Describing spatial transitions using mereotopological relations over histories*

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Abstract

Muller (1998) develops a language of motion and shape change in terms of topological relations and temporal order relations between regions of space-time (histories). He uses this language to state and prove the transition rules developed in (Randell, Cui, and Cohn, 1992) that constrain the changes in spatial relations possible for objects whose shape changes continuously. Unfortunately, Muller's statement of the transition rules is inadequate. This paper presents an alternative statement of these transition rules.

In an important and elegant paper, Philippe Muller (1998) develops a theory of motion based on the geometry of four-dimensional regions of space-time, called "histories". This approach was suggested in Hayes' "Naive Physics Manifesto" (1979), but Muller's paper was the first to explore it systematically.

Muller's language is a first-order language over the universe of histories. Muller's paper admits both regions that are regular and those that are "regular and open"; i.e. the interior of regular regions. I will modify this here to include only regular[1] regions; this restriction does not affect the issues under discussion here, and it simplifies the presentation.

The language contains three primitive binary relations:[2]

$\mathrm{C}xy$ – Regions x and y are connected; that is, they share at least one point.
$x < y$ — Region x strictly precedes region y temporally.
$x \lessgtr y$ — The temporal projections of regions x and y share at least one instant.

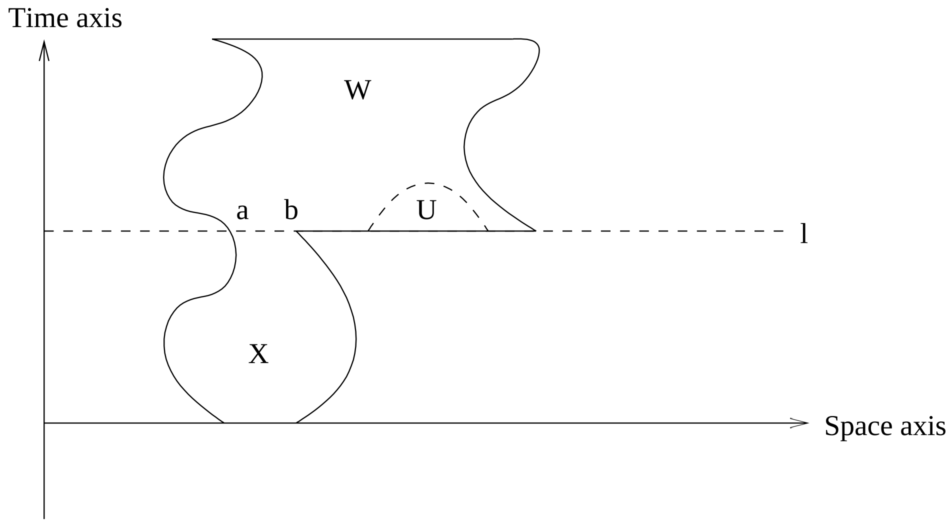
Other relations between regions are defined in terms of non-recursive first-order formulas over these.

Muller proposes the following definition (D4.2) for continuity: Region w represents a continuous function from time to space if it is satisfies the following:

*This research was supported in part by NSF grant #IRI-9625859. This paper began as the last section of (Davis, 2001). However, the reviewers of that paper felt, with some justification, that this material was peripheral to the primary purposes of that paper, and should be deleted. That left me in a quandary as to what to do with this. The material here is not really substantive enough to warrant separate publication; on the other hand, it would be a pity to discard it entirely. I decided, therefore, to publish it just as a technical note on the Web. I have not attempted to turn it into an independent work; rather, it should be considered as an addendum to (Davis, 2001).

[1]A region is regular if it is equal to the closure of its interior

[2]This paper follows Muller's notational conventions: a variable is a single lower-case letter; predicates are either prefix or infix; atomic formulas are strings of symbols without further punctuation.



W is the whole history in the solid boundary. X is the portion of W below the dotted line.
U is the semi-circle on the left, also a subregion of W.

