DUAL-PRIMAL FETI METHODS FOR THREE-DIMENSIONAL ELLIPTIC PROBLEMS WITH HETEROGENEOUS COEFFICIENTS

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Abstract. In this paper, certain iterative substructuring methods with Lagrange multipliers are considered for elliptic problems in three dimensions. The algorithms belong to the family of dual–primal FETI methods which have recently been introduced and analyzed successfully for elliptic problems in the plane. The family of algorithms for three dimensions is extended and a full analysis is provided for the new algorithms. Particular attention is paid to finding algorithms with a small primal subspace since that subspace represents the only global part of the dual–primal preconditioner. It is shown that the condition numbers of several of the dual–primal FETI methods can be bounded polylogarithmically as a function of the dimension of the individual subregion problems and that the bounds are otherwise independent of the number of subdomains, the mesh size, and jumps in the coefficients. These results closely parallel those for other successful iterative substructuring methods of primal as well as dual type.

Key words. domain decomposition, Lagrange multipliers, FETI, dual–primal methods, preconditioners, elliptic equations, finite elements, heterogeneous coefficients

AMS subject classifications. 65F10,65N30,65N55

1. Introduction. The FETI methods are domain decomposition methods of iterative substructuring type. They are thus a special type of preconditioned conjugate gradient methods which have been developed for solving the often huge algebraic systems of equations which arise in finite element computations. The dual-primal FETI (FETI–DP) methods were introduced recently by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [9]. Their work was followed by a significant contribution to the theory of two dimensional second and fourth order problems by Mandel and Tezaur [16], by a paper by Farhat, Lesoinne, and Pierson [10] which specifically addresses an algorithm for three–dimensional problems, and by Pierson's doctoral dissertation [18]. The algorithm presented in [10], [18], uses constraints on the averages over edges and faces, similarly to those of the algorithms considered in this paper. Our contribution is to the extension of the family of algorithms for problems in three dimensions and to the analysis. We also show that good convergence bounds can be maintained even for quite general coefficients such as those that model highly heterogeneous materials. Our work has been inspired by that of Mandel and Tezaur and it is also based on our own earlier work, in particular [5], [6], and [13].

It is well known that domain decomposition algorithms cannot be scalable, i.e., have a rate of convergence which is independent of the number of subregions, unless a coarse space component is included. We note that the underlying coarse spaces

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for three dimensional problems are often more complicated than the quite simple constructions that work well for problems in the plane; see [23] for a discussion. We will construct several of our FETI–DP methods using relatively exotic coarse spaces. Thus, our Algorithms B and C are closely related to certain interpolation operators and coarse spaces known from earlier work on primal iterative substructuring methods; see [5, 6]. Both these methods have relatively large global, primal subspaces.

The term dual–primal refers to the idea of enforcing some continuity constraints, across the interface between the subregions, throughout the iteration, as in a primal method, while all other constraints are enforced by using dual variables, i.e., Lagrange multipliers, as in a dual method. We will see that the FETI–DP methods differ in several important respects from the strictly dual FETI methods, in particular, the one-level FETI method which is described in section 3. In fact, both from an algorithmic and analytic point of view, the FETI–DP methods are closer to the primal iterative substructuring methods than the FETI methods previously developed. While the global part of the preconditioner for a strictly dual FETI method is directly associated with the dual variables, that of a FETI–DP method is not.

We note that primal iterative substructuring methods have been studied quite extensively, see, e.g., [6], [8], and [5], well before a similarly complete, and quite challenging, mathematical theory was developed for the FETI methods, see [15], [20], and [13]; FETI algorithms using inexact subdomain solvers have also been developed and analyzed by two of the authors in [12]. We note that primal iterative substructuring methods have been developed extensively even for elliptic systems, e.g., in [17], and that we believe that we have all or almost all the tools necessary to extend our current results and algorithms to the systems of linear elasticity; cf. also [12]. We also note that algorithmically some of the FETI–DP methods that we consider have certain features in common with very early work on iterative substructuring methods for problems with many substructures; cf. the studies on Neumann–Dirichlet algorithms by Dryja, Proskurowski, and Widlund [4], and two contributions to the first international symposium on domain decomposition methods, [3] and [22]. We note, in particular, that the Neumann subsystems of these early algorithms are nonsingular; there are no floating subregions because of a device very similar to that used in the FETI–DP methods. The use of Lagrange multipliers, in a special context, was also suggested in [22].

The remainder of this paper is organized as follows. In section 2, we introduce our scalar elliptic equation which can have very different coefficients in different subregions and which has served as a standard, non-trivial model problem in many studies of iterative substructuring methods. We also introduce a simple finite element space, the decomposition of our region, and our variational problem. In section 3, we give a brief description of a one-level FETI method to provide a necessary background. In section 4, we introduce our four FETI-DP methods. In section 5, we provide, with few proofs, some auxiliary results many of which have previously been developed for the analysis of primal iterative substructuring methods. In section 6, we prove almost optimal bounds on the condition number of three of the methods. They are independent of the number of substructures and grow only polylogarithmically with the number of degrees of freedom associated with the individual substructures.

2. Elliptic model problem, finite elements, and geometry. Let $\Omega \subset \mathbf{R}^3$ be a bounded, polyhedral region, let $\partial\Omega_D \subset \partial\Omega$ be a closed set of positive measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. We impose homogeneous Dirichlet and general Neumann boundary conditions, respectively, on these two subsets and introduce the Sobolev space $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$.

For simplicity, we will only consider a piecewise linear, conforming finite element approximation of the following scalar, second order model problem:

Find $u \in H_0^1(\Omega, \partial\Omega_D)$, such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \tag{1}$$

where

$$a(u, v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v dx, \;\; f(v) := \int_\Omega f v dx \; + \; \int_{\partial\Omega_N} g_N v ds, \tag{2}$$

and where g_N is the Neumann boundary data defined on $\partial\Omega_N$; it provides, together with the volume load f, the contributions to the load vector of the finite element problem. The coefficient $\rho(x) > 0$ for $x \in \Omega$.

We decompose Ω into non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$, also known as substructures, and each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := \overline{\left(\bigcup_{i=1}^N \partial\Omega_i\right)} \setminus \partial\Omega$. The interface Γ is composed of subdomain faces, regarded as open sets, which are shared by two subregions, edges which are shared by more than two subregions, and vertices which are endpoints of edges. If Γ intersects $\partial\Omega_N$ along an edge common to the boundaries of only two subdomains, we will regard it as part of the face common to this pair of subdomains. We denote the faces of Ω_i by $\mathcal{F}^{ij}$, its edges by $\mathcal{E}^{ik}$, and its vertices by $\mathcal{V}^{i\ell}$.

We denote the standard finite element space of continuous, piecewise linear functions on Ω_i by $W^h(\Omega_i)$; we always assume that these functions vanish on $\partial\Omega_D$. For simplicity, we assume that the triangulation of each subdomain is quasi uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $W_i := W^h(\partial\Omega_i \cap \Gamma), i = 1, \ldots, N$, and by $W := \prod_{i=1}^N W_i$ the associated product space. We will often consider elements of W which are discontinuous across the interface.

The finite element approximation of the elliptic problem is continuous across Γ and we denote the corresponding subspace of W by $\widehat{W}$. We note that while the stiffness matrix K and Schur complement S, which correspond to the product space W, generally are singular, those of $\widehat{W}$ are not.

We will also use additional, intermediate subspaces $\widetilde{W}$ of W for which only a relatively small number of continuity constraints are enforced across the interface. One of the benefits of working in $\widetilde{W}$, rather than in W, will be that certain related Schur complements, $\widetilde{S}$ and S_Δ, are strictly positive definite; see further sections 3 and 4.

We assume that possible jumps of $\rho(x)$ are aligned with the subdomain boundaries and, for simplicity, that on each subregion Ω_i, $\rho(x)$ has the constant value $\rho_i > 0$. Our bilinear form and load vector can then be written, in terms of contributions from individual subregions, as

$$a(u, v) = \sum_{i=1}^N \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v dx, \quad f(v) = \sum_{i=1}^N \big(\int_{\Omega_i} f v dx \; + \; \int_{\partial\Omega_i \cap \partial\Omega_N} g_N v ds \big). \tag{3}$$

In our theoretical analysis, we assume that each subregion Ω_i is the union of a number of shape regular tetrahedral coarse elements and that the number of such tetrahedra is uniformly bounded for each subdomain. Thus, the subregions are not very thin and we can also easily show that the diameters of any pair of neighboring subdomains are comparable. We also assume that if a face of a subdomain intersects $\partial\Omega_D$, then the measure of this set is comparable to that of the face. Similarly, if an edge of a subdomain intersects $\partial\Omega_D$, we assume that the length of this intersection is bounded from below in terms of the length of the edge as a whole. The sets of nodes on $\partial\Omega, \partial\Omega_i$, and Γ are denoted by $\partial\Omega_h, \partial\Omega_{i,h}$, and Γ_h, respectively.

