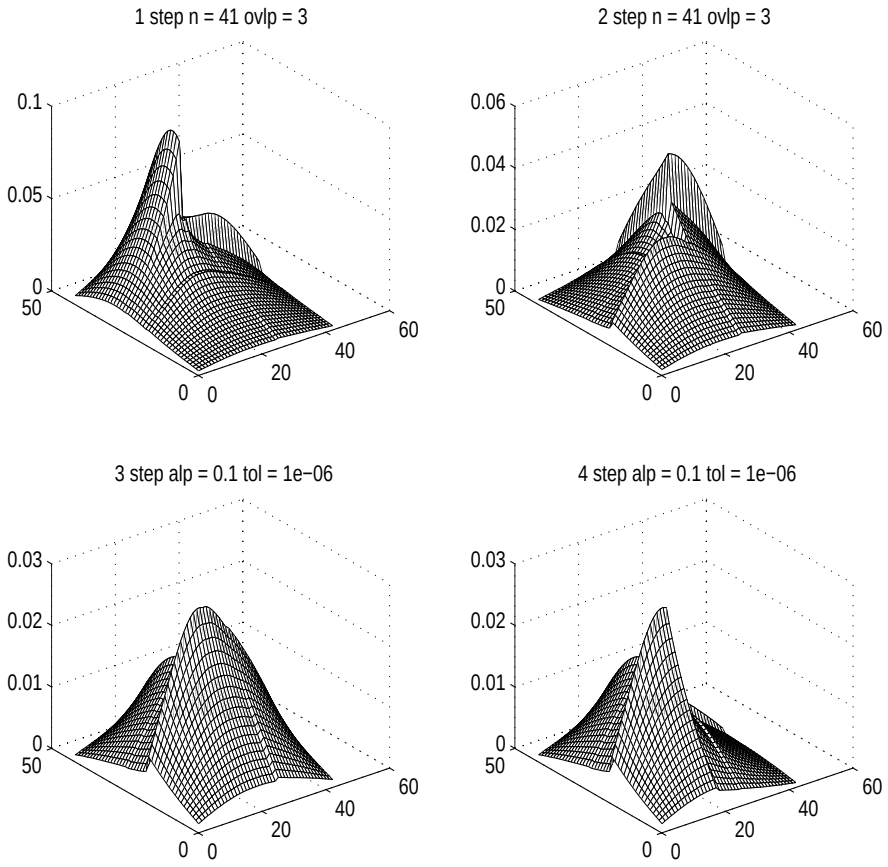

Figure 7.4: An example of error vectors on four overlapping subdomains with $n = 37$, $\alpha = 0.1$, $ovlp = 3$.

operator. To compare the numerical and exact solutions, we used two different exact solutions with $\alpha = \tilde{\alpha} = 0.01, 1$, and 10. From the computational results, we find.

- If we apply one-level method with GMRES, Algorithm 2 converges for $\alpha = 0.01$ and diverge for the other values of α. A larger α makes divergence faster. In the cases of $\alpha = 0.1, 1$ and 10 we have four eigenvalues which have negative real part. Their eigenvectors have four spikes at the four cross points. The