Figure 1: Muller's definition of discontinuity

$$\text{D4.2: CONTINU}w \triangleq \text{CON}_t w \wedge \forall_x \forall_u \; ((\text{TS}xw) \wedge x \gtrless u \wedge \text{P}uw) \Rightarrow \text{C}xu.$$

Here "$\text{CON}_t w$" means that the temporal projection of w is a connected time interval. "Puw" means that u is a subregion of w. "TSxw" means that x is a "time-slice" of w. This predicate is a little tricky. For a normal regions w and x, it asserts that x is the normalization of the restriction of w to a time period i, where i is a regular subset of the time-line. The definition of these predicates in terms of "C" "<" and "$\gtrless$" is given in (Muller, 1998).

Figure 1 shows how a discontinuous function of time fails to satisfy definition D4.2. Note that the only part of line l contained in x is the segment between points a and b.

In (Davis, 2001) we show that the graph of a function from time to regions satisfies definition D4.2 if and only if it is continuous with respect to the Hausdorff metric. More precisely, we state and prove the following theorem:

Theorem 1: Let w be a bounded normal history whose temporal projection is a connected time interval I, and let $w(t)$ be the cross-section of w at time $t \in I$. Then w satisfies Muller's definition D4.2 iff $w(t)$ is continuous in the Hausdorff distance.

However, Theorem 1 above leads to a conflict between Muller's analysis of transitions and the analysis that we give in (Davis, 2001). Muller claims to show that it follows from his definition that the only transitions possible are the ones in figure 2. In (Davis, 2001) we show, on the contrary, that functions continuous in the Hausdorff distance can execute any of the transitions in figure 3. Indeed, note that the transition from EQ to DS described in (Davis, 2001) satisfies Muller's of continuity. Where, then, is the difference between Muller's account of the transitions and ours?

(In figure 3, the significance of the arrow from the dashed circle on the right to the dashed circle on the left is that *every* relation on the right can undergo a transition to *any* relation on the left. That is, there should be a arrow from each of the five states on the right to each of the three states on the left; however, we have summarized these in terms of the dashed circles, in order to simplify the diagram.)

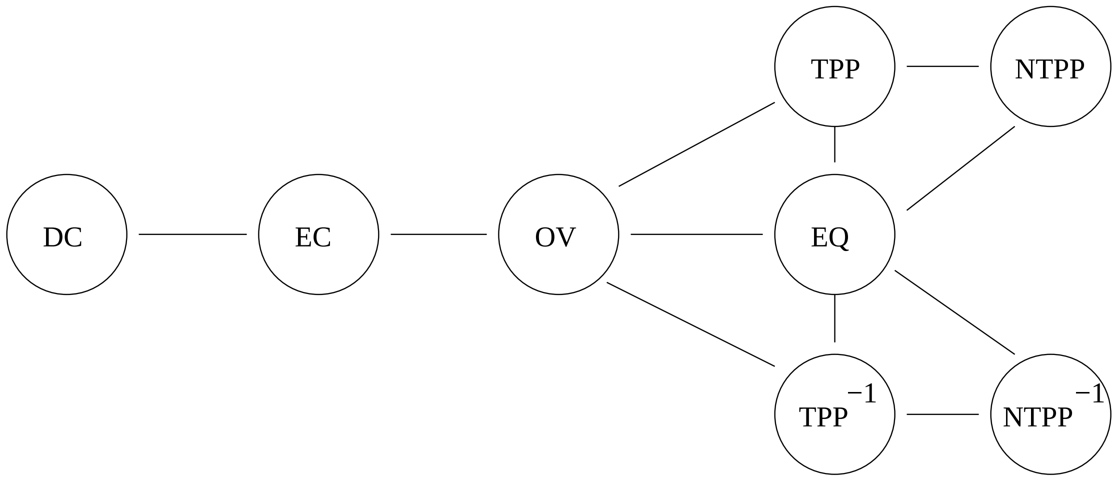


Figure 2: Undirected transition graph from (Randell, Cui, and Cohn, 1992)