As in previous work on Neumann–Neumann and FETI algorithms, a crucial role is played by *the weighted counting functions* μ_i, which are associated with the individual subdomain boundaries $\partial\Omega_i$; cf. [5, 8, 14, 19]. In this paper they will be used in the definition of certain diagonal scaling matrices. These functions are defined, for $\gamma \in [1/2, \infty)$, and for $x \in \Gamma_h \cup \partial\Omega_h$, by a sum of contributions from Ω_i, and its relevant next neighbors

$$\mu_i(x) = \begin{cases} \sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x) & x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}, \\ \rho_i^\gamma(x) & x \in \partial\Omega_{i,h} \cap (\partial\Omega_h \setminus \Gamma_h), \\ 0 & x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h}. \end{cases} \tag{4}$$

Here, $\mathcal{N}_x$ is the set of indices of the subregions which have x on its boundary. We note that any node of Γ_h belongs either to two faces, to at least three edges, or is a vertex of several substructures. The μ_i are continuous, piecewise discrete harmonic functions; for a definition see section 3. The pseudo inverses $\mu_i^\dagger$, which belong to the same class of functions, are defined, for $x \in \Gamma_h \cup \partial\Omega_h$, by

$$\mu_i^\dagger(x) = \begin{cases} \mu_i^{-1}(x) & \text{if } \mu_i(x) \neq 0, \\ 0 & \text{if } \mu_i(x) = 0. \end{cases} \tag{5}$$

We note that these functions provide a partition of unity:

$$\sum_i \rho_i^\gamma(x)\mu_i^\dagger(x) \equiv 1 \quad \forall x \in \Gamma_h \cup \partial\Omega_h. \tag{6}$$

3. One-level FETI methods. In this section, we will introduce some notations and certain other aspects of the older one-level FETI methods which we will use in the rest of this paper. We begin by defining a stiffness matrix K for the entire product space $\prod_{i=1}^N W^h(\Omega_i)$. K is a direct sum of local stiffness matrices $K^{(i)}$ which correspond to the subdomains $\Omega_i, i = 1, \ldots, N$, and to the appropriate terms in the first formula of (3). The local load vectors are obtained similarly; see the second formula of (3).

Any nodal variable, not associated with Γ_h, is called interior and it only belongs to one substructure; the nodal values on $\partial\Omega_N \setminus \Gamma$ also belong to this set. The interior variables of any subdomain can be eliminated by block Gaussian elimination in work which can clearly be parallelized across the subdomains. The resulting reduced matrices are the Schur complements

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} K_{I\Gamma}^{(i)}, \quad i = 1, \ldots, N.$$

Here, Γ and I represent the interface and interior, respectively. We note that the $S^{(i)}$, and their inverses or pseudo inverses, are only needed in terms of matrix-vector products and that their elements therefore need not be explicitly computed. We also obtain reduced load vectors for each subdomain. The one originating in Ω_i is denoted by f_i and the local vectors of interface nodal values, which can be regarded as components of an element of the product space W, by u_i.

The elimination of the interior variables of a substructure can also be viewed in terms of an orthogonal projection, with respect to the bilinear form $\langle K^{(i)}\cdot,\cdot\rangle$, onto the subspace of vectors with components that vanish at all the nodes of $\overline{\partial\Omega_i \setminus \partial\Omega_N}$. Here $\langle\cdot,\cdot\rangle$ denotes the ℓ_2−inner product. We note that these vectors represent elements of $W^h(\Omega_i) \cap H_0^1(\Omega_i, \overline{\partial\Omega_i \setminus \partial\Omega_N})$. These local subspaces are orthogonal, in this energy inner product, to the space of discrete harmonic vectors which represent discrete harmonic finite element functions: With v_Γ and w_Γ vectors of interface values, such a vector, $w = (w_I, w_\Gamma)$, is defined by

$$\langle K^{(i)} w, v\rangle = 0 \quad \forall v \text{ such that } v_\Gamma = 0, \tag{7}$$

on the subdomain Ω_i, or, equivalently, by

$$K_{II}^{(i)} w_I + K_{I\Gamma}^{(i)} w_\Gamma = 0. \tag{8}$$

We can regard w_Γ as a vector of Dirichlet data given on $\partial\Omega_{i,h} \cap \Gamma_h$ and note that a piecewise discrete harmonic function is completely defined by its values on the interface.

The Schur complement $S^{(i)}$ satisfies the following minimum property: $\forall w \in W_i$

$$\langle S^{(i)} w, w\rangle = \min \langle K^{(i)} v, v\rangle, \tag{9}$$

where the minimum is taken over all $v = (v_I, v_\Gamma) \in W^h(\Omega_i)$ such that $v_\Gamma = w$.

We note that we can view the Schur complement $S^{(i)}$ as the restriction of the stiffness matrix $K^{(i)}$ to the space of discrete harmonic functions. In what follows, we will almost exclusively work with functions in the trace spaces W_i and, whenever convenient, consider such an element as representing a discrete harmonic function in Ω_i. We also note that it is this piecewise discrete harmonic part of the solution, representing an element of $\widehat{W}$, that is determined by any iterative substructuring method; the other, interior, parts of the solution are computed locally as indicated above.

We now briefly review a part of the derivation of the traditional FETI methods prior to showing, in the next section, how matters change in the FETI–DP case. We begin by reformulating the finite element problem, reduced to the interface Γ, as a minimization problem with constraints given by the requirement of continuity across Γ :

Find $u \in W$, such that

$$\left.\begin{array}{c} J(u) := \frac{1}{2}\langle Su, u\rangle - \langle f, u\rangle \to \min \\ Bu = 0 \end{array}\right\} \tag{10}$$

where $u = [u_1, \ldots, u_N]^t, f = [f_1, \ldots, f_N]^t$, and $S = diag_i(S^{(i)})$.

The matrix $B = [B^{(1)}, \ldots, B^{(N)}]$ is constructed from $\{0, 1, -1\}$ such that the values of the solution u, associated with more than one subdomain, coincide when $Bu = 0$. Here, as in [13, Sections 5 and 4], we can either work with fully redundant or non–redundant constraints, i.e., with either all possible or the smallest possible number of constraints for each node of Γ_h. The local Schur complement matrices $S^{(i)}$ are positive semidefinite and, in fact, in many cases, there are floating subdomains, i.e., subregions for which the $S^{(i)}$ are singular. The problem (10) is uniquely solvable if and only if $ker\,(S) \cap ker\,(B) = \{0\}$, i.e., S is invertible on the null space of B. This condition holds since the original finite element model is elliptic.

In a standard one–level FETI method a vector of Lagrange multipliers λ is introduced to enforce all the constraints $Bu = 0$ and we obtain a saddle point formulation of (10):

Find $(u, \lambda) \in W \times U$, such that

$$\left.\begin{array}{rcccl} Su & + & B^t\lambda & = & f \\ Bu & & & = & 0 \end{array}\right\} . \tag{11}$$

In this article, we will exclusively work with fully redundant sets of Lagrange multipliers. The matrix B^t then has a null space and to assure uniqueness it is appropriate to restrict the choice of Lagrange multipliers to $range\,(B)$. In fact, in the one-level FETI methods the space of Lagrange multipliers is chosen as a subspace of $range\,(B)$, since further constraints on the Lagrange multipliers must be introduced in order to assure the solvability of the first equation of (11); see, e.g., [11, 15, 13].

We will also use a full column rank matrix R built from all of the null space elements of S; these elements are associated with individual subdomains (the rigid body motions in the case of elasticity). Thus, $range\,(R) = ker\,(S)$. We note that no subdomain with a boundary which intersects $\partial\Omega_D$ contributes to R.

The solution of the first equation in (11) exists if and only if $f - B^t\lambda \in range\,(S)$; this constraint leads to the introduction of an orthogonal projection P from U onto $ker\,(G^t)$ with $G := BR$. We note that we do not need any such projection in the dual-primal FETI methods defined in the next section.

Eliminating the primal variables from (11) and considering the component orthogonal to G, we obtain

$$\left.\begin{array}{rcl} P^tF\lambda & = & P^td \\ G^t\lambda & = & e \end{array}\right\} \tag{12}$$

with $F := BS^{\dagger}B^t, d := BS^{\dagger}f$, $S^{\dagger}$ a pseudoinverse of S, and $e := R^tf$.

The original FETI method is a conjugate gradient method applied to

$$P^tF\lambda = P^td, \qquad \lambda \in \lambda_0 + range\,(P), \tag{13}$$

with an initial approximation λ_0 chosen such that $G^t\lambda_0 = e$.

We will not describe the preconditioners used in the solution of this dual problem but will postpone this topic to the next section; there are no essential differences between the two cases as far as preconditioners are concerned. For a more detailed description and analysis of a number of one–level FETI algorithms, see Klawonn and Widlund [13].

4. Dual-Primal FETI methods. In previous studies of FETI-DP methods for problems in two dimensions, see Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [9] and Mandel and Tezaur [16], the constraints on the degrees of freedom associated with the vertices of the substructures are enforced in each iteration, i.e., the corresponding degrees of freedom belong to the primal set of variables, while all the constraints associated with the edge nodes are fully enforced only at the convergence of the iterative method. A linear system of algebraic equations is solved exactly in each step of the iteration. All unknowns except those of the subdomain vertices can be eliminated at a modest expense, and in parallel across the subdomains, resulting in a Schur complement for the vertex variables. In this first step, we can take full advantage of a high quality sparse matrix Cholesky solver when solving the individual subdomain problems, which in fact are Neumann problems on the individual subregions except for a Dirichlet condition at the subdomain vertices. The order of the Schur complement equals the number of subdomain vertices which do not belong to $\partial\Omega_D$. It is sparse since it can be shown quite easily that no nonzero off–diagonal elements exist in the reduced system matrix except those that correspond to pairs of vertices which belong to the same substructure.