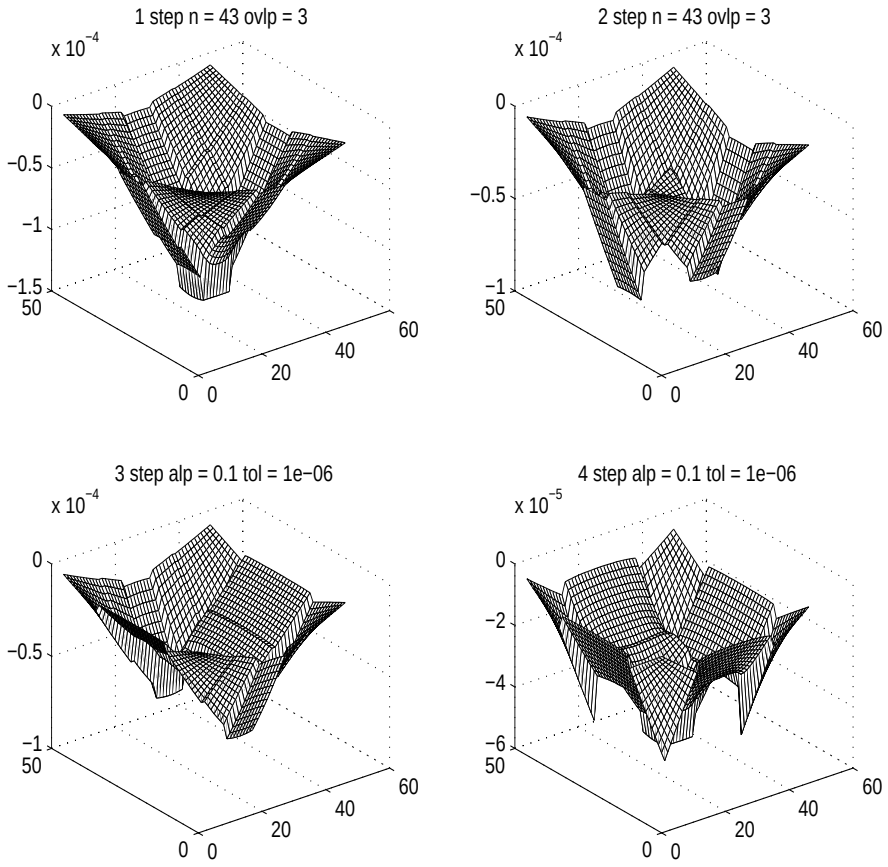

Figure 7.5: An example of error vectors on nine overlapping subdomains of Algorithm 3 (OSM-D) with $n = 43$, $\alpha = 0.1$, $ovlp = 3$.

associated eigenvalues do not change with different exact solutions.

- If we apply two level methods with GMRES, the results are similar to those of one-level with GMRES. The coarse grid correction does not affect the four eigenvalues which have negative real part and their eigenvectors very much.

An example of the four approximate eigenvectors corresponding to the eigenvalues with negative real part of two level Algorithm 2 (OSM-C) with GMRES is given in Figure 7.1.

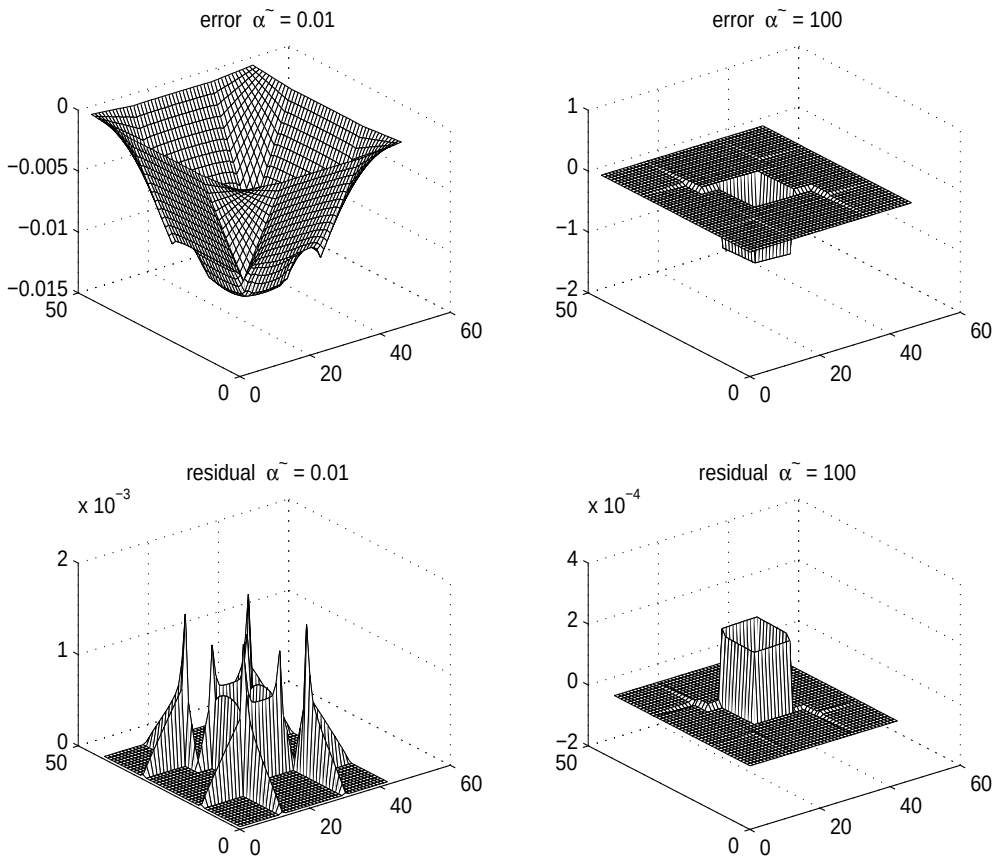

Figure 7.6: A comparison between residual and error vector of Algorithm 3 (OSM-D) of 4-th iteration (on four coloring) in 11-th step with n=43, $ovlp$=2, $\tilde{\alpha} = 100$ and $\tilde{\alpha} = 0.01$ for $\alpha = 0.01$.