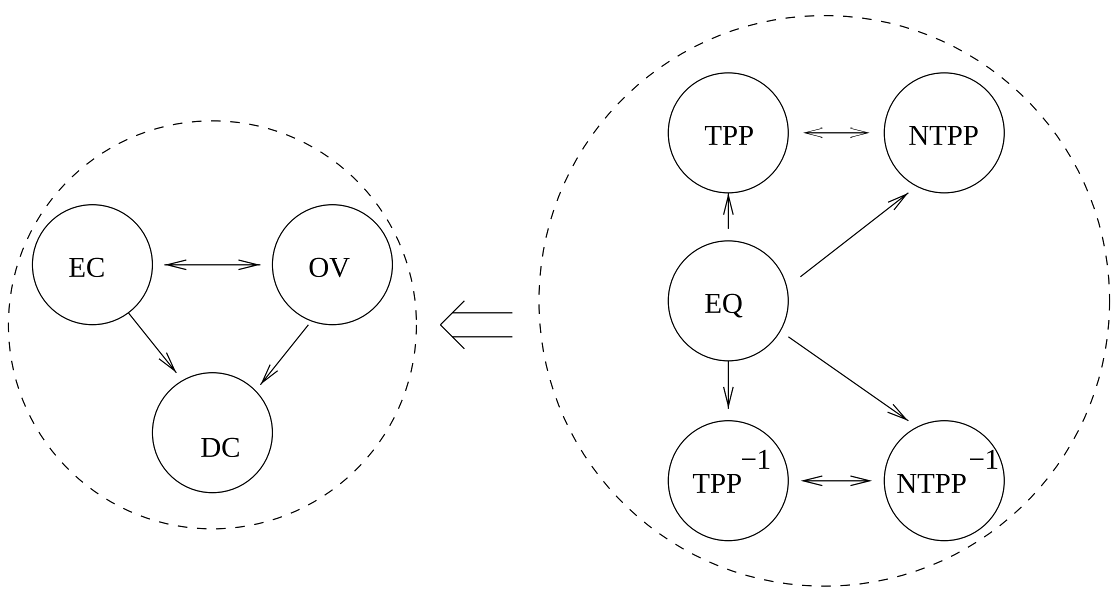


Figure 3: Transition graph for the Hausdorff metric (Davis, 2001).

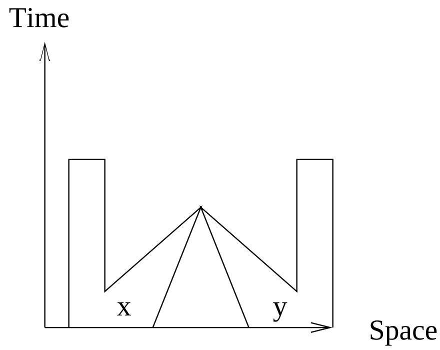


Figure 4: The normalized cross-sections of x and y are always disconnected

The resolution is that Muller is using a different, and, we believe, flawed formal definition of a "transition". That is, the formal theorems Th 4.3 – 4.6 that Muller proves are, indeed, true, but his interpretation of these theorems as expressing transition relations is incorrect.

Consider, for example, the form of Th. 4.3:

$$\mathrm{TS}xw \wedge \mathrm{P}yw \wedge \mathrm{DC}xz \wedge \mathrm{OV}_{sp}yz \wedge x \gtrless y \Rightarrow \neg \mathrm{CONTINU}w$$

Muller claims that this theorem expresses the impossibility of a transition from DC to OV by a continuous function w.

The relation $\mathrm{OV}_{sp}yz$ is intended to mean that y and z overlap spatially in every temporal slice of their common domain. Muller's formal definition is

$$\mathrm{OV}_{sp}xy \triangleq \mathrm{OV}xy \wedge x \subseteq_t x \cdot y$$

An analogous spatial relation is defined for each of the RCC relations.

There are a couple of peculiarities with the above form of Thm. 4.3. First, it is not at all obvious why this axiom should express the non-existence of a transition relation. Second, the relations DC and OV occupy non-symmetric logical positions in this formula. (Muller is not trying to identify the "directionality" or "dominance" of transitions, in the sense of (Galton, 1995).)