In their recent paper, Mandel and Tezaur [16] established a condition number bound of the form $C(1 + \log(H/h))^2$ for the resulting FETI–DP method, in two dimensions, if it is equipped with a Dirichlet preconditioner which is very similar to those used for some of the older FETI methods; cf. Farhat, Mandel, and Roux [11]. This preconditioner is built from local solvers on the subregions with zero Dirichlet conditions at the vertices of the subregions. This algorithm is scalable with the constant C independent of the number of subregions, the subdomain diameters, as well as the mesh size h of the finite element model. Mandel and Tezaur also established a corresponding result for a fourth-order elliptic problem in the plane. Their proof in [16], for the second order equation, uses linear algebra arguments and a lemma from a classical paper by Bramble, Pasciak, and Schatz [2, Lemma 3.5].

The same algorithm, Algorithm A, can also be defined for the three dimensional case but it does not perform well; see Farhat et al. [9, sect. 5]. This is undoubtedly related to the poor performance of *vertex-based* iterative substructuring methods; see [6, Section 6.1]. A condition number estimate for this algorithm is given in Remark 2 at the end of the paper.

In the present study, as well as in others of FETI–DP methods, we work with subspaces $\widetilde{W} \subset W$ for which sufficiently many constraints are enforced so that the resulting leading diagonal block matrix of the saddle point problem, though no longer block diagonal, is strictly positive definite. We also introduce two subspaces, $\widehat{W}_\Pi \subset \widehat{W}$ and $\widetilde{W}_\Delta$, corresponding to a primal and a dual part of the space $\widetilde{W}$. These subspaces will play an important role in the description and analysis of our iterative method. The direct sum of these spaces equals $\widetilde{W}$, i.e.,

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta. \tag{14}$$

The second subspace, $\widetilde{W}_\Delta$, is the direct sum of local subspaces $\widetilde{W}_{\Delta,i}$ of $\widetilde{W}$ where each subdomain Ω_i contributes a subspace $\widetilde{W}_{\Delta,i}$; only its $i-th$ component in the sense of the product space $\widetilde{W}$ is non trivial.

In the description of our algorithms, we will need certain standard finite element cutoff functions $\theta_{\mathcal{E}^{ik}}$, $\theta_{\mathcal{F}^{ij}}$, and $\theta_{\mathcal{V}^{i\ell}}$. The first two are the discrete harmonic functions which equal 1 on $\mathcal{E}_h^{ik}$ and $\mathcal{F}_h^{ij}$, respectively, and which vanish elsewhere on Γ_h; $\theta_{\mathcal{V}^{i\ell}}$ denotes the piecewise discrete harmonic extension of the standard nodal basis function associated with the vertex $\mathcal{V}^{i\ell}$. These cutoff functions will also be used in the analysis of the methods; see sections 5 and 6.

We are now ready to define our algorithms in terms of pairs of subspaces.

Algorithm A: The primal subspace, $\widehat{W}_\Pi$, is spanned by the nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$. The local subspace $\widetilde{W}_{\Delta,i}$ is defined as the subspace of W_i of elements which vanish at the subdomain vertices, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0 \ \forall \mathcal{V}^{i\ell} \in \partial\Omega_i\}.$$

Hence, $\widetilde{W} = \widetilde{W}_A$ is the subspace of W of functions that are continuous at the subdomain vertices.

Algorithm B: The primal subspace, $\widehat{W}_\Pi$, is spanned by the vertex nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ associated with all the edges and faces, respectively, of the interface. The local subspace $\widetilde{W}_{\Delta,i}$ is defined as the subspace of W_i where the values at the subdomain vertices vanish together with the averages $\overline{u}_{\mathcal{E}^{ik}}$ and $\overline{u}_{\mathcal{F}^{ij}}$, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0, \overline{u}_{\mathcal{E}^{ik}} = 0, \overline{u}_{\mathcal{F}^{ij}} = 0 \ \forall \mathcal{V}^{i\ell}, \mathcal{E}^{ik}, \mathcal{F}^{ij} \subset \partial\Omega_i\}.$$

Here,

$$\overline{u}_{\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} u ds}{\int_{\mathcal{E}^{ik}} 1 ds} \quad \text{and} \quad \overline{u}_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} u dx}{\int_{\mathcal{F}^{ij}} 1 dx}. \tag{15}$$

Hence, $\widetilde{W} = \widetilde{W}_B$ is the subspace of W of functions that are continuous at the subdomain vertices and have the same values of $\overline{u}_{\mathcal{E}^{ik}}$ and $\overline{u}_{\mathcal{F}^{ij}}$ independently of which component of $u \in \widetilde{W}_B$ is used in the evaluation of these averages.

Algorithm C: The primal subspace, $\widehat{W}_\Pi$, is spanned by the vertex nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ of all the edges of Γ. The local subspace $\widetilde{W}_{\Delta,i}$ is defined as the subspace of W_i where the values at the subdomain vertices vanish together with the averages $\overline{u}_{\mathcal{E}^{ik}}$, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0, \overline{u}_{\mathcal{E}^{ik}} = 0 \ \forall \mathcal{V}^{i\ell}, \mathcal{E}^{ik} \subset \partial\Omega_i\}.$$

Hence, $\widetilde{W} = \widetilde{W}_C$ is the subspace of W of functions that are continuous at the subdomain vertices and have common averages $\overline{u}_{\mathcal{E}^{ik}}$ for all the edges. The number of degrees of freedom of the corresponding primal subspace $\widehat{W}_\Pi$ is therefore equal to the sum of the number of vertices and the number of edges; this $\widehat{W}_\Pi$ will be of lower dimension than the primal space of Algorithm B.

The number of constraints enforced in all the iterations of Algorithms B and C is substantially larger than when only the vertex constraints are satisfied as in Algorithm A, but we are still able to work with a uniformly bounded number of such constraints for each substructure. In order to put this in perspective, we consider Algorithms B and C in the very regular case of cubic substructures. There are then seven global variables for each interior substructure in the case of Algorithm B since there are eight vertices, each shared by eight cubes, twelve edges, each shared by four, and six faces each shared by a pair of substructures. The count for Algorithm C is four. We note that the counts would be different, relative to the number of substructures, in the case of tetrahedral subregions.

It is useful to distinguish between the continuity constraints at the vertices and the other constraints. The latter are sometimes called optional constraints since they are not needed to guarantee solvability of the subproblems if there are enough vertex constraints. The optional constraints could be handled as the vertex variables, after a change of basis. Another possibility, which we advocate, is to introduce an additional set of Lagrange multipliers which are computed exactly in each iteration to enforce the required optional constraints of the primal subspace; see Farhat, Lesoinne, and Pierson [10], where this approach is used; for a more detailed description, see section 4.2, especially formulae (24)-(28), of that paper.

We are able to show as strong a result for Algorithm C as for Algorithm B. It is therefore natural to attempt to drop additional constraints, i.e., further decrease the primal subspace $\widehat{W}_\Pi$ while preserving the fast convergence of the FETI-DP method. This leads to the introduction of our final algorithm.

Algorithm D: The primal subspace $\widehat{W}_\Pi$, is defined in terms of constraints associated with a subset of the edges and vertices of the interface. Our recipe for selecting such *primal* edges and vertices is relatively complicated and can only be fully understood by reading the proof of Lemma 10 carefully.

We first describe the requirements on a minimal set of primal constraints which we have found necessary to give a complete proof of a good bound for Algorithm D. For each face, we should have at least one designated, primal edge. Additionally, for all pairs of substructures Ω_i, Ω_j, which have an edge in common, we must have an acceptable *edge path* between the two subdomains. An acceptable edge path is a path from Ω_i to Ω_j, possibly via several other subdomains, Ω_k, which have the edge $\mathcal{E}^{ij}$ in common and such that their coefficients satisfy $TOL{*}\rho_k \geq \min(\rho_i, \rho_j)$ for some chosen tolerance TOL. The path can only pass from one subdomain to another through an edge designated as primal. Finally, we consider all pairs of substructures which have a vertex $\mathcal{V}^{i\ell}$ but not a face or an edge in common. Then, we assume that either $\mathcal{V}^{i\ell}$ is a primal vertex or that we have an acceptable edge path of the same nature as above, except that we can be more lenient and only insist on $TOL{*}\rho_k \geq (h_k/H_k)\min(\rho_i, \rho_j)$. We also note, that we could allow our edge paths to stray somewhat further away from the edge $\mathcal{E}^{ij}$, or the vertex $\mathcal{V}^{i\ell}$, and that in fact a careful examination of the proof of Lemma 10 would reveal that alternative, more liberal rules concerning the paths could be adopted.

