Domain Decomposition Methods for Mortar Finite Elements

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Bunicului meu

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Aceasta teza este dedicata Bunicului meu.

ABSTRACT

Domain decomposition methods are powerful iterative methods for solving systems of algebraic equations arising from the discretization of partial differential equations by, e.g., finite elements. The computational domain is decomposed into overlapping or nonoverlapping subdomains. The problem is divided into, or assembled from, smaller subproblems corresponding to these subdomains.

In this dissertation, we focus on domain decomposition methods for *mortar finite elements*, which are nonconforming finite element methods that allow for a geometrically nonconforming decomposition of the computational domain into subregions and for the optimal coupling of different variational approximations in different subregions.

We introduce a FETI method for mortar finite elements, and provide numerical comparisons of FETI algorithms for mortar finite elements when different preconditioners, given in the FETI literature, are considered. We also analyze the complexity of the preconditioners for the three dimensional versions of the algorithms.

We formulate a variant of the balancing method for mortar finite elements, which uses extended local regions to account for the nonmortar sides of the subregions. We prove a polylogarithmic condition number estimate for our algorithm in the geometrically nonconforming case. Our estimate is similar to those for other Neumann–Neumann and substructuring methods for mortar finite elements.

In addition, we establish several fundamental properties of mortar finite elements: the existence of the nonmortar partition of any interface, the L^2 stability of the mortar projection for arbitrary meshes on the nonmortar side, and prove Friedrichs and Poincaré inequalities for geometrically nonconforming mortar elements.

Contents

Chapter 1

Introduction

1.1 An Overview

Domain decomposition methods are powerful iterative methods for solving systems of algebraic equations arising from the discretization of partial differential equations (PDEs) by, e.g., finite elements. The computational domain, i.e., the domain where the PDE is defined and must be solved, is decomposed into overlapping or nonoverlapping subdomains. The problem is divided into, or assembled from, smaller subproblems corresponding to these subdomains. More generally, we can consider the decomposition of the finite element space into a sum of subspaces, which are often related to the subdomains or to other sets. The subproblems are solved directly or iteratively, and the exchange of information between subspaces is handled by an iterative method. These algorithms can be regarded as methods for building Krylov space preconditioners for the linear systems.

The best of the domain decomposition methods converge in relatively few iteration steps. The iteration count is independent of the number of subregions and depends only weakly on the dimension of the finite element spaces used in the discretization. These scalability properties are due in part to the use of a *coarse space*, which has few degrees of freedom compared with the original finite element problem. At each iteration step, the corresponding coarse solver smoothes out the low frequency components of the iteration error, while the local solvers smooth out the high frequency ones.

The first domain decomposition algorithm is believed to be the alternating Schwarz method, which was developed in the 19th century by Schwarz [115]. It is an iterative method and was used by Schwarz for proving the existence of solutions for elliptic PDEs on domains which are the union of subregions with non-void intersection, if the PDE can be solved in each subregion. The interest in this algorithm was revived in the 1980's by P. L. Lions [87, 88], who analyzed a variational formulation of the Schwarz method. Numerous domain decomposition methods have been designed, studied, and implemented in the last decade; see Section 3.3 for a brief overview. Two monographs by Smith, Bjørstad, and Gropp [122] and Quarteroni and Valli [108] have appeared recently, and yearly international conferences are being held; see [67, 37, 38, 68, 73, 107, 74, 69, 16, 92, 82].

In this thesis, we concentrate on *mortar finite elements*, which are nonconforming finite element methods that allow for a geometrically nonconforming decomposition of the computational domain into subregions and, at the same time, for the optimal coupling of different variational approximations in different subregions. Here, optimality means that the global error is bounded by the sum of the local approximation errors on each subregion.

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [13, 14], for low-order and spectral finite elements, and were extended to three dimensional elements in [8, 9].

Mortar finite elements have several advantages over the conforming finite elements. For example, the mesh generation is more flexible and can be made quite simple on individual subregions. This also makes it possible to move different parts of the mesh relative to each other, e.g., in a study of time dependent problems. The same feature is most valuable in optimal design studies, where the relative position of parts of, e.g., an automobile or an electrical machine, is not fixed a priori. The mortar methods also allow for local refinement of finite element models in only certain subregions of the computational domain, and are well suited for parallel computing; cf. [71].

Let us briefly describe the mortar finite element space V^h for the two dimensional case. The computational domain Ω is decomposed into a nonoverlapping polygonal partition $\{\Omega_i\}_{i=1:N}$, possibly made of curvilinear polygons. The parti-

tion is called geometrically conforming if the intersection between the closure of any two subregions is either empty, a vertex, or an entire edge, and it is called nonconforming otherwise. The restriction of V^h to any subregion Ω_i is a conforming finite element or spectral element space. These spaces can be independent of each other, and can differ in degree and type.

Across the interface Γ, i.e., the set of points that belong to the boundaries of at least two subregions, pointwise continuity is not required. We partition Γ into a union of nonoverlapping subregion edges called nonmortars. The edges across Γ, not chosen to be nonmortars, are called mortars. We note that the partition and the choice of nonmortars are not unique; however, any choice can be treated similarly. On the two sides of a nonmortar edge γ, there are two distinct traces of the mortar functions and we only require that their difference is L^2-orthogonal to a space of test functions defined on γ. This space is generally a subspace of codimension two of the restriction of V^h to γ; this allows the values of the mortar function at the end points of any nonmortar to be genuine degrees of freedom.

As we have just seen, when introducing a mortar finite element space on Ω, the first step is to decompose Ω into nonoverlapping subregions. Since the domain decomposition methods are also based on such a decomposition of the computational domain, there exists a natural connection between mortars and domain decomposition. In the recent past, this connection has been the basis for extensive studies of domain decomposition methods for mortars; see [3, 44, 45, 46, 84, 85, 81, 126, 2, 1, 36, 134], and also Chapter 2 for more details.

This thesis is also focused on *domain decomposition methods* for *mortar finite elements*. Our main goal has been to obtain numerical and theoretical performance estimates for these methods of the same form as in the conforming finite element case.

In Chapter 5, we analyze the Finite Element Tearing and Interconnecting (FETI) method of Farhat and Roux [61] applied to mortar finite elements. The FETI method is an iterative substructuring method where Lagrange multipliers are used to enforce continuity conditions across the subdomain interface. Therefore, we can apply this method to mortar finite elements without changing the algorithm. The main difference is that the Lagrange multipliers matrix is used to

enforce mortar conditions across the interface, instead of pointwise continuity.

Our numerical experiments, which were performed both in two and three dimensions, show that the original preconditioner for the FETI method with conforming elements used in [61] does not perform well in the mortar case. However, a new preconditioner recently suggested by Klawonn and Widlund [78] performs satisfactory. The number of iterations required to achieve convergence depends only weakly on the number of nodes in each subregion and is independent of the number of subregions. Thus, by these experiments with the mortar case, we were led to believe that the new preconditioner might always have a better performance than the original preconditioner, even for the conforming case. Our numerical tests confirm this hypothesis. The number of iterations for the conforming finite elements FETI method with the new preconditioner is roughly half of that required with the original preconditioner. We also analyze the extra computational effort, due to the complexity of the mortar conditions, required for the implementation of the FETI algorithm with new preconditioner, in the three dimensional case. We conclude that the improvement of the iteration count offsets this extra cost.

In Chapter 6, we formulate a variant of the balancing method of Mandel and Brezina [89] for mortar finite elements. The balancing method is a hybrid nonoverlapping Schwarz method of Neumann-Neumann type. An important role is played by the counting functions associated with the boundary nodes of each subregion. They indicate to how many local spaces a node on the interface belongs, and are used to construct the local approximate solvers and a coarse space of minimal dimension. In the mortar case, the values of a mortar function on a nonmortar side depend on its values on the mortar sides opposite the nonmortar. In our algorithm, we choose to extend the local spaces to capture this feature. We also change the counting functions, which are now associated to the number of local spaces to which a boundary node belongs. We prove an upper bound for the condition number of our balancing algorithm for geometrically nonconforming mortar case, which is of the same form as the condition number estimates for other domain decomposition methods for mortars; see, e.g., Achdou, Maday, and Widlund [3] and Dryja [46].

Another part of our work is concerned with establishing several fundamental

properties of mortar finite elements. We present these results in Chapter 4.

By using a combinatorial argument, we prove the existence of the nonmortar partition of any interface, in two and three dimensions, for general geometrically nonconforming partitions.

The mortar projection is the operator which gives the values of the mortar finite element function at the interior nodes of a nonmortar as obtained from the mortar conditions. We show that the mortar projection is stable in L^2 for arbitrary meshes on the nonmortar. The constant in the stability inequality depends only on the polynomial degree of the finite element space on the nonmortar, and does not depend on the properties of the nonmortar mesh.

Another important issue in the study of mortar methods is to compare the condition numbers of the stiffness matrices of the discretized system, obtained when using mortar and continuous elements. For a geometrically conforming partition, it has been shown by Bernardi and Maday [10, 11] that the condition number behaves as in the continuous case. We show that the same result holds for the geometrically nonconforming case, by proving a Friedrichs-type inequality for the mortar functions. The same type of arguments also lead to a proof of a Poincaré-type inequality for mortar finite element methods.

The L^2 stability result of Section 4.2 has appeared as a technical report [123]. The Friedrichs and Poincaré inequalities for mortars have also appeared as a technical report [124] which has been submitted for publication. The numerical results for two dimensional problems of Chapter 5 appear in papers written jointly with Axel Klawonn [125, 126], the first of which has been accepted for publication in [82].

In the remainder of this chapter, we give a brief presentation of certain Sobolev spaces and some of their properties. In Section 1.3, we present the variational formulation of our elliptic boundary values problem, and, in Section 1.4, we describe the finite element spaces we considered in this thesis. We conclude the chapter by presenting, in Section 1.5, some technical results on Sobolev norms for finite element functions which are often used in the analysis of domain decomposition methods.

1.2 Sobolev Spaces

Sobolev spaces are essential tools in solving elliptic partial differential equations. Using the variational formulation of the PDE, the existence of a generalized solution in the appropriate Sobolev space can be established by using variational methods, in particular the Lax-Milgram lemma. Regularity results are also expressed by bounding the Sobolev norm of the solution of the PDE in terms of the Sobolev norm of the boundary data and the right hand side; cf. [29, 86].

In this section, we present some basic results on Sobolev spaces, which are used in the study of domain decomposition methods, and are relevant for this thesis. We restrict our presentation to the spaces we need in our work. For a description of the general spaces and their properties, see [4, 86, 95].

Let $\Omega \subset \mathtt{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain with smooth boundary. The space $L^2(\Omega)$ is defined as the space of square integrable functions,

$$L^2(\Omega) \;=\; \{u \;:\; ||u||_{L^2(\Omega)} = \Big(\int_\Omega |u|^2 dx \Big)^{1/2} < \infty \}.$$

Let k be a positive integer. The Sobolev space $H^k(\Omega)$ is the Hilbert space of functions with weak derivatives of all orders less than and equal to k in the space $L^2(\Omega)$. In particular, the inner product on $H^1(\Omega)$ is

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv \; dx \;+\; \int_\Omega \nabla u \cdot \nabla v \; dx.$$

The H^1-seminorm and norm of $u \in H^1(\Omega)$ are, respectively,

$$\begin{aligned} |u|^2_{H^1(\Omega)} &= \int_\Omega |\nabla u|^2 \; dx; \\ ||u||^2_{H^1(\Omega)} &= |u|^2_{H^1(\Omega)} \;+\; ||u||^2_{L^2(\Omega)}. \end{aligned}$$

Of particular interest for domain decomposition methods, see, e.g., [52], is the scaled norm obtained by dilation of a domain of unit diameter,

$$||u||^2_{H^1(\Omega)} \;=\; |u|^2_{H^1(\Omega)} \;+\; \frac{1}{diam(\Omega)^2} ||u||^2_{L^2(\Omega)}, \tag{1.1}$$

where $diam(\Omega)$ is the diameter of Ω.

The Sobolev spaces can also be defined as the closure of $C^\infty(\Omega)$ in the corresponding norm, e.g., $H^1(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to $||\cdot||_{H^1(\Omega)}$. Let $C_0^\infty(\Omega) \subset C^\infty(\Omega)$ be the set of smooth functions with compact support. The subspace $H_0^1(\Omega) \subset H^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to $||\cdot||_{H^1(\Omega)}$, and consists of all the functions from $H^1(\Omega)$ which vanish on $\partial\Omega$ in the L^2 sense.

The following lemma gives necessary and sufficient conditions for a piecewise H^1 function to be H^1 on the entire domain. The proof is elementary, and is based on Green's formula; cf. [41].

Lemma 1.1. *Let $\{\Omega_i\}_{i=1:N}$ be a nonoverlapping partition of the domain Ω, and let u be a function defined on Ω such that $u \in H^1(\Omega_i)$, $i = 1 : N$. Then $u \in H^1(\Omega)$ if and only if the jump of u across the interface between the subregions $\{\Omega_i\}_{i=1:N}$ vanishes in the L^2 sense.*

On the entire space $\mathtt{R}^d$, another way to introduce the Sobolev spaces is by using the Fourier transforms; see [63] for more details. In this setting, let $\mathcal{F}u$ be the Fourier transform of u, and let

$$(\mathcal{F}u)(\zeta) \;=\; (1 + |\zeta|^2)^{s/2}\hat{u}(\zeta).$$

Then $u \in H^s(\mathtt{R}^d)$ if and only if $\mathcal{F}u \in L^2(\mathtt{R}^d)$, and

$$||u||_{H^s(\mathtt{R}^d)} \;=\; ||\mathcal{F}u||_{L^2(\mathtt{R}^d)}.$$

We note that this definition holds for every real number $s \in \mathtt{R}$. For negative numbers, the space $H^{-s}(\mathtt{R}^d)$ is isomorphic to the dual space of $H^s(\mathtt{R}^d)$ with respect to the inner product

$$< u, v > \;=\; (\mathcal{F}u, \mathcal{F}v)_{L^2(\mathtt{R}^d)}. \tag{1.2}$$

The fractional Sobolev spaces on bounded domains are introduced in a slightly different way. If k is a positive integer, then $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$in the L^2 inner product; see (1.2). Let $s \in \mathtt{R}$ such that s is not an integer. Then, $H^s(\Omega)$ can be obtained from $H^{\lfloor s \rfloor}(\Omega)$ and $H^{\lceil s \rceil}(\Omega)$ by using the K-method of interpolation; cf., e.g., Lions and Magenes [86] and Triebel [131]. Alternatively, for s positive, it follows from an extension theorem that any $u \in H^s(\Omega)$ can be obtained

as the restriction of $\tilde{u} \in H^s(\mathbb{R}^d)$. It also follows from a density result that $C_0^\infty(\mathbb{R}^d)$ is dense in $H^s(\Omega)$.

For the purpose of this thesis, we restrict our attention to the spaces $H^s(\Omega)$, $-1 \leq s \leq 1$. For these spaces, the smoothness conditions imposed on the boundary of Ω can be relaxed to Ω being a *bounded Lipschitz domain*, which is satisfied, e.g., by polygonal and polyhedral domains.

1.2.1 Trace Theorems

The trace theorems are results concerning the restriction of elements of Sobolev spaces on a domain to the boundary of the domain. Their duals are the extension theorems. The following trace theorem will be useful later on; see [4] for the general theory.

Theorem 1.1. *If Ω is a Lipschitz domain and $u \in H^s(\Omega)$, $1/2 < s \leq 1$, then,*

$$\gamma_0 u = u \mid_{\partial\Omega} \in H^{s-1/2}(\partial\Omega).$$

Moreover, the restriction operator from $H^s(\Omega)$ to $H^{s-1/2}(\partial\Omega)$ is onto and continuous,

$$||\gamma_0 u||_{H^{s-1/2}(\partial\Omega)} \leq C(s,\Omega)||u||_{H^s(\Omega)},$$

where $C(s,\Omega)$ is a constant that depends only on s and Ω.

The next theorem is a variant of Theorem 1.1, for functions in $H^1(\Omega)$. We consider the norms given by (1.1), such that the dependence of the constants on the domain Ω can be specified.

Theorem 1.2. *If Ω is a Lipschitz domain, then*

$$|u|^2_{H^{1/2}(\partial\Omega)} \leq C|u|^2_{H^1(\Omega)},$$

and

$$||u||^2_{L^2(\partial\Omega)} \leq C\left(diam(\Omega)|u|^2_{H^1(\Omega)} + \frac{1}{diam(\Omega)}||u||^2_{L^2(\Omega)}\right).$$

Let $\Lambda \subseteq \partial\Omega$ be a subset of the boundary of Ω of positive measure. Then $H^{1/2}(\Lambda)$ is the trace space of $H^1(\Omega)$ on Λ. As we mentioned before, the fractional Sobolev spaces can also be obtained by using the K-method of interpolation. Thus, $H^{1/2}(\Lambda) = [L^2(\Lambda), H^1(\Lambda)]_{1/2}$. The scaled norm on $H^{1/2}(\Lambda)$ given by K-method interpolation is equivalent to

$$||u||^2_{H^{1/2}(\Lambda)} \;=\; |u|^2_{H^{1/2}(\Lambda)} + \frac{1}{\mathrm{diam}(\Omega)}||u||^2_{L^2(\Lambda)}.$$

Let $H^{1/2}_{00}(\Lambda) = [L^2(\Lambda), H^1_0(\Lambda)]_{1/2}$ be a space obtained by the K-method of interpolation. This space plays an important role in the analysis of the domain decomposition methods. It can also be characterized as the maximal subspace of $H^{1/2}(\Lambda)$ of the functions which will belong to $H^{1/2}(\partial\Omega)$ when extended by zero to the rest of $\partial\Omega$. We note that the embedding $H^{1/2}_{00}(\Lambda) \subset H^{1/2}(\Lambda)$ is strict. The appropriate norm for this space is

$$||u||^2_{H^{1/2}_{00}(\Lambda)} \;=\; ||u||^2_{H^{1/2}(\Lambda)} + \int_\Lambda \frac{u^2(x)}{d(x,\partial\Lambda)}\, d\sigma_x.$$

The last term comes from the part of the $H^{1/2}$-seminorm of u corresponding to $\partial\Omega \setminus \Lambda$, where u vanishes by definition. From the intrinsic formula for the $H^{1/2}$ seminorm, it follows that

$$||u||^2_{H^{1/2}_{00}(\Lambda)} \;=\; ||u||^2_{L^2(\Lambda)} \;+\; \int_\Lambda\int_\Lambda \frac{|u(x)-u(y)|^2}{|x-y|^d}\, d\sigma_x d\sigma_y \;+\; \int_\Lambda \frac{u^2(x)}{d(x,\partial\Lambda)}\, d\sigma_x. \quad (1.3)$$

1.2.2 Poincaré and Friedrichs Inequalities

The Poincaré and Friedrichs inequalities provide simple equivalent norms for spaces like H^1_0 and H^1, and are used to derive convergence and condition number estimates for finite element methods. They can be proven using the Rellich compactness theorem and the completeness of Sobolev spaces; see [41, 96].

We are interested in formulations of the inequalities specifying the dependence of the constants on the domain Ω; see [96] for elementary proofs. We introduce the following notations: Let $\widehat{\Omega} \subset \mathtt{R}^d$, $d \in \{2, 3\}$, be a reference Lipschitz domain of unit diameter, and let Ω be a domain of diameter $diam(\Omega)$ obtained by a uniform dilation of $\widehat{\Omega}$.

Theorem 1.3. (Poincaré's Inequality) *There exists a constant C that depends only on $\widehat{\Omega}$, such that*

$$||u||^2_{L^2(\Omega)} \;\leq\; C\Big(diam(\Omega)^2|u|^2_{H^1(\Omega)} \;+\; \frac{1}{diam(\Omega)^d}\Big|\int_\Omega u dx\Big|^2\Big), \quad \forall u \in H^1(\Omega).$$

Theorem 1.4. (Friedrichs' Inequality) *Let $c > 0$ and let $\Lambda \subset \partial\Omega$ such that*

$$c\mu(\partial\Omega) \;\leq\; \mu(\Lambda),$$

where μ is the Lebesgue measure. Then,

$$||u||^2_{L^2(\Omega)} \;\leq\; C\Big(diam(\Omega)^2|u|^2_{H^1(\Omega)} \;+\; \frac{1}{c^2 diam(\Omega)^{d-2}}\Big|\int_\Lambda u d\sigma\Big|^2\Big), \quad \forall u \in H^1(\Omega),$$

where C is a constant that does not depend on u, Ω, Λ, or c.

Let $\Lambda \subset \partial\Omega$ and let

$$H^1_{0,\Lambda}(\Omega) \;=\; \{u \in H^1(\Omega) \mid u_{|_\Lambda} = 0\}. \tag{1.4}$$

The next corollary follows directly from Theorem 1.4.

Corollary 1.1. *If $\mu(\Lambda) > 0$, then the H^1-seminorm is an equivalent norm on $H^1_{0,\Lambda}(\Omega)$, i.e.,*

$$||u||_{H^1(\Omega)} \;\leq\; C|u|_{H^1(\Omega)}, \quad \forall u \in H^1_{0,\Lambda}(\Omega).$$

1.3 Variational Formulation of Elliptic Problems

The variational methods are used extensively in solving self-adjoint elliptic PDEs, see, e.g., [29, 63, 96], and they are fundamental for the finite element methods; see [27, 41]. To simplify our presentation, we only discuss the Poisson problem on a domain Ω, with mixed Neumann-Dirichlet boundary conditions.

Let $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$, where $\partial\Omega_N$ and $\partial\Omega_D$ are the parts of the boundary where Neumann and Dirichlet boundary conditions are imposed, respectively. For unique solvability, we require that $\partial\Omega_D$ has positive Lebesgue measure.

Let $f \in L^2(\Omega)$, and let $H^1_{0,\partial\Omega_D}(\Omega)$ be defined as in 1.4. We look for a solution $\mathtt{u} \in H^1_{0,\partial\Omega_D}(\Omega)$ of the mixed boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta \mathtt{u} & = & f & \text{on} \quad \Omega \\ \mathtt{u} & = & 0 & \text{on} \quad \partial\Omega_D \\ \frac{\partial \mathtt{u}}{\partial n} & = & 0 & \text{on} \quad \partial\Omega_N. \end{array} \right. \tag{1.5}$$

A simple application of Green's theorem will result in

$$a(\mathtt{u}, v) \;=\; f(v), \quad \forall\; v \in H^1_{0,\partial\Omega_D}(\Omega), \tag{1.6}$$

where

$$a(v, w) \;=\; \int_\Omega \nabla v \cdot \nabla w \; dx \quad \text{and} \quad f(v) \;=\; \int_\Omega f v \; dx.$$

A solution u of (1.6) is called a weak solution of the mixed problem (1.5). The main ingredient in the proof of the existence and uniqueness of a weak solution is the Lax-Milgram Lemma.

Lemma 1.2. *(Lax–Milgram [83]) Let X be a Hilbert space and let $a_X : X \times X \to \mathtt{R}$ be a bilinear form. If $a_X(\cdot,\cdot)$ is continuous and coercive, i.e.,*

$$\begin{aligned} c||x||^2_X &\leq a_X(x,x), \quad \forall\; x \in X, \\ a_X(x,y) &\leq C a_X(x,x)^{1/2} a_X(y,y)^{1/2} \quad \forall\; x, y \in X, \end{aligned}$$

where $c, C > 0$, then for every continuous functional $T : X \to \mathtt{R}$ there exists a unique solution $x_T \in X$ such that

$$a_X(x_T, x) \;=\; T(x), \quad \forall x \in X.$$

The continuity of $a_X(\cdot,\cdot)$ follows from the Schwarz inequality. Using Corollary 1.1, we obtain that $a(\cdot,\cdot)$ is coercive. The existence of the weak solution u of (1.6) can therefore be established by using Lemma 1.2. From a classical regularity result, see [86], we obtain that, if Ω is bounded and regular enough, e.g., a C^2 domain, then $u \in H^2(\Omega)$ and

$$||u||_{H^2(\Omega)} \;\leq\; C||f||_{L^2(\Omega)},$$

where C depends only on Ω. Using once again Green's theorem we obtain that any weak solution $u \in H^2(\Omega)$ is a solution of the mixed problem (1.5).

1.4 Finite Element Methods

The main difficulty in any study of mortar finite elements comes from the fact mortar elements are discontinuous across subdomain interface. To concentrate on this problem, and simplify our presentation, we consider only low order mortar elements. The underlying conforming finite elements are Q_1 or P_1 elements on rectangular domains. In this section, we give a brief presentation of these elements.

Let $\mathcal{K} \subset \mathtt{R}^2$. A rectangular grid on $\mathcal{K}$ provides a triangulation of $\mathcal{K}$, which is required to be shape regular, i.e., the ratio of the largest and smallest side of each rectangle is uniformly bounded from above. The restriction of $Q_1(\mathcal{K})$ to each rectangle is a bilinear function, with four degrees of freedom. Enforcing pointwise continuity at the nodes of the rectangular grid results in uniquely determined continuous functions.

The three dimensional elements are defined in a similar fashion, using parallelipipeds, and can be extended to affine deformations of parallelipipeds.

The P_1 elements can be defined, in two dimensions, on arbitrary polygonal domains $\mathcal{K}$ partitioned into triangles. The restriction of $P_1(K)$ to each triangle is a linear function with three degrees of freedom, i.e., the values of the finite element function at the vertices of the triangle. As for Q_1 functions, enforcing pointwise continuity at the nodes of the partition of $\mathcal{K}$ results into uniquely determined continuous functions. In three dimensions, the P_1 elements are defined similarly, using tetrahedra for the partition of $\mathcal{K}$.