Furthermore, the spatial RCC relations are defined *ad hoc*; it is not clear what precisely Muller intends and therefore not clear whether his formal definitions achieve his intentions. For example, the above definition of OV_{sp} has at least three apparent formal flaws. First, it is asymmetric in x and y. Second, Muller does not define the notation $x \cdot y$ and it is not clear whether he means the intersection or the normalized intersection. Third, the condition $\mathrm{OV}xy$ is redundant if $x \cdot y$ means the normalized intersection, and does not accomplish anything reasonable if $x \cdot y$ means the intersection. Another example: Muller claims that DC_{sp} is equivalent to DC; however, figure 4 shows two histories x and y that are not DC but whose normalized spatial cross sections are always disconnected.

A more systematic approach to representing transitions can be accomplished along the following lines: Let $x(t)$ and $y(t)$ be two functions from time intervals to normal spatial regions, and let R and P be two spatial relations. We say that x and y transition from R to P if, for some t_1, t_2 in the domains of both x and y, if $\mathrm{R}(x(t_1), y(t_1))$, $\mathrm{P}(x(t_2), y(t_2))$ and for all $t \in [t_1, t_2]$ either $\mathrm{R}(x(t), y(t))$ or $\mathrm{P}(x(t), y(t))$.

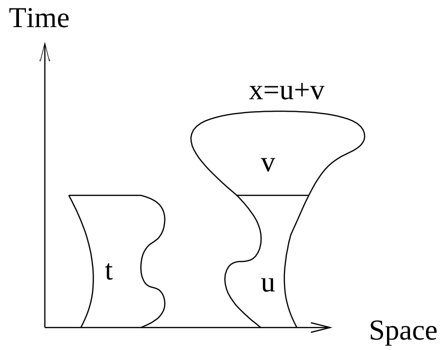


Figure 5: The relation SPLIT$txuv$

Now, turning to the language of histories, let x be a normal history whose temporal projection is a connected interval i. Define the corresponding function $x(t)$ to be the normalization of the cross-section of x at t, which we will abbreviate Nx(x,t). (This is not, of course, a function in the formal language, as it maps to a purely spatial region; it is just for our informal discussion.) x is then equal to the closure of the set $\{\langle t,p\rangle \mid p \in x(t)\}$. We can then define two normal histories x and y as transitioning transition from R to P if Nx(x,t) and Nx(y,t) transition from R to P as defined above. Our task, then, is to give a definition of the concept "The normalized cross-sections of x and y at time t have spatial relation R," in Muller's language of histories.

We proceed as follows:

I. We will identify a time instant t with any history that ends at t. The following temporal predicates will be useful:

$$\text{MEET}xy \stackrel{\Delta}{=} x \between y \wedge \neg x\sigma y \wedge \forall_{wz}(\text{P}wx \wedge \text{P}zy) \Rightarrow w < z \vee w \between z$$

The notation $x\sigma y$ means that the interiors of the time-projections of x and y overlap. Muller gives the following definition:

$$x\sigma y \stackrel{\Delta}{=} \exists_z z \subseteq_t x \wedge z \subseteq_t y$$

Muller also gives a definition of MEET, but his requires the use of open histories, which we have excluded.

$$\text{SAME-END}xy \stackrel{\Delta}{=} \exists_z \text{MEET}xz \wedge \text{MEET}yz$$

(x and y end at the same time.)

II. If time t is in the interior of the temporal projection of history x, then x can be split into two histories u and v such that u ends at the end of t and v begins at the beginning of t, using the following formula.

$$\text{SPLIT}txuv \stackrel{\Delta}{=} x = u + v \wedge \text{SAME-END}tu \wedge \text{MEET}tv$$

Note that the cross-section of x at t is equal to the union of the cross-sections of u and v, and likewise for the normalized cross-sections (Figure 5).

III. A history a "touches" history u from above if u meets a and u is externally connected to a. History a touches history v from below if a meets v and a is externally connected to v.