We now give a description of a possible way of selecting the set of primal constraints. We start by choosing enough edges so that for each face of the interface there is at least one designated, primal edge which is part of the boundary of the face. In addition, we can exercise an option of designating some of the vertices of the substructures as primal; this is not strictly necessary but if constraints are enforced at enough vertices throughout the computation, then the related Schur complement can be made invertible even without any edge constraints. As pointed out above, this can be an advantage in the implementation of the method.

After this initial phase, which in the case of hexagonal substructures can involve as few as three edge constraints per subdomain, and hence a very small primal space, we turn to considering the effects of the possibly very large variation of the coefficients ρ_i; if there are no great variations in the coefficients, we need do nothing more. We examine each edge $\mathcal{E}^{ij}$ not previously designated as primal, one by one. We consider all pairs of subdomains that have this edge in common and try to find an acceptable edge path between the two subdomains Ω_i and Ω_j. If no such path can be found, we add the edge $\mathcal{E}^{ij}$ to the set of designated edges; a trivial, acceptable edge path is then created. We also note that since two subdomains that share a face, always have at least one designated edge in common, we need not consider any such pairs of subdomains in this step.

Finally, we consider, one by one, all vertices which so far have not been designated as primal. We consider pairs of substructures that have such a vertex $\mathcal{V}^{i\ell}$ in common but which do not have a face or edge in common. For each vertex inspected, we try to find an acceptable edge path subject only to the more lenient condition on the coefficients. If we fail in finding such a path, we mark the vertex $\mathcal{V}^{i\ell}$ as primal, i.e., a vertex where the constraints should be exactly satisfied throughout the FETI iteration.

We note that we are free to add any other vertex, edge, or face constraints to our definition of the primal space; the bounds on the condition numbers will only improve. If all edges and vertices are primal, we are back to Algorithm C.

We can now formulate our FETI–DP algorithms. Each of them is expressed in terms of a Schur complement $\widetilde{S}$ related to the dual space $\widetilde{W}_\Delta$. We can arrive at this reduced problem by eliminating the primal variables associated with the interior nodes, the vertex nodes designated as primal, as well as the Lagrange multipliers related to the optional constraints. This Schur complement $\widetilde{S}$ can equally well be defined by a variational problem: $\forall w_\Delta \in \widetilde{W}_\Delta$,

$$\langle \widetilde{S} w_\Delta, w_\Delta \rangle = \min \langle S w, w \rangle, \tag{16}$$

where we take the minimum over all $w \in \widetilde{W}$ of the form $w = w_\Pi + w_\Delta, w_\Pi \in \widehat{W}_\Pi$. We note that any Schur complement of a positive definite, symmetric matrix is always associated with such a variational problem. We also obtain, analogously, a reduced right hand side $\tilde{f}_\Delta$, from the load vectors associated with the individual subdomains.

We now reformulate the original finite element problem, reduced to the degrees of freedom of the second subspace $\widetilde{W}_\Delta$, as a minimization problem with constraints given by the requirement of continuity across all of Γ_h:

Find $u_\Delta \in \widetilde{W}_\Delta$, such that

$$\left. \begin{array}{c} J(u_\Delta) := \frac{1}{2}\langle \widetilde{S} u_\Delta, u_\Delta \rangle - \langle \tilde{f}_\Delta, u_\Delta \rangle \to \min \\ B_\Delta u_\Delta = 0 \end{array} \right\} . \tag{17}$$

The matrix B_Δ is constructed from $\{0, 1, -1\}$, in a way very similar to the matrix B discussed in section 3, and in such a way that the values of the solution u_Δ, associated with more than one subdomain, coincide when $B_\Delta u_\Delta = 0$. Again these constraints are very simple and just express that the nodal values coincide across the interface; in comparison with the FETI method described in the previous section, we can drop some of the constraints, in particular those associated with the vertex nodes of the primal space. However, we will otherwise use all possible constraints and thus work with a fully redundant set of Lagrange multipliers as in [13, section 5].

By introducing a set of Lagrange multipliers $\lambda \in V := \mathit{range}\,(B_\Delta)$, to enforce the constraints $B_\Delta u_\Delta = 0$, we obtain a saddle point formulation of (17), as in (11). Since $\widetilde{S}$ is invertible, we can eliminate the subvector u_Δ, and we obtain the following system for the dual variables:

$$F\lambda = d := B_\Delta \widetilde{S}^{-1} \tilde{f}_\Delta, \tag{18}$$

where

$$F := B_\Delta \widetilde{S}^{-1} B_\Delta^t .$$

Algorithmically, the matrix $\widetilde{S}$ is only needed in terms of $\widetilde{S}^{-1}$ times a vector and such an operation can be computed relatively inexpensively. While it is natural to describe a Schur complement in terms of a second set of variables and resulting from the elimination of a first set, the action of its inverse on a vector can often advantageously be obtained by solving the entire linear system from which it originates after augmenting the given right hand side with zeros. Full advantage can then be taken of algorithms that symmetrically reorder the larger matrix so as to preserve sparsity. In the case at hand, it is thus advantageous to group all the interior and dual variables of each subdomain together and to factor the resulting blocks in parallel across the subdomains using a good ordering algorithm. The contributions to the remaining Schur complement, of the primal variables, can also be computed locally prior to subassembly and factorization of this final, global part of the linear system of equations.

The operator F will obviously depend on the choice of the subspaces $\widehat{W}_\Pi$ and $\widetilde{W}_\Delta$ and we denote the operators of the resulting linear systems by F_A, F_B, F_C, and F_D, respectively. To define the FETI–DP Dirichlet preconditioner, we need to introduce an additional set of local Schur complement matrices, $S_\Delta^{(i)}$, which is obtained by restricting $S^{(i)}$ to the space $\widetilde{W}_{\Delta,i}$; in the case of Algorithm A, we simply remove the rows and columns corresponding to the subdomain vertices from $S^{(i)}$. The associated block–diagonal matrix is given by

$$S_\Delta := diag_{i=1}^N(S_\Delta^{(i)}).$$

We can compute S_Δ times a vector $w_\Delta \in \widetilde{W}_\Delta$ by solving local Dirichlet problems with solutions in $\widetilde{W}_{\Delta,i},\ i = 1, \dots, N$, and then multiplying them by the stiffness matrix of their respective subdomain. These solutions are constrained to vanish at designated subdomain vertices and to have zero edge and face averages, as required by the algorithm in question.

We also introduce diagonal scaling matrices $D_\Delta^{(i)}$ that operate on the Lagrange multiplier spaces. Each of their diagonal elements corresponds to a Lagrange multiplier which enforces continuity between the nodal values of some $w_i \in \widetilde{W}_i$ and $w_j \in \widetilde{W}_j$ at some point $x \in \Gamma_h$; it is given by $\rho_j^\gamma(x)\mu_j^\dagger(x)$. Finally, we define a scaled jump operator by

$$B_{D,\Delta} := [D_\Delta^{(1)} B_\Delta^{(1)}, \dots, D_\Delta^{(N)} B_\Delta^{(N)}].$$

As in Klawonn and Widlund [13, section 5], we solve the dual system (18) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := B_{D,\Delta} S_\Delta B_{D,\Delta}^t. \tag{19}$$

The FETI-DP method is the standard preconditioned conjugate gradient algorithm for solving the preconditioned system

$$M^{-1} F \lambda = M^{-1} d.$$

This definition of M clearly depends on the choice of the subspaces $\widehat{W}_\Pi$ and $\widetilde{W}_\Delta$ for the different algorithms. The resulting preconditioners are denoted by $M_A^{-1}, M_B^{-1}, M_C^{-1}$, and M_D^{-1}, respectively.

5. Some auxiliary lemmas. The purpose of this section is to provide, in most cases without proofs, the few auxiliary results that are required for a complete proof of Lemmas 9 and 10, which provide the core of the proofs of our main results. Some of these results are borrowed from [6, 8, 7]. Here, we formulate them using trace spaces on the subdomain boundaries, i.e., $H^{1/2}(\partial\Omega_i)$ instead of the spaces $H^1(\Omega_i)$ and discrete harmonic extensions; given the well–known equivalence of the norms, nothing essentially new needs to be proven. In our proofs, we will work with the S–norm defined by $|u|_S^2 = \sum_{i=1}^N |u_i|_{S^{(i)}}^2$ and $|u_i|_{S^{(i)}}^2 = \langle S^{(i)} u_i, u_i\rangle$. A proof of the equivalence of the $S^{(i)}-$ and the $H^{1/2}(\partial\Omega_i)-$semi–norms of elements of W_i can be found in [1] for the case of piecewise linear elements and two dimensions and the tools necessary to extend this result to more general finite elements are provided in [21]; in our case, we of course have to multiply $|u_i|_{H^{1/2}(\partial\Omega_i)}^2$ by the factor ρ_i.

We also recall that we can define the $H_{00}^{1/2}(\tilde{\Gamma})-$norm, $\tilde{\Gamma} \subset \partial\Omega_i$, of an element of W_i which is supported in $\tilde{\Gamma}$, as the $H^{1/2}(\partial\Omega_i)-$norm of the function extended by zero onto $\partial\Omega_i \setminus \tilde{\Gamma}$.