We note that, in two dimensions, both P_1 or Q_1 elements results into identical projections across the interface, since the restriction of either $P_1(\mathcal{K})$ or $Q_1(\mathcal{K})$ to $\partial\mathcal{K}$ is a piecewise linear function of the mesh partition of $\partial\mathcal{K}$.

1.5 Cutoff Estimates

A result which is often needed in the analysis of domain decomposition methods, e.g., of substructuring methods, is a bound for the energy of a finite element function which has been cut down to zero at all nodes except for one vertex, or for the nodes on one side of the domain, in terms of the energy of the original function;

see, e.g. [23, 48, 89]. In the next lemma we present a classical polylogarithmic bound of this type, which will be used in Chapter 5 and in Chapter 6.

Theorem 1.5. *Let D be a rectangle of diameter H, and let V^h be a low order finite element space over D, with mesh size h. Let $w \in V^h$, and $\widetilde{w} \in V^h(\partial D)$ such that $\widetilde{w}$ is equal to $w\,|_{\partial D}$ at all the interior nodes of one side of D and vanishes at all the other nodes. Then,*

$$\begin{aligned} ||\widetilde{w}||^2_{H^{1/2}(\partial D)} &\leq C(1+log(H/h))^2 \left(|w|^2_{H^{1/2}(\partial D)} + \frac{1}{H}||w||^2_{L^2(\partial D)}\right); \\ ||\widetilde{w}||^2_{H^{1/2}(\partial D)} &\leq C(1+log(H/h))^2 \left(|w|^2_{H^1(D)} + \frac{1}{H^2}||w||^2_{L^2(D)}\right). \end{aligned}$$

If $\widetilde{w}$ vanishes at all the nodes of ∂D, except at one vertex, where it is equal to w, then

$$\begin{aligned} ||\widetilde{w}||^2_{H^{1/2}(\partial D)} &\leq C(1+log(H/h)) \left(|w|^2_{H^{1/2}(\partial D)} + \frac{1}{H}||w||^2_{L^2(\partial D)}\right); \\ ||\widetilde{w}||^2_{H^{1/2}(\partial D)} &\leq C(1+log(H/h)) \left(|w|^2_{H^1(D)} + \frac{1}{H^2}||w||^2_{L^2(D)}\right). \end{aligned}$$

We note that, in each case, the second inequality follows form the first one, by using the trace Theorem 1.2.

Chapter 2

Mortar Finite Elements

2.1 Introduction

Mortar finite elements are nonconforming finite element methods that allow for nonconforming decomposition of the computational domain into subregions and for the optimal coupling of different variational approximations in different subregions. Because of these features, the mortar elements can be used effectively in solving large classes of problems.

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [13, 14], for low-order and spectral finite elements. A three dimensional version was developed by Ben Belgacem and Maday in [9], and was further analyzed for three dimensional spectral elements in [8]. See also Seshaiyer and Suri [117, 118] for mortar *hp* finite elements, Hoppe [72], Ben Belgacem, Buffa, and Maday [5], and Buffa, Maday, and Rapetti [30] for mortar $H(curl)$ elements. Cai, Dryja, and Sarkis [33] have extended the mortar methods to overlapping decompositions, and another family of mortar elements has recently been introduced by Wohlmuth [138].

Several domain decomposition methods for mortar finite elements have been shown to perform similarly to the case of conforming finite elements; cf. Achdou, Maday, and Widlund [3] and Dryja [44, 45] for iterative substructuring methods, Widlund [134] for additive Schwarz algorithms, Dryja [46], Le Tallec [84], and Le Tallec, Sassi, and Vidrascu [85] for Neumann-Neumann algorithms, and Lacour

and Maday [81] and Klawonn and Stefanica [126] for the FETI method. For other studies of preconditioners for the mortar method, see Casarin and Widlund [36] for a hierarchical basis preconditioner, Achdou, Kuznetsov, and Pironneau [2], and Achdou, and Kuznetsov [1] for iterative substructuring preconditioners. Multigrid methods have also been used to solve mortar problems; cf. Braess, Dahmen, and Wieners [18], Braess, Dryja, and Hackbusch [19], Wieners and Wohlmuth [136, 135], and Wohlmuth [139].

2.2 2–D Low Order Mortar Finite Elements

To introduce a mortar finite element space, the computational domain Ω is decomposed using a nonoverlapping partition $\{\Omega_i\}_{i=1:N}$, consisting of polygons,

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i, \qquad \Omega_j \bigcap \Omega_k = \emptyset \quad \text{if} \quad 1 \leq j \neq k \leq N.$$

As in Section 1.3, let $\partial\Omega_D$ be the part of $\partial\Omega$ where Dirichlet conditions are imposed. If an edge of a polygon intersects $\partial\Omega_D$, we require that the entire edge belongs to $\partial\Omega_D$. The partition is said to be geometrically conforming if the intersection between the closure of any two subregions is either empty, a vertex, or an entire edge, and it is nonconforming otherwise.

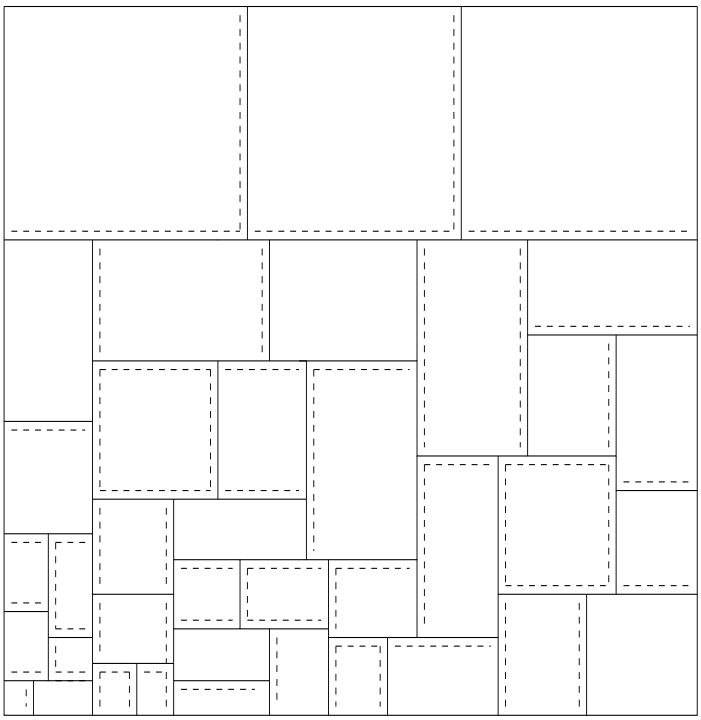
The interface between the subregions $\{\Omega_i\}_{i=1:N}$, denoted by Γ, is defined as the closure of the union of the parts of $\{\partial\Omega_i\}_{i=\overline{1:N}}$ that are interior to Ω:

$$\Gamma = \overline{\cup_{i=1}^{N}(\partial\Omega_i \setminus \partial\Omega)}.$$

Alternatively, Γ can be defined as the set of points that belong to the boundaries of at least two subregions.

We denote by V^h be the space of low order mortar finite elements, and by $V^h(S)$ the restriction of V^h to a set S. For every subregion Ω_i, $V^h(\Omega_i)$ is a conforming element space. The mortar elements are nonconforming finite elements, since pointwise continuity is not required across Γ. Instead, we choose a set of open edges $(\gamma_l)_{l=\overline{1:L}}$ of the subregions $\{\Omega_i\}_{i=1:N}$, called nonmortars, which form a

Figure 2.1: Partition of Γ into nonmortars (dotted)



disjoint partition of the interface,

$$\Gamma = \bigcup_{l=1}^{L} \overline{\gamma}_l, \quad \gamma_m \cap \gamma_n = \emptyset \ \text{ if } \ 1 \leq m \neq n \leq L;$$

see Figure 2.1. We impose weak continuity conditions for the mortar finite element functions, in the sense that the jump of a mortar function across each nonmortar is required to be orthogonal to a space of test functions.

In Section 4.1, we prove that a nonmortar partition of the interface is always possible. The partition is not unique, but any choice can be treated the same from a theoretical point of view.

The edges of $\{\Omega_i\}_{i=1:N}$ that are part of Γ and were not chosen to be nonmortars are called mortars and are denoted by $\{\zeta_m\}_{m=1}^{M}$. We note that the mortars also

cover the interface:

$$\Gamma = \bigcup_{m=1}^{M} \overline{\zeta}_m, \quad \zeta_m \cap \zeta_n = \emptyset \ \text{ if } \ 1 \leq m \neq n \leq M.$$

Each nonmortar γ_l belongs to exactly one subregion, denoted by $\Omega_{i(l)}$. Let Γ_l be the union of the parts of the mortars that coincides geometrically with $\overline{\gamma}_l$.

$$\Gamma_l \ = \ \bigcup_{i=1}^{q(\gamma_l)} (\overline{\zeta}_{l,i} \cap \overline{\gamma}_l). \tag{2.1}$$

For each γ_l, we will choose a space of test functions $\Psi^h(\gamma_l)$ which is a subspace of $V^h(\gamma_l)$, the restriction of $V^h(\Omega_{i(l)})$ to γ_l, and which is of codimension two. Thus, when the space $V^h(\Omega_{i(l)})$ is piecewise linear, i.e., $P_1(\Omega_{i(l)})$ or $Q_1(\Omega_{i(l)})$, $\Psi^h(\gamma_l)$ is given by the restriction of $V^h(\Omega_{i(l)})$ to $\overline{\gamma}_l$, subject to the constraints that these continuous, piecewise linear functions are constant in the first and last mesh intervals of $\overline{\gamma}_l$.

The mortar projection on γ_l is defined on all of $L^2(\Gamma_l)$ and takes values in $V^h(\gamma_l)$. For two arbitrary values q_1 and q_2, and for $u_l \in L^2(\Gamma_l)$, the function $\pi_{q_1,q_2}(u_l) \in V^h(\gamma_l)$ equals q_1 and q_2 at the two end points of γ_l, and satisfies

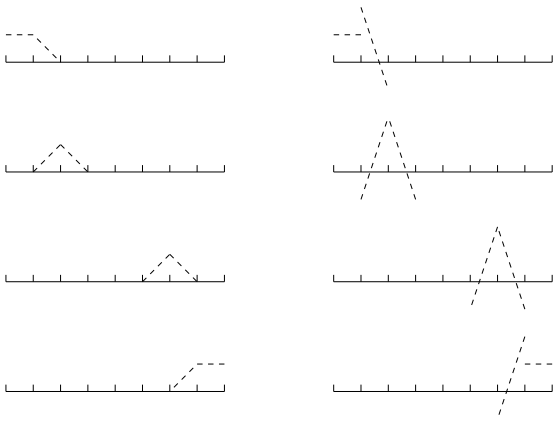
$$\int_{\gamma_l} (u_l - \pi_{q_1,q_2}(u_l))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma_l). \tag{2.2}$$

We are now able to define the mortar finite element space V^h fully. Any mortar function $v \in V^h$, vanishes at all the nodes on $\partial\Omega_D$. The restriction of v to any Ω_i is a P_1 or a Q_1 finite element function. Across the interface, we require v to satisfy the mortar conditions for each nonmortar γ_l, i.e., $v_{|\overline{\gamma}_l}$ is equal to the mortar projection of $v_{|\Gamma_l}$. The values of v at the end points of $\overline{\gamma}_l$ (denoted by A_l and B_l) are genuine degrees of freedom:

$$v_{|\overline{\gamma}_l} \ = \ \pi_{v_l(A_l),v_l(B_l)}(v_{|\Gamma_l}).$$

Since $V^h(\Omega_i) \subset H^1(\Omega_i)$, we obtain, from Theorem 1.1, that the restriction of a mortar function v to any nonmortar γ_l belongs to the space $H^{1/2}(\gamma_l)$. Then, the test functions space $\Psi^h(\gamma_l)$ may be embedded in the dual space of $H^{1/2}(\gamma_l)$ with respect to the L^2 inner product, and therefore $\Psi^h(\gamma_l) \subset H^{-1/2}(\gamma_l)$.

Figure 2.2: Test functions; classical mortars–left; new mortars–right



Based on this observation, a space of discontinuous piecewise linear test functions $\Psi^h_{new}(\gamma_l)$ for low order mortars has been developed by Wohlmuth [138].

There, the test function associated to the first interior node on γ_l is constant 1 on the first mesh interval, decreases linearly from 2 to -1 on the second mesh interval, and vanishes everywhere else. A similar test function is considered for the last interior node on γ_l. For any other node on γ_l, the test function corresponding to such a node has the support on the two mesh intervals having the node as an end point; it increases linearly from -1 to 2 on the first interval and decreases from 2 to -1 on the second interval; cf. Figure 2.2.

The new mortar space has similar approximation properties as the classical mortar space; cf. [138]. In a recent paper [140], multigrid methods for the new mortar methods are introduced and analyzed.

A major advantage of the new mortar finite element space is that the mortar projection can be represented by a banded matrix, as opposed to the classical mortar finite element method, where the mortar projection matrix is, in general, a full matrix; see Section 2.2.1 for more details.

2.2.1 Matrix Formulation of the Mortar Conditions

From the mortar conditions (2.2), it follows that the interior nodes of the nonmortar sides are not associated with genuine degrees of freedom in the finite element space V^h. To emphasize this aspect, we present here the matrix formulation of the mortar conditions.

Let γ be an arbitrary nonmortar side, and let u be a mortar function. Let u_γ be the vector of the interior nodal values of u on γ. For simplicity, we assume that the mesh is uniform on γ, of mesh size h. Let $u_{\Gamma(\gamma)}$ be the vector of the values of u at the end points of γ and at all the nodes on the edges opposite γ, such that the intersection of γ and the support of the corresponding nodal basis functions is not empty. Then u_γ is uniquely determined by $u_{\Gamma(\gamma)}$, and the mortar conditions (2.2) can be written in matrix form as

$$M_\gamma u_\gamma - N_\gamma u_{\Gamma(\gamma)} \;=\; 0, \tag{2.3}$$

solving for u_γ,

$$u_\gamma \;=\; P_\gamma u_{\Gamma(\gamma)}, \tag{2.4}$$

with $P_\gamma = M_\gamma^{-1} N_\gamma$.

We note that N_γ is a banded matrix with a band of similar size for both the classical and the new mortars. For the classical mortar method, M_γ is a tridiagonal matrix,

$$M_\gamma \;=\; \frac{h}{6} \begin{pmatrix} 5 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 5 \end{pmatrix},$$

and the mortar projection matrix P_γ is a full matrix. The projection of a nodal basis function from the mortar side, i.e. $u_{l,\Gamma}$ has all entries equal to 0, except for one, which is equal to 1, results in a function with support equal to γ, the nodal values of which decay exponentially to 0 at the end points of γ.

For the new mortar method, M_γ is a diagonal matrix,

$$M_\gamma \;=\; h\, I,$$

and $P_\gamma = N_\gamma/h$ is banded. Therefore, the mortar projection of a nodal basis function on the mortar side vanishes outside the mesh intervals on the nonmortar which intersect its support.

2.3 The 3-Dimensional Case

In this case, the mortars and nonmortars are open faces of subregions, and the mortar projections are defined differently than in the two dimensional case.

To introduce the mortar finite element space, we follow the outline from the previous section. Let $\{\Omega_i\}_{i=1:N}$ be a nonoverlapping polyhedral partition of Ω. If a face or an edge of a polyhedron intersects $\partial\Omega_D$ at an interior point, then the entire face or edge is assumed to belong to $\partial\Omega_D$. The partition is said to be geometrically conforming if the intersection between the closures of any two subregions is either empty, a vertex, an entire edge, or an entire face, and it is nonconforming otherwise.

The nonmortars $\{\mathcal{F}_l\}_{l=1}^L$ are faces of the subregions $\{\Omega_i\}_{i=1:N}$, such that

$$\Gamma = \bigcup_{l=1}^{L} \overline{\mathcal{F}}_l, \quad \mathcal{F}_m \cap \mathcal{F}_n = \emptyset \ \text{ if } \ 1 \le m \ne n \le L;$$

see Section 4.1 for a proof of the existence of such partition. The mortars are the faces which have not been chosen to be nonmortars, and their closures also cover the interface.

Across each nonmortar face $\mathcal{F}_l$, we impose mortar conditions as follows. Opposite $\overline{\mathcal{F}_l}$ we find a union of parts of mortar faces which we denote by $\mathcal{G}_l$. We note that $\mathcal{G}_l$ coincides geometrically with $\overline{\mathcal{F}_l}$. Let $\Psi^h(\mathcal{F}_l) \subset V^h(\mathcal{F}_l)$ be the test function space. The value of a test function at a node on $\partial\mathcal{F}_l$ is a convex combination of its values at the neighboring interior nodes of $\mathcal{F}_l$. Thus, the weights are positive and their sum equals one, but they are otherwise arbitrary. The dimension of $\Psi^h(\mathcal{F}_l)$ is equal to the number of interior nodes of $\mathcal{F}_l$.

The mortar projection on $\mathcal{F}_l$ is defined on all of $L^2(\mathcal{G}_l)$ and takes values in $V^h(\mathcal{F}_l)$. Let $u_l \in L^2(\mathcal{G}_l)$, and let w_l be a piecewise linear function defined on $\partial\mathcal{F}_l$. Then $\pi_{w_l}(u_l) \in V^h(\mathcal{F}_l)$ is the mortar projection of u_l if

$$\pi_{w_l}(u_l)\,|_{\partial\mathcal{F}_l} \;=\; w_l, \quad \text{and} \tag{2.5}$$

$$\int_{\mathcal{F}_l} (u_l - \pi_{w_l}(u_l))\ \psi ds \ = \ 0, \quad \forall \psi \in \Psi^h(\gamma_l). \tag{2.6}$$

The mortar finite element space V^h consists of functions v which vanish at all the points of $\partial\Omega_D$ Its restriction to any Ω_i is a P_1 or a Q_1 finite element function. We also require v to satisfy the mortar conditions for each nonmortar $\mathcal{F}_l$, i.e.,

$$v_{|\overline{\mathcal{F}_l}} \ = \ \pi_{v|_{\partial\mathcal{F}_l}}(v_{|_{\mathcal{F}_l}}).$$

From the definition of the mortar projection, cf. (2.5), (2.6), it follows that the values of v at all the boundary nodes of the nonmortars are genuine degrees of freedom.

We conclude this section by noting that a version of the new mortars for the 3–D case has been developed by Wohlmuth and Krause [140].

2.4 Stability Properties of Mortar Projections

The discontinuity of the mortar functions across the interface is the main difference between mortar and conforming finite elements. To show that mortars perform similarly to the conforming finite elements, we need estimates for the jump of the mortar function across an arbitrary nonmortar side. Since the jump is given in terms of the mortar projection, it is important to establish stability properties of the mortar projections in several norms, for different mortar finite elements.

In this section, we present some classical stability results for low order mortars, which are valid in both two and three dimensions. In Section 4.2, we show that the mortar projection with zero values at the end points of the nonmortar side is stable in the L^2 norm for arbitrary nonuniform meshes on the nonmortar.

Let γ be a nonmortar side, and let $V^h(\gamma)$ be the continuous piecewise linear space which is the restriction of V^h to γ. Let $\pi_\gamma : L^2(\gamma) \to V^h(\gamma) \cap H_0^1(\gamma)$ be the mortar projection operator which vanishes at the end points; cf. (2.2) with $q_1 = q_2 = 0$ and $\pi_\gamma = \pi_{0,0}$. The following stability properties of π_γ were first proven for uniform meshes in [13, 14], and by Ben Belgacem [6, 7] for quasiuniform meshes.

Theorem 2.1. *The mortar projection π_γ which assumes values 0 at the end points of γ is stable in the L^2–norm and the H^1_0–seminorm,*

$$\begin{aligned} ||\pi_\gamma(\chi)||_{L^2(\gamma)} &\leq C||\chi||_{L^2(\gamma)}, \quad \forall \chi \in L^2(\gamma); \\ |\pi_\gamma(\chi)|_{H^1_0(\gamma)} &\leq C|\chi|_{H^1_0(\gamma)}, \quad \forall \chi \in H^1_0(\gamma), \end{aligned}$$

where C is a constant independent of h. By interpolation, a similar bound can be obtained for the $H^{1/2}_{00}$ norm, i.e.,

$$||\pi_\gamma(\chi)||_{H^{1/2}_{00}(\gamma)} \leq C||\chi||_{H^{1/2}_{00}(\gamma)} \quad \forall \chi \in H^{1/2}_{00}(\gamma).$$

Similar results have been proven for mesh-dependent norms in Braess, Dahmen, Wieners [18], and for the case when the ratio of any two neighboring mesh intervals over γ is uniformly bounded in Seshaiyer and Suri [117]. In the three dimensional case, the L^2 stability of the mortar projection which vanishes on the boundary of the nonmortar side was established by Ben Belgacem and Maday [9].

We note that proofs similar to those for the results of Theorem 2.1 provide stability inequalities for the more general case when $\lambda \in H^1(\gamma)$, and $\pi_\gamma(\lambda)$ vanishes at the end points of γ.

2.5 Variational Formulation of Mortar Problems

In this section, we modify the variational formulation of the elliptic problems discretized with conforming finite elements, see Section 1.3, and apply it to the mortar finite element case. The main difference comes from the fact that mortar elements are only piecewise H^1 functions.

As before, we discuss the Poisson problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& f & \text{on} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega_D \\ \frac{\partial u}{\partial n} &=& 0 & \text{on} \quad \partial\Omega_N. \end{array} \right. \tag{2.7}$$

with $f \in L^2(\Omega)$ and $u \in H^1_{0,\partial\Omega_D}(\Omega)$.

We discretize (2.7) using a low order mortar finite element space V^h, corresponding to the nonoverlapping partition $\{\Omega_i\}_{i=1:N}$ of Ω. Using the fact that

$V^h(\Omega_k) \in H^1(\Omega_k)$, $k = 1 : K$, we obtain

$$a^\Gamma(u, v) \;=\; f(v), \quad \forall v \in H_0^1(\Omega), \tag{2.8}$$

where the bilinear form $a^\Gamma(\cdot, \cdot)$ is defined as the sum of contributions from the individual subregions

$$a^\Gamma(v, w) \;=\; \sum_{k=1}^{K} \int_{\Omega_k} \nabla v \cdot \nabla w \; dx,$$

and

$$f(v) \;=\; \int_\Omega f v \; dx.$$

The discrete problem is then:

$$\text{Find } u_h \in V^h \text{ such that } a^\Gamma(u_h, v_h) = f(v_h), \;\; \forall \; v_h \in V^h. \tag{2.9}$$

We note that, for $w_h = v_h$, we obtain the square of what is often called a broken norm,

$$a^\Gamma(v_h, v_h) \;=\; \sum_{k=1}^{K} |v_h|^2_{H^1(\Omega_k)}.$$

Here the norm has been broken along Γ and it is finite for any mortar function u_h even if u_h is discontinuous across Γ.

The existence and uniqueness of the solution of problem (2.9) follows from the Lax-Milgram Lemma 1.2 as soon as we have proven the coercivity of the broken norm with respect to the L^2 norm,

$$c||v_h||^2_{L^2(\Omega)} \;\leq\; a^\Gamma(v_h, v_h), \quad \forall \; v_h \in V^h,$$

where c is a positive constant. This inequality can be regarded as a Friedrichs inequality for mortar finite elements, and was first established in [13, 14]; see also Section 4.3.

2.6 Error Estimates

An important feature of mortar finite elements is that the mortar solution u_h approximates the exact solution u with the same accuracy as a corresponding

conforming finite element solution. The proof we present here follows the same steps as the proofs in [9, 13], which are given for both the two and three dimensional elements.

We assume that the exact solution satisfies $u \in H^2(\Omega)$. From the second Strang lemma, see [127],

$$\begin{aligned}\sum_{k=1}^{K} ||u-u_h||_{H^1(\Omega_k)} &\leq C\Big(\inf_{v_h \in V^h} \sum_{k=1}^{K} ||u-v_h||_{H^1(\Omega_k)} + \sup_{w_h \in V^h} \frac{|a^\Gamma(u,w_h) - f(w_h)|}{\sum_k ||w_h||_{H^1(\Omega_k)}} \Big) \\ &= C\Big(\inf_{v_h} \sum_{k=1}^{K} ||u-v_h||_{H^1(\Omega_k)} + \sup_{w_h} \frac{\int_\Gamma \frac{\partial u}{\partial n}[w_h]\ d\sigma}{\sum_k ||w_h||_{H^1(\Omega_k)}} \Big), \quad (2.10)\end{aligned}$$

where $[w_h]$ is the jump of w_h across the interface. We note that the first term of (2.10) is the best approximation error, while the second term is the consistency error.

The best approximation error can be estimated by using interpolation inequalities for conforming finite elements, see [41], and stability properties of the mortar projections, see Section 2.4. Thus,

$$\inf_{v_h \in V^h} \sum_{k=1}^{K} ||u-v_h||_{H^1(\Omega_k)} \ \leq\ Ch||u||_{H^2(\Omega)}.$$

To estimate the consistency error, we use the fact that the jump of a mortar function is orthogonal to the space of test functions. We find

$$\int_\Gamma \frac{\partial u}{\partial n}[w_h]\ d\sigma \ \leq\ Ch||u||_{H^2(\Omega)} \sum_{k=1}^{K} ||w_h||_{H^1(\Omega_k)}.$$

The following upper bound is therefore established,

$$\sum_{k=1}^{K} ||u-u_h||_{H^1(\Omega_k)} \ \leq\ Ch||u||_{H^2(\Omega)},$$

and it has the same form as for the conforming finite element case.