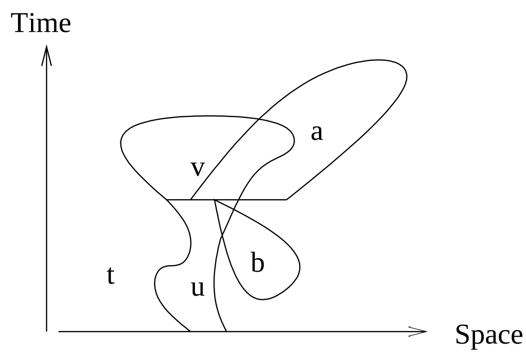


Figure 6: The relation SAME-FACEuv

$$\text{TOUCHX1}au \triangleq \text{MEET}ua \wedge \text{EC}ua$$

$$\text{TOUCHX2}av \triangleq \text{MEET}av \wedge \text{EC}av$$

IV. If history u meets history v, then they have the same meeting face if every history a that touches u is externally connected to a history b that touches v and vice versa. (Figure 6)

$$\begin{aligned}&\text{SAME-FACE}uv \triangleq \\ &\text{MEET}uv \wedge [[\forall_a \text{TOUCHX1}au \Rightarrow \exists_b \text{TOUCHX2}bv \wedge \text{EC}ab] \wedge \\ &\qquad [\forall_b \text{TOUCHX2}bv \Rightarrow \exists_a \text{TOUCHX1}au \wedge \text{EC}ab]]\end{aligned}$$

V. We can now simplify the analysis of the cross section of history x at time t by creating a new history r that consists of: the union of [the part of x before t] with [a "reflection" of the part of x after t]. (Figure 7.)

$$\text{REFLECT}xtr \triangleq \exists_{u,v,w} \text{SPLIT}xtuv \wedge \text{SAME-FACE}vw \wedge r = u + w$$

VI. History a touches the boundary of the end of history u if a has the same end as u, and no matter how thin you slice a, the later half is EC to u (Figure 8).

$$\text{TOUCHBD}au \triangleq \text{SAME-END}au \wedge \forall_s \text{SPLIT}sabc \Rightarrow \text{EC}uc$$

VI. History a touches the interior of the end of history u if a touches u but is disconnected from any history that touches the boundary of the end of u (Figure 9.)

$$\text{TOUCHINT}au \triangleq \text{TOUCH1}au \wedge \forall_b \text{TOUCHBD}bu \Rightarrow \text{DC}ba$$

VII. The normalized cross-section of history x at time t is a subset of the normalized cross-section of history y at time t if the following holds: Let q and r be reflections of x and y at t. Then

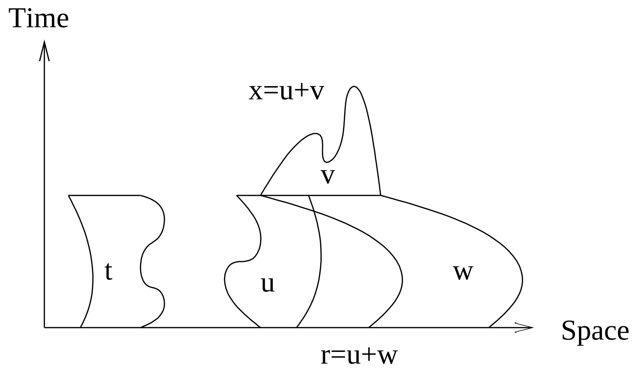


Figure 7: The relation REFLECTxtr

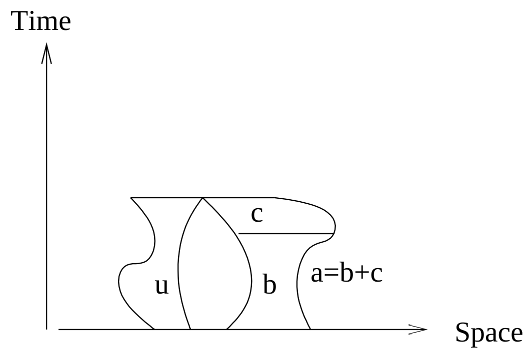


Figure 8: The relation TOUCHBDau

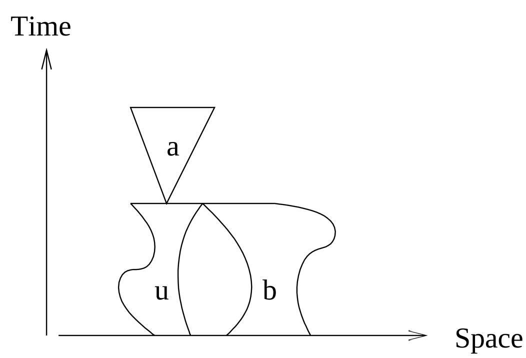


Figure 9: The relation TOUCHINTau

any region that touches the interior of q also touches the interior of r. (We will subscript c to RCC relations to indicate spatial relations that hold at an instantaneous cross-section.)