The first lemma can, essentially, be found in Dryja, Smith, and Widlund [6, Lemma 4.4].

LEMMA 1. *Let $\theta_{\mathcal{F}^{ij}}$ be the finite element function that is equal to* 1 *at the nodal points on the face $\mathcal{F}^{ij}$, which is common to two subregions Ω_i and Ω_j, and that vanishes on $(\partial\Omega_{i,h} \cup \partial\Omega_{j,h}) \setminus \mathcal{F}_h^{ij}$. Then,*

$$|\theta_{\mathcal{F}^{ij}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C(1+\log(H_i/h_i))H_i.$$

The same bounds also hold for the other subregion Ω_j.

The following result can, essentially, be found in Dryja, Smith, and Widlund [6, Lemma 4.5] or in Dryja [3, Lemma 3].

LEMMA 2. *Let $\theta_{\mathcal{F}^{ij}}$ be the function introduced in Lemma 1 and let I^h denote the interpolation operator onto the finite element space $W^h(\Omega_i)$. Then, $\forall u \in W_i$,*

$$\|I^h(\theta_{\mathcal{F}^{ij}}u)\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \leq C(1+\log(H_i/h_i))^2\big(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\big).$$

We will also need two additional results which are used to estimate the contributions to our bounds from the edges of Ω_i. For the next lemma, see Dryja, Smith, and Widlund [6, Lemma 4.7].

LEMMA 3. *Let $\theta_{\mathcal{E}^{ik}}$ be the cutoff function associated with the edge $\mathcal{E}^{ik}$. Then, $\forall u \in W_i$,*

$$|I^h(\theta_{\mathcal{E}^{ik}}u)|^2_{H^{1/2}(\partial\Omega_i)} \leq C\|u\|^2_{L_2(\mathcal{E}^{ik})}.$$

This result follows by an elementary estimate of the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy.

We will also need a Sobolev-type inequality for finite element functions, see Dryja and Widlund [7, Lemma 3.3] or Dryja [3, Lemma 1].

LEMMA 4. *Let $\mathcal{E}^{ik}$ be any edge of Ω_i which forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then, $\forall u \in W_i$,*

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq C(1+\log(H_i/h_i))\big(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\big).$$

We also state a nonstandard version of Friedrichs' inequality that is given in a somewhat different form in [8, Lemma 6].

LEMMA 5. *Let $\mathcal{E}^{ik}$ be an edge of $\mathcal{F}^{ij}$. Then, $\forall u \in W_i$ that vanish on $\mathcal{E}^{ik}$,*

$$\|u\|^2_{L_2(\mathcal{F}^{ij})} \leq CH_i(1+\log(H_i/h_i))|u|^2_{H^{1/2}(\mathcal{F}^{ij})}.$$

The proof of the main results in Mandel and Tezaur [16] is based on a bound for a certain interpolation operator. In our proofs, we could also use a different interpolation operator for each of our algorithms. Although these operators now play no direct role in the proofs of our main results, they are nevertheless of independent interest. They also illustrate how in the case of Algorithms B and C, we can approximate an arbitrary element in $\widetilde{W}_B$ and $\widetilde{W}_C$, respectively, by a continuous interpolant which is almost uniformly stable in the energy norm; concerning $\widetilde{W}_D$, see Remark 1.

The first interpolation operator, I_A^h, is given by the continuous piecewise linear interpolant on the coarse triangulation of Γ, used in the definition of the Ω_i.

Our second interpolation operator I_B^h is defined, $\forall u \in \widetilde{W}_B$, by sums over all the vertices, edges, and faces of Γ,

$$I_B^h u(x) = \sum_{\mathcal{V}^{i\ell} \in \Gamma} u(\mathcal{V}^{i\ell}) \theta_{\mathcal{V}^{i\ell}}(x) + \sum_{\mathcal{E}^{ik} \subset \Gamma} \overline{u}_{\mathcal{E}^{ik}} \theta_{\mathcal{E}^{ik}}(x) + \sum_{\mathcal{F}^{ij} \subset \Gamma} \overline{u}_{\mathcal{F}^{ij}} \theta_{\mathcal{F}^{ij}}(x). \tag{20}$$

The operator I_B^h, a modification of an operator introduced in [6, p. 1690], has almost optimal stability properties. We note that the values of $I_B^h u(x)$ on $\partial\Omega_i$ depend only on the W_i component of u.

We also introduce a third interpolation operator, I_C^h, which provides an alternative to I_B^h.

$$I_C^h u(x) = \sum_{\mathcal{V}^{i\ell} \in \Gamma} u(\mathcal{V}^{i\ell}) \theta_{\mathcal{V}^{i\ell}}(x) + \sum_{\mathcal{E}^{ik} \subset \Gamma} \overline{u}_{\mathcal{E}^{ik}} \theta_{\mathcal{E}^{ik}}(x) + \sum_{\mathcal{F}^{ij} \subset \Gamma} \overline{u}_{\partial\mathcal{F}^{ij}} \theta_{\mathcal{F}^{ij}}(x). \tag{21}$$

Here the average $\overline{u}_{\mathcal{E}^{ik}}$ is defined as in (15) and $\overline{u}_{\partial\mathcal{F}^{ij}}$ is given by

$$\overline{u}_{\partial\mathcal{F}^{ij}} = \frac{\int_{\partial\mathcal{F}^{ij}} u ds}{\int_{\partial\mathcal{F}^{ij}} 1 ds}.$$

This average is a convex combination of the values of the $\overline{u}_{\mathcal{E}^{ik}}$ of the face in question. This interpolant is well defined for any element $u \in \widetilde{W}_C$.

The next lemma provides L_2- and $H^{1/2}-$estimates for the vertex based interpolation operator I_A^h. This is essentially Dryja, Smith, and Widlund [6, Lemma 4.1]. The proof follows directly from Poincaré's inequality and a standard discrete Sobolev inequality; see also [6, section 4].

LEMMA 6. *The vertex based interpolation operator I_A^h satisfies*

$$|I_A^h u|^2_{H^{1/2}(\mathcal{F}^{ij})} \leq C\,(H_i/h_i)\,|u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i,$$

and

$$\|u - I_A^h u\|^2_{L_2(\mathcal{F}^{ij})} \leq C\,(H_i/h_i)\,H_i\,|u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i.$$

Here the constant C is independent of the diameter H_i of Ω_i, and the mesh size h_i.

We have better results for the interpolation operators I_B^h and I_C^h, introduced in (20) and (21), respectively. A bound for I_B^h can be found in a somewhat different form in Dryja, Smith, and Widlund [6, pp. 1689–90]. We note that our L_2-estimate is now improved in comparison to [6, p. 1690] since our estimate of the interpolation error contains no logarithmic factor.

LEMMA 7. *The interpolation operators I_B^h and I_C^h, defined in (20) and (21), respectively, satisfy*

$$|I_B^h u|^2_{H^{1/2}(\mathcal{F}^{ij})} \leq C(1 + \log(H_i/h_i))|u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i,$$
$$|I_C^h u|^2_{H^{1/2}(\mathcal{F}^{ij})} \leq C(1 + \log(H_i/h_i))|u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i,$$

and

$$\|u - I_B^h u\|^2_{L_2(\mathcal{F}^{ij})} \leq C H_i |u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i,$$
$$\|u - I_C^h u\|^2_{L_2(\mathcal{F}^{ij})} \leq C H_i |u|^2_{H^{1/2}(\mathcal{F}^{ij})} \quad \forall u \in W_i.$$

Here the constant C is independent of the diameter H_i of Ω_i, and the mesh size h_i.

6. Convergence analysis. Our analysis borrows ideas from the recent paper by Mandel and Tezaur [16], and our own paper [13]. In the latter, fast one-level FETI algorithms and a theory for the elliptic problem of the class defined by (3) were developed for an arbitrary choice of the ρ_i.

As in [16], the two different Schur complements, $\widetilde{S}$ and S_Δ, introduced in section 4, play an important role in the analysis of the dual–primal iterative algorithm. Both operate on the second subspace $\widetilde{W}_\Delta$ and we also recall that $\widetilde{S}$ represents a global problem while S_Δ does not.

Let $V := \mathit{range}\,(B_\Delta)$ be the space of Lagrange multipliers. As in [13, Section 5], we introduce a projection

$$P_\Delta := B^t_{D,\Delta} B_\Delta.$$

A simple computation shows, see [13, Lemma 4.2], that P_Δ preserves the jump of any function $u_\Delta \in \widetilde{W}_\Delta$, i.e., $B_\Delta P_\Delta u_\Delta = B_\Delta u_\Delta$ and we also have $P_\Delta u = 0 \;\; \forall u \in \widehat{W}$.

Analogously to [13, Lemma 5.2], we have

LEMMA 8. *For any $\mu \in V$, there exists a $w_\Delta \in \mathit{range}\,(P_\Delta)$, such that $\mu = B_\Delta w_\Delta$.*

Proof: We note that for any $\mu \in V = \mathit{range}\,(B_\Delta)$, there exists a w'_Δ, such that $\mu = B_\Delta w'_\Delta$. Choosing $w_\Delta := P_\Delta w'_\Delta$, we have $B_\Delta w_\Delta = B_\Delta w'_\Delta = \mu$.