2.7 Saddle Point Formulation

We use the same notations as in Section 2.2, and concentrate our attention on the mortar conditions (2.2). Let $[v_l]$ be the jump of $v_h \in V^h$ across γ_l. The test

functions from the mortar conditions can also be regarded as Lagrange multipliers. In other words, a function u is in the mortar space V^h if and only if

$$\int_{\gamma_l} [v_l]\mu_l ds = 0, \tag{2.11}$$

for all nonmortar sides γ_l and for all Lagrange multipliers μ_l which form a basis for $\Psi^h(\gamma_l)$. From the trace theorem 1.2, $[v_l] \in H^{1/2}(\gamma_l)$. Therefore the dual space $H^{-1/2}(\gamma_l)$ is a natural embedding space for μ_l.

Let $M^h = \prod_l \Psi^h(\gamma_l) \subset \prod_l H^{-1/2}(\gamma_l)$, and $\mu_h \in M^h$, with $\mu_h = (\mu_l)_{l=1:L}$. The bilinear form $b(\cdot,\cdot)$ is given by

$$b(v_h, \mu_h) \;=\; \sum_{l=1}^{L} \int_{\gamma_l} [v_h]\mu_l ds.$$

Then v_h is a mortar function if and only if $b(v_h, \mu_h) = 0,\ \forall \mu_h \in M^h$. The discrete problem (2.9) can be written in a saddle point (mixed) formulation as follows: Find $(u_h, \lambda_h) \in V^h \times M^h$ such that

$$\begin{array}{lcll} a^\Gamma(u_h, v_h) \;+\; b(v_h, \lambda_h) & = & f^\Gamma(v_h), & \forall\ v_h \in V^h \\ b(u_h, \mu_h) & = & 0, & \forall\ \mu_h \in M^h. \end{array}$$

The saddle point formulation for mortars has been introduced by Ben Belgacem [7], where the Babuška-Brezi condition is also established. Similar results for mesh dependent norms [137] have been proven by Braess and Dahmen [17], Braess, Dahmen, and Wieners [18] and Wohlmuth [138].

The saddle point formulation is useful in the practical implementation of mortar finite elements, since it decouples the nodes on opposite sides across the interface, and provides a straightforward way to enforce the mortar conditions. The stiffness matrix is then block diagonal, with each block corresponding to a Neumann problem on one subregion. A mixed formulation along the same lines is possible for conforming finite elements, when continuity at the matching nodes is required. One of the domain decomposition methods we present in this thesis, FETI, uses this saddle point formulation. The primal variables are eliminated and a preconditioned problem is solved for the Lagrange multipliers. Therefore, a FETI method for mortars can be developed just by replacing the pointwise matching

conditions across Γ of the standard finite element case by the mortar conditions; see Chapter 5.

Chapter 3

Domain Decomposition Methods

3.1 Direct vs. Iterative Methods

When solving a strongly elliptic, self-adjoint PDE, the stiffness matrix of the resulting linear system is sparse, banded, symmetric, and positive definite. This system may be solved by using direct or iterative methods. For two dimensional problems, the work and storage required for direct methods grows moderately with the number of degrees of freedom. To be more precise, assume we want to solve a Poisson problem on a rectangular domain using a Q_1 finite element with N nodes on each side. Let $n = N^2$ be the number of the degrees of freedom. Nested dissection provides an asymptotically optimal ordering of the nodes, and it is therefore an efficient direct method. The work and storage required are $O(n^{3/2})$ and $O(n \log n)$, respectively; cf. [66]. Therefore, this and other direct methods are well-suited for solving two dimensional problems.

For three dimensional problems, the work and storage required by the direct methods grow much faster, which makes them less attractive. For nested dissection, the work and storage are $O(n^2)$ and $O(n^{4/3})$, respectively; see [114]. Therefore, in the three dimensional case, iterative methods, where the solution is obtained as the limit of a sequence of approximate solutions which are computed recursively, are more practical.

A class of such iterative methods are the domain decomposition methods, which provide Krylov space preconditioners for the linear system. The computational

domain is decomposed into overlapping or nonoverlapping subdomains, and local spaces are introduced by restricting the finite element space used to discretize the PDE on each subdomain. In each iteration, a problem similar to the original one is solved on each subdomain, either exactly or approximately.

Many domain decomposition methods have good parallelization properties since the local problems can be solved in parallel. If a coarse space with a few degrees of freedom in each subdomain is used in the design of the algorithms, together with the local spaces, then convergence can be achieved in relatively few iterations. For a good method the number of iterations is independent of the number of subdomains and depends only polylogarithmically on the number of nodes in each subdomain.

3.1.1 Preconditioned Conjugate Gradient Methods

The conjugate gradient (CG) method is an efficient iterative method for solving linear systems of equations. It requires little storage and, for well-conditioned problems, converges in relatively few iterations.

When an elliptic self-adjoint PDE is discretized using a finite element space, the resulting linear system,

$$Ax = b, \tag{3.1}$$

has a symmetric positive definite stiffness matrix A. A domain decomposition method will provide a symmetric, positive definite preconditioner M of A. The preconditioned version of (3.1),

$$MAx = Mb, \tag{3.2}$$

is then solved by a conjugate gradient method.

Following [130], we now present the preconditioned CG algorithm for solving (3.2). We note that the (unpreconditioned) conjugate gradient algorithm can be obtained from the preconditioned version if we let $M = I$.

Preconditioned Conjugate Gradient Iteration

$x_0 = 0$, $r_0 = b$, $p_0 = Mr_0$, $n = 1$
while $(Mr_{n-1}, r_{n-1}) \geq tol$
  $z_{n-1} = Mr_{n-1}$
  $\beta_n = (z_{n-1}, r_{n-1})/(z_{n-2}, r_{n-2})$ $\qquad (\beta_1 = 0)$
  $p_n = z_{n-1} + \beta_n p_{n-1}$ $\qquad (p_1 = z_0)$
  $\alpha_n = (z_{n-1}, r_{n-1})/(Ap_n, p_n)$
  $x_n = x_{n-1} + \alpha_n p_n$
  $r_n = r_{n-1} - \alpha_n A p_n$
  $n = n + 1$
end

Here, $(\cdot, \cdot)$ is the l_2 inner product. Let $||e||_A^2 = (Ae, e)$, and let $\kappa(MA)$ be the l_2 condition number of the preconditioned matrix MA, i.e.,

$$\kappa(MA) = \frac{\lambda_{max}(MA)}{\lambda_{min}(MA)}.$$

Then, by a standard error estimate,

$$||x_n - x||_A^2 \;\leq\; 2\left(\frac{\sqrt{\kappa(MA)} - 1}{\sqrt{\kappa(MA)} + 1}\right)^{2n} \; ||x_0 - x||_A^2.$$

Many domain decomposition methods result in good and robust preconditioners, with small condition number $\kappa(MA)$, and require a small number of iterations until satisfactory convergence is achieved. Also, the preconditioner M is never formed explicitly, since it is needed only in a matrix vector multiplication. In domain decomposition, this is done by solving problems similar to the original problem on small subdomains of the computational domain. In the CG algorithm, the local problems can be solved in parallel

Modified versions of the CG algorithm are used to solve symmetric indefinite problems or some special positive definite nonsymmetric problems; see, e.g., Szyld and Widlund [128]. Nonsymmetric problems can also be solved, e.g., by the GMRES method of Saad and Schultz [113] which we will not discuss here.

3.2 Abstract Schwarz Theory

The general concepts of the Schwarz theory were first introduced by Dryja and Widlund in [49, 50]. Since then, many domain decomposition methods have been formulated and analyzed in this framework. We will use this elegant framework throughout this thesis. In this section, we present the abstract theory, as developed in Smith, Bjørstad, and Gropp [122]; see also [70, 141, 142] for different presentations.

Let V be a finite dimensional space, with inner product $a : V \times V \to \mathtt{R}$, and let $f : V \to \mathtt{R}$, be a continuous operator. We want to find a solution $u \in V$ for the model problem

$$a(u, v) \;=\; f(v), \quad \forall\, v \in V. \tag{3.3}$$

The space V is written as a sum of subspaces $V_i \subset V$, $i = 0 : N$,

$$V \;=\; V_0 + V_1 + \ldots + V_N.$$

We note that this sum is not necessarily direct.

Let $\tilde{a}_i : V_i \times V_i \to \mathtt{R}$, be bilinear forms which are symmetric, continuous, and coercive. Let $\tilde{T}_i : V \to V_i$ be the corresponding projection-like operators, defined by

$$\tilde{a}_i(\tilde{T}_i v, v_i) \;=\; a(v, v_i), \quad \forall\, v_i \in V_i, \tag{3.4}$$

where $v \in V$. Let $I_i : V_i \to V$, be the embedding operator. Using the operators $T_i : V \to V$, $T_i = I_i \tilde{T}_i$, several methods for solving (3.3) can be introduced, which result in a preconditioned equation with the same solution u as (3.3).

The **additive Schwarz method** requires solving

$$T_{as} u = g_{as}, \tag{3.5}$$

where

$$T_{as} = T_0 + T_1 + \ldots + T_N \quad \text{and} \quad g_{as} = \sum_{i=0}^{N} I_i g_i,$$

with $g_i = \tilde{T}_i u$. From (3.3) and (3.4) it follows that

$$\tilde{a}_i(g_i, v_i) \;=\; \tilde{a}_i(\tilde{T}_i u, v_i) \;=\; a(u, v_i) \;=\; f(v_i), \quad \forall\, v_i \in V_i.$$

Therefore, g_{as} is obtained by solving a set of local problems,

$$\tilde{a}_i(g_i, v_i) \;=\; f(v_i), \quad \forall\; v_i \in V_i,$$

which do not require any knowledge of u.

The equation (3.5) is the preconditioned version of (3.3). Therefore (3.5) is solved directly by a CG iteration, without any further preconditioning; cf. Section 3.1.1 with $M = I$.

The **multiplicative Schwarz method** consists of solving

$$T_{ms} u = g_{ms}, \tag{3.6}$$

where

$$T_{ms} = I - (I - T_N) \ldots (I - T_1)(I - T_0),$$

and g_{ms} is computed without knowing u, by solving N problems of the same form as (3.4). Since T_{ms} is a nonsymmetric operator, the equation (3.6) is generally solved by a GMRES iteration. Symmetric version of the multiplicative method have also been suggested, corresponding to the operators

$$\begin{aligned} T_{sms} &= T_{ms} + T_{ms}^T - T_{ms}^T T_{ms}; \\ \bar{T}_{sms} &= T_{ms} + T_{ms}^T, \end{aligned}$$

where the transpose operator is computed with respect to $a(\cdot, \cdot)$. Because of symmetry, the CG iteration can be used with these methods.

We note that, in this abstract setting, the alternating Schwarz method [115] can be regarded as a multiplicative method.

In practice, the multiplicative algorithms converge faster than the additive ones, which, in turn, have more potential for parallelization. The **hybrid methods** combine good features of the additive and multiplicative methods to obtain good convergence and parallelization properties. Cai [31] advocates the use of the operator

$$\gamma T_0 + I - (I - T_N) \ldots (I - T_1)(I - T_0),$$

with $\gamma > 0$ a balancing parameter, and Mandel [89] suggests the operator

$$T_0 + (I - T_0)(T_1 + \ldots + T_N),$$

which is used in the balancing method; see Mandel and Brezina [91].

3.2.1 Abstract Convergence Estimates

The convergence analysis of the Schwarz methods relies on three assumptions which must be satisfied, and the parameters therein.

The first assumption requires a stable splitting of V in terms of the subspaces $(V_i)_{i=0:N}$.

Assumption 1: There exists a minimum constant C_0 with the property that, for all $u \in V$, there exist $u_i \in V_i$ such that

$$u = \sum_{0=1}^{N} u_i,$$

and

$$\sum_{i=0}^{N} \tilde{a}_i(u_i, u_i) \ \leq \ C_0^2 a(u, u).$$

The next assumption quantifies the orthogonality of the subspaces $V_1, \ldots, V_N$. We note that the space V_0 is not included in this assumption.

Assumption 2: There exist minimal constants $0 \leq \epsilon_{ij} \leq 1$ such that

$$|a(I_i v_i, I_j v_j)|^2 \ \leq \ \epsilon_{ij} a(I_i v_i, I_i v_i) a(I_j v_j, I_j v_j), \quad \forall \ v_i \in V_i, \ v_j \in V_j, \ i, j = 1 : N,$$

and let $\rho(\epsilon)$ be the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

The final assumption requires the existence of a lower bound for the norm given by $\tilde{a}_i(\cdot, \cdot)$ in terms of the norm induced by the inner product $a(\cdot, \cdot)$.

Assumption 3: There exists a minimal parameter $\omega \in [1, 2)$ such that

$$a(I_i v_i, I_i v_i) \ \leq \ \omega \tilde{a}_i(v_i, v_i), \quad \forall \ v_i \in \text{Range}(T_i), \quad \forall \ i = 0 : N. \tag{3.7}$$

The form of the Assumption 3 presented here has been introduced by Widlund in [133]. The classical form assumption, see [122], required the bound (3.7) to hold for all $v_i \in V_i$. The importance of the modified assumption is emphasized in the analysis of the balancing method, for both conforming and mortar finite elements; see Section 6.2.2 and Section 6.5 of Chapter 6.

The classical estimate for $\kappa(T_{as})$, the condition number of the additive Schwarz method, follows from Assumptions 1–3; see [122] for a proof.

Theorem 3.1. *If Assumptions 1-3 are satisfied, then*

$$C_0^{-2} a(v,v) \;\leq\; a(T_{as} v, v) \;\leq\; \omega(1+\rho(\epsilon)) a(v,v) \quad \forall\, v \in V, \tag{3.8}$$

and therefore

$$\kappa(T_{as}) \;\leq\; C_0^2 \omega (1+\rho(\epsilon)).$$

We note that ω and C_0^2 balance each other, in the following sense: If the approximate bilinear forms $\tilde{a}_i(\cdot,\cdot)$ are all multipied by a constant factor $\tilde{\mu}$, then Assumptions 1–3 hold if the parameters ω and C_0^2 are replaced with $\omega/\tilde{\mu}$ and $\tilde{\mu} C_0^2$, respectively. Thus, the effect of scaling does not change the value of the product $C_0^2 \omega$, which is important given the estimate for $\kappa(T_{as})$ from Theorem 3.1.

The relative scalability of these parameters offers flexibility in choosing the local approximate bilinear forms $\tilde{a}_i(\cdot,\cdot)$ and local solvers $\tilde{T}_i$.

For the multiplicative method, the convergence estimate from the next theorem has been obtained by Bramble et al. [24]; see also Griebel and Oswald [70] for a more detailed analysis.

Theorem 3.2. *If Assumptions 1-3 are satisfied, then*

$$\kappa(T_{sms}) \;\leq\; \frac{[1+2\omega^2\rho^2(\epsilon)] C_0^2}{2-\omega}. \tag{3.9}$$

It is easy to see that if the product $C_0^2 \omega$ is constant, then the right hand side of (3.9) is minimal for the smallest possible ω. We note that this optimality condition has already been enforced in Assumption 3.

3.3 Several Domain Decomposition Methods

In this section, we present several examples of how the abstract Schwarz theory can be used to give convergence estimates for various domain decomposition methods. We restrict the discussion to results obtained for the Poisson problem (2.7) discretized by conforming finite elements or spectral elements. The changes which have to be made for the mortar case, and some results we have obtained for those algorithms are presented in Chapters 5 and 6.

We begin by specifying the spaces and bilinear forms for some general methods. The space V is the finite element space or the spectral element method used to discretize the PDE, and $a(\cdot,\cdot)$ is the bilinear form obtained from the variational formulation (2.8) of the original problem.

The spaces V_i, $i = 1 : N$, are local spaces. To construct them, the computational domain Ω is decomposed into subdomains $\{\mathcal{D}_i\}_{i=1:N}$, not necessarily disjoint,

$$\Omega \;=\; \bigcup_{i=1}^{N} \mathcal{D}_i.$$

For many domain decomposition methods, V_i is a subspace of V consisting of functions which vanish at all the degrees of freedom outside $\mathcal{D}_i$, Then, I_i is the embedding operator and $\tilde{a}_i(\cdot,\cdot)$ may equal the restriction of $a(\cdot,\cdot)$ to $V_i \times V_i$, in which case Assumption 3 is satisfied with $\omega = 1$.

The operators $\tilde{T}_i$ are given by (3.4). We note that computing $\tilde{T}_i v$ for $v \in V$ is equivalent to solving a problem similar to the original one on the smaller subspace V_i. For all the methods of Section 3.2, in each step of the CG iteration N local problems must be solved each time the preconditioned operator, e.g., T_{as} or T_{ms}, is applied to a vector.

The space V_0 is a coarse space which has a small number of degrees of freedom in each subdomain. It plays a crucial role in domain decomposition methods, since algorithms without a coarse space have a condition number, and therefore a rate of convergence, which depends on the number of subdomains; see [122] for a detailed explanation. A good coarse space solver provides the mechanism for global communication of information between subdomains in each iteration. It is therefore possible for the condition number of the resulting algorithm to be independent of the number of subregions.

Choosing the appropriate coarse space V_0 and a coarse solver $a_0(\cdot,\cdot)$ for each method is critical for obtaining a successful method. In the some cases, the coarse space is embedded in V, $V_0 \subset V$, and $a_0(\cdot,\cdot) = a(\cdot,\cdot)$; see [35, 52], while in other cases, V_0 might be unrelated to the decomposition $\{\mathcal{D}_i\}_{i=1:N}$; see [32, 39, 40]. Then, an operator $Q_H : V \to V_0$ with good approximation properties is needed in establishing that the constant C_0 of Assumption 1 is independent of the properties

of the discretization and of the number of subdomains N.

3.3.1 Overlapping Methods

Assume that the decomposition $\{\mathcal{D}_i\}_{i=1:N}$ of Ω is assembled from subdomains which intersect their neighbors, and let δ be the minimal overlap of any two subregions with nonempty intersection. The methods using such a decomposition are called overlapping, and have been proven to have very good convergence properties. In [52], Dryja and Widlund construct the overlapping subdomains by extending the subregions of a coarse triangulation partition of Ω. Then, the coarse space can be chosen to be a low order finite element space on the coarse mesh defined by the nonoverlapping partition of Ω. The condition number of the two-level additive and multiplicative Schwarz methods is bounded from above by $C(1 + H/\delta)$, where H is the diameter of the coarse mesh, and C is a constant independent of the number of subdomains and the number of nodes in each subdomain. This result is sharp; cf. Brenner [26].

For non–embedded meshes with a coarse space independent of the overlapping partition, a similar bound exists; cf. Chan, Smith, and Zou [39]. The same bound holds for the p-version finite element method spectral elements; cf. Pavarino [101, 102], Casarin [34, 35], and Pahl [98].

3.3.2 Nonoverlapping Methods

If $\{\mathcal{D}_i\}_{i=1:N}$ is a nonoverlapping partition of Ω, there are several nonoverlapping domain decomposition methods with good convergence properties. The variables corresponding to the interior nodes of the subdomains are eliminated, which results in a Schur complement problem for the unknowns corresponding to the interface nodes.

The substructuring methods, which have been studied extensively by Bramble, Pasciak, and Schatz [20, 21, 22, 23], involve using a preconditioner where some of the couplings between nodes are dropped.

Several iterative substructuring method have also been analyzed by Smith [119, 120]. For the vertex-based algorithms, a preconditioner is obtained by dropping the

coupling between the faces, edges, and vertices of the subdomains $\{\mathcal{D}_i\}_{i=1:N}$, while keeping the coupling between vertices. For the wire basket algorithms, we group the vertices and edges into a wire basket, and drop the coupling between the faces and the wire basket, and between pairs of faces. For both algorithms the condition number is bounded by $C(1+log(H/h))^2$; cf. Dryja, Smith, and Widlund [48]. For spectral elements, a polylogarithmic bound, $C(1+\log p)^2$, has been established by Pavarino and Widlund [103, 104]; see also Bică [15] for a similar result for p-version finite elements.

The Neumann-Neumann methods are nonoverlapping additive Schwarz methods where the local problems are Neumann problems on each floating subdomain, i.e. subdomain the boundary of which does not intersect $\partial\Omega_D$. The local solvers can be defined by weighted H^1 inner products on each subdomain. In [53], Dryja and Widlund proved that the condition number for optimal Neumann-Neumann methods is bounded by $C(1+log(H/h))^2$, and in [28], Brenner and Sung have proved that this bound is sharp. The balancing method of Mandel and Brezina [89, 91] is a hybrid method of Neumann-Neumann type using a special space of so-called balanced functions; see Chapter 6. The same condition number estimate, $C(1+log(H/h))^2$, also holds for the balancing method.

The FETI method of Farhat and Roux [61, 62] is a Lagrange multiplier based iterative substructuring method. We will present it in great detail in Chapter 5. The condition number of this method is of order $(1+log(H/h))^3$ for a general partition, and $(1+log(H/h))^2$ for a partition without crosspoints, i.e., points which belong to the boundary of more than two subdomains; cf. Mandel and Tezaur [93]. Another FETI method which has a condition number of order $(1+log(H/h))^2$ for arbitrary partitions has been recently suggested by Klawonn and Widlund [78].

Other nonoverlapping methods exist, but they will not be discussed in any detail in this thesis; see, e.g., the multilevel Schwarz methods [51, 47, 143, 144].

Many decomposition methods have been extended to other elliptic problems, like linear elasticity and plate and shell problems; see, e.g., [25, 55, 94, 75, 76, 77, 105, 106, 121].

Chapter 4

Some Properties of Mortar Finite Elements

4.1 The Existence of the Mortar Partition

In the geometrically conforming case, when each edge or face has exactly one edge or face opposite, the interface can be partitioned into nonmortars by choosing either one of those two sides as a nonmortar.

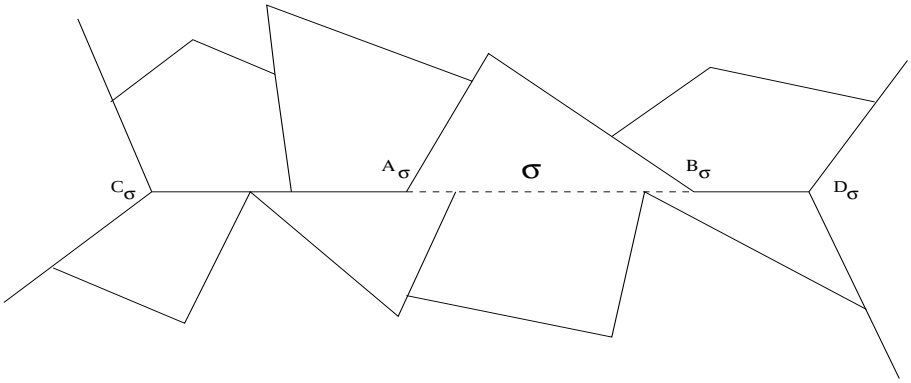
In this section, we prove the existence of the nonmortar partition in the geometrically nonconforming case. We first discuss the two dimensional case, and then indicate how our proof can be extended to three dimensions. We use the same notations as in Section 2.2. Let Γ be the interface between the subregions $\{\Omega_i\}_{i=1:N}$, and let σ_j^k, $1 \leq j \leq n_k$ be the open sides of the subregion Ω_k which do not intersect $\partial\Omega$. Let $\mathfrak{S}$ be the set of all the sides, and let

$$\mathcal{S} \;=\; \bigcup_{k=1}^{K} \bigcup_{j=1}^{n_k} \sigma_j^k.$$

Since $\Gamma = \overline{\mathcal{S}}$, it is enough to find a nonmortar partition of $\mathcal{S}$.

Let $\sigma \in \mathfrak{S}$ be an arbitrary side. We assume, for clarity reasons, that σ is parallel to the x-axis. Let A_σ and B_σ be the left and right end points of σ; cf. Figure 4.1. The support of σ, supp(σ), is defined as follows: let d_σ be the line passing through σ, and let C_σ be the leftmost point on d_σ such that $A_\sigma C_\sigma$ is a union of sides from $\mathfrak{S}$. Similarly, let D_σ be the rightmost point on d_σ such that

Figure 4.1: Support of side σ



$B_\sigma D_\sigma$ is a union of sides from $\mathcal{S}$. The points C_σ and D_σ exist since Ω is bounded and there is a finite number of sides in $\mathfrak{S}$. The open segment $C_\sigma D_\sigma$ is called the support of σ. We note that it is possible for several sides to have the same support, but the support of any side is unique.