7.3 Numerical results for Algorithm 3 (OSM-D)

7.3.1 Numerical examples

In this subsection, we will illustrate the behavior of the fractional error vectors on two, four, nine and a general number of subdomains.

Two overlapping subdomains

In a unit rectangular domain, we compare each error vector on the artificial in-

terfaces. There are two artificial interfaces in this case. On each interface, we need both old and new values. Figure 7.2 shows that the numerical error on each artificial boundary decreases to zero. Another illustration, in Figure 7.3, shows the behavior of the l^2 norm for four parts of two artificial interfaces, $g1$ and $g2$ for Γ_2 and $g3$ and $g4$ for Γ_1.

Four and Nine overlapping subdomains

We have a similar behavior of the error vector for four and nine overlapping subdomains. Figure 7.4 is for four overlapping subdomains and Figure 7.5 for nine overlapping subdomains.

7.3.2 Discontinuity and $\tilde{\alpha}$

From the numerical results, we see that the error and residual vectors depend on the values of $\tilde{\alpha}$. As we can see in Figure 7.6, a larger $\tilde{\alpha}$ makes the jumps of the error vectors larger across the artificial interfaces.

7.3.3 Numerical results of one-level Algorithm 3 (OSM-D)

Table 7.1 is a table of the number of iterations for a residual reduction of 10^{-6}. We used the 501 as a maximum number of iterations. The value $(**)$ means that the residuals are still decreasing but has not reached a reduction of 10^{-6}. For a fixed α, the numerical results deteriorate with significantly smaller $\tilde{\alpha}$. We note that a given α, the condition in (6.26) in the two overlapping case might not hold for significantly smaller $\tilde{\alpha}$. It seems that we have a similar condition for the more general case. We have tested with overlap $ovlp = 1$ and $ovlp = 2$ and the cases of overlap $ovlp = 2$ gives better results than with overlap $ovlp = 1$ in most cases.

7.4 Numerical results for Two-level Algorithm 3 (OSM-D)

Figure 7.7 and 7.8 are results for two different sizes of the coarse grid correction. To check the effect of the coarse grid correction, we vary Θ, the relaxation factor, from 0 to 1. We have used $\Theta = 0, 0.25, 0.5, 0.75, 1$. The results change continuously with the value of Θ. The case of small $\tilde{\alpha}$ improve much with even the smallest relaxation factor $\Theta = 0.25$. This means that if we use the Robin boundary condition which is close to the Dirichlet condition ($\tilde{\alpha} \to 0$), then the results with the coarse grid corrections behave as in the classical theory of overlapping Schwarz methods. However, with a larger value of $\tilde{\alpha}$, the numerical results are different. It seems that a coarse grid correction does not improve the convergence in such cases.

We also have numerical results with $Nc = 0, 4, 6, 11, 16$, different sizes of the coarse grid correction. Table 7.2 shows some of them. The results show that with small $\tilde{\alpha}$, the best results appear with $Nc = 16$ but with larger $\tilde{\alpha}$, the results can be different. From our experiments, we find that two-level Algorithm 3 (OSM-D) does not always converge with a large $\tilde{\alpha}$.

7.5 Numerical results for One-level Algorithm 3 (OSM-D) with GMRES

Table 7.3 is a table of the number of iterations with GMRES with restart after every 30 iterations for a residual reduction of 10^{-6}. We used the 150 as the maximum number of iterations. The value $(**)$ means that the residual is still decreasing but has not reached a reduction of 10^{-6}.

Table 7.1: Number of iterations of Algorithm 3 (OSM-D) for a residual reduction of 10^{-6} versus α, $\tilde{\alpha}$, number of grid points (N), number of subdomains, and overlapping size(ovlp).