$\square$

Let $x \in \Gamma_h$ and let $w_\Delta \in \widetilde{W}_\Delta$. We borrow the following formula from [13]:

$$P_\Delta w_\Delta(x) \;=\; \sum_{j \in \mathcal{N}_{\Delta,x}} \rho_j^\gamma \mu_j^\dagger (w_{\Delta,i}(x) - w_{\Delta,j}(x)), x \in \partial\Omega_{i,h} \cap \Gamma_h. \tag{22}$$

Here, $\mathcal{N}_{\Delta,x}$ is the set of indices of the subregions which have the node x on its boundary. We note that the coefficients in this expression are constant on the set of the nodal points of each face and each edge of $\partial\Omega_i$, and that this formula is independent of the particular choice of B_Δ.

We first analyze Algorithm B and begin by proving the following core estimate.

LEMMA 9 (Algorithm B). *For all $w_\Delta \in \widetilde{W}_{\Delta,B}$, we have,*

$$|P_\Delta w_\Delta|^2_{S_\Delta} \le C\,(1 + \log(H/h))^2 |w_\Delta|^2_{\widetilde{S}},$$

where $C > 0$ is independent of h, H, ρ_i, and γ.

Proof. We consider an arbitrary $w_\Delta \in \widetilde{W}_{\Delta,B}$. In order to compute its $\widetilde{S}-$norm, cf. (16), we determine the element $w = w_\Pi + w_\Delta \in \widetilde{W}_B, w_\Pi \in \widehat{W}_{\Pi,B}$, with the correct minimal property. Then, by the definition of $\widetilde{S}$, $|w_\Delta|_{\widetilde{S}} = |w|_S$. We next note that we can subtract any continuous function from w_Δ without changing the values of $P_\Delta w_\Delta$; thus, $P_\Delta w = P_\Delta w_\Delta$. It is also easy to see, by carrying out a simple computation and by using formula (22), that $P_\Delta w_\Delta \in \widetilde{W}_{\Delta,B}$. We also recall that the $S_\Delta-$norm of any element of $\widetilde{W}_\Delta$ equals its $S-$norm.

We model our proof on [13, Lemmas 4.7, 5.4] but note that the arguments need to be modified to some extent. We also note that we only have contributions from faces and edges since all elements in $\widetilde{W}_B$ are continuous at the vertices. Here, in contrast to the proof in [13], we do not need to assume that there are not any subdomains with boundaries which only intersects $\partial\Omega_D$ only in isolated points.

We introduce the notation $(v_i)_{i=1,\ldots,N} := P_\Delta w$. Then, we have to estimate

$$|P_\Delta w|^2_S = \sum_{i=1}^N |v_i|^2_{S^{(i)}}.$$

We can therefore focus on the estimate of the contribution from a single subdomain Ω_i. We first assume that its boundary and the boundaries of its relevant neighbors do not intersect $\partial\Omega_D$.

We cut the function v_i using the functions $\theta_{\mathcal{F}^{ij}}$ and $\theta_{\mathcal{E}^{ik}}$ and write it as a sum of terms which vanish at all the interface nodes outside individual faces and edges; cf., e.g., [6, 8, 7]. We then have, since the v_i vanish at the subdomain vertices,

$$v_i = \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} I^h(\theta_{\mathcal{F}^{ij}} v_i) + \sum_{\mathcal{E}^{ik} \subset \partial\Omega_i} I^h(\theta_{\mathcal{E}^{ik}} v_i).$$

We find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger (w_i - w_j))$$

and we have to estimate its $H_{00}^{1/2}(\mathcal{F}^{ij})-$norm; this formula follows from (22).

With $\gamma \geq 1/2$, we can easily prove that

$$\rho_i(\rho_j^\gamma \mu_j^\dagger)^2 \leq \min(\rho_i, \rho_j). \tag{23}$$

We note that $\rho_j^\gamma \mu_j^\dagger$ is constant on $\mathcal{F}_h^{ij}$ and that w has common face averages, i.e., $\overline{w}_{i,\mathcal{F}^{ij}} = \overline{w}_{j,\mathcal{F}^{ij}}$. Using inequality (23), these observations, and Lemma 2, we obtain,

$$\begin{array}{rl} & \rho_i \| I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger (w_i - w_j))\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\ = & \rho_i \| I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger ((w_i - \overline{w}_{i,\mathcal{F}^{ij}}) - (w_j - \overline{w}_{j,\mathcal{F}^{ij}})))\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\ \leq & C\,(1+\log(H_i/h_i))^2 \min(\rho_i,\rho_j) \left(|w_i - w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} + \right. \\ & \qquad \left. + \frac{1}{H_i} \|(w_i - \overline{w}_{i,\mathcal{F}^{ij}}) - (w_j - \overline{w}_{j,\mathcal{F}^{ij}})\|^2_{L_2(\mathcal{F}^{ij})} \right). \end{array} \tag{24}$$

We can estimate this expression by

$$C\,(1+\log(H_i/h_i))^2 \left(\rho_i |w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \rho_j |w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} \right),$$

as desired, by applying a Poincaré inequality. We note that, by assumption, H_j and H_i are comparable and so are h_j and h_i, since the triangulations of Ω_i and Ω_j are quasi uniform.

By using Lemma 3, we can estimate the contributions of the edges of Ω_i to the energy of v_i in terms of L_2-norms over the edges. These L_2-terms are then estimated by using Lemma 4. If four subdomains, e.g., $\Omega_i, \Omega_j, \Omega_k$, and Ω_ℓ, have an edge $\mathcal{E}^{ik}$ in common, then, according to (22), there are three contributions to the estimate of the contribution of Ω_i to $|P_\Delta w|_S^2$, namely

$$\begin{array}{l} \rho_i \, \| I^h(\rho_j^\gamma \, \mu_j^\dagger \, \theta_{\mathcal{E}^{ik}} (w_i - w_j))\|^2_{L_2(\mathcal{E}^{ik})} \; + \; \rho_i \, \| I^h(\rho_k^\gamma \, \mu_k^\dagger \, \theta_{\mathcal{E}^{ik}} (w_i - w_k))\|^2_{L_2(\mathcal{E}^{ik})} + \\ \qquad\qquad + \rho_i \, \| I^h(\rho_\ell^\gamma \, \mu_\ell^\dagger \, \theta_{\mathcal{E}^{ik}} (w_i - w_\ell))\|^2_{L_2(\mathcal{E}^{ik})}. \end{array} \tag{25}$$

We first consider the second term in detail assuming that Ω_i shares a face with each of Ω_j and Ω_ℓ, but only an edge with Ω_k. In the next estimate, we use $|\overline{w}_{i,\mathcal{E}^{ik}}|^2 \leq 1/H_i \|w_i\|^2_{L_2(\mathcal{E}^{ik})}$ and $\|\theta_{\mathcal{E}^{ik}}\|^2_{L_2(\mathcal{E}^{ik})} \leq C\, H_i$. Using formula (23), Lemma 4, and that

w has common edge averages, i.e., $\overline{w}_{i,\mathcal{E}^{ik}} = \overline{w}_{k,\mathcal{E}^{ik}}$, we obtain,

$$
\begin{aligned}
& \rho_i \, \| I^h(\rho_k^\gamma \mu_k^\dagger \theta_{\mathcal{E}^{ik}}(w_i - w_k)) \|^2_{L_2(\mathcal{E}^{ik})} \\
= \; & \rho_i \, \| I^h(\rho_k^\gamma \mu_k^\dagger (\theta_{\mathcal{E}^{ik}}(w_i - \overline{w}_{i,\mathcal{E}^{ik}}) - \theta_{\mathcal{E}^{ik}}(w_k - \overline{w}_{k,\mathcal{E}^{ik}}))) \|^2_{L_2(\mathcal{E}^{ik})} \\
\leq \; & 2 \left(\rho_i \| I^h(\theta_{\mathcal{E}^{ik}}(w_i - \overline{w}_{i,\mathcal{E}^{ik}})) \|^2_{L_2(\mathcal{E}^{ik})} + \rho_k \| I^h(\theta_{\mathcal{E}^{ik}}(w_k - \overline{w}_{k,\mathcal{E}^{ik}})) \|^2_{L_2(\mathcal{E}^{ik})} \right) \\
\leq \; & C \left(\rho_i \| w_i \|^2_{L_2(\mathcal{E}^{ik})} + \rho_k \| w_k \|^2_{L_2(\mathcal{E}^{ik})} \right) \qquad (26) \\
\leq \; & C(1 + \log(H/h)) \left(\rho_i \left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \tfrac{1}{H_i} \| w_i \|^2_{L_2(\mathcal{F}^{ij})} \right) + \right. \\
& \qquad \left. + \rho_k \left(|w_k|^2_{H^{1/2}(\mathcal{F}^{kj})} + \tfrac{1}{H_k} \| w_k \|^2_{L_2(\mathcal{F}^{kj})} \right) \right) \\
\leq \; & C(1 + \log(H/h)) \left(\rho_i |w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \rho_k |w_k|^2_{H^{1/2}(\mathcal{F}^{kj})} \right),
\end{aligned}
$$

with $\mathcal{F}^{ij}$ a face of Ω_i and $\mathcal{F}^{kj}$ a face of Ω_k, which have the edge $\mathcal{E}^{ik}$ in common. The last inequality follows from the shift invariance of the expressions on the third line, i.e., we can add constants to w_i and w_k without changing the value of the expressions and then use Poincaré's inequality.