Theorem 4.1. *For any partition $\{\Omega_i\}_{i=1:N}$ of Ω, there exists a decomposition of the interface Γ into nonmortars.*

Proof. Let σ_1 be an arbitrary side, and $\text{supp}(\sigma_1)$ its support. Let $\mathcal{S}_1 = \mathcal{S} \backslash \text{supp}(\sigma_1)$. Using an inductive process, we may choose an arbitrary side σ_{p+1} from $\mathcal{S}_p$, $p \geq 1$, and define

$$\mathcal{S}_{p+1} \;=\; \mathcal{S}_p \setminus \text{supp}(\sigma_{p+1}).$$

Then,

$$\mathcal{S}_{p+1} \;=\; \mathcal{S} \setminus \bigcup_{j=1}^{p+1} \text{supp}(\sigma_j).$$

Since the number of sides from $\mathfrak{S}$ is finite, after a finite number of steps, P, the set $\mathcal{S}_P$ will be empty, $\mathcal{S}_P = \emptyset$. Therefore,

$$\mathcal{S} \;=\; \bigcup_{p=1}^{P} \text{supp}(\sigma_p). \tag{4.1}$$

We now show that the union in (4.1) is disjoint. We prove, by contradiction, that

$$\text{supp}(\sigma_r) \cap \text{supp}(\sigma_q) \;=\; \emptyset, \quad \forall\, 1 \leq q < r \leq P. \tag{4.2}$$

If (4.2) does not hold, then there exist q and r such that

$$\tilde{\sigma} \;=\; \text{supp}(\sigma_r) \cap \text{supp}(\sigma_q) \;\neq\; \emptyset,$$

where $1 \leq q < r \leq P$. Since all the support segments are open, $\tilde{\sigma}$ is an open segment. Note that $\tilde{\sigma} \subset \mathcal{S}$. Then there exists $\sigma \in \mathfrak{S}$ such that $\sigma \cap \tilde{\sigma} \neq \emptyset$, and therefore

$$\sigma \cap \text{supp}(\sigma_r) \neq \emptyset; \quad \sigma \cap \text{supp}(\sigma_q) \neq \emptyset.$$

From the definition of $\text{supp}(\sigma)$, it follows that

$$\text{supp}(\sigma) \;=\; \text{supp}(\sigma_r) \;=\; \text{supp}(\sigma_q). \tag{4.3}$$

Since $q < r$,

$$\text{supp}(\sigma_r) \;\subset\; \mathcal{S}_{r-1} \;\subseteq \mathcal{S}_q \;\subset\; \mathcal{S} \setminus \text{supp}(\sigma_q). \tag{4.4}$$

From (4.3) and (4.4),

$$\text{supp}(\sigma_q) \;\subset\; \mathcal{S} \setminus \text{supp}(\sigma_q),$$

which is a contradiction. Our claim (4.2) is therefore proven.

Since $\Gamma = \overline{\mathcal{S}}$, from (4.1) and (4.2), it is enough if we prove that a consistent choice of nonmortars can be made for every $\text{supp}(\sigma_p)$, $p = 1 : P$. This can be done, e.g., by choosing as nonmortars all the sides from $\text{supp}(\sigma_p)$ which belong to subregions which are on the same side of $\text{supp}(\sigma_p)$. □

In the three dimensional case, the support of a face σ is the union of all the faces which are coplanar with σ, such that $\text{supp}(\sigma)$ is connected. Then $\text{supp}(\sigma)$ is a polygon, not necessarily convex, and the supports of two different faces are either disjoint, or they coincide. The rest of the proof and the choice of nonmortars follow as in the two dimensional case.

4.2 L^2 Stability of the 1-D Mortar Projection

In Section 2.4, we have presented some stability properties for the low order mortar projections. In this section, we address the issue of the L^2 stability of the mortar projection for a more general class of mortar finite elements.

Following the notations of Section 2.2, let V^h be the a mortar finite element space whose restriction to any subregion Ω_k is a conforming P_{m_k} or Q_{m_k} finite element space. The mortar conditions (2.2) across the nonmortars of the interface correspond to a different mortar projection. More formally, if γ is a nonmortar side, let $V^h(\gamma)$ be the continuous piecewise polynomial space which is the restriction of V^h to γ. We then define the mortar projection operator $\pi_\gamma : L^2(\gamma) \to V^h(\gamma) \cap H_0^1(\gamma)$ by the following L^2-orthogonality condition:

$$\int_{\overline{\gamma}} (\chi - \pi_\gamma(\chi))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma).$$

Here, $\chi \in L^2(\gamma)$ and the test function space $\Psi^h(\gamma)$ is the subspace of $V^h(\gamma)$, whose restriction to the first and last mesh intervals are polynomials of degree 1 less than the corresponding degree from $V^h(\gamma)$.

As we mentioned in Section 2.4, the stability of the mortar projection has been established for various meshes; see Bernardi, Maday, and Patera [13, 14] for uniform meshes, Ben Belgacem [7] and Braess, Dahmen, Wieners [18] for quasiuniform meshes. In a more general case, Seshaiyer and Suri [116, 117] give a proof of the L^2 stability of the mortar projection, if the ratio of any two neighboring mesh intervals over γ is uniformly bounded. The constant in their bound depends on the maximum value of that ratio and on m, the polynomial degree.

In this section, we prove that the mortar projection is uniformly stable in L^2 for arbitrary meshes, with the constant in the bound depending only on m. Our result is obtained by refining a method used in Ben Belgacem and Maday [9] for mortar projections with uniform meshes for three dimensional mortar finite elements.

Before we begin our analysis of the L^2 stability of the mortar projection, let us comment on the issue of the H_0^1 stability. In [117], Seshaiyer and Suri prove that the mortar projection is stable in the H_0^1 norm, if the ratio of any two neighboring mesh intervals is uniformly bounded. In [43], Crouzeix and Thomée prove a similar

result for the L_2-projection from $L^2(\gamma)$ onto $V^h(\gamma) \cap H_0^1(\gamma)$. They also show that the projection is not stable in the H_0^1 norm for arbitrary meshes. Therefore, it is reasonable to believe that some condition on the mesh of γ is necessary in order to obtain the H_0^1 stability of the mortar projection π_γ.

4.2.1 Technical Tools

The most important result of this section gives a good L^2 approximation (see Lemma 4.3 for the precise result) of a polynomial from $V^h(\gamma)$ which vanishes at the end points of γ by another polynomial from $\Psi^h(\gamma)$. To do so, we need some results about minimizing the L^2 norm of polynomials satisfying certain constraints. The main idea of the proofs is to use Legendre polynomial expansions and Lagrange multipliers methods; see, e.g., [12] for elementary properties of the Legendre polynomials.

For simplicity, we only work with odd degree polynomials. Similar estimates and results can also be derived for even degree polynomials.

Lemma 4.1. *Let P be a polynomial of degree $2n+1$ on $[-1,\ 1]$, with $P(-1) = c_1$ and $P(1) = c_2$. Then*

$$\inf_P ||P||^2_{L^2(-1,1)} = \frac{2(c_1^2 + c_2^2)(n+1) + 2c_1c_2}{(n+1)(2n+1)(2n+3)}.$$

Proof. We write P in the basis of Legendre polynomials,

$$P(x) = \sum_{k=0}^{2n+1} a_k L_k(x). \tag{4.5}$$

Since $L_k(1) = 1$ and $L_k(-1) = (-1)^k$, the conditions $P(-1) = c_1$ and $P(1) = c_2$ can be expressed as

$$\sum_{k=0}^{2n+1} a_k = c_2, \quad \text{and} \quad \sum_{k=0}^{2n+1} (-1)^k a_k = c_1,$$

or, equivalently:

$$\sum_{k=0}^{n} a_{2k} = \frac{c_1 + c_2}{2} \tag{4.6}$$

$$\sum_{k=0}^{n} a_{2k+1} \;=\; \frac{c_2 - c_1}{2}. \tag{4.7}$$

The Legendre polynomials are orthogonal in the L^2 inner product, and

$$||L_k||^2_{L^2(-1,1)} \;=\; \frac{1}{k+1/2}.$$

Therefore

$$||P||^2_{L^2(-1,1)} \;=\; \sum_{k=0}^{2n+1} \frac{a_k^2}{k+1/2}. \tag{4.8}$$

Therefore, we can split our minimization problem into two subproblems, corresponding to the even and odd degree coefficients, respectively, which will be solved in a similar fashion.

For the subproblem corresponding to $\{a_{2k}\}_{k=0:n}$, we want to minimize

$$\sum_{k=0}^{n} \frac{a_{2k}^2}{2k+1/2},$$

subject to the constraint (4.6). From Schwarz's inequality, we find

$$\begin{aligned}\Big(\sum_{k=0}^{n} \frac{a_{2k}^2}{2k+1/2}\Big)\,\Big(\sum_{k=0}^{n} 2k+1/2\Big) \;&=\; \Big(\sum_{k=0}^{n} \frac{a_{2k}^2}{2k+1/2}\Big)\,\frac{(n+1)(2n+1)}{2} \\ &\geq\; \Big(\sum_{k=0}^{n} a_{2k}\Big)^2 \;=\; \frac{(c_1+c_2)^2}{4}.\end{aligned}$$

We note that, in Schwarz's inequality there exist coefficients a_{2k}, $k = 0 : n$, such that equality is realized. Therefore,

$$\min_{\sum a_{2k} = (c_1+c_2)/2} \sum_{k=0}^{n} \frac{a_{2k}^2}{2k+1/2} \;=\; \frac{(c_1+c_2)^2}{2(n+1)(2n+1)}. \tag{4.9}$$

Similarly, for the problem corresponding to $\{a_{2k+1}\}_{k=0:n}$, we obtain:

$$\min_{\sum a_{2k+1} = (c_2-c_1)/2} \sum_{k=0}^{n} \frac{a_{2k+1}^2}{2k+3/2} \;=\; \frac{(c_2-c_1)^2}{2(n+1)(2n+3)}. \tag{4.10}$$

Adding (4.9) and (4.10), we obtain:

$$\inf_P ||P||^2_{L^2(-1,1)} \;=\; \frac{2(c_1^2+c_2^2)(n+1)+2c_1c_2}{(n+1)(2n+1)(2n+3)}.$$

□

Lemma 4.2. *Let P be a polynomial of degree $2n+1$ on $[-1,\ 1]$, with $P(-1) = 0$ and $P(1) = c_2$. Expand P in the Legendre polynomials basis and assume that a_{2n+1}, the coefficient of the highest degree term in the expansions, is given. Then*

$$\inf_P ||P||^2_{L^2(-1,1)} \ = \ a^2_{2n+1}\frac{2(2n+3)(n+1)}{n(2n+1)(4n+3)} \ - \ \frac{2c_2a_{2n+1}}{n(2n+1)} \ + \ \frac{c_2^2}{2n(2n+1)}. \quad (4.11)$$

If the value of P at -1 is no longer required to be 0, then

$$\inf_P ||P||^2_{L^2(-1,1)} \ = \ a^2_{2n+1}\frac{8(n+1)^2}{(4n+3)(2n+1)^2} \ - \ \frac{4c_2a_{2n+1}}{(2n+1)^2} \ + \ \frac{2c_2^2}{(2n+1)^2}. \quad (4.12)$$

Proof. Writing P in the Legendre basis as in (4.5), and imposing $P(-1) = 0$ and $P(1) = c_2$, we obtain

$$a_{2n+1} + \sum_{k=0}^{2n} a_k \ = \ c_2, \quad \text{and} \quad -a_{2n+1} + \sum_{k=0}^{2n}(-1)^k a_k \ = \ 0.$$

Since a_{2n+1} is fixed, we can solve for $\sum a_{2k}$ and $\sum a_{2k+1}$:

$$\sum_{k=0}^{n} a_{2k} \ = \ \frac{c_2}{2}, \quad \text{and} \quad \sum_{k=0}^{n-1} a_{2k+1} \ = \ \frac{c_2 - 2a_{2n+1}}{2}.$$

In Lemma 4.1, we have solved the problem of minimizing $\sum_{k=0}^{2n+1} \frac{a_k^2}{k+1/2}$, when the sums of the odd and even terms, respectively, are kept constant. Using that result and (4.8), we obtain

$$\begin{aligned}\inf_P ||P||^2_{L^2(-1,1)} \ &= \ a^2_{2n+1}\frac{1}{(2n+1)+1/2} \ + \ \frac{c_2^2}{2(n+1)(2n+1)} \ + \ \frac{(c_2-2a_{2n+1})^2}{2n(2n+1)} \\ &= \ a^2_{2n+1}\frac{2(n+1)(2n+3)}{n(2n+1)(4n+3)} \ - \ \frac{2c_2a_{2n+1}}{n(2n+1)} \ + \ \frac{c_2^2}{2n(n+1)}.\end{aligned}$$

If the value of P at -1 is no longer fixed, then the only condition that the coefficients $\{a_k\}_{k=0:2n}$ must satisfy is

$$\sum_{k=0}^{2n} a_k \ = \ c_2 - a_{2n+1}.$$

Using once again Schwarz's inequality, and without splitting the problem into two cases, we obtain:

$$\Big(\sum_{k=0}^{2n}\frac{a_k^2}{k+1/2}\Big)\,\Big(\sum_{k=0}^{2n}k+1/2\Big) \;=\; \Big(\sum_{k=0}^{2n}\frac{a_k^2}{k+1/2}\Big)\,\frac{(2n+1)^2}{2} \;\geq\; (\sum_{k=0}^{2n}a_k)^2,$$

which can be written as:

$$\sum_{k=0}^{2n}\frac{a_k^2}{k+1/2} \;\geq\; \frac{2}{(2n+1)^2}\,(\sum_{k=0}^{2n}a_k)^2 \;=\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}.$$

Therefore:

$$||P||^2_{L^2(-1,1)} \;=\; \sum_{k=0}^{2n+1}\frac{a_k^2}{k+1/2} \;\geq\; \frac{2a_{2n+1}^2}{4n+3} \;+\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}.$$

Since in Schwarz's inequality there exist coefficients a_k, $k=0:2n$ such that the equality is realized, we conclude that there exists a polynomial P such that

$$\begin{aligned}\inf_P ||P||^2_{L^2(-1,1)} &= \frac{2a_{2n+1}^2}{4n+3} \;+\; \frac{2(c_2-a_{2n+1})^2}{(2n+1)^2}\\ &= a_{2n+1}^2\frac{8(n+1)^2}{(4n+3)(2n+1)^2} \;-\; \frac{4c_2a_{2n+1}}{(2n+1)^2} \;+\; \frac{2c_2^2}{(2n+1)^2}.\end{aligned}$$

□

The next lemma is the main result of this section. We introduce the following notations. Let $\gamma=[a,b]$ be a segment partitioned into intervals $\{I_j\}_{j=1:(N+1)}$, $I_j=(x_{j-1},x_j)$, with $a=x_0<x_1<\ldots<x_{N+1}=b$, and let $h_j=x_j-x_{j-1}$, for $j=1:N+1$. Let $\{m_j\}_{j=1:(N+1)}$ be a set of positive integers. We define the piecewise polynomial spaces $V^h(\gamma)$ and $\Psi^h(\gamma)$ as follows:

$$V^h(\gamma)=\{v\in C(0,1);\qquad v|_{I_j}\in P_{m_j}(I_j),\ \forall\ j=1:N+1\},\tag{4.13}$$

$$\begin{aligned}\Psi^h(\gamma)=\{v\in C(0,1);\qquad & v|_{I_j}\in P_{m_j}(I_j),\ \forall\ j=2:N;\\ & v|_{I_j}\in P_{m_j-1}(I_j),\ j\in\{1,N+1\}\}.\end{aligned}\tag{4.14}$$

Lemma 4.3. *Let $\pi\chi \in V^h(\gamma) \cap H_0^1(\gamma)$. Then there exists a function $\chi_h \in \Psi^h(\gamma)$ and a constant $0 < C(m) < 1$, depending only on m_1 and m_{N+1}, and not on the partition of γ, such that*

$$||\pi\chi - \chi_h||_{L^2(\gamma)} \ \leq \ C(m)||\pi\chi||_{L^2(\gamma)}. \tag{4.15}$$

More precisely,

$$C(m) \ = \ max\left\{ \left(1 - \frac{1}{m_1^2}\right)^{1/2}, \ \left(1 - \frac{1}{m_{N+1}^2}\right)^{1/2} \right\}.$$

Proof. We will choose χ_h to be equal to $\pi\chi$ on all of the partition intervals except for the first two and the last two intervals. If we look for χ_h equal to $\pi\chi$ on all the intervals except the first and last ones, it can be proven that the best constant M in (4.15) would depend on h_0 and h_1, which we want to avoid. Since χ_h will be defined in a similar way at both ends of γ, we only present the construction of χ_h on $I_1 = (x_0, x_1)$ and $I_2 = (x_1, x_2)$.

We may assume, without any loss of generality, that m_1 and m_2 are odd, i.e., $m_1 = 2n_1 + 1$ and $m_2 = 2n_2 + 1$; similar results can be obtained for all the other cases. Let $\beta_1 = \pi\chi(x_1)$ and $\beta_2 = \pi\chi(x_2)$. Note that $\pi\chi(x_0) = 0$, since $\pi\chi$ vanishes at the end points of γ. We require that $\chi_h(x_2) = \beta_2$, and denote the value of χ_h at x_1 by α_1, which will be different than β_1: $\chi_h(x_1) = \alpha_1 \neq \beta_1$. We will look for $\chi_h \in \Psi^h(\gamma)$ such that $||\pi\chi - \chi_h||_{L^2(\gamma)}$ is minimal, and then choose α_1 such that relation (4.15) will hold on the two intervals I_1 and I_2.

On I_2, $\pi\chi - \chi_h$ is a polynomial of degree $2n_2 + 1$ which takes the values $\beta_1 - \alpha_1$ and 0, respectively, at the left and right end points. After a suitable change of variables, which maps I_2 into $(-1, 1)$, and using Lemma 4.1, we can find χ_h on I_2 such that

$$\frac{8}{h_2}||\pi\chi - \chi_h||_{L^2(I_2)}^2 \ = \ \frac{2(\beta_1 - \alpha_1)^2}{(2n_2 + 1)(2n_2 + 3)}. \tag{4.16}$$

On I_1, $\pi\chi - \chi_h$ is a polynomial of degree $2n_1 + 1$ which takes the value $\beta_1 - \alpha_1$ at x_1, the left end point of I_1. Let a be the coefficient of L_{2n_1+1} in the Legendre expansion of $\pi\chi$ over I_1. Since χ_h is a polynomial of degree $2n_1$ over I_1, a is also the coefficient of L_{2n_1+1} in the Legendre expansion of $\pi\chi - \chi_h$. After a suitable

change of variables, which maps I_1 into $(-1,1)$, and using (4.12) from Lemma 4.2, there exists χ_h, satisfying all the above mentioned properties, such that

$$\frac{8}{h_1}||\pi\chi-\chi_h||^2_{L^2(I_1)} \;=\; a^2\frac{8(n_1+1)^2}{(4n_1+3)(2n_1+1)^2} - \frac{4a(\beta_1-\alpha_1)}{(2n_1+1)^2} + \frac{2(\beta_1-\alpha_1)^2}{(2n_1+1)^2}. \quad (4.17)$$

We now find lower bounds for $||\pi\chi||_{L^2(I_2)}$ and $||\pi\chi||_{L^2(I_1)}$. On I_2, $\pi\chi$ takes the values β_1 and β_2 at the end points. After a change of variables and using Lemma 4.1, we obtain

$$\frac{8}{h_2}||\pi\chi||^2_{L^2(I_2)} \;\geq\; \frac{2(\beta_1^2+\beta_2^2)(n_2+1)+2\beta_1\beta_2}{(n_2+1)(2n_2+1)(2n_2+3)}. \quad (4.18)$$

The minimal value of the right hand side of (4.18) is obtained for $\beta_2 = -\beta_1/2(n_2+1)$, and therefore:

$$\frac{8}{h_2}||\pi\chi||^2_{L^2(I_2)} \;\geq\; \Big(1-\frac{1}{4(n_2+1)^2}\Big)\frac{2\beta_1^2}{(2n_2+1)(2n_2+3)}. \quad (4.19)$$

On I_1, $\pi\chi$ takes the values 0 and β_1 at the end points, and in its Legendre expansion the coefficient of L_{2n_1+1} is a. After a change of variables, and using (4.11) from Lemma 4.2, we obtain

$$\frac{8}{h_1}||\pi\chi||^2_{L^2(I_1)} \;\geq\; a^2\frac{2(2n_1+3)(n_1+1)}{n_1(2n_1+1)(4n_1+3)} \;-\; \frac{2\beta_1 a}{n_1(2n_1+1)} \;+\; \frac{\beta_1^2}{2n_1(2n_1+1)}. \quad (4.20)$$

We choose $\alpha_1 = \beta_1/2$, and compare the L^2 norms of $\pi\chi-\chi_h$ and $\pi\chi$ separately on I_1 and I_2.

On I_2, we obtain by using (4.16) and (4.19), that

$$||\pi\chi-\chi_h||^2_{L^2(I_2)} \;\leq\; \frac{2}{3}\;||\pi\chi||^2_{L^2(I_2)}. \quad (4.21)$$

On I_1, we obtain by using (4.17) and (4.20), that

$$||\pi\chi-\chi_h||^2_{L^2(I_1)} \;\leq\; (1-1/(2n_1+1)^2)\;||\pi\chi||^2_{L^2(I_1)}. \quad (4.22)$$

We make a similar construction for χ_h on I_{N-1} and I_N. Since $\pi\chi-\chi_h$ vanishes outside the first and last two mesh intervals of γ, we can conclude, by using (4.21) and (4.22), that

$$||\pi\chi-\chi_h||_{L^2(\gamma)} \;\leq\; C(m)||\pi\chi||_{L^2(\gamma)},$$

where $C(m)$ depends only on m_1 and m_{N+1}, and not on the particular properties of the partition of γ. □

4.2.2 Stability property of the mortar projection

We are now able to prove the uniform stability of the mortar projection onto $V^h(\gamma) \cap H_0^1(\gamma)$, i.e., that the bound is independent of the mesh. In Theorem 4.2, the spaces $V^h(\gamma)$ and $\Psi^h(\gamma)$ are those defined in Section 4.2.

Theorem 4.2. *Let γ be a nonmortar side, and let m be the degree of the piecewise polynomial restriction of the mortar function to γ. Let π_γ be the mortar projection of $L^2(\gamma)$ into $V^h(\gamma) \cap H_0^1(\gamma)$, which satisfies*

$$\int_{\overline{\gamma}} (\chi - \pi_\gamma(\chi))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma).$$

Then there exists a constant $\tilde{C}(m)$ depending only on m such that

$$||\pi_\gamma(\chi)||_{L^2(\gamma)} \ \leq \ \tilde{C}(m)||\chi||_{L^2(\gamma)}, \quad \forall \chi \in L^2(\gamma).$$

More precisely,

$$C(m) \ = \ max \left\{ m_1^2, m_{N+1}^2 \right\}.$$

Proof. Let $p_h : L^2(\gamma) \to \Psi^h(\gamma)$ be the L^2 projection into the space $\Psi^h(\gamma)$:

$$\int_{\overline{\gamma}} (\chi - p_h(\chi))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma),$$

where $p_h(\chi) \in \Psi^h(\gamma)$. Then

$$\int_{\overline{\gamma}} (\pi_\gamma(\chi) - p_h(\chi))\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma),$$

and therefore $p_h(\chi)$ is the projection of $\pi_\gamma(\chi)$ into $\Psi^h(\gamma)$. Then:

$$||\pi_\gamma(\chi) - p_h(\chi)||_{L^2(\gamma)} \ = \ \inf_{\chi_h \in \Psi^h(\gamma)} ||\pi_\gamma(\chi) - \chi_h||_{L^2(\gamma)} \ \leq \ C(m)||\pi_\gamma(\chi)||_{L^2(\gamma)},$$

with $M < 1$, according to Lemma 4.3, applied for the case when all the degrees m_j are equal to m.

A simple computation will lead us to the desired conclusion:

$$\begin{aligned} ||\pi_\gamma(\chi)||^2_{L^2(\gamma)} \ &= \ \int_{\overline{\gamma}} (\pi_\gamma(\chi) - p_h(\chi))\pi_\gamma(\chi) ds \ + \ \int_{\overline{\gamma}} p_h(\chi)\pi_\gamma(\chi) ds \\ &= \ \int_{\overline{\gamma}} (\pi_\gamma(\chi) - p_h(\chi))^2 ds \ + \ \int_{\overline{\gamma}} p_h(\chi)\pi_\gamma(\chi) ds \\ &\leq \ C(m)^2||\pi_\gamma(\chi)||^2_{L^2(\gamma)} \ + \ ||\pi_\gamma(\chi)||_{L^2(\gamma)}||p_h(\chi)||_{L^2(\gamma)}. \end{aligned}$$

Since $C(m) < 1$, and since $p_h(\chi)$ is an L^2 projection of χ:

$$||\pi_\gamma(\chi)||_{L^2(\gamma)} \leq \frac{1}{1-C(m)^2}||p_h(\chi)||_{L^2(\gamma)} \leq \frac{1}{1-C(m)^2}||\chi||_{L^2(\gamma)}.$$

□

4.3 Poincaré and Friedrichs Inequalities for Mortars

As we have mentioned in Section 2.2, a mortar finite elements function is only piecewise H^1, but not in $H^1(\Omega)$. Therefore, the Friedrichs and Poincaré inequalities, as presented in Section 1.2.2, cannot be applied to mortar functions. In this section, we show that these inequalities also hold for mortars, if the H^1 seminorm in replaced by the broken seminorm

$$a^\Gamma(v_h, v_h) = \sum_{i=1}^{N} |v_h|^2_{H^1(\Omega_i)}.$$

It is important to note that the constants in the inequalities we derive depend only on the diameter of Ω, and neither on the properties of the partition $\{\Omega_i\}_{i=1:N}$, nor on those of the mortar finite element. Using the Friedrichs inequality, we prove, in Section 4.3.4, that the condition number of the unpreconditioned mortar finite element method has the same upper bound as in the continuous finite element case, and does not depend on the number of the subregions in the partition of Ω. This is a refinement of a result of Bernardi, Maday, and Patera [13, 14], where a Friedrichs inequality is proven using the Rellich compactness theorem. This leads to an estimate of the condition number which depends on the number of the subregions in the partition $\{\Omega_i\}_{i=1:N}$ and their diameters.