N	$(51)^2$		$(101)^2$		$(101)^2$		$(201)^2$	
$\sharp$ of subdomains	100		100		400		100	
α, $\tilde{\alpha}$ \ ovlp	1	2	1	2	1	2	1	2
α=0.01, $\tilde{\alpha}$=100	65	48	104	61	166	135	215	111
α=0.01, $\tilde{\alpha}$=10	53	41	84	51	152	113	143	87
α=0.01, $\tilde{\alpha}$=1	28	23	40	28	72	52	53	40
α=0.01, $\tilde{\alpha}$=0.1	14	12	18	15	24	20	20	18
α=0.01, $\tilde{\alpha}$=0.01	58	31	91	57	150	90	126	91
α=0.1, $\tilde{\alpha}$=100	86	62	148	85	237	174	295	156
α=0.1, $\tilde{\alpha}$=10	67	53	109	66	204	147	174	113
α=0.1, $\tilde{\alpha}$=1	29	25	40	30	73	60	53	41
α=0.1, $\tilde{\alpha}$=0.1	33	24	41	34	66	52	46	41
α=0.1, $\tilde{\alpha}$=0.01	141	75	229	143	399	234	321	232
α=1, $\tilde{\alpha}$=100	116	74	238	139	462	290	363	253
α=1, $\tilde{\alpha}$=10	59	60	87	67	169	145	106	90
α=1, $\tilde{\alpha}$=1	22	17	28	22	52	39	34	29
α=1, $\tilde{\alpha}$=0.1	126	91	157	130	273	215	179	161
α=1, $\tilde{\alpha}$=0.01	(**)	293	(**)	(**)	(**)	(**)	(**)	(**)
α=10, $\tilde{\alpha}$=100	80	81	125	95	244	206	150	131
α=10, $\tilde{\alpha}$=10	35	20	61	37	114	75	95	63
α=10, $\tilde{\alpha}$=1	91	97	85	88	174	185	116	97
α=10, $\tilde{\alpha}$=0.1	(**)	(**)	(**)	(**)	(**)	(**)	(**)	(**)
α=10, $\tilde{\alpha}$=0.01	(**)	(**)	(**)	(**)	(**)	(**)	(**)	(**)
α=100, $\tilde{\alpha}$=100	25	20	47	27	89	45	82	50
α=100, $\tilde{\alpha}$=10	106	106	113	106	229	220	144	116
α=100, $\tilde{\alpha}$=1	464	(**)	405	446	(**)	(**)	(**)	(**)
α=100, $\tilde{\alpha}$=0.1	(**)	(**)	(**)	(**)	(**)	(**)	(**)	(**)
α=100, $\tilde{\alpha}$=0.01	(**)	(**)	(**)	(**)	(**)	(**)	(**)	(**)