Since Ω_i and Ω_j, as well as Ω_i and Ω_ℓ, have a face in common, the argument given above could be simplified for the first and third edge contributions; they can be reduced to estimates for face terms directly.

We finally have to consider boundary subregions which have a nonempty intersection with $\partial\Omega_D$ and show that we can obtain bounds of the same quality. We then need different arguments to eliminate the $L_2(\mathcal{F}^{ij})-$terms. In case this intersection is a face or an edge, we can use exactly the same arguments as in [13, p. 71] which includes using Lemma 5. If the boundary of a substructure intersects $\partial\Omega_D$ in just one or a few single points, the shifting can be done exactly as above for the face and edge terms of an interior subregion.

□

We now prove our condition number estimate for Algorithm B, which only depends polylogarithmically on the dimension of the subproblems.

THEOREM 1 (Algorithm B). *The condition number satisfies*

$$\kappa(M_B^{-1} F_B) \leq C\,(1 + \log(H/h))^2.$$

Here, C is independent of h, H, γ, and the values of the ρ_i.

Proof: We have to estimate the smallest eigenvalue $\lambda_{min}(M_B^{-1}F_B)$ from below and the largest eigenvalue $\lambda_{max}(M_B^{-1}F_B)$ from above. We will show that

$$\langle M_B \lambda, \lambda \rangle \leq \langle F_B \lambda, \lambda \rangle \leq C\,(1 + \log(H/h))^2 \langle M_B \lambda, \lambda \rangle \quad \forall \lambda \in V. \qquad (27)$$

Lower bound: This bound is derived using purely algebraic arguments. As in the analysis of the one–level FETI methods, we can use the following formula, see Mandel and Tezaur [15] or Klawonn and Widlund [13, p. 73],

$$\langle F_B \lambda, \lambda \rangle = \sup_{0 \neq v_\Delta \in \widetilde{W}_\Delta} \frac{\langle \lambda, B_\Delta v_\Delta \rangle^2}{|v_\Delta|^2_{\widetilde{S}}}.$$

Let $\mu \in V$ be arbitrary. It then follows from Lemma 8 that there exists a $w_\Delta \in$ *range* (P_Δ) with $\mu = B_\Delta w_\Delta$. Since $w_\Delta = P_\Delta w_\Delta$ and $|u_\Delta|_{\widetilde{S}} \leq |u_\Delta|_{S_\Delta} \;\; \forall u_\Delta \in \widetilde{W}_\Delta$, we obtain

$$\langle F_B \lambda, \lambda \rangle \geq \frac{\langle \lambda, B_\Delta w_\Delta \rangle^2}{|w_\Delta|^2_{\widetilde{S}}} \geq \frac{\langle \lambda, B_\Delta w_\Delta \rangle^2}{|w_\Delta|^2_{S_\Delta}} = \frac{\langle \lambda, \mu \rangle^2}{|B_{D,\Delta}^t \mu|^2_{S_\Delta}} = \frac{\langle \lambda, \mu \rangle^2}{\langle M_B^{-1} \mu, \mu \rangle}.$$

The left inequality of (27) follows by choosing $\mu := M_B\lambda$.

Upper bound: Using Lemma 9, we obtain $\forall \lambda \in V$,

$$\begin{aligned}\langle F_B\lambda,\lambda\rangle &= \sup_{0\neq w_\Delta\in\widetilde{W}_\Delta} \frac{\langle\lambda, B_\Delta w_\Delta\rangle^2}{|w_\Delta|^2_{\widetilde{S}}}\\ &\le C\,(1+\log(H/h))^2 \sup_{w_\Delta\neq 0}\frac{\langle\lambda, B_\Delta w_\Delta\rangle^2}{|P_\Delta w_\Delta|^2_{S_\Delta}}\\ &= C\,(1+\log(H/h))^2 \sup_{w_\Delta\neq 0}\frac{\langle\lambda, B_\Delta w_\Delta\rangle^2}{\langle M_B^{-1}B_\Delta w_\Delta, B_\Delta w_\Delta\rangle}\\ &= C\,(1+\log(H/h))^2 \sup_{\mu\in V}\frac{\langle\lambda,\mu\rangle^2}{\langle M_B^{-1}\mu,\mu\rangle} = C\,(1+\log(H/h))^2\langle M_B\lambda,\lambda\rangle.\end{aligned}$$

□

We now turn to the analysis of Algorithms C and D.

LEMMA 10 (Algorithms C, D).

For all $w_\Delta \in \widetilde{W}_{\Delta,C}$, we have,

$$|P_\Delta w_\Delta|^2_{S_\Delta} \le C\,(1+\log(H/h))^2|w_\Delta|^2_{\widetilde{S}}.$$

For all $w_\Delta \in \widetilde{W}_{\Delta,D}$, we have,

$$|P_\Delta w_\Delta|^2_{S_\Delta} \le C\,\max(1,TOL)\,(1+\log(H/h))^2|w_\Delta|^2_{\widetilde{S}}.$$

In both cases, $C>0$ is independent of h, H, ρ_i, and γ.

Proof. We can proceed as in the proof of Lemma 9; we will use the same notation and only discuss details that are technically different. We note that in Algorithm D all vertices are not necessarily constrained and that therefore we have to estimate terms of $P_\Delta w(x)$ related to the vertices which are not primal.

We cut the function v_i using the functions $\theta_{\mathcal{F}^{ij}}$, $\theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ and write it as a sum of terms which vanish at all the interface nodes outside individual faces, edges, and vertices, respectively; cf., e.g., [6, 8, 7]. We then have

$$v_i = \sum_{\mathcal{F}^{ij}\subset\partial\Omega_i} I^h(\theta_{\mathcal{F}^{ij}}v_i) + \sum_{\mathcal{E}^{ik}\subset\partial\Omega_i} I^h(\theta_{\mathcal{E}^{ik}}v_i) + \sum_{\mathcal{V}^{il}\in\partial\Omega_i}\theta_{\mathcal{V}^{il}}v_i(\mathcal{V}^{il}).$$

As in [13] and the proof of Lemma 9, we find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}}\rho_j^\gamma\mu_j^\dagger(w_i-w_j))$$

and we have to estimate its $H^{1/2}_{00}(\mathcal{F}^{ij})-$norm. Using inequality (23) and that $\rho_j^\gamma\mu_j^\dagger$ is constant on $\mathcal{F}_h^{ij}$, we obtain,

$$\begin{aligned} & \rho_i\|I^h(\theta_{\mathcal{F}^{ij}}\rho_j^\gamma\mu_j^\dagger(w_i-w_j))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\ = \;& \rho_i\|I^h(\theta_{\mathcal{F}^{ij}}\rho_j^\gamma\mu_j^\dagger((w_i-\overline{w}_{i,\mathcal{F}^{ij}})-(w_j-\overline{w}_{j,\mathcal{F}^{ij}})+\\ &\qquad\qquad +(\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{F}^{ij}})))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\ \le\;& 2\min(\rho_i,\rho_j)\Big(\|I^h(\theta_{\mathcal{F}^{ij}}((w_i-\overline{w}_{i,\mathcal{F}^{ij}})-(w_j-\overline{w}_{j,\mathcal{F}^{ij}})))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}+\\ &\qquad\qquad +\|(\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{F}^{ij}})\theta_{\mathcal{F}^{ij}}\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\Big).\end{aligned} \tag{28}$$

The first term can be estimated as in (24) by

$$C\left(1+\log(H_i/h_i)\right)^2\left(\rho_i|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})}+\rho_j|w_j|^2_{H^{1/2}(\mathcal{F}^{ij})}\right),$$

as desired, by applying a Poincaré inequality. There remains to estimate $\|((\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{F}^{ij}})\theta_{\mathcal{F}^{ij}}\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}$. Let $\mathcal{E}^{ik}\subset\partial\mathcal{F}^{ij}$ be a designated, primal edge. Then, we have

$$|\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{F}^{ij}}|^2\leq 2\left(|\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{i,\mathcal{E}^{ik}}|^2+|\overline{w}_{j,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{E}^{ik}}|^2\right).$$

It is sufficient to consider the first term on the right hand side. The shift invariance allows us to assume that $\overline{w}_{i,\mathcal{F}^{ij}}=0$. Using $|\overline{w}_{\mathcal{E}^{ik}}|^2\leq C/H_i\|w_i\|^2_{L_2(\mathcal{E}^{ik})}$ and Lemmas 1 and 4, we obtain

$$\|(\overline{w}_{i,\mathcal{F}^{ij}}-\overline{w}_{j,\mathcal{F}^{ij}})\theta_{\mathcal{F}^{ij}}\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\leq C\left(1+\log(H/h)\right)^2\left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})}+|w_j|^2_{H^{1/2}(\mathcal{F}^{ij})}\right).$$

The remainder of the proof of the result for Algorithm C can be carried out as in the proof of Lemma 9. However, for Algorithm D, we need to do some further work.