In the geometrically conforming case, a variant of the Friedrichs inequality for mortars was proven by Bernardi and Maday [10, 11].

To keep the presentation simple, our model problem will be Poisson's equation with Dirichlet boundary conditions on Ω, a bounded open polygon in R^2. Our results can also be obtained, using the same methods, for any second order

self-adjoint elliptic problems with mixed boundary conditions, and for the three dimensional case.

Following the construction from Section 2.2, let $\{\Omega_i\}_{i=1:N}$ be a geometrically nonconforming polygonal partition of Ω, and V^h a mortar finite element defined on this partition. Let H_i be the diameter of Ω_i, h_i the smallest diameter of any of the elements of Ω_i, and $h = \min h_i$, $i = 1 : N$. We do not require that all the H_k are of the same order of magnitude, but only require that the diameters of any two adjacent subregions Ω_r and Ω_s ($\partial\Omega_r \cap \partial\Omega_s \neq \emptyset$) are comparable, i.e., $c \leq H_r/H_s \leq C$, where c and C are positive constants independent of the subregions considered. We assume that all the subregions are generated from a finite number of reference domains $\widehat{\Omega}_j$, $j = 1 : J$, by mappings F_i, such that $\Omega_i = F_i(\widehat{\Omega}_j)$, and

$$||\partial F_i|| \leq CH_i, \quad \forall\, i = 1 : N; \quad ||\partial F_i^{-1}|| \leq CH_i^{-1}, \quad \forall\, i = 1 : N.$$

As a consequence, we note that the length of every side of Ω_i is bounded from below by a uniform fraction of H_i.

4.3.1 Technical Results

We begin with a version of the Friedrichs inequality on a reference subregion $\widehat{\Omega}$.

Lemma 4.4. *Let $\widehat{\Omega} \subset R^2$ be a fixed, open, bounded domain with Lipschitz boundary. Let $c_0 > 0$ and let $\widehat{\Lambda} \subset \partial\widehat{\Omega}$ be a part of the boundary of $\widehat{\Omega}$ such that*

$$c_0\mu(\partial\widehat{\Omega}) \leq \mu(\widehat{\Lambda}),$$

where μ is the Lebesgue measure. Then,

$$||w||^2_{L^2(\widehat{\Omega})} \leq C(\ |w|^2_{H^1(\widehat{\Omega})} + \frac{1}{c_0}|\int_{\widehat{\Lambda}} w d\sigma|^2\ ), \quad \forall\, w \in H^1(\widehat{\Omega}),$$

where C is a constant that depends only on $\widehat{\Omega}$, and not on w, $\widehat{\Lambda}$, or c_0.

We also need a generalized version of the Friedrichs inequality. We note that Lemma 4.4 follows from Lemma 4.5.

Lemma 4.5. *Let $\widehat{\Omega} \subset R^2$ be a fixed, open, bounded domain with a Lipschitz boundary, let $c_0 \in (0,1)$ be a constant, and let $\widehat{\Lambda} \subset \partial\widehat{\Omega}$ such that*

$$c_0 \mu(\partial\widehat{\Omega}) \ \leq \ \mu(\widehat{\Lambda}). \tag{4.23}$$

Let $\widehat{\psi}$ be a bounded positive function defined on $\widehat{\Lambda}$ with the following properties:

$$0 \leq \widehat{\psi} \leq 2 \quad \text{and} \quad \frac{1}{2}\mu(\widehat{\Lambda}) \ \leq \ \mu(\{x \in \widehat{\Lambda} \ : \ \widehat{\psi}(x) \geq 1\}). \tag{4.24}$$

Then, the following inequality holds,

$$||w||^2_{L^2(\widehat{\Omega})} \ \leq \ C \ (\ (diam(\widehat{\Omega}))^2 |w|^2_{H^1(\widehat{\Omega})} \ + \ \frac{1}{c_0^2} \left| \int_{\widehat{\Lambda}} w\widehat{\psi} d\sigma \right|^2 \ ), \quad \forall w \in H^1(\widehat{\Omega}), \tag{4.25}$$

where C is a constant independent of c_0, w, $\widehat{\Lambda}$, and $\widehat{\psi}$.

Proof. We may assume that $\mathrm{diam}(\widehat{\Omega}) = 1$. The general inequality is obtained easily by a scaling argument.

Suppose that Lemma 4.5 is not true. Then there exists a sequence $\{w_n\}_{n=1:\infty}$ of functions in $H^1(\widehat{\Omega})$, a sequence of boundary parts $(\widehat{\Lambda}_n)_{n=1:\infty}$ satisfying (4.23), and a sequence of functions $(\widehat{\psi}_n)_{n=1:\infty}$ defined on $(\widehat{\Lambda}_n)_{n=1:\infty}$ and satisfying property (4.24), such that

$$||w_n||_{L^2(\widehat{\Omega})} \ = \ 1, \quad \forall \ n = 1:\infty; \tag{4.26}$$

$$|w_n|^2_{H^1(\widehat{\Omega})} \ + \ \frac{1}{c_0^2} \left| \int_{\widehat{\Lambda}_n} w_n \widehat{\psi}_n d\sigma \right|^2 \ \leq \ \frac{1}{n}, \quad \forall \ n = 1:\infty. \tag{4.27}$$

For $n \to \infty$, w_n converges to 0 in the $H^1(\widehat{\Omega})$-seminorm. Therefore, the sequence $\{w_n\}$ is bounded in $H^1(\widehat{\Omega})$, and we obtain, from the Rellich theorem, the existence of a subsequence of $\{w_n\}$ that converges in the $L^2(\widehat{\Omega})$ norm. For simplicity, we also denote this subsequence by $\{w_n\}$. Since $|w_n|^2_{H^1(\widehat{\Omega})} \to 0$, we also have convergence of $\{w_n\}$ in the $H^1(\widehat{\Omega})$ norm. The limit function is a constant function, $\tilde{c}$.

From (4.27), we know that $|\int_{\widehat{\Lambda}_n} w_n \widehat{\psi}_n d\sigma| \to 0$. Since $w_n \to \tilde{c}$ in $H^1(\widehat{\Omega})$, and the functions $\widehat{\psi}_n$ are uniformly bounded by assumption, we obtain, using a trace theorem, that

$$\left| \int_{\widehat{\Lambda}_n} \widehat{\psi}_n d\sigma \ \right| \ |\tilde{c}| \ \to \ 0. \tag{4.28}$$

From (4.23) and (4.24):

$$\left| \int_{\widehat{\Lambda}_n} \widehat{\psi}_n d\sigma \right| \;\geq\; \frac{1}{2}\mu(\widehat{\Lambda}) \;\geq\; \frac{c_0}{2}\mu(\partial\widehat{\Omega}). \tag{4.29}$$

Finally, from (4.28) and (4.29), we obtain that $\tilde{c} = 0$, which implies $w_n \to 0$ in $L^2(\widehat{\Omega})$. This contradicts assumption (4.26), and the proof is completed. □

The next lemma is purely geometrical, and a version of it has first appeared in Bernardi and Maday [10].

Lemma 4.6. *Let Ω be a bounded domain in the plane and let $\{\Omega_i\}_{i=1:N}$ be a shape regular partition of Ω. where Ω_k is a polygon of diameter H_k. The ratio of the diameters of neighboring subregions is uniformly bounded.*

Let ℓ be a line passing through Ω and let $(\Omega_{i,\ell})_{i=1:n(\ell)}$ be the subregions with interiors intersecting ℓ. Then,

$$\sum_{i=1}^{n(\ell)} H_{i,\ell} \;\leq\; C\,diam(\Omega),$$

where C is a constant which depends only on the minimal angle of the polygonal subregions $\{\Omega_i\}_{i=1:N}$, and not on their diameters $(H_k)_{k=1:K}$.

Proof. The proof is a straightforward generalization of that of Lemma 2.2 from [10]. It is based on the assumptions that neighboring subregions have comparable diameters and that the minimal angle for the polygonal subregions is uniformly bounded. Each subregion can be decomposed into triangles with a common vertex in the centroid of the subregion, and having one of the sides of the polygon. Then, the same proof given in [10] by Bernardi and Maday for the case of a triangular partition of Ω will work for our problem. □

4.3.2 An Estimate of the L^2 Norm of Jumps Across Non-mortars

An important step in the proofs of the Friedrichs and Poincaré inequalities for mortar finite elements is an estimate of the L^2 norm of the jump of the mortar

finite element function v over the nonmortars. We restrict the technical discussion of this section to the case when all subregions are rectangles. In the last section, we explain how our results can be extended for a general polygonal partition.

Let γ be a nonmortar side of the subregion Ω_l, and let v_l be the restriction of the mortar function v to Ω_l. Let $\zeta_{l,i}$, $i = 1 : q(\gamma)$ be the mortars opposite γ across the interface Γ. Let $\Omega_{l,i}$ be the subregion which has $\zeta_{l,i}$ as a side, and let

$$\delta_{i,i+1} = \partial\Omega_{l,i} \cap \partial\Omega_{l,i+1}, \quad \forall\ i = 1 : (q(\gamma) - 1).$$

Since every subregion $\Omega_{l,i}$ has a diameter on the order of H_l, $q(\gamma)$ is uniformly bounded by a constant C which depends only on the lower and upper bounds of the ratios of the diameters of adjacent subregions.

Let $\tilde{v}$ be the function that is equal to $v_{l,i}$ (the restriction of v to $\Omega_{l,i}$) on $\zeta_{l,i}$. We note that $\tilde{v}$ can have two values at the vertices on the interface Γ that are interior to γ.

Lemma 4.7. *Let $[v] = v_l - \tilde{v}$ be the jump of v across the nonmortar γ. Then,*

$$\int_\gamma [v]^2 d\sigma \ \leq\ C\ \mu(\gamma)(\ |v_l|^2_{H^1(\Omega_l)}\ +\ \sum_{i=1}^{q(\gamma)} |v_{l,i}|^2_{H^1(\Omega_{l,i})}\ ), \tag{4.30}$$

where C is a constant that does not depend on γ, and μ is the Lebesgue measure.

Proof. By definition,

$$\int_\gamma [v]^2 d\sigma = \int_\gamma |v_l - \tilde{v}|^2 d\sigma. \tag{4.31}$$

Since any space of test functions $\Psi^h(\gamma)$ contains the constant functions, and since the functions v_l and $\tilde{v}$ satisfy the mortar conditions (2.2), we find that the averages of v_l and $\tilde{v}$ over γ are equal, i.e.,

$$\int_\gamma v_l d\sigma = \int_\gamma \tilde{v} d\sigma = \overline{v}_\gamma \mu(\gamma).$$

Opposite Ω_l, we construct a rectangle Ω_{new} with one side equal to γ and with sides of length $\min\{\mu(\delta_{i,i+1})\}$, $i = 1 : (q(\gamma) - 1)$, perpendicular to γ.

Let v_{new} be an extension of $\tilde{v}$ from γ to Ω_{new} with the following properties:

$$v_{new} \in H^1(\Omega_{new}), \tag{4.32}$$

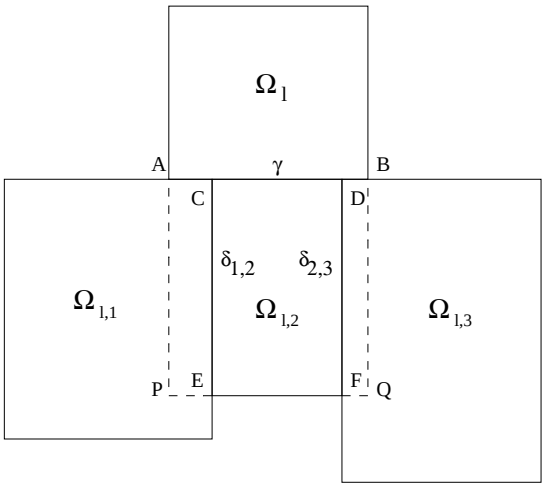


Figure 4.2: Local nonconforming situation

$$\int_\gamma v_{new} d\sigma \;=\; \int_\gamma \tilde{v} d\sigma \;=\; \overline{v}_\gamma \mu(\gamma). \tag{4.33}$$

Note that v_{new} need not be a finite element function, and its trace on γ need not be equal to $\tilde{v}$. We just require that the average of v_{new} over γ is equal to that of $\tilde{v}$.

We now provide the details of the construction of v_{new}. Without loss of generality, we consider the case of only three rectangles $\Omega_{l,1}$, $\Omega_{l,2}$, $\Omega_{l,3}$. Let $v_{l,1}$, $v_{l,2}$, and $v_{l,3}$ be the restrictions of v to $\Omega_{l,1}$, $\Omega_{l,2}$, $\Omega_{l,3}$, respectively. These functions may be discontinuous across $\delta_{1,2}$ and $\delta_{2,3}$; cf. Figure 4.2, where $\Omega_{new} = ABQP$, $\delta_{1,2} = CE$ and $\delta_{2,3} = DF$.

For every segment $\delta_{i,i+1}$, we construct in one of the rectangles $\Omega_{l,i}$ or $\Omega_{l,i+1}$ (say on $\Omega_{l,i}$) a function $\chi_i \in H^1(\Omega_{l,i})$ such that χ_i is equal to $v_{l,i+1} - v_{l,i}$ on $\delta_{i,i+1}$ and vanishes on the side opposite $\delta_{i,i+1}$

The choice of whether to construct the function χ_i on $\Omega_{l,i}$ or $\Omega_{l,i+1}$ is made according to which of $\partial\Omega_{l,i} \cap \gamma$ and $\partial\Omega_{l,i+1} \cap \gamma$ is the largest. If $\mu(\partial\Omega_{l,i} \cap \gamma) \geq \mu(\partial\Omega_{l,i+1} \cap \gamma)$, we choose subregion $\Omega_{l,i}$; otherwise $\Omega_{l,i+1}$. As a consequence, since the subregions $\Omega_{l,i}$ and $\Omega_{l,i+1}$ have diameters on the order of H_l, we find that the length of the intersection of γ with the boundary of the chosen domain is on the order of H_l.

This choice avoids potential difficulties that occur in the study of the geometrically nonconforming case, due to small intersections of boundaries of two subre-

gions. Such a configuration can appear naturally, e.g., from a small perturbation of a geometrically conforming partition.

In our case, we construct the functions χ_2 and χ_3 as follows:

$$\begin{aligned}
\chi_2 : \Omega_{l,2} \to \mathtt{R}; \quad \chi_{2|\Omega_{new}\cap\delta_{1,2}} &= (v_{l,1} - v_{l,2})_{|\Omega_{new}\cap\delta_{1,2}}; \\
\chi_{2|\Omega_{new}\cap\delta_{2,3}} &= 0; \\
\int_{CD} \chi_2 \, d\sigma &= 0; \\
\chi_3 : \Omega_{l,2} \to \mathtt{R}; \quad \chi_{3|\Omega_{new}\cap\delta_{2,3}} &= (v_{l,3} - v_{l,2})_{|\Omega_{new}\cap\delta_{2,3}}; \\
\chi_{3|\Omega_{new}\cap\delta_{1,2}} &= 0; \\
\int_{CD} \chi_3 \, d\sigma &= 0.
\end{aligned}$$

For this purpose, we use extension and trace theorems on the unit square $\widehat{\Omega}$; see, e.g., Nečas [97] and Section 1.2.1.

Let $\hat{s}$, $\hat{s}_1$, $\hat{s}_2$, and $\hat{s}_3$ be the sides of $\widehat{\Omega}$, in consecutive order. If $\hat{\psi} \in H^{\frac{1}{2}}(\hat{s})$, we can, using several reflections, extend it to a function $E(\hat{\psi}) \in H^{\frac{1}{2}}(\partial\widehat{\Omega})$ such that

$$||E(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}.$$

Let ϕ_1 and ϕ_2 be positive $C^\infty(\partial\widehat{\Omega})$ functions with the following properties: ϕ_1 is 1 on $\hat{s}$ and 0 on $\hat{s}_2$ (the side opposite to $\hat{s}$ in the rectangle $\widehat{\Omega}$) and is bounded from above by 1 on $\partial\widehat{\Omega}$; ϕ_2 is supported in $\hat{s}_1$, and $\int_{\hat{s}_1} \phi_2 dx = 1$. The function

$$E_0(\hat{\psi}) \;=\; \phi_1 E(\hat{\psi}) - \phi_2 \int_{\hat{s}_1} \phi_1 E(\hat{\psi}) dx$$

is an extension of $\hat{\psi}$ satisfying

$$\begin{aligned}
E_0(\hat{\psi})_{|\hat{s}} = \hat{\psi}; \qquad & E_0(\hat{\psi})_{|\hat{s}_2} = 0; \\
||E_0(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} &\leq C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}; \\
\int_{\hat{s}_1} E_0(\hat{\psi}) d\sigma &= 0.
\end{aligned}$$

Since $E_0(\hat{\psi}) \in H^{\frac{1}{2}}(\partial\widehat{\Omega})$, there exists a harmonic extension of $E_0(\hat{\psi})$ to the unit square $\widehat{\Omega}$, $\hat{u} \in H^1(\widehat{\Omega})$, which does not have to be a finite element function, such that

$$|\hat{u}|_{H^1(\widehat{\Omega})} \leq C||E_0(\hat{\psi})||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C||\hat{\psi}||_{H^{\frac{1}{2}}(\hat{s})}.$$

There exists a diffeomorphism $F : \widehat{\Omega} \rightarrow \Omega_{l,2}$ that induces a natural mapping from the functions defined on $\Omega_{l,2}$ into the functions defined on $\widehat{\Omega}$. We construct χ_2 in the following steps:

1. let $\hat{\psi}_2 := (v_{l,1} - v_{l,2})_{|\Omega_{new} \cap \delta_{1,2}} \circ F$;
2. let $\hat{u}_2$ be the extension of $E_0(\hat{\psi}_2)$ from $\partial\widehat{\Omega}$ to $\widehat{\Omega}$ described above;
3. let $\chi_2 := \hat{u}_2 \circ F^{-1}$.

From the properties of F and the extensions on $\widehat{\Omega}$, we obtain

$$\begin{aligned}
|\chi_2|_{H^1(\Omega_{l,2})} &\leq C|\hat{u}_2|_{H^1(\widehat{\Omega})} \leq C||E_0(\hat{\psi}_2)||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C||\hat{\psi}_2||_{H^{\frac{1}{2}}(\hat{s})} \\
&= C||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}, \\
||\chi_2||_{L^2(CD)} &\leq C\mu(CD)||\hat{u}_2||_{L^2(\partial\widehat{\Omega})} \leq C\mu(\gamma)||\hat{u}_2||_{L^2(\partial\widehat{\Omega})} \leq C\mu(\gamma)||\hat{u}_2||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \\
&= C\mu(\gamma)||E_0(\hat{\psi}_2)||_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \leq C\mu(\gamma)||\hat{\psi}_2||_{H^{\frac{1}{2}}(\hat{s})} \\
&= C\mu(\gamma)||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}.
\end{aligned}$$

Therefore, χ_2 has the following properties:

$$\begin{aligned}
\chi_{2|\Omega_{new} \cap \delta_{1,2}} &= (v_{l,1} - v_{l,2})_{|\Omega_{new} \cap \delta_{1,2}}; \\
\chi_{2|\Omega_{new} \cap \delta_{2,3}} &= 0; \\
\int_{CD} \chi_2 d\sigma &= 0; \qquad (4.34) \\
|\chi_2|_{H^1(\Omega_{l,2})} &\leq C||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}; \qquad (4.35) \\
||\chi_2||_{L^2(CD)} &\leq C\mu(\gamma)||\hat{v}_{l,1} - \hat{v}_{l,2}||_{H^{\frac{1}{2}}(\widehat{\Omega} \cap \hat{\delta}_{1,2})}. \qquad (4.36)
\end{aligned}$$

The function χ_3 is constructed similarly.

Finally, v_{new} is defined as follows:

$$v_{new} = \begin{cases} v_{l,1} & \text{on } \overline{\Omega}_{l,1} \cap \overline{\Omega}_{new}; \\ v_{l,2} + \chi_2 + \chi_3 & \text{on } \overline{\Omega}_{l,2} \cap \overline{\Omega}_{new}; \\ v_{l,3} & \text{on } \overline{\Omega}_{l,3} \cap \overline{\Omega}_{new}. \end{cases} \qquad (4.37)$$

We note that v_{new} is continuous by construction.

We now show that v_{new} satisfies (4.32) and (4.33). Since v_{new} is piecewise H^1 and continuous, it follows from Lemma 1.1 that $v_{new} \in H^1(\Omega_{new})$. From (4.34), it results that

$$\begin{aligned}\int_\gamma v_{new} d\sigma &= \int_{AB} v_{new} d\sigma = \int_{AB} \tilde{v} d\sigma + \int_{CD} \chi_2 + \int_{CD} \chi_3 dx \\ &= \int_{AB} \tilde{v} d\sigma = \overline{v}_\gamma \mu(\gamma).\end{aligned}$$

Using the construction of v_{new} and Ω_{new}, we begin the proof of (4.30):

$$\begin{aligned}\int_\gamma [v]^2 d\sigma &= \int_\gamma |v_l - \tilde{v}|^2 d\sigma \leq 2\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma + 2\int_\gamma |\tilde{v} - \overline{v}_\gamma|^2 d\sigma \\ &\leq 2\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma + 4\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma + 4\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma \quad (4.38)\end{aligned}$$

We now estimate the three terms of (4.38). For the first two, we use Lemma 4.4 and Lemma 4.5, both for the unit square $\hat{\Omega}$, which is the reference subregion. We will apply the Friedrichs inequality, trace theorems, and inverse inequalities only on the reference unit square $\widehat{\Omega}$, since we look for results independent of the partition $\{\Omega_i\}_{i=1:N}$ of Ω.

Estimate of $\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma$:

$$\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma = \mu(\gamma) \int_{\hat{s}} |\hat{v}_l - \overline{v}_\gamma|^2 d\sigma \leq \mu(\gamma) ||\hat{v}_l - \overline{v}_\gamma||^2_{L^2(\partial\widehat{\Omega})} \leq C\mu(\gamma) ||\hat{v}_l - \overline{v}_\gamma||^2_{H^1(\widehat{\Omega})},$$

where the last inequality follows from the trace Theorem 1.2. Since,

$$\int_{\hat{s}} (\hat{v}_l - \overline{v}_\gamma) d\sigma = 0,$$

we obtain, from the Friedrichs inequality, see Theorem 1.4,

$$||\hat{v}_l - \overline{v}_\gamma||_{H^1(\widehat{\Omega})} \leq C|\hat{v}_l|^2_{H^1(\widehat{\Omega})},$$

and therefore,

$$\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma \leq C\mu(\gamma)|\hat{v}_l|^2_{H^1(\widehat{\Omega})} \leq C\mu(\gamma)|v_l|^2_{H^1(\Omega_l)}. \quad (4.39)$$

Estimate of $\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma$:
From (4.33) we obtain that $\int_\gamma (v_{new} - \overline{v}_\gamma) d\sigma = 0$. Applying the same method as for the first term, we get

$$\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma \ \leq \ C\mu(\gamma) |v_{new}|^2_{H^1(\Omega_{new})}. \tag{4.40}$$

The last inequality holds since, by assumption, the diameters of adjacent subregions are uniformly comparable. From the construction of v_{new}, we find:

$$|v_{new}|^2_{H^1(\Omega_{new})} \ \leq \ 3\Big(\sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})} \Big) \ + \ 3|\chi_2|^2_{H^1(\Omega_{l,2})} \ + \ 3|\chi_3|^2_{H^1(\Omega_{l,2})} \tag{4.41}$$

The estimates of $|\chi_2|^2_{H^1(\Omega_{l,2})}$ and $|\chi_3|^2_{H^1(\Omega_{l,2})}$ are similar, so we derive only one of them. One of the sides of the subregions $\Omega_{l,1}$ and $\Omega_{l,2}$ that intersect $\Omega_{new} \cap \delta_{1,2}$ is a nonmortar side. For our proof it does not make any difference which it is, and we can assume that it is a side of $\Omega_{l,1}$. To this nonmortar will correspond a space of test functions $\Psi_{1,2}$. Denote by $\psi_{1,2} \in \Psi_{1,2}$ the test function that is equal to 1 at all the nodes on the nonmortar side that are also in $\Omega_{new} \cap \delta_{1,2}$ except for the last one, and is equal to 0 at all the other nodes. We replace $\Omega_{new} \cap \delta_{1,2}$ with CE. From the mortar condition (2.2), we obtain:

$$\int_{CE} v_{l,1} \psi_{1,2} d\sigma \ = \ \int_{CE} v_{l,2} \psi_{1,2} d\sigma.$$

Let α denote the following congruent terms:

$$\alpha \ = \ \frac{\int_{CE} v_{l,1} \psi_{1,2} d\sigma}{\int_{CE} \psi_{1,2} d\sigma} \ = \ \frac{\int_{CE} v_{l,2} \psi_{1,2} d\sigma}{\int_{CE} \psi_{1,2} d\sigma}.$$

Then it is easy to see that:

$$\int_{CE} (v_{l,1} - \alpha) \psi_{1,2} d\sigma \ = \ \int_{CE} (v_{l,2} - \alpha) \psi_{1,2} d\sigma \ = \ 0. \tag{4.42}$$