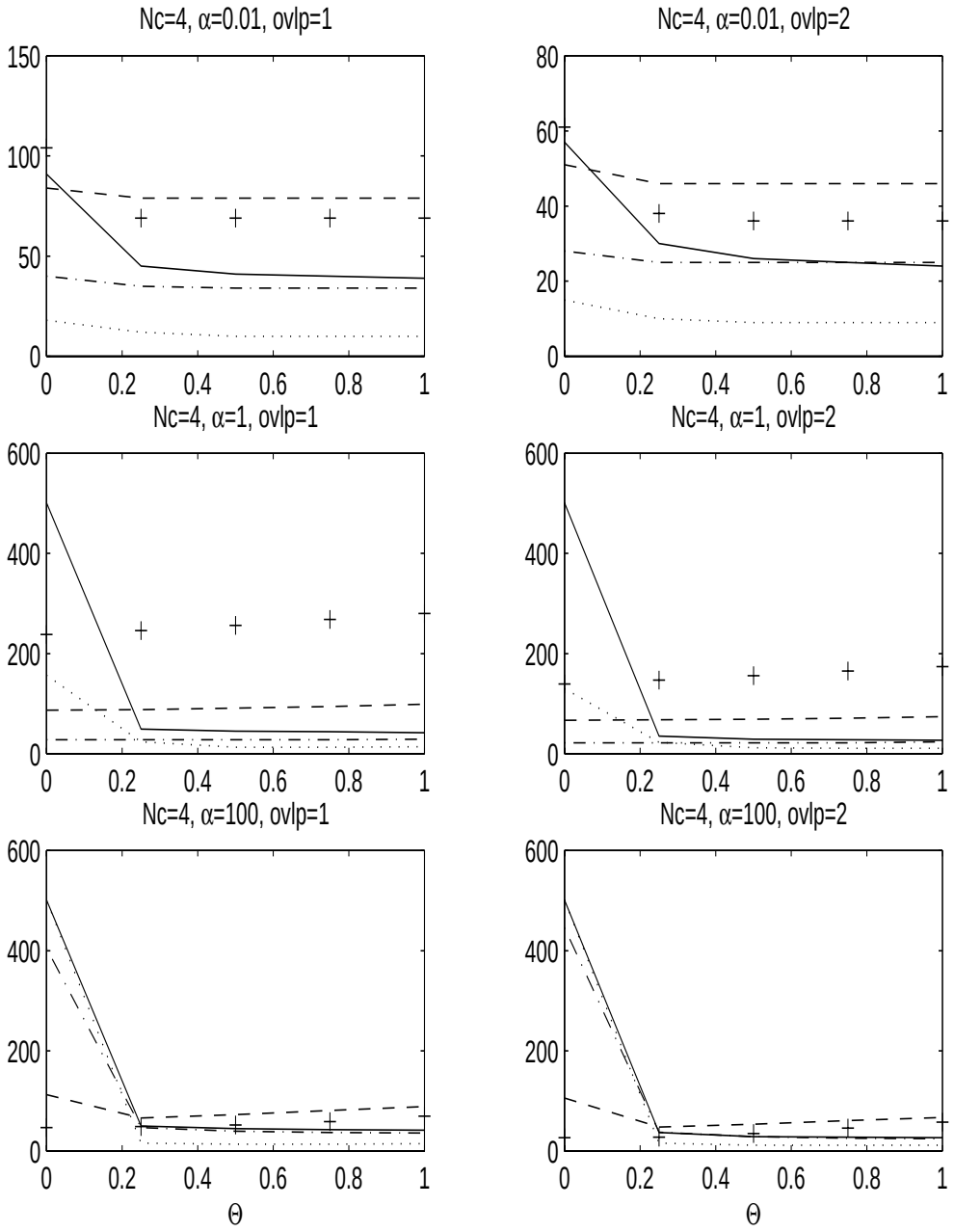

Figure 7.7: Number of iterations of two-level Algorithm 3 with $N = (101)^2$, $Nc = 4$ and 100 subdomains with relaxation factor Θ=0, 0.25, 0.5, 0.75, 1; solid line : $\tilde{\alpha} = 0.01$, dotted line : $\tilde{\alpha} = 0.1$, dash-dot line : $\tilde{\alpha} = 1$, dashed line : $\tilde{\alpha} = 10$, plus : $\tilde{\alpha} = 100$.