Proceeding as in the proof of Lemma 9, we can estimate the contributions of the edges of Ω_i to the energy of v_i in terms of L_2-norms over the edges. We first consider the second term of (25) in detail again assuming that Ω_i shares a face with each of Ω_j and Ω_ℓ, but only an edge with Ω_k. If we have a trivial, acceptable edge path, i.e., the common edge is designated as primal, we can proceed exactly as in (26). Otherwise assume that we have a non-trivial, acceptable edge path through the subdomain Ω_j via the edges $\mathcal{E}^{ij}$ and $\mathcal{E}^{jk}$; in general the acceptable edge path could be more complicated but such a case could be analyzed similarly. We obtain

$$\begin{aligned}
&\rho_i\,\|\rho_k^{\gamma}\mu_k^{\dagger}I^h(\theta_{\mathcal{E}^{ik}}(w_i-w_k))\|^2_{L_2(\mathcal{E}^{ik})}\\
=\ &\rho_i\,\|\rho_k^{\gamma}\mu_k^{\dagger}\left(I^h(\theta_{\mathcal{E}^{ik}}(w_i-\overline{w}_{i,\mathcal{E}^{ij}}))+\theta_{\mathcal{E}^{ik}}(\overline{w}_{j,\mathcal{E}^{ij}}-\overline{w}_{j,\mathcal{E}^{jk}})-\right.\\
&\qquad\left.-I^h(\theta_{\mathcal{E}^{ik}}(w_k-\overline{w}_{k,\mathcal{E}^{jk}}))\right)\|^2_{L_2(\mathcal{E}^{ik})}\\
\leq\ &C\min(\rho_i,\rho_k)\left(\|I^h(\theta_{\mathcal{E}^{ik}}(w_i-\overline{w}_{i,\mathcal{E}^{ij}}))\|^2_{L_2(\mathcal{E}^{ik})}+H_j\,|\overline{w}_{j,\mathcal{E}^{ij}}-\overline{w}_{j,\mathcal{E}^{jk}}|^2+\right.\\
&\qquad\left.+\|I^h(\theta_{\mathcal{E}^{ik}}(w_k-\overline{w}_{k,\mathcal{E}^{jk}}))\|^2_{L_2(\mathcal{E}^{ik})}\right).
\end{aligned}\tag{29}$$

The terms of the last expression can be estimated as before in (26). The only difference is that additionally, we have to use $TOL*\rho_j\geq\min(\rho_i,\rho_k)$. We obtain

$$\begin{aligned}
\rho_i\,\|\rho_k^{\gamma}\mu_k^{\dagger}I^h(\theta_{\mathcal{E}^{ik}}(w_i-w_k))\|^2_{L_2(\mathcal{E}^{ik})}&\leq C(1+\log(H/h))\left(\rho_i|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})}+\right.\\
&\left.+\rho_k|w_k|^2_{H^{1/2}(\mathcal{F}^{ik})}+TOL*\rho_j\left(|w_j|^2_{H^{1/2}(\mathcal{F}^{ij})}+|w_j|^2_{H^{1/2}(\mathcal{F}^{jk})}\right)\right).
\end{aligned}$$

Since Ω_i and Ω_j, as well as Ω_i and Ω_ℓ, have a face in common, the argument given above could be simplified for the first and third edge contributions, see (25); they can be reduced to estimates of face terms.

Finally, we consider the terms resulting from the vertices. We have, according to (22),

$$\begin{aligned}
&\rho_i|\theta_{\mathcal{V}^{i\ell}}v_i(\mathcal{V}^{i\ell})|^2_{H^{1/2}(\partial\Omega_i)}\\
&\leq C\sum_{j\in\mathcal{N}_{\Delta,\mathcal{V}^{i\ell}}}\rho_i(\rho_j^{\gamma}\mu_j^{\dagger})^2|\theta_{\mathcal{V}^{i\ell}}|^2_{H^{1/2}(\partial\Omega_i)}|w_i(\mathcal{V}^{i\ell})-w_j(\mathcal{V}^{i\ell})|^2\\
&\leq C\sum_{j\in\mathcal{N}_{\Delta,\mathcal{V}^{i\ell}}}\min(\rho_i,\rho_j)h_i|w_i(\mathcal{V}^{i\ell})-w_j(\mathcal{V}^{i\ell})|^2.
\end{aligned}$$

We now consider each pair of substructures separately. Let Ω_i, Ω_l be such a pair and assume that we have an acceptable edge path through Ω_j, via the edges $\mathcal{E}^{ij}$ and $\mathcal{E}^{jl}$ with the condition

$$TOL * \rho_j \geq \frac{h_j}{H_j} \min(\rho_i, \rho_l). \tag{30}$$

We can proceed as in the analysis of the edge terms and obtain

$$\begin{aligned} &\min(\rho_i, \rho_l) h_i |w_i(\mathcal{V}^{il}) - w_l(\mathcal{V}^{il})|^2 \\ &\leq 3 \min(\rho_i, \rho_l) h_i \left(|w_i(\mathcal{V}^{il}) - \overline{w}_{i,\mathcal{E}^{ij}}|^2 + |\overline{w}_{j,\mathcal{E}^{ij}} - \overline{w}_{j,\mathcal{E}^{jl}}|^2 + |w_l(\mathcal{V}^{il}) - \overline{w}_{l,\mathcal{E}^{jl}}|^2 \right). \end{aligned}$$

It is sufficient to estimate the first term on the last line; the third term can be treated in exactly the same way, and the second term can be estimated as above with the only difference of an additional factor h_j/H_j which is accounted for in (30). Using $h_i |w_i(\mathcal{V}^{il})|^2 \leq C \|w_i\|^2_{L_2(\mathcal{E}^{ij})}$ and Lemma 4, and estimating $|\overline{w}_{i,\mathcal{E}^{ij}}|$ as before, we obtain

$$\begin{aligned} |(w_i(\mathcal{V}^{il}) - \overline{w}_{i,\mathcal{E}^{ij}}|^2 &\leq 2 \left(|(w_i(\mathcal{V}^{il})|^2 + |\overline{w}_{i,\mathcal{E}^{ij}}|^2 \right) \\ &\leq C(1 + \log(H_i/h_i)) h_i^{-1} \left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + 1/H_i \|w_i\|^2_{L_2(\mathcal{F}^{ij})} \right) \\ &\leq C(1 + \log(H_i/h_i)) h_i^{-1} |w_i|^2_{H^{1/2}(\mathcal{F}^{ij})}. \end{aligned}$$

Here, the last line follows again from the shift invariance of the first expression. Using (30), we finally obtain

$$\begin{aligned} &\min(\rho_i, \rho_l) h_i |w_i(\mathcal{V}^{il}) - w_l(\mathcal{V}^{il})|^2 \\ &\leq C \left(1 + \log(H/h)\right) \left(\rho_i |w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \rho_\ell |w_\ell|^2_{H^{1/2}(\mathcal{F}^{j\ell})} + \right. \\ &\qquad \left. + TOL * \rho_j \left(|w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} + |w_j|^2_{H^{1/2}(\mathcal{F}^{j\ell})} \right) \right). \end{aligned}$$

The boundary subregions can again be treated as in the proof of Lemma 9.

□

We can now prove our condition number estimates for Algorithms C and D, which are as strong as those in Theorem 1. The proof can be carried out exactly as for Theorem 1, using Lemma 10 instead of Lemma 9.

THEOREM 2 (Algorithms C, D). *The condition numbers satisfy*

$$\kappa(M_C^{-1} F_C) \leq C \left(1 + \log(H/h)\right)^2.$$

and

$$\kappa(M_D^{-1} F_D) \leq C \max(1, TOL) \left(1 + \log(H/h)\right)^2.$$

Here, C is independent of h, H, γ, and the values of the ρ_i.

REMARK 1. *It is possible to define a fourth interpolation operator I_D^h, based on the weights ρ_i, the pseudoinverses $\mu_i^\dagger$, and the averages over the subdomain boundaries, by*

$$I_D^h u(x) = \sum_i \overline{u}_{\partial\Omega_i} \rho_i^\gamma(x) \mu_i^\dagger(x). \tag{31}$$

Here the average $\overline{u}_{\partial\Omega_i}$ is defined by

$$\overline{u}_{\partial\Omega_i} = \frac{\int_{\partial\Omega_i} u ds}{\int_{\partial\Omega_i} 1 ds},$$

where we use the component in W_i when computing this average. This operator naturally appears in studies of Neumann-Neumann algorithms. We can establish the same type of bounds as in Lemma 7, provided that we introduce the same constraints as for Algorithm D.

REMARK 2. *It is already known from the numerical results in [9, 10] that Algorithm A is not competitive. We can prove that the condition number of Algorithm A satisfies the weaker bound,*

$$\kappa(M_A^{-1} F_A) \leq C\,(H/h)\,(1 + \log(H/h))^2,$$

in the same way as Theorem 1, using Lemma 6 and a variant of Lemma 10. Here, C is independent of h, H, γ, and the values of the ρ_i.

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