Since the rectangles $(\Omega_{l,i})_{i=1:3}$ are neighbors of Ω_l, their diameters are of the same order as H_l, the diameter of Ω_l. Moreover, CE has a length on the order of H_l since it is a side of the rectangle $\Omega_{l,2}$, and the following estimate holds,

$$c_0 \mu(\partial\Omega_{l,1}) \ \leq \ \tilde{c}_0 H_l \ \leq \ \mu(CE).$$

Here, c_0 is a constant that does not depend on the subregion $\Omega_{l,1}$. It is easy to see, from the definition of $\psi_{1,2}$, that

$$\frac{1}{2}\mu(CE) \;\leq\; \mu(\{x \in CE \;:\; \psi_{1,2}(x) \geq 1\}).$$

Therefore, conditions (4.23) and (4.24) of Lemma 4.5 are satisfied. Let $F : \widehat{\Omega} \rightarrow \Omega_{l,2}$ be the diffeomorphism between $\Omega_{l,2}$ and the reference unit square $\widehat{\Omega}$. The induced mapping takes a function u defined on $\Omega_{l,2}$ into the function $\hat{u} = u \circ F$ defined on $\widehat{\Omega}$. It is easy to see that conditions (4.23) and (4.24) are also satisfied on $\widehat{\Omega}$. From Lemma 4.5, we obtain:

$$||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^1(\widehat{\Omega})} \leq C\Big(|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})} \;+\; \frac{1}{c_0^2}\Big|\int_{\hat{s}_1}(\hat{v}_{l,1} - \hat{\alpha})\hat{\psi}_{1,2}d\sigma\Big|^2\Big) = C|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})}, \quad (4.43)$$

since, by (4.42), the integral over $\hat{s}_1$ vanishes. Using a trace theorem, inequalities of Sobolev norms on affine equivalent domains, and (4.43), we obtain:

$$\begin{aligned} ||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} &\leq C||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\partial\widehat{\Omega})} \;\leq\; C||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^1(\widehat{\Omega})} \;\leq\; C|\hat{v}_{l,1}|^2_{H^1(\widehat{\Omega})} \\ &\leq C|v_{l,1}|^2_{H^1(\Omega_{l,1})}. \end{aligned}$$

Once again, C is a constant that does not depend on the subregion $\Omega_{l,1}$. A similar estimate holds for $v_{l,2} - \alpha$. Therefore,

$$\begin{aligned} ||\hat{v}_{l,1} - \hat{v}_{l,2}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} &\leq 2||\hat{v}_{l,1} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} \;+\; 2||\hat{v}_{l,2} - \hat{\alpha}||^2_{H^{\frac{1}{2}}(\widehat{\Omega}\cap\hat{\delta}_{1,2})} \\ &\leq C(|v_{l,1}|^2_{H^1(\Omega_{l,1})} + |v_{l,2}|^2_{H^1(\Omega_{l,2})}). \end{aligned} \quad (4.44)$$

From (4.35) and (4.44), we obtain,

$$|\chi_2|^2_{H^1(\Omega_{l,2})} \;\leq\; C(|v_{l,1}|^2_{H^1(\Omega_{l,1})} + |v_{l,2}|^2_{H^1(\Omega_{l,2})}).$$

Since the estimate of χ_3 is similar, we obtain, using (4.41), an estimate of v_{new},

$$|v_{new}|^2_{H^1(\Omega_{new})} \leq C\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}. \quad (4.45)$$

From (4.40) and (4.45),

$$\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma \;\leq\; C\mu(\gamma)\sum_{i=1}^{3}|v_{l,i}|^2_{H^1(\Omega_{l,i})}. \quad (4.46)$$

Estimate of $\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma$:
From the definitions of $\tilde{v}$ and v_{new}, we find

$$\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma \;=\; \int_{CD} |\chi_2 + \chi_3|^2 d\sigma \;\leq\; 2||\chi_2||^2_{L^2(CD)} + 2||\chi_3||^2_{L^2(CD)}.$$

From (4.36) and (4.44), we obtain,

$$||\chi_2||^2_{L^2(CD)} \;\leq\; C\mu(\gamma)||\hat{v}_{l,1} - \hat{v}_{l,2}||^2_{H^{\frac{1}{2}}(\Omega_{new}\cap\delta_{1,2})} \;\leq\; C(|v_{l,1}|^2_{H^1(\Omega_{l,1})} + |v_{l,2}|^2_{H^1(\Omega_{l,2})}),$$

and therefore,

$$\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma \leq C\mu(\gamma) \sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})}. \tag{4.47}$$

We now complete the proof of our lemma. Substituting the estimates (4.39), (4.46), and (4.47) into (4.38), it results that

$$\begin{aligned}
\int_\gamma [v]^2 d\sigma \;&\leq\; 3\int_\gamma |v_l - \overline{v}_\gamma|^2 d\sigma \;+\; 3\int_\gamma |v_{new} - \overline{v}_\gamma|^2 d\sigma \;+\; 3\int_\gamma |\tilde{v} - v_{new}|^2 d\sigma \\
&\leq\; C\mu(\gamma)|v_l|^2_{H^1(\Omega_l)} \;+\; C\mu(\gamma)\sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})} \;+\; C\mu(\gamma)\sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})} \\
&\leq\; C\mu(\gamma)\;(\;|v_l|^2_{H^1(\Omega_l)} \;+\; \sum_{i=1}^{3} |v_{l,i}|^2_{H^1(\Omega_{l,i})}\;),
\end{aligned}$$

where C is a constant not depending on the length of γ. □

4.3.3 Proofs of the main results

We are now ready to state and prove the Poincaré and Friedrichs inequalities for mortar finite elements.

Theorem 4.3. *(Friedrichs inequality) For every $v \in V^h$,*

$$||v||^2_{L^2(\Omega)} \;\leq\; C(diam(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)},$$

where C is a constant independent of $(H_k)_{k=1:K}$, $(h_k)_{k=1:K}$, and K.

Proof. We may assume that no edge of the subregions is parallel to the x- or y-axis. Otherwise, since the number of support lines for the edges is finite for any partition of Ω, we can rotate Ω to obtain the desired property.

Let ℓ_0 be a parallel to the x-axis passing through Ω. The intersection of ℓ_0 and the interface Γ consists of a finite number, $n(y_0)-1$, of points, denoted by P_1, P_2, $\ldots$, $P_{n(y_0)-1}$ in increasing order of their x-coordinates. Let $\{P_0, P_{n(y_0)}\} = \ell \cap \partial\Omega$, such that P_0 is the leftmost of the two points, and let $(\alpha_i(y_0), y_0)$ be the coordinates of P_i, for $i = 0 : n(y_0)$. Then,

$$\begin{aligned} \{(\alpha_0(y_0), y_0),\ (\alpha_{n(y_0)}(y_0), y_0)\} &= \ell \cap \partial\Omega, \\ (\alpha_i(y_0), y_0) \in \Gamma \cap \ell_0, &\quad \forall \quad i = 1 : n(y_0) - 1, \\ \alpha_i(y_0) \le \alpha_{i+1}(y_0), &\quad \forall \quad i = 0 : n(y_0) - 1. \end{aligned}$$

Let $\gamma_{\ell,i}$, for $i = 1 : n(y_0) - 1$, be the nonmortar to which P_i belongs; if P_i is vertex of a subregion, and there are several nonmortars ending at P_i, we can choose $\gamma_{\ell,i}$ arbitrarily among them.

Let $(x, y_0) \in \ell_0$ be an arbitrary point on ℓ_0. Denote by $n(x, y_0)$ the well-defined index with the property:

$$\alpha_{n(x,y_0)}(y_0) \ \le \ x \ < \ \alpha_{n(x,y_0)+1}(y_0).$$

By integrating $\partial v/\partial x$ along ℓ_0 from $(\alpha_0(y_0), y_0)$ to (x, y_0), we obtain:

$$|v(x,y_0) - v(\alpha_0(y_0), y_0)| \ \le \ \sum_{i=0}^{n(x,y_0)} \left| \int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \frac{\partial v}{\partial x}(t, y_0) dt \right| \ + \ \sum_{i=1}^{n(x,y_0)} |[v](\alpha_i(y_0), y_0)|.$$

Since $v \in V^h$ and $(\alpha_0(y_0), y_0) \in \partial\Omega$, we find that $v(\alpha_0(y_0), y_0) = 0$. Using the Schwarz inequality and the previous formula, we obtain:

$$\begin{aligned} |v(x,y_0)| \ \le \ & \sum_{i=0}^{n(x,y_0)} (\alpha_{i+1}(y_0) - \alpha_i(y_0))^{\frac{1}{2}} \Big(\int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)} \left| \frac{\partial v}{\partial x} \right|^2 dt \Big)^{\frac{1}{2}} \\ + \ & \Big(\sum_{i=1}^{n(x,y_0)} \mu(\gamma_{l,i}) \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n(x,y_0)} \frac{1}{\mu(\gamma_{l,i})} \|[v](\alpha_i(y_0), y_0)|^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.48}$$

Squaring both sides of (4.48) and apply Schwarz inequality, the inequality becomes:

$$\begin{aligned} |v(x,y_0)|^2 &\leq 2\Big(\sum_{i=0}^{n(x,y_0)}(\alpha_{i+1}(y_0)-\alpha_i(y_0))\Big)\Big(\sum_{i=0}^{n(x,y_0)}\int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)}\Big|\frac{\partial v}{\partial x}(t,y_0)\Big|^2\,dt\Big) \\ &+ 2\Big(\sum_{i=1}^{n(x,y_0)}\mu(\gamma_{l,i})\Big)\Big(\sum_{i=1}^{n(x,y_0)}\frac{1}{\mu(\gamma_{l,i})}|[v](\alpha_i(y_0),y_0)|^2\Big). \end{aligned}$$

Since

$$\sum_{i=0}^{n(x,y_0)}(\alpha_{i+1}(y_0)-\alpha_i(y_0)) = \alpha_{n(x,y_0)+1}(y_0)-\alpha_0(y_0) \leq \mathrm{diam}(\Omega),$$

and, since, from Lemma 4.6,

$$\sum_{i=1}^{n(x,y_0)}\mu(\gamma_{l,i}) \leq \sum_{i=1}^{n(y_0)}\mu(\gamma_{l,i}) \leq \mathrm{diam}(\Omega),$$

we obtain,

$$\begin{aligned} |v(x,y_0)|^2 &\leq 2\,\mathrm{diam}(\Omega)\sum_{i=0}^{n(x,y_0)}\int_{\alpha_i(y_0)}^{\alpha_{i+1}(y_0)}\Big|\frac{\partial v}{\partial x}(t,y_0)\Big|^2\,dt \\ &+ C\,\mathrm{diam}(\Omega)\sum_{i=1}^{n(x,y_0)}\frac{1}{\mu(\gamma_{l,i})}|[v](\alpha_i(y_0),y_0)|^2. \end{aligned} \tag{4.49}$$

Integrate (4.49) over Ω. The first term is then bounded from above by

$$2(\mathrm{diam}(\Omega))^2\sum_{k=1}^{K}|v_k|^2_{H^1(\Omega_k)}.$$

Let γ be a nonmortar with endpoints of coordinates (x_1,y_1) and (x_2,y_2), and slope λ. Since no edge of the subregions $\{\Omega_i\}_{i=1:N}$ is parallel to the x- or y-axis, then $\lambda\neq 0$ and $\lambda\neq\infty$ and we can write the equation for γ as $x=\frac{y}{\lambda}+b$. When the second term of (4.49) is integrated, the jump of v across γ is integrated over $y_1\leq y\leq y_2$ and $x\geq\frac{y}{\lambda}+b$. Its contribution is equal to

$$\frac{1}{\mu(\gamma)}\int_{y_1}^{y_2}\int_{x\geq\frac{y}{\lambda}+b}\Big|[v](\frac{y}{\lambda}+b,y)\Big|^2\,dxdy \leq \frac{1}{\mu(\gamma)}\,\mathrm{diam}(\Omega)\int_{y_1}^{y_2}\Big|[v](\frac{y}{\lambda}+b,y)\Big|^2\,dy$$

$$= \frac{1}{\mu(\gamma)}\ \mathrm{diam}(\Omega)\sqrt{\frac{\lambda^2}{1+\lambda^2}}\int_\gamma [v]^2 d\sigma$$
$$\leq \mathrm{diam}(\Omega)\ \frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma.$$

As a consequence, after integrating (4.49) over Ω, we find

$$||v||^2_{L^2(\Omega)} \leq 2\ (\mathrm{diam}(\Omega))^2\ \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)} + C\ (\mathrm{diam}(\Omega))^2 \sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma. \tag{4.50}$$

If γ is a side of the subregion Ω_l, then, from Lemma 4.7 and using the notations therein, we have

$$\frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma \ \leq\ C(|v_l|^2_{H^1(\Omega_l)} + \sum_{i=1}^{q(l)} |v_{l,i}|^2_{H^1(\Omega_{l,i})}). \tag{4.51}$$

Recall that $\Omega_{l,i}$ are the subregions with a side opposite γ. When we add (4.51) over all nonmortar sides γ, every term $|v_{l,i}|^2_{H^1(\Omega_{l,i})}$ appears a finite number of times, which is bounded from above independently of $(H_k)_{k=1:K}$, and $(h_k)_{k=1:K}$. Then,

$$\sum_{s\ nonmortar} \frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma \ \leq\ C\sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)}. \tag{4.52}$$

Substituting (4.52) into (4.50), we obtain

$$||v||^2_{L^2(\Omega)} \ \leq\ C\ (\mathrm{diam}(\Omega))^2\ \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)}.$$

□

The next theorem is a variant of the Poincaré inequality. The proof is similar to that of Theorem 4.3 and it is based on Lemma 4.7.

Theorem 4.4. *(Poincaré inequality) For every $v \in V^h$,*

$$||v||^2_{L^2(\Omega)} \ \leq\ C(diam(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)} + C\frac{1}{\sigma(\Omega)}|\int_\Omega v dx|^2,$$

where C is a constant independent of $(H_k)_{k=1:K}$, $(h_k)_{k=1:K}$, and K, and $\sigma(\Omega)$ is the area of Ω.

Proof. The proof follows the steps of the proof of the Friedrichs inequality. We can again assume that no edge of the subregions is parallel to the x- or y-axis.

Let (x_1, y_1) and (x_2, y_2) be arbitrary points in Ω. We evaluate $v(x_1, y_1) - v(x_2, y_2)$ by adding the integral of $\frac{\partial v}{\partial y}$ from (x_1, y_1) to (x_1, y_2) and the integral of $\frac{\partial v}{\partial x}$ from (x_1, y_2) to (x_2, y_2), taking the jumps of v across the interface Γ into account.

We square both sides of the resulting inequality and integrate them twice over Ω, once with respect to (x_1, y_1) and once with respect to (x_2, y_2).

The left hand side becomes

$$\int_\Omega \int_\Omega |v(x_1, y_1) - v(x_2, y_2)|^2 dx_1 dy_1 \, dx_2 dy_2 \;=\; 2\sigma(\Omega)||v||^2_{L^2(\Omega)} - 2|\int_\Omega v dx|^2, \quad (4.53)$$

with $\sigma(\Omega)$ on the order of $(\mathrm{diam}(\Omega))^2$.

The right hand side is bounded from above by the sum of the H^1-seminorms of the restrictions of the finite element function v to the subregions $\{\Omega_i\}_{i=1:N}$ and the result of the integration of the squares of all the jumps of v over nonmortar sides. Reasoning as in the previous proof, we obtain a bound for the right hand side,

$$4(\mathrm{diam}(\Omega))^2\sigma(\Omega)\sum_{k=1}^{K}|v_k|^2_{H^1(\Omega_k)} \;+\; C\,(\mathrm{diam}(\Omega))^4 \sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma. \quad (4.54)$$

We use Lemma 4.7 to estimate the second term of (4.54). As in the previous proof, the number of appearances for any term $|v_k|^2_{H^1(\Omega_k)}$ does not depend on $(H_k)_{k=1:K}$, $(h_k)_{k=1:K}$, and K,

$$\sum_{\gamma\ nonmortar} \frac{1}{\mu(\gamma)}\int_\gamma [v]^2 d\sigma \;\le\; C\sum_{k=1}^{K}|v_k|^2_{H^1(\Omega_k)}. \quad (4.55)$$

From (4.53), (4.54), and (4.55), we obtain,

$$\begin{aligned} 2\sigma(\Omega)||v||^2_{L^2(\Omega)} - 2|\int_\Omega v dx|^2 \;&\le\; 4(\mathrm{diam}(\Omega))^2\sigma(\Omega)\sum_{k=1}^{K}|v_k|^2_{H^1(\Omega_k)} \\ &+\; C(\mathrm{diam}(\Omega))^4\sum_{k=1}^{K}|v_k|^2_{H^1(\Omega_k)}. \end{aligned}$$

Dividing both sides by $\sigma(\Omega) \cong (\text{diam}(\Omega))^2$, we find,

$$||v||^2_{L^2(\Omega)} \ \leq \ C(\text{diam}(\Omega))^2 \sum_{k=1}^{K} |v_k|^2_{H^1(\Omega_k)} \ + \ C\frac{1}{\sigma(\Omega)}|\int_{\Omega} v dx|^2.$$

□

4.3.4 Condition Number Estimate

A consequence of the Friedrichs inequality is that the condition number of the Poisson problem solved using mortar finite elements has the same form as in the conforming case.

For simplicity, we assume that, for all the subregions in the partition of the computational domain, H_k is of order H, and h_k is of order h.

Theorem 4.5. *For any $u \in V^h$,*

$$c\sum_{k=1}^{K} ||u||^2_{L^2(\Omega_k)} \ \leq \ a^{\Gamma}(u,u) \ \leq \ C\frac{1}{h^2}\sum_{k=1}^{K} ||u||^2_{L^2(\Omega_k)}, \tag{4.56}$$

where c and C are constants that do not depend on the partition of Ω. The condition number of the stiffness matrix K_{mortar} corresponding to the discrete mortar problem satisfies

$$\kappa(K_{mortar}) \ \leq \ \frac{C}{h^2},$$

where C is independent of the partition of Ω.

Proof. From the definition of the broken norm, $a^{\Gamma}(u,u) = \sum_{k=1}^{K} |u_k|^2_{H^1(\Omega_k)}$. The right inequality of (4.56) follows from the inverse inequality. The left inequality follows from Theorem 4.3, since $||u||^2_{L^2(\Omega)} = \sum_{k=1}^{K} ||u||^2_{L^2(\Omega_k)}$. The estimate of the condition number is a direct consequence of (4.56). □

4.3.5 Extensions to more general geometries

In this section, we extend the construction from Section 4.3.2 to a general partition. The assumption that all subregions are rectangles was only used in Lemma 4.7, to

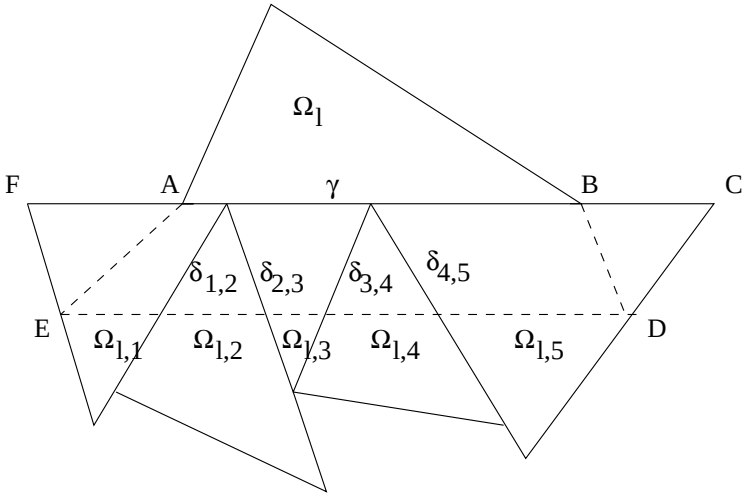


Figure 4.3: Triangular subregions case

estimate of the L^2-norm of the jumps of a mortar function across nonmortars. In particular, we used the assumption in the construction of v_{new}.

We use the same notations, and make a similar construction, as in Section 4.3.2. Once again, we require that the partition of Ω has all the properties required in Section 4.3. Thus, the length of every side of Ω_k is bounded from below by a uniform fraction of H_k, each subregion is obtained from a finite number of reference domains by a uniformly bounded mapping, and the ratio of the diameters of any two adjacent subregions is uniformly bounded. For each partition, a (finite) number of different reference domains might be required.

Opposite the nonmortar γ, we construct a polygon by cutting off part of the union of the subregions, the boundaries of which intersect γ, by a line parallel to γ; see Figure 4.3. Because of the properties just reviewed, we can choose that line such that the length of the side parallel to γ of $\Omega_{l,i} \cap \Omega_{new}$ is bounded from below by a uniform fraction of $H_{l,i}$. As before, we extend the jump of v from $\delta_{i,i+1} \cap \Omega_{new}$ to $\Omega_{l,i}$ or $\Omega_{l,i+1}$ (say to $\Omega_{l,i}$), according to which of $\partial\Omega_{l,i} \cap \gamma$ and $\partial\Omega_{l,i+1} \cap \gamma$ is the largest. We can do this uniformly, using the corresponding reference domain. We obtain a function χ_i vanishing on the sides opposite $\delta_{i,i+1}$, the average of which

over $\gamma \cap \Omega_{l,i}$ is 0, and which satisfies properties similar to (4.35) and (4.36); cf. Section 4.3.2. After this step, the proof can be completed as before.

The construction presented above must be changed slightly, if there are triangles among the subregions $\{\Omega_i\}_{i=1:N}$. For a triangle with only one vertex on γ, we cannot uniformly extend a function defined on one side so that it vanishes on an opposite side; cf. Figure 2, for $\Omega_{l,2} \cap \Omega_{new}$. Instead, we can construct an extension χ_2 of $v_{l,2} - v_{l,1}$ from $\delta_{1,2} \cap \Omega_{new}$ to $\Omega_{l,2} \cap \Omega_{new}$ satisfying

$$||\chi_2||_{H^{\frac{1}{2}}(\delta_{2,3} \cap \Omega_{new})} \leq ||\chi_2||_{H^{\frac{1}{2}}(\delta_{1,2} \cap \Omega_{new})}.$$

Then, we extend $v_{l,3} - v_{l,2} + \chi_2$ from $\delta_{2,3} \cap \Omega_{new}$ to $\Omega_{l,3} \cap \Omega_{new}$, resulting in a function, χ_3, which vanishes on $\delta_{3,4}$, by using the usual construction. We can do this since $\Omega_{l,3} \cap \Omega_{new}$ is a quadrilateral, and the length of the side parallel to $\gamma \cap \Omega_{l,3}$ is, by construction, uniformly bounded from below by $H_{l,3}$. A similar extension, χ_4, is made for the jump of v across $\delta_{4,5}$ on $\Omega_{l,4} \cap \Omega_{new}$. Thereafter, $v_{l,4} - v_{l,3} + \chi_4$ is extended from $\delta_{3,4} \cap \Omega_{new}$ to $\Omega_{l,3} \cap \Omega_{new}$, resulting in a function which vanishes on $\delta_{2,3}$. Finally, the function v_{new} is obtained by adding all the auxiliary functions χ_i to $v\,|_{\Omega_{new}}$.

Chapter 5

The FETI Method for Mortar Finite Elements

5.1 Introduction

The FETI method is an iterative substructuring method using Lagrange multipliers which is actively used in industrial–size parallel codes for solving difficult computational mechanics problems. This method was introduced by Farhat and Roux [61]; a detailed presentation is given in [62], a monograph by the same authors. Originally used to solve second order, self-adjoint elliptic equations, it has later been extended to many other problems, e.g., time-dependent problems, cf. Farhat, Chen, and Mandel [54], plate bending problems, cf. Farhat et al. [55, 59, 94], heterogeneous elasticity problems with composite materials, cf. Farhat and Rixen [109, 110], acoustic scattering and Helmholtz problems, cf. Farhat et al. [57, 58] and Franca et al. [64, 65], and linear elasticity with inexact solvers, cf. Klawonn and Widlund [77]. We note that an algebraic version of the FETI method, the A–FETI method, has also been studied; cf. Park et al. [99, 100]. Recent results have shown that this method is mathematically equivalent to one instance of the FETI method; see Rixen, Farhat, Tezaur, and Mandel [112].

The FETI method has been designed for conforming finite elements, and is based on the decomposition of the computational domain Ω into non-overlapping subdomains. Pointwise continuity across the interface is enforced by using Lagrange multipliers. After eliminating the subdomain variables, the dual problem,

given in terms of Lagrange multipliers, is solved by a projected conjugate gradient (PCG) method. Once an accurate approximation for the Lagrange multipliers has been obtained, the values of the primal variables are obtained by solving a local problem for each subdomain; see Section 5.2 for more details.

It was shown experimentally by Farhat, Mandel, and Roux [60] that applying a certain projection operator in the PCG solver plays a role similar to that of a coarse problem for other domain decomposition algorithms, and that certain variants of the FETI algorithm are numerically scalable with respect to both the problem size and the number of subdomains. Mandel and Tezaur later showed that for a FETI method which employs a Dirichlet preconditioner the condition number grows at most in proportion to $(1 + \log(H/h))^2$, if the decomposition of Ω does not have crosspoints, i.e. the points that belong to the closure of more than two subdomains, and as $(1 + \log(H/h))^3$ in the general case; cf [93, 129]. Here, H is the subdomain diameter and h is the mesh size. Using a different preconditioner, Klawonn and Widlund obtained a FETI method which converges in fewer iterations than the classical FETI method, and proved an upper bound for the condition number of their method which is on the order of $(1 + \log(H/h))^2$; cf.[78].