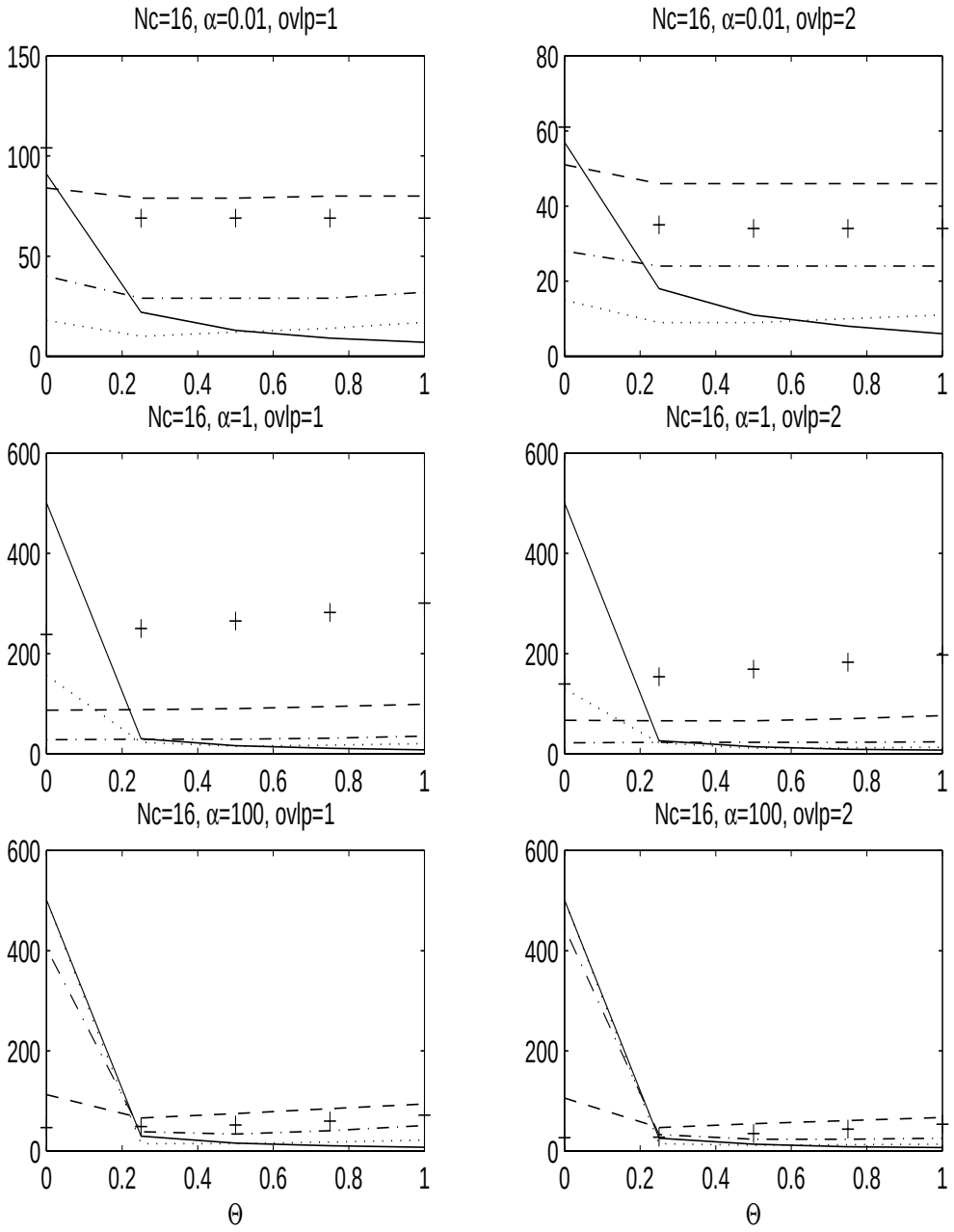

Figure 7.8: Number of iterations of two-level Algorithm 3 with $N = (101)^2$, $Nc = 16$ and 100 subdomains with relaxation factor Θ=0, 0.25, 0.5, 0.75, 1; solid line : $\tilde{\alpha} = 0.01$, dotted line : $\tilde{\alpha} = 0.1$, dash-dot line : $\tilde{\alpha} = 1$, dashed line : $\tilde{\alpha} = 10$, plus : $\tilde{\alpha} = 100$.

Table 7.2: Number of iterations of Two-level Algorithm 3 (OSM-D) for a residual reduction of 10^{-6} with $N = (101)^2$ and 100 overlapping subdomains versus α, $\tilde{\alpha}$, the size of coarse grid correction Nc, and overlapping size(ovlp).

Nc	0		4		6		11		16	
α, $\tilde{\alpha}$ \ ovlp	1	2	1	2	1	2	1	2	1	2
α=0.01, $\tilde{\alpha}$=100	104	61	69	36	68	34	69	34	69	34
α=0.01, $\tilde{\alpha}$=10	84	51	79	46	78	46	78	45	80	46
α=0.01, $\tilde{\alpha}$=1	40	28	34	25	28	24	30	23	32	24
α=0.01, $\tilde{\alpha}$=0.1	18	15	10	9	10	9	16	10	17	11
α=0.01, $\tilde{\alpha}$=0.01	91	57	39	24	23	15	7	7	7	6
α=0.1, $\tilde{\alpha}$=100	148	85	122	60	141	93	143	92	140	89
α=0.1, $\tilde{\alpha}$=10	109	66	106	64	108	82	107	74	110	79
α=0.1, $\tilde{\alpha}$=1	40	30	37	28	30	27	35	26	36	28
α=0.1, $\tilde{\alpha}$=0.1	41	34	12	10	11	10	18	11	19	12
α=0.1, $\tilde{\alpha}$=0.01	229	143	43	27	26	17	8	8	8	7
α=1, $\tilde{\alpha}$=100	238	139	280	174	285	193	300	201	301	197
α=1, $\tilde{\alpha}$=10	87	67	99	74	90	74	95	78	99	76
α=1, $\tilde{\alpha}$=1	28	22	29	24	29	23	36	23	35	24
α=1, $\tilde{\alpha}$=0.1	157	130	14	11	13	11	19	12	20	13
α=1, $\tilde{\alpha}$=0.01	(**)	(**)	42	27	26	17	8	8	8	7
α=10, $\tilde{\alpha}$=100	125	95	149	114	140	120	154	129	154	124
α=10, $\tilde{\alpha}$=10	61	37	87	64	75	53	80	38	91	60
α=10, $\tilde{\alpha}$=1	85	88	33	24	30	24	39	24	40	25
α=10, $\tilde{\alpha}$=0.1	(**)	(**)	15	12	13	11	20	14	22	14
α=10, $\tilde{\alpha}$=0.01	(**)	(**)	42	27	26	17	8	8	8	7
α=100, $\tilde{\alpha}$=100	47	27	70	58	56	51	60	41	72	54
α=100, $\tilde{\alpha}$=10	113	106	89	67	78	61	86	54	94	67
α=100, $\tilde{\alpha}$=1	405	446	36	25	31	24	44	24	51	26
α=100, $\tilde{\alpha}$=0.1	(**)	(**)	15	12	13	11	21	14	22	14
α=100, $\tilde{\alpha}$=0.01	(**)	(**)	42	27	26	17	8	8	8	7