$$\mathrm{PP}_c txy \triangleq \exists_{q,r} \mathrm{REFLECT} xtq \land \mathrm{REFLECT} ytr \land \forall_a \mathrm{TOUCHINT} aq \Rightarrow \mathrm{TOUCHINT} ar$$

VII. The boundary of normalized cross-section of history x at time t is disjoint from the boundary of the normalized cross-section of history y at time t if the following holds: Let q and r be reflections of x and y at t. Let a and b be histories that touch the boundaries of the ends of q and r. Then there exist subhistories c in a and d in b such that c touches the boundary of the end of q, d touches the boundary of the end of r, but c and d are disconnected.

$$\begin{array}{l} \mathrm{DISJ\text{-}BD} txy \triangleq \\ \exists_{q,r} \mathrm{REFLECT} xtq \land \mathrm{REFLECT} ytr \land \\ \quad \forall_{a,b} [\mathrm{TOUCH\text{-}BD} aq \land \mathrm{TOUCH\text{-}BD} br] \Rightarrow \\ \quad\; \exists_{c,d} \mathrm{PP} ca \land \mathrm{PP} db \land \mathrm{TOUCH\text{-}BD} cq \land \mathrm{TOUCH\text{-}BD} da \land \mathrm{DC} cd. \end{array}$$

VIII. The remaining spatial relations over normalized cross-sections of x and y can be defined from PP_c and $\mathrm{DISJ\text{-}BD}_c$.

$\mathrm{TPP}_c txy \triangleq \mathrm{PP}_c txy \land \neg \mathrm{DISJ\text{-}BD} txy$.
$\mathrm{NTPP}_c xy \triangleq \mathrm{PP}_c txy \land \mathrm{DISJ\text{-}BD} txy$.
$\mathrm{DS}_c txy \triangleq \neg \exists_w \mathrm{PP}_c twx \land \mathrm{PP}_c twy$.
(The cross-sections of x and y at t have no common interior points.)
$\mathrm{OV}_c txy \triangleq \quad \exists_{a,b,c} \mathrm{PP}_c tax \land \mathrm{PP}_c tay \land \mathrm{PP}_c tbx \land \mathrm{PP}_c tcy \land \mathrm{DS}_c tbx \land \mathrm{DS}_c tcy$.
$\mathrm{EC}_c txy \triangleq \mathrm{DS}_c txy \land \neg \mathrm{DISJ\text{-}BD} txy$.
$\mathrm{DC}_c txy \triangleq \mathrm{DS}_c txy \land \mathrm{DISJ\text{-}BD} txy$.
$\mathrm{EQ}_c txy \triangleq \mathrm{PP}_c txy \land \mathrm{PP}_c tyx$.

IX. Following the discussion above, we can state the existence of a transition from relation R_c to S_c in the statement

$$\begin{array}{l} \exists_{a,b,t,w}\ \mathrm{CONTINU} a \land \mathrm{CONTINU} b \land \mathrm{R}_c tab \land \mathrm{S}_c wab \land t \subseteq_t w \land \\ \qquad \forall_u\ t \subseteq u \subseteq w \Rightarrow [\mathrm{R}_c uab \lor \mathrm{S}_c uab]. \end{array}$$

It can reasonably be objected to this analysis that, though it observes the letter of the mereotopological enterprise, it violates the spirit, as it achieves its ends by using the very great expressive power of first-order logic over histories to, in effect, define time instants and spatio-temporal points. Certainly, proving the correctness of rules that state the non-existence of transitions, or worse, those that state the existence of transitions, from plausible mereotopological axioms, would seem to be daunting if not hopeless in this expression of these rules. One might hope, therefore, that a more natural mereotopological expression of transition rules could be found that could indeed be proved in a mereotopological theory. It seems doubtful to me, however, that such a characterization could be found that would be entirely satisfying.

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