In Section 5.3, we discuss how the FETI method can be applied for solving self-adjoint elliptic equations discretized by low order mortar finite elements. We introduce three algorithms, corresponding to three different preconditioners for the dual problem suggested in the FETI literature: The Dirichlet preconditioner, which has been used successfully for conforming finite elements, see, e.g., Farhat, Mandel, and Roux [55], a block–diagonal preconditioner used by Lacour [79, 80], and the new preconditioner introduced by Klawonn and Widlund in [78]. We use geometrically nonconforming mortar finite elements of the second generation, for which no continuity conditions are imposed at the vertices of the subdomains.

In Section 5.5, we present numerical comparisons of the performances of the three algorithms, which were implemented in both two and three dimensions. We use geometrically nonconforming mortar finite elements of the second generation, for which no continuity conditions are imposed at the vertices of the subdomains. We conclude that the new preconditioner of Klawonn and Widlund performs best in terms of both iteration and flop counts, and has scalability properties similar to

those of the algorithm with the Dirichlet preconditioner in a conforming case.

For other work on FETI and Lagrange multiplier based substructuring methods for problems with non–matching grids, see Farhat and Géradin [56], and Rixen, Farhat, and Géradin [111] and the references therein.

5.2 The Classical FETI Algorithm

In this section, we review the original FETI method of Farhat and Roux for elliptic problems discretized by conforming finite elements. Our presentation is similar to that of Farhat, Mandel, and Tezaur [94].

For simplicity, we only discuss the Poisson equation with mixed boundary conditions,

$$\left\{ \begin{array}{rcccc} -\Delta u & = & f & \text{on} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega_D \\ \frac{\partial u}{\partial n} & = & 0 & \text{on} & \partial\Omega_N. \end{array} \right. \tag{5.1}$$

On Ω, we consider P_1 or Q_1 finite elements with mesh size h. The finite element mesh is partitioned along mesh lines into N non–overlapping subdomains $\Omega_i \subset \Omega, i = 1 : N$. Since the finite element mesh is conforming, the boundary nodes of the subdomains match across the interface. A subdomain Ω_i is said to be floating if $\partial\Omega_i \cap \partial\Omega_D = \emptyset$, and non–floating otherwise.

For each Ω_i, let K_i and $\hat{f}_i$ be the local stiffness matrix and right hand side, respectively. As in other substructuring methods, the first step of the FETI method consists in eliminating the interior subdomain variables. If K_i is written using blocks obtained by ordering the interior nodes first, and the boundary nodes last, then

$$K_i \;=\; \left(\begin{array}{cc} K_{II,i} & K_{IB,i} \\ K_{BI,i} & K_{BB,i} \end{array} \right),$$

where $K_{BI,i}$ is the transpose matrix of $K_{IB,i}$. Similarly,

$$\hat{f}_i \;=\; \left(\begin{array}{c} \hat{f}_{I,i} \\ \hat{f}_{B,i} \end{array} \right).$$

The Schur complement matrix $S^{(i)}$ and the corresponding right hand side f_i are given by

$$S^{(i)} \;=\; K_{BB,i} - K_{BI,i} K_{II,i}^{-1} K_{IB,i};$$

$$f_i \;=\; \hat{f}_{B,i} - K_{BI,i}K_{II,i}^{-1}\hat{f}_{I,i}.$$

Let $S = diag_{i=1}^{N} S^{(i)}$ be a block–diagonal matrix, and let f be the vector $[f_1, \ldots, f_N]$. Similarly, we denote by u_i the vector of nodal values on $\partial\Omega_i$ and by u the vector $[u_1, \ldots, u_N]$.

If Ω_i is a floating subdomain, then $S^{(i)}$ is a singular matrix and its kernel is generated by a vector Z_i which is equal to 1 at the nodes of $\partial\Omega_i$ and vanishes at all the other interface nodes. Let Z consisting of all the column vectors Z_i. Then

$$\mathrm{Ker}S \;=\; \mathrm{Range}Z. \tag{5.2}$$

Let B be the matrix of constraints which measures the jump of a given vector u across the interface; B will also be referred to as the Lagrange multiplier matrix. Each row of the matrix B is associated to two matching nodes across the interface, and has values 1 and -1, respectively at the two nodes, and zero entries everywhere else. A finite element function with corresponding vector values u is continuous if and only if $Bu = 0$.

For a method without redundant constraints and multipliers, the number of pointwise continuity conditions required at crosspoints, i.e., the points that belong to the closure of more than two subdomains, and therefore the number of corresponding rows in the matrix B, is one less then the number of the subdomains meeting at the crosspoint. There exist several different ways of choosing which conditions to enforce at a crosspoint, all of them resulting in algorithms with similar properties.

An alternative suggested in [109, 110] is to connect all the degrees of freedom at the crosspoints by Lagrange multipliers and use a special scaling, resulting in a method with redundant multipliers.

Let W_i be the space of the degrees of freedom associated with $\partial\Omega_i \setminus \partial\Omega_D$, and let W be the direct sum of all spaces W_i. If $U = \mathrm{Range}B$ is the space of the Lagrange multipliers, then

$$S : W \to W, \quad B : W \to U.$$

By introducing Lagrange multipliers λ for the constraint $Bu = 0$, we obtain a

saddle point Schur formulation of (5.1),

$$\begin{array}{rcl} Su + B'\lambda & = & f \\ Bu & = & 0, \end{array} \tag{5.3}$$

where B' denotes the transpose of B.

5.2.1 Algebraic Formulation

In the FETI method, the primal variable u is eliminated from (5.3) and the resulting equation for the dual variable λ is solved by a projected conjugate gradient method.

We note that S is singular if there exist at least one floating subdomains among the subdomains Ω_i, $i = 1 : N$. Let $S^\dagger : W \to W$ be the pseudoinverse of S, such that $S^\dagger b \in \text{Range}S$, for any $b \perp \text{Ker}S$. A solution for the first equation in (5.3) exists if and only if

$$f - B'\lambda \ \perp \ \text{Ker}S. \tag{5.4}$$

If (5.4) is satisfied, then

$$u \ = \ S^\dagger(f - B'\lambda) + Z\alpha, \tag{5.5}$$

where $Z\alpha$ is an element of $\text{Ker}S = \text{Range}Z$; cf. (5.2) to be determined.

Let $G = BZ$. Substituting (5.5) into the second equation in (5.3), it follows that

$$BS^\dagger B'\lambda \ = \ BS^\dagger f + G\alpha. \tag{5.6}$$

An important role in the FETI algorithm is played by $V \subset U$ defined by $V = \text{Ker}G'$. In other words,

$$V \ = \ \text{Ker}G' \ \perp \ \text{Range}G \ = \ B\text{Range}Z \ = \ B\text{Ker}S. \tag{5.7}$$

Let $P \ = \ I - G(G'G)^{-1}G'$ be the projection onto V. Since $P(G\alpha) = 0$, if P is applied to (5.6), then

$$PBS^\dagger B'\lambda \ = \ PBS^\dagger f. \tag{5.8}$$

It is easy to see that $G'G$ is non–singular, by using the fact that

$$\text{Ker}B \cap \text{Range}Z \ = \ \text{Ker}B \cap \text{Ker}S \ = \ \emptyset.$$

We now return to the necessary condition (5.4). From (5.2), we obtain that (5.4) is equivalent to $f - B'\lambda \perp \text{Range}Z$, which leads to

$$Z'(f - B'\lambda) \;=\; 0$$

and therefore to

$$G'\lambda \;=\; Z'f. \tag{5.9}$$

Let $F = BS^{\dagger}B'$, $d = BS^{\dagger}f$, and $e = Z'f$. We concluded that we have to solve the dual problem (5.8) for λ, subject to the constraint (5.9); with the new notations,

$$PF\lambda \;=\; Pd; \tag{5.10}$$

$$G'\lambda \;=\; e. \tag{5.11}$$

After that an approximate solution for λ is found, the primal variable u can be obtained as follows: Solving for α in (5.6),

$$\alpha \;=\; (G'G)^{-1}G'(F\lambda - d).$$

Then u can be obtained from (5.5) after solving a Neumann or a mixed boundary problem on each floating and nonfloating subdomain, respectively, corresponding to a vector multiplication by $S^{\dagger}$.

The main part of the FETI algorithm consists of solving (5.10) for the dual variable λ, which is done by a projected conjugate gradient (PCG) method. Since λ must also satisfy the constraint (5.11) let

$$\lambda_0 = G(G'G)^{-1}e \tag{5.12}$$

be the initial approximation. Then $G'\lambda_0 = e$ and $\lambda - \lambda_0 \in \text{Ker}G' = V$. If all the increments $\lambda_k - \lambda_{k-1}$, i.e., the search directions, are in V, then (5.11) will be satisfied.

One possible preconditioner for (5.10) is of the form PM, where

$$M \;=\; BSB'. \tag{5.13}$$

When a vector multiplication by M is performed, N independent Dirichlet problems have to be solved in each iteration step. Therefore, M is known as the Dirichlet preconditioner. We note that the Schur complement matrix S is never computed explicitly, since only the action of S on a vector is needed.

Mandel and Tezaur [93] have shown that the condition number of this FETI method has a condition number which grows polylogarithmically with the number of nodes in each subdomain,

$$\kappa(PMPF) \leq C \left(1 + \log \frac{H}{h}\right)^3,$$

where C is a positive constant independent of h, H. If there are no crosspoints in the partition of Ω, then the bound improves to $(1 + \log(H/h))^2$.

We conclude this section by presenting the PCG algorithm:

Projected Preconditioned Conjugate Gradient Iteration (PCG)

$\lambda_0 = G(G'G)^{-1}e$, $r_0 = Pd - PF\lambda_0$, $n = 1$
while $(Mr_{n-1}, r_{n-1}) \geq tol$
  $w_{n-1} = Pr_{n-1}$
  $z_{n-1} = Mw_{n-1}$
  $y_{n-1} = Pz_{n-1}$
  $\beta_n = (y_{n-1}, r_{n-1})/(y_{n-2}, r_{n-2})$ $\qquad (\beta_1 = 0)$
  $p_n = y_{n-1} + \beta_n p_{n-1}$ $\qquad (p_1 = y_0)$
  $\alpha_n = (y_{n-1}, r_{n-1})/(Ap_n, p_n)$
  $\lambda_n = \lambda_{n-1} + \alpha_n p_n$
  $r_n = r_{n-1} - \alpha_n A p_n$
  $n = n + 1$
end

In contrast to the CG algorithm of Section 3.1.1, in each iteration step of the PCG algorithm, the residual and the search directions are projected onto the space V, i.e. $w_{n-1} = Pr_{n-1}$ and $y_{n-1} = Pz_{n-1}$. This projection step plays the role of a coarse problem which is solved in each iteration, and is the reason why the FETI method is numerically scalable.

5.3 The FETI Algorithm for Mortars

As we have seen in Section 5.2, in the classical FETI algorithm the computational domain Ω is partitioned into nonoverlapping subregions, multiple degrees of freedom are introduced for the matching nodes across the interface, and pointwise continuity across the interface is enforced by a Lagrange multiplier matrix B. This methodology is very similar to that used in Section 2.7, where a saddle point formulation for the mortar finite element method has been introduced.

Therefore, the FETI method can be applied without any algorithmic changes for a mortar finite element discretization of Ω, using the nonoverlapping partition $\{\Omega_i\}_{i=1:N}$ introduced for the mortar element in Section 2.2. To keep the presentation clear, we assume each subregion Ω_i has $diam(\Omega_i)$ of order H and has diameter of the mesh size of order h. The matrix S is again a block–diagonal matrix $diag_{i=1}^{N} S^{(i)}$, where the local Schur complement matrices $S^{(i)}$ are obtained from the finite element discretizations on individual subregions. As in the case of conforming finite elements, we have to solve the problem

$$\begin{array}{lcl} Su + B'\lambda = f, \\ Bu = 0. \end{array}$$

We note that, in contrast to the conforming finite element case, the matrices $S^{(i)}$ can now be built from different discretizations on different subregions Ω_i. Also, the matrix B should enforce mortar conditions across the interface, instead of pointwise continuity.

The dual problem is obtained as in Section 5.2.1. It results in solving

$$PF\lambda = Pd, \tag{5.14}$$

with a PCG method, with the initial approximation λ_0 given by (5.12) and with all the search direction in V. The price we pay for the inherent flexibility of the mortar finite elements is related to the fact that the matrix B is more complicated in the mortar case, compared to that of the classical FETI method with conforming finite elements.

The matrix B has one block, B_γ, for each nonmortar side γ. Using the matrix formulation of the mortar conditions (2.3) and (2.4), we let M_γ and N_γ be the

matrices which multiply the nonmortar and mortar nodal values in the mortar conditions across γ, respectively. Then B_γ consists of the columns of M_γ and $-N_\gamma$ for the nodes of γ and those on the mortars opposite γ, and has zero columns corresponding to all the other nodes.

We note that all the mortar conditions are associated with the interior nodes on the nonmortar sides. Therefore, the problem of choosing the crosspoints constraints which existed for conforming elements does not exist in the mortar case.

In our numerical experiments, we have implemented three different preconditioners suggested in the FETI literature for the dual problem (5.14). In Sections 5.3.1–5.3.3. we present each of them briefly.

5.3.1 The Dirichlet Preconditioner

A possible preconditioner for the dual problem (5.14) is the Dirichlet preconditioner (5.13) which performs very well for conforming finite elements,

$$PM = PBSB'.$$

5.3.2 A Block-diagonal Preconditioner

In [79, 80], Lacour suggested another preconditioner designed specifically for a mortar version of the FETI algorithm, and which does not have counterpart in the conforming case.

Let $diagB_\gamma B'_\gamma$ be the block–diagonal matrix which has a block $B_\gamma B'_\gamma$ of size equal the number of interior nodes on γ for each nonmortar γ. We note that $diagB_\gamma B'_\gamma$ is the block–diagonal part of the matrix BB'. The non-zero entries of $B_\gamma B'_\gamma$ which do not belong to the diagonal blocks are of two types. Some correspond to Lagrange multipliers associated to the first and last interior points of the nonmortars. Others occur because there are nodal basis functions associated to points on the mortar sides, the support of which intersects more than one nonmortar. However, there are relatively few such non-zero entries.

The preconditioner $P\overline{M}$ is defined as follows:

$$P\overline{M} \;=\; P(diagB_\gamma B'_\gamma)^{-1}BSB'(diagB_\gamma B'_\gamma)^{-1}. \tag{5.15}$$

5.3.3 A New Preconditioner

Recently, Klawonn and Widlund [78] have discussed a FETI method for elliptic problems with heterogeneous coefficients, discretized by conforming finite elements. In the case of no coefficient jump, which reduces to our Poisson problem, the new preconditioner suggested in [78] has the form

$$P\widehat{M} \;=\; P(BB')^{-1}BSB'(BB')^{-1}. \tag{5.16}$$

They also established the following upper bound for the condition number of their FETI method, which is valid for all cases, including when the partition contains crosspoints:

$$\kappa(P\widehat{M}PF) \;\leq\; C\left(1+\log\frac{H}{h}\right)^2.$$

The new preconditioner $P\widehat{M}$ also provides new insight about the connection between the FETI method and the Neumann-Neumann methods, in particular the balancing method.

In the same paper, it is proven that the preconditioner $\widehat{M}$ with a minimal number of pointwise continuity conditions at the crosspoints, and therefore of Lagrange multipliers, results in a similar algorithm as the FETI method with redundant Lagrange multipliers of Farhat and Rixen [109, 110]. Since the Lagrange multipliers in the mortar case are not associated with the vertices of the subregions, the method with redundant multipliers cannot be implemented for mortars.

The new preconditioner of Klawonn and Widlund, which depends only on the Lagrange multiplier matrix B, can be used for the mortar FETI algorithm. We note that the matrix BB' is non–singular in the mortar case, since the rank of B is equal to the number of Lagrange multipliers. To see this, it is enough to take the minor of B consisting of the columns corresponding to the interior nodes of the nonmortars. This results in a block–diagonal matrix $diagM_\gamma$ which is non–singular, since each block M_γ is a diagonally dominant matrix.

5.4 Convergence Analysis of the New Preconditioner for Conforming Finite Elements

In [78], Klawonn and Widlund give a convergence analysis for a FETI method with a new preconditioner, for solving elliptic problems with heterogeneous coefficients. In this section, we present a slightly changed version of their theory for the much simpler case of the Poisson equation. Since the proofs presented here require only algebra, this theory can also be applied for the FETI method with mortars, if the matrix B is changed accordingly; see Section 5.3.

The first step is an elementary functional analysis result. Let $< \cdot, \cdot >$ be the euclidean inner product on the Lagrange multiplier space U. From (5.7), we find that an equivalent definition for V is $V \subset U$ is $V \perp B\mathrm{Ker}S$, i.e.,

$$V \;=\; \{v \in U \mid \; < v, B\theta >= 0, \;\; \forall\, \theta \in \mathrm{Ker}S\}. \tag{5.17}$$

Let $(\cdot, \cdot)_V$ be an inner product on V, and let $|| \cdot ||_V$ be the corresponding norm on V. We denote by V' be dual space of V with respect to $(\cdot, \cdot)_V$.

Lemma 5.1. *Let $P : U \to V$ be the orthogonal projection onto V with respect to $< \cdot, \cdot >$, and let $F, \widehat{M} : U \to V$ be linear symmetric operators on V. Assume there exists positive constants C_i, $i = 1 : 4$, such that*

$$C_1||\lambda||^2_{V'} \;\leq\; < F\lambda, \lambda > \;\leq\; C_2||\lambda||^2_{V'}, \;\forall\, \lambda \in V'; \tag{5.18}$$

$$C_3||v||^2_V \;\leq\; < \widehat{M}v, v > \;\leq\; C_4||v||^2_V, \;\forall\, v \in V. \tag{5.19}$$

Then

$$\kappa(P\widehat{M}PF) \;\leq\; \frac{C_2C_4}{C_1C_3}.$$

Proof. This is Lemma 3.1 of Mandel and Tezaur [93], and a proof of it can be found therein. □

Choosing the appropriate norm on V is crucial. Following [78], let

$$||v||^2_V =< \widehat{M}v, v >=< SB'(BB')^{-1}v, B'(BB')^{-1}v >= |B'(BB')^{-1}v|^2_S. \tag{5.20}$$

The matrix BB' is invertible since B has full rank, and has more columns, i.e., the total number of nodes on the interface, than rows, i.e., the number of interior nodes on the nonmortar sides.

To prove that $||\cdot||_V$ is a norm, let $v \in V$ such that $||v||_V = 0$. Then, from (5.20), it follows that $B'(BB')^{-1}v = \zeta \in \text{Ker}S$. From (5.17) we obtain $< v, B\zeta >= 0$. Since $B\zeta = v$ we obtain that $< v, v >= 0$, and therefore $v = 0$, which proves that $|| \cdot ||_V$ is indeed a norm.

The next step is to show that the left inequality of (5.18) holds for $C_1 = 1$, i.e.,

$$||\lambda||^2_{V'} \;\leq\; < F\lambda, \lambda > \quad \forall\, \lambda \in V'.$$

The proof which we present here is somewhat different than that from [78], and highlights the relationship between $||\lambda||^2_{V'}$ and $< F\lambda, \lambda >$. We begin by deriving an equivalent formula for $< F\lambda, \lambda >$.

Lemma 5.2. *Let $F = BS^{\dagger}B'$. Then,*

$$< F\lambda, \lambda > \;=\; \sup_{v \in V} \frac{< \lambda, v >^2}{\inf_{Bu=0} |B'(BB')^{-1}v + u|^2_S} \tag{5.21}$$

Proof. In [93], the following elementary result has been proven:

$$< F\lambda, \lambda > \;=\; \sup_{w \perp KerS} \frac{< \lambda, Bw >^2}{|w|^2_S}; \tag{5.22}$$

see also [78]. For completeness, we include a short proof of (5.22), as appears in [78].

We note that $S^{\dagger} = S^{-1/2}S^{-1/2}$ on Range$S$. Since $\lambda \in V'$ means that $B'\lambda \in$ Range$S$, we obtain that $S^{-1/2}B'\lambda \in$ Range$S$. Then, from the definition of F, we find that

$$\begin{aligned} < F\lambda, \lambda > \;&=\; < S^{\dagger}B'\lambda, B'\lambda > \;=\; ||S^{-1/2}B'\lambda||^2 \\ &=\; \sup_{\hat{w} \in RangeS} \frac{< B'\lambda, S^{-1/2}\hat{w} >^2}{< \hat{w}, \hat{w} >} \;=\; \sup_{w \in RangeS} \frac{< \lambda, Bw >^2}{|w|^2_S} \\ &=\; \sup_{w \perp KerS} \frac{< \lambda, Bw >^2}{|w|^2_S}. \end{aligned}$$

From (5.22), it follows easily that

$$< F\lambda, \lambda > \;\; = \;\; \sup_{w \in W} \frac{< \lambda, Bw >^2}{|w|_S^2}, \tag{5.23}$$

since W is an orthogonal sum of subspaces Ker$S$ and (Ker$S$)$_\perp$, and $\lambda \in V \perp B$Ker$S$.

We now use (5.23) to prove (5.21). Let $\lambda \in V$ and $w \in W$. In another step originally used in [93], we prove the existence of $z \in$ Ker$S$ such that $B\tilde{w} \in V$, where $\tilde{w} = w + z$. In other words, $B\tilde{w} = Bw + Bz \perp BKerS$, which leads to

$$< Bz, B\theta > \;\; = \;\; - < Bw, B\theta > \quad \forall\ \theta \in \text{Ker}S. \tag{5.24}$$

Applying the Riesz representation theorem to the quotient space Ker$S$/(Ker$S \cap$ Ker$B$) with inner product $< B\ \cdot, B\ \cdot >$, we obtain that a solution $z \in$ Ker$S$ for (5.24) exists, and satisfies $||Bz|| \leq ||Bw||$.

Let $v = B\tilde{w} \in V$. Since $< \lambda, Bz >= 0$, we obtain

$$< \lambda, Bw > \;\; = \;\; < \lambda, B\tilde{w} > - < \lambda, Bz > \;\; = \;\; < \lambda, B\tilde{w} > \;\; = \;\; < \lambda, v > .$$

Since $z \in$ Ker$S$, it follows that $< Sw, w >=< S\tilde{w}, \tilde{w} >$, and therefore

$$\frac{< \lambda, Bw >^2}{|w|_S^2} \;\; = \;\; \frac{< \lambda, B\tilde{w} >^2}{|\tilde{w}|_S^2}.$$

From (5.23) we obtain that

$$\begin{aligned} < F\lambda, \lambda > \;\; &= \;\; \sup_{\tilde{w} \in W, B\tilde{w} \in V} \frac{< \lambda, B\tilde{w} >^2}{|\tilde{w}|_S^2} \\ &= \;\; \sup_{v \in V} \frac{< \lambda, v >^2}{\inf_{B\tilde{w}=v} |\tilde{w}|_S^2} \\ &= \;\; \sup_{v \in V} \frac{< \lambda, v >^2}{\inf_{Bu=0} |B'(BB')^{-1}v + u|_S^2}. \end{aligned}$$

For the last step, we used the fact that $B(B'(BB')^{-1}v) = v$, and that therefore

$$\tilde{w} - B'(BB')^{-1}v \;\; = \;\; u \;\; \in \;\; \text{Ker}B,$$

for all $\tilde{w} \in W$ such that $B\tilde{w} = v$. □

Using the result of the previous lemma, it is easy to see that three of the four inequalities from (5.18) and (5.19) are satisfied.

Lemma 5.3. *The following bound holds,*

$$||\lambda||^2_{V'} \;\leq\; <F\lambda, \lambda> \quad \forall\ \lambda \in V'. \tag{5.25}$$

Proof. By definition,

$$||\lambda||^2_{V'} \;=\; \sup_{v\in V} \frac{<\lambda, v>^2}{|B'(BB')^{-1}v|^2_S}. \tag{5.26}$$

The inequality (5.25) follows from (5.21) and (5.26) if we note that

$$\inf_{Bu=0} |B'(BB')^{-1}v + u|^2_S \;\leq\; |B'(BB')^{-1}v|^2_S.$$

□

In the next theorem, we present the only estimate needed to obtain an upper bound for the condition number of the FETI method.

Theorem 5.1. *The bound*

$$|B'(BB')^{-1}v|^2_S \;\leq\; C(1+log(H/h))^2 \inf_{Bu=0} |B'(BB')^{-1}v + u|^2_S, \tag{5.27}$$

is equivalent to

$$|B'(BB')^{-1}Bw|^2_S \;\leq\; C(1+log(H/h))^2|w|^2_S, \quad \forall\ w \in W \text{ such that } Bw \in V. \tag{5.28}$$

If these bounds hold, then the condition number of the FETI method satisfies

$$\kappa(P\widehat{M}PF) \;\leq\; C(1+log(H/h))^2. \tag{5.29}$$

Proof. To show that (5.27) and (5.28) are equivalent, we first assume that (5.27) is satisfied. Let $w \in W$ such that $Bw = v \in V$. Then $w = B'(BB')^{-1}v + u_w$, with $u_w \in W$ and $Bu_w = 0$. It is easy to see that (5.28) follows from

$$\begin{aligned} |B'(BB')^{-1}Bw|^2_S &= |B'(BB')^{-1}v|^2_S \\ &\leq C \inf_{Bu=0} |B'(BB')^{-1}v + u|^2_S \\ &\leq C|B'(BB')^{-1}v + u_w|^2_S \\ &= C(1+log(H/h))^2|w|^2_S. \end{aligned}$$

Reciprocally, assume that (5.28) holds. Let $v \in V$ arbitrary. Let $u_v \in W$ satisfy $Bu_v = 0$ and

$$\inf_{Bu=0} |B'(BB')^{-1}v + u|_S^2 \;=\; |B'(BB')^{-1}v + u_v|_S^2.$$

Then (5.27) follows from (5.28) for $w = B'(BB')^{-1}v + u_v$.