Table 7.3: Number of iterations of Algorithm 3 (OSM-D) with GMRES (30) for a residual reduction of 10^{-6} versus α, $\tilde{\alpha}$, number of grid points (N), number of subdomains, and overlapping size(ovlp).

N	$(51)^2$		$(101)^2$		$(101)^2$		$(201)^2$		$(201)^2$	
♯ of subdomains	100		100		400		100		400	
α, $\tilde{\alpha}$ \ ovlp	1	2	1	2	1	2	1	2	1	2
α=0.01, $\tilde{\alpha}$=100	24	19	30	24	50	38	46	30	58	49
α=0.01, $\tilde{\alpha}$=10	26	22	36	26	55	43	47	36	80	55
α=0.01, $\tilde{\alpha}$=1	22	19	26	22	43	36	29	26	49	43
α=0.01, $\tilde{\alpha}$=0.1	11	9	12	11	20	19	13	12	23	20
α=0.01, $\tilde{\alpha}$=0.01	17	12	21	17	29	22	25	21	49	29
α=0.1, $\tilde{\alpha}$=100	25	20	40	25	56	42	51	40	60	54
α=0.1, $\tilde{\alpha}$=10	27	24	40	27	59	47	51	40	88	60
α=0.1, $\tilde{\alpha}$=1	21	19	25	21	43	38	27	25	50	43
α=0.1, $\tilde{\alpha}$=0.1	12	11	14	12	21	19	15	14	24	21
α=0.1, $\tilde{\alpha}$=0.01	19	14	23	19	35	25	28	23	47	34
α=1, $\tilde{\alpha}$=100	29	26	52	28	89	60	81	53	139	90
α=1, $\tilde{\alpha}$=10	27	27	40	29	77	55	49	40	108	78
α=1, $\tilde{\alpha}$=1	18	16	21	18	36	27	24	21	43	36
α=1, $\tilde{\alpha}$=0.1	17	17	19	17	32	30	20	19	40	32
α=1, $\tilde{\alpha}$=0.01	23	17	28	23	47	32	39	28	58	46
α=10, $\tilde{\alpha}$=100	36	29	53	39	140	85	77	53	(**)	141
α=10, $\tilde{\alpha}$=10	21	18	26	21	43	36	28	25	56	43
α=10, $\tilde{\alpha}$=1	35	28	42	35	71	71	52	41	56	71
α=10, $\tilde{\alpha}$=0.1	22	20	24	22	54	52	23	24	55	54
α=10, $\tilde{\alpha}$=0.01	27	21	32	26	59	54	55	32	81	60
α=100, $\tilde{\alpha}$=100	18	17	21	18	52	30	23	21	59	53
α=100, $\tilde{\alpha}$=10	42	30	55	42	107	113	51	55	128	108
α=100, $\tilde{\alpha}$=1	54	34	50	53	141	129	53	51	101	98
α=100, $\tilde{\alpha}$=0.1	25	22	27	25	50	50	28	27	55	50
α=100, $\tilde{\alpha}$=0.01	29	23	51	29	62	49	50	51	150	63

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