To prove (5.29), we use Lemma 5.1. From (5.20), it follows that the inequality (5.19) holds for $C_3 = C_4 = 1$, and, from Lemma 5.3, we find that the left inequality of (5.18) holds for $C_1 = 1$.

From (5.27), and using (5.18) and (5.26), we conclude that

$$< F\lambda, \lambda > \;\leq\; C(1 + log(H/h))^2 ||\lambda||_{V'}^2,$$

Thus, the right inequality of (5.18) holds for $C_2 = C(1 + log(H/h))^2$.

Therefore, from Lemma 5.1, we obtain

$$\kappa(P\widehat{M}PF) \;\leq\; C(1 + log(H/h))^2.$$

□

A bound of the form (5.28) was proven, for conforming finite elements, by Klawonn and Widlund [78]. The key for their proof was that the operator $I - B'(BB')^{-1}B$ takes any function $w \in W$, not necessarily continuous, into a continuous function, $w - B'(BB')^{-1}Bw$, the nodal values of which are the average of the nodal values of v at the matching nodes. The same operator appears in the study of the balancing method; see [78]. This led to new connections between the FETI and the balancing methods.

For mortar finite elements, we note that $w - B'(BB')^{-1}Bw$ is a mortar function, for any $w \in W$, not necessarily a mortar function. However, we do not have a similar interpretation of the nodal values of $w - B'(BB')^{-1}Bw$ in terms of those of w, as in the conforming finite element case. In addition, the operator $I - B'(BB')^{-1}B$ is no longer local, since, for the mortar case, the matrix $(BB')^{-1}$ may be a full matrix.

5.5 Numerical Comparisons

In this section, we present numerical results for the FETI method for mortar finite elements. We have tested each of the three preconditioners of Sections 5.3.1–5.3.3. Our experiments were done in MATLAB, and for two and three dimensions.

Our interests were two–fold:

- to compare the convergence performances of the different preconditioners for the FETI algorithms with mortars, based on the iteration counts;
- to study the differences between the FETI methods for conforming finite elements using pointwise continuity conditions and mortar conditions across the interface, being interested in the iteration counts, as well as the computational costs;

5.5.1 Experiments in 2–D

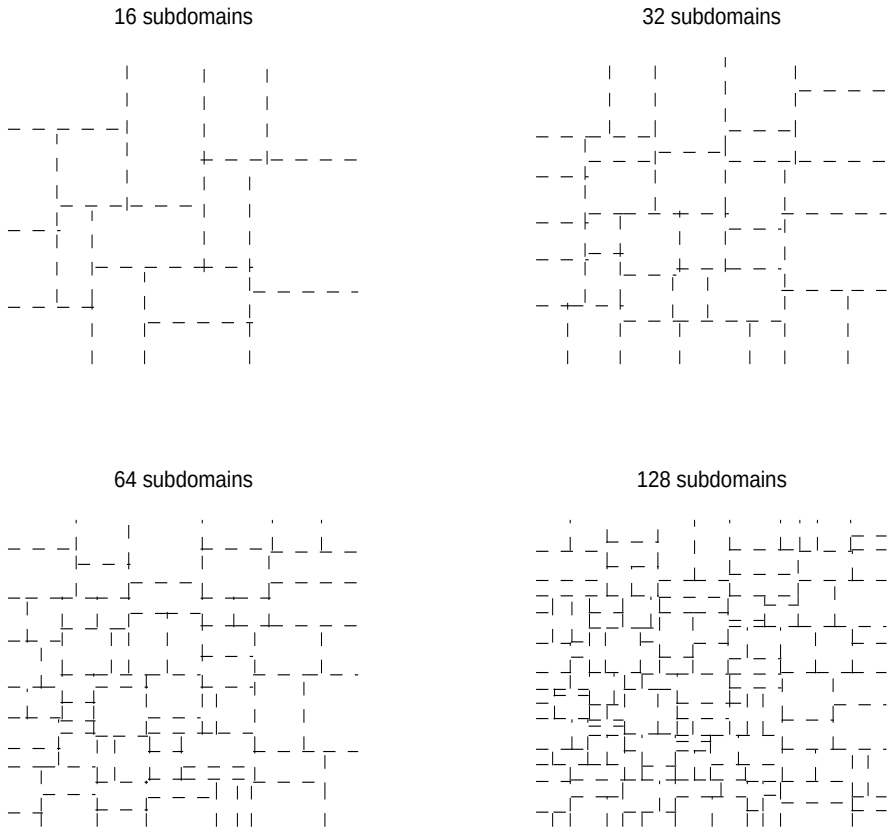
As the model problem in 2–D, we chose the Poisson equation on the unit square $\Omega = [0,1]^2$ with zero Dirichlet boundary conditions. The right hand was chosen such that an exact solution of the problem was known.

The computational domain Ω was partitioned into 16, 32, 64, and 128 geometrically nonconforming rectangular subregions, respectively; see Figure 5.1. On each subregion, we considered Q_1 elements of mesh size h, and, to make the comparisons easier, all the subregions considered had diameters of the same order, H. For each partition, the number of nodes on each edge, H/h, has been taken to be, *on average*, 4, 8, 16, and 32. Across the partition interface Γ the meshes do not necessarily match. A saddle point formulation of the problem has been used, and mortar conditions have been enforced across Γ.

For our tests with conforming finite elements, Ω was partitioned into 4×4, 6×6, 8×8, and 11×11 congruent squares, and Q_1 elements were used in each square. The meshes match across Γ, and, non-redundant pointwise continuity conditions, or mortar conditions, are used across Γ for comparison purposes.

We report the iteration count and the flop count of the algorithms. The PCG iteration was stopped when the residual norm had decreased by a factor of 10^{-7}.

Figure 5.1: Geometrically nonconforming partitions of Ω



We begin by discussing the differences between the new preconditioner $\widehat{M}$, cf. (5.16), the preconditioner $\overline{M}$, cf. (5.15), and the Dirichlet preconditioner M, cf. (5.13), and by presenting some implementation details.

In general, the preconditioners $\widehat{M}$ and $\overline{M}$ require some extra work in comparison to M. In each iteration step, when we multiply a vector by the preconditioner, we have to solve two systems with the matrix BB', and $diag B_\gamma B'_\gamma$, respectively. The non-zero entries outside the diagonal blocks are relatively few; see Section 5.3.2 for details on the nature of their occurrences.

It is easy to see that $diag B_\gamma B'_\gamma$ has a band of order H/h, the number of interior nodes on an arbitrary nonmortar. The matrix BB' is also banded, but in this case the band depends on the ordering of the nodes on the interface, and it is possible to have order $1/h$. Therefore, multiplying a vector by $(BB')^{-1}$ is potentially an expensive operation.

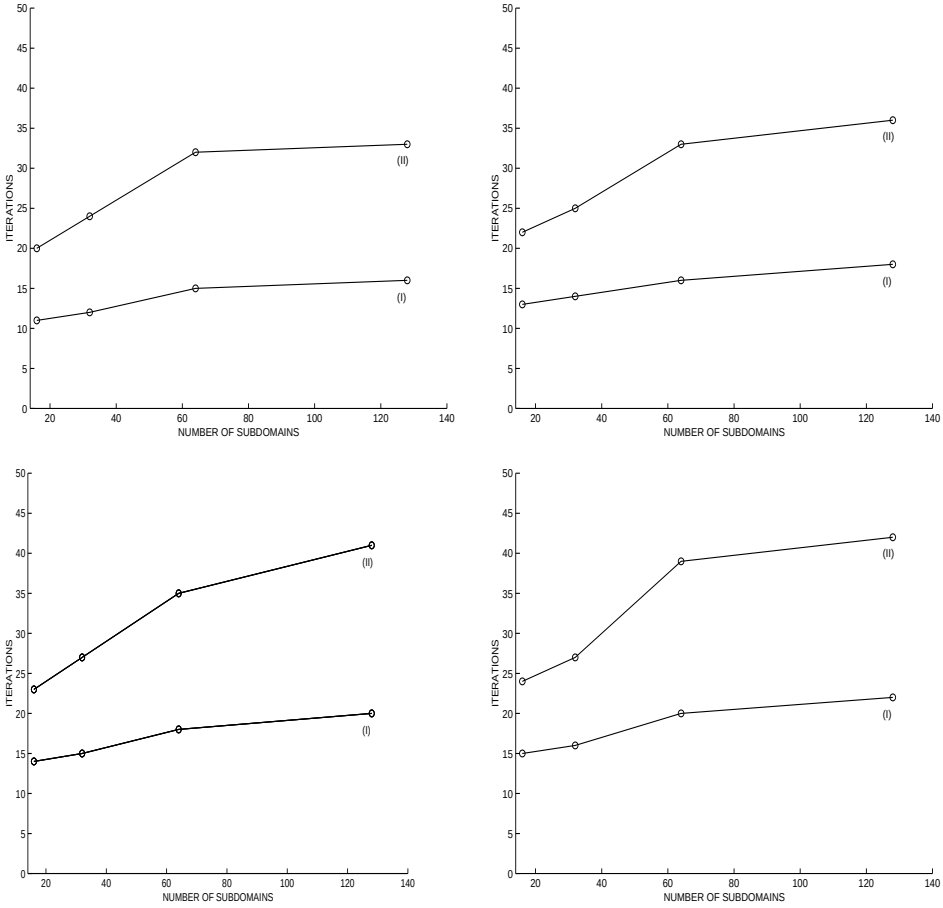
Table 5.1: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

		(I)		(II)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS	Iter	MFLOPS
16	4	11	9.0e–1	20	1.5e+0	108	7.4e+0
16	8	13	1.2e+1	22	1.9e+1	290	2.3e+2
16	16	14	2.2e+2	23	3.4e+2	406	5.6e+3
16	32	15	4.4e+3	24	6.8e+3	486	1.3e+5
32	4	12	2.4e+0	24	4.3e+0	223	3.7e+0
32	8	14	2.7e+1	25	4.6e+1	438	7.4e+2
32	16	15	4.9e+2	27	8.5e+2	620	1.8e+4
32	32	16	1.1e+4	27	1.9e+4	692	4.4e+5
64	4	15	7.2e+0	32	1.3e+1	487	1.9e+2
64	8	16	7.4e+1	33	1.4e+2	1071	4.3e+3
64	16	18	1.3e+3	35	2.4e+3	1725	1.1e+5
64	32	20	3.0e+4	39	5.7e+4	2130	2.9e+6
128	4	16	1.6e+1	33	2.9e+1	1107	9.0e+2
128	8	18	1.7e+2	36	3.2e+2	1413	2.0e+4
128	16	20	3.1e+3	41	6.0e+3	1761	2.5e+5
128	32	22	7.1e+4	42	1.3e+5	–	–

However, the sparsity pattern of BB' plays a very important role. Our results show that the costs for applying $(BB')^{-1}$ and $(diagB_\gamma B'_\gamma)^{-1}$ are comparable and relatively small compared to the costs for other operations performed during one iteration, e.g., multipling a vector by the Schur complement, or by its pseudoinverse. These costs further decrease if the Cholesky factorizations of BB' and $diagB_\gamma B'_\gamma$ are computed only once, and the Cholesky factors are stored. Then, solving systems with BB' or $diagB_\gamma B'_\gamma$ only amounts to one back and one forward solve. Moreover, the improvement of the iteration count offsets this extra cost easily.

We note that we do not compute the Schur complements explicitly, nor their pseudoinverses, but only the stiffness matrices for each subdomain.

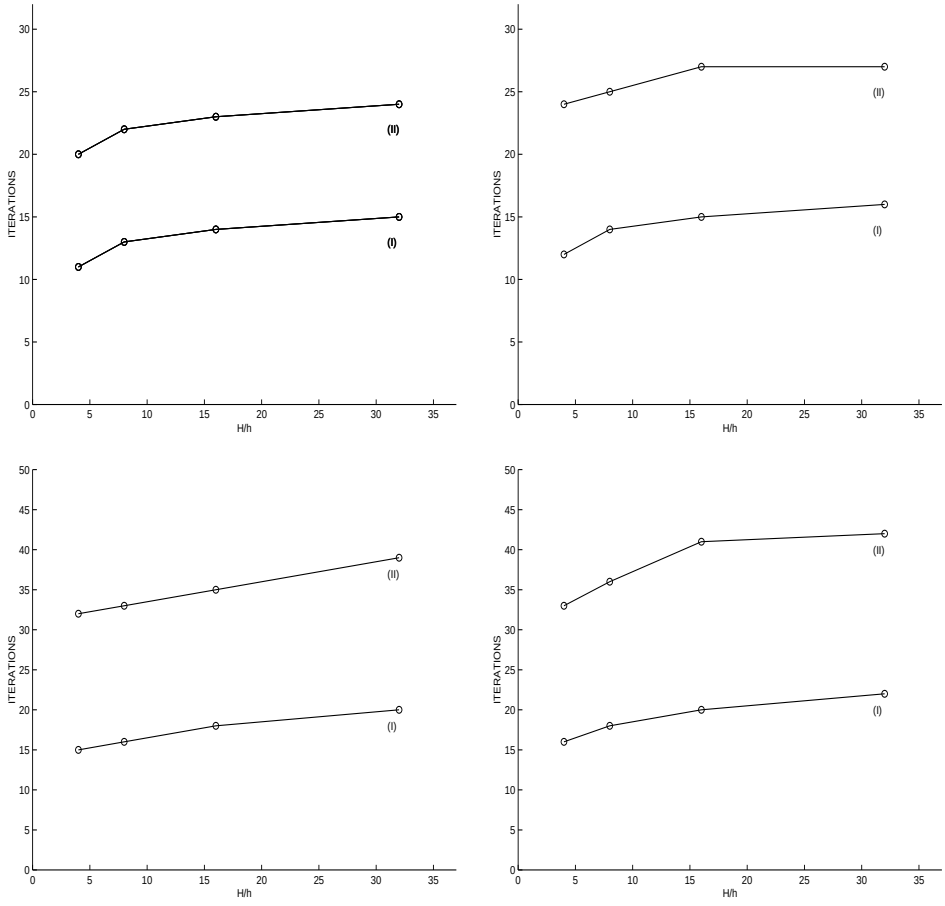
Figure 5.2: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner. Upper left : $H/h = 4$, Upper right : $H/h = 8$, Lower left : $H/h = 16$, Lower right : $H/h = 32$.



To multiply a vector by a Schur complement matrix, we solve, in each subregion, a Poisson problem with Dirichlet boundary conditions.

To multiply a vector by $S^{\dagger}$, we solve one Poisson problem with mixed boundary conditions in each non–floating subregion, and with Neumann boundary conditions in each floating subregion; see, e.g., [48]. We note that we only need to store the interior–boundary and boundary–boundary blocks of the local stiffness matrix and the Cholesky factor of the interior–interior block, which is symmetric and positive

Figure 5.3: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner. Upper left : $Ns = 16$, Upper right : $Ns = 32$, Lower left : $Ns = 64$, Lower right : $Ns = 128$, $Ns =$ Number of subdomains.



definite.

To have a uniquely solvable problem on the floating subregions, we require the solution of the local Neumann problem to be orthogonal to Ker$S$, i.e. to the constant functions on the subregion. In two dimensions, a simple way of enforcing this orthogonality condition is by adding a Lagrange multiplier, and storing the LU components of the extended stiffness matrix.

In Table 5.1, we report the iteration and flop counts for the new preconditioner

Table 5.2: Geometrically conforming partition, Matching grids and continuity constraints across the interface: (I) = New Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

		(I)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS
16	4	7	5.9e–1	18	1.3e+0
16	8	9	7.7e+0	19	1.5e+1
16	16	10	1.6e+2	20	3.0e+2
16	32	11	3.7e+3	21	6.6e+3
36	4	9	2.3e+0	23	5.2e+0
36	8	10	2.3e+1	24	5.0e+1
36	16	11	4.2e+2	26	9.1e+2
36	32	13	1.1e+4	28	2.2e+4
64	4	9	4.6e+0	25	1.2e+1
64	8	10	4.7e+1	25	1.1e+2
64	16	11	1.0e+3	27	2.2e+3
64	32	13	2.1e+4	28	4.6e+4
121	4	9	9.7e+0	25	2.5e+1
121	8	10	8.1e+1	25	1.8e+2
121	16	11	1.4e+3	27	3.1e+3
121	32	13	5.1e+4	28	1.0e+5

$\widehat{M}$, cf. (5.16), the preconditioner $\overline{M}$, cf. (5.15), and the Dirichlet preconditioner M, cf. (5.13). The Dirichlet preconditioner M does not yield a numerically scalable method and converges only in hundreds of iterations. The iteration count appears to be linear in H/h, and the computational costs are one to two orders of magnitude greater than for the other preconditioners. The new preconditioner $\widehat{M}$ has scalability properties similar to those of M in the conforming case. When the number of nodes on each subdomain edge (i.e. H/h) is fixed and the number of subdomains, N, is increased, the iteration count shows only a slight growth, cf. also Figure 5.2. When H/h is increased while the partition is kept unchanged, the increase in the number of iterations is quite satisfactory and very similar to that of the conforming case, cf. also Figure 5.3. Note that the number of iterations and the

computational cost for $\widehat{M}$ are about half of that for $\overline{M}$. This suggests that dropping the non-zero diagonal terms of BB' relaxes the weak continuity conditions for mortar finite elements more than is optimal.

As a comparison, we also present iteration counts for a geometrically conforming case. The computational domain Ω is partitioned in a geometrically conforming fashion into 16, 36, 64, and 121 squares.

Table 5.3: Geometrically conforming partition, matching grids and mortar conditions across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

		(I)		(II)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS	Iter	MFLOPS
16	4	6	5.4e–1	6	5.4e–1	6	1.0e+0
16	8	6	5.6e+0	7	6.4e+0	18	2.0e+1
16	16	6	1.1e+2	8	1.4e+2	38	6.8e+2
16	32	7	2.6e+3	8	2.9e+3	47	1.7e+4
36	4	8	2.1e+0	9	2.3e+0	14	5.6e+0
36	8	8	1.9e+1	10	2.3e+1	33	9.1e+1
36	16	9	3.6e+2	10	3.9e+2	49	2.0e+3
36	32	11	9.5e+3	12	1.0e+4	56	5.3e+4
64	4	8	4.4e+0	10	5.2e+0	20	1.5e+1
64	8	9	4.4e+1	11	5.2e+1	25	1.1e+2
64	16	11	1.0e+3	11	1.0e+3	52	5.2e+3
64	32	12	1.9e+4	13	2.1e+4	61	1.1e+5
121	4	10	1.1e+1	13	1.4e+1	48	5.9e+1
121	8	11	8.9e+1	13	1.0e+2	78	6.5e+2
121	16	12	1.5e+3	15	1.8e+3	116	1.4e+4
121	32	14	5.4e+4	17	6.4e+4	121	4.2e+5

Across the interface Γ, we can use pointwise continuity conditions, as in the classical FETI method, and the preconditioners $\widehat{M}$ and M, or we can use mortar conditions and the preconditioners $\widehat{M}$ and $\overline{M}$.

When pointwise continuity is enforced, the new preconditioner $\widehat{M}$ converges in less than half the number of iterations required for the Dirichlet preconditioner.

In this case, BB' is very close to twice the identity matrix, and therefore almost no extra work is required when a system with the matrix BB' is solved. This observation is supported by a comparison of the flop counts; cf. Table 5.2. Also, the vector matrix multiplication by $B'(BB')^{-1}B$ is very easy to compute, since it is close to an operator from the balancing algorithm; see Section 5.4. The PCG algorithm with the new preconditioner can be written such that only the product of a vector by $B'(BB')^{-1}B$ and not by $(BB')^{-1}$ needs to be computed.

When mortar conditions are used, computing the matrix B is simple for matching nodes, in particular no computations of integrals resulting from the mortar conditions are necessary. From Table 5.3, it is easy to see that $\widehat{M}$ and $\overline{M}$ behave similarly in terms of computational costs per iteration and iteration counts, which are just slightly higher for the block diagonal preconditioner $\overline{M}$. However, the Dirichlet preconditioner M performs, once again, poorly, when mortar conditions are used across the interface, and does not seem to yield a scalable method.

We finally note that there is little difference in terms of iteration count and computational costs between the use of the new preconditioner $\widehat{M}$ when continuity constraints or mortar conditions are used; cf. Table 5.2 and Table 5.3.

5.5.2 Experiments in 3–D

As the model problem in 3–D, we choose the Poisson equation on the unit cube $\Omega = [0,1]^3$ with zero Dirichlet boundary conditions. The right hand side is chosen such that the exact solution is known.

The computational domain Ω has been partitioned into 8, 16, and 32 nonconforming parallelipipeds, respectively; see Figure 5.5 and Figure 5.6, at the end of this section, for the 8 subdomains partition of Ω, and the corresponding choice of the nonmortar faces. We chose these partitions such that in each case there exist floating subdomains, i.e., interior subdomains.

The subdomains have diameter of order H, and Q_1 elements of mesh size h are used in each subdomain. The number of nodes on each edge is, *on average*, 4, 8, and 16. Across the partition interface Γ the meshes do not match, and mortar conditions for three dimensional elements are enforced by using a Lagrange

multipliers matrix; see Section 2.3.

We report the iteration count and the flop count of the algorithms. The PCG iteration was stopped when the residual norm had decreased by a factor of 10^{-7}.

Table 5.4: Geometrically nonconforming partition, non–matching grids across the interface, 3–D: (I) = New Preconditioner, (II) = Block–diagonal Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

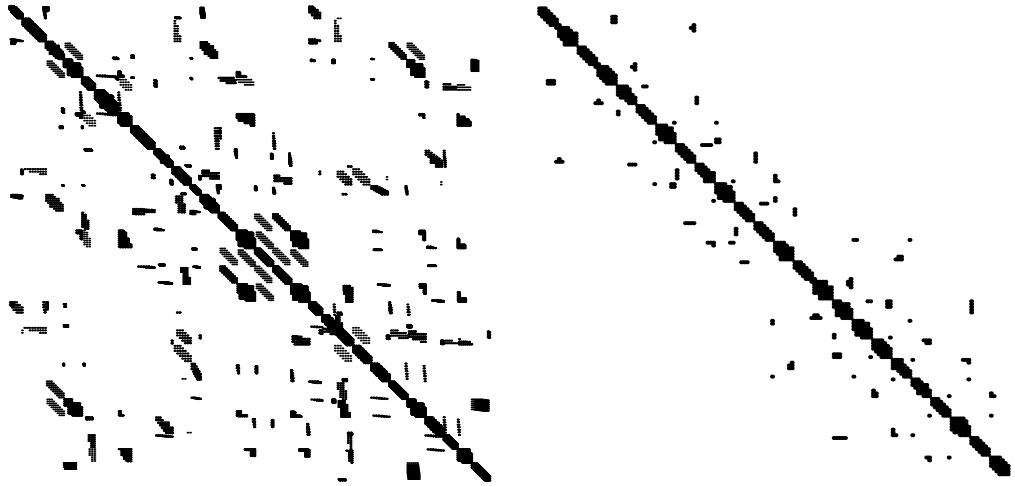
		(I)		(II)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS	Iter	MFLOPS
8	4	12	1.7e+0	37	4.6e+0	797	9.8e+1
8	8	15	4.0e+1	40	9.1e+1	11104	2.4e+4
8	16	18	1.2e+3	45	3.0e+3	–	–
16	4	14	5.5e+0	42	1.2e+1	2978	8.0e+2
16	8	17	1.1e+2	45	2.2e+2	–	–
16	16	20	3.0e+3	56	7.4e+3	–	–
32	4	17	2.1e+1	52	3.2e+7	4751	2.8e+3
32	8	19	4.0e+2	64	6.5e+2	–	–

We tested the performance of the three preconditioners for mortars, the new preconditioner $\widehat{M}$, cf. (5.16), the preconditioner $\overline{M}$, cf. (5.15), and the Dirichlet preconditioner M, cf. (5.13); cf. Table 5.4.

We do not compute the Schur complements explicitly, but only store those components of the stiffness matrices which are relevant for the multiplication of a vector by the Schur complement matrix and by the pseudoinverse of the Schur complement. In the mortar case, the stiffness matrices for each subdomain may be different, unless there are repeated subdomains. Therefore, in the three dimensional case, the memory requirements when implementing mortar finite element methods are significant. However, a similar situation also occurs for the conforming finite element case, if the partition of the computational domain into subdomains is unstructured and if there are few repeated subdomains.

As in the 2–D case, the Dirichlet preconditioner M does not yield a numerically scalable method, requiring hundreds or thousands of iterations to converge.

Figure 5.4: Non–zero entries of BB'. Left: 3–D partition, 16 subdomains, 8 nodes on subdomain edge. Right: 2–D partition, 16 subdomains, 8 nodes on subdomain edge.



The new preconditioner $\widehat{M}$ seems to be numerically scalable. The number of iterations grows slowly when the number of nodes on each subdomain edge is increased, for a fixed number of subdomains. A similar behavior is noticed when the number of nodes in each subdomain if fixed, and the number of subdomains is increased.

Using the preconditioner $\overline{M}$ results into a method which converges in about three times as many iterations than when $\widehat{M}$ is used. We recall that, in the 2–D case, the number of iterations for the method with $\overline{M}$ was only about twice as large as that for $\widehat{M}$. This result is due to the fact that, in the three dimensional case, there are many nodes, e.g., the nodes on the wire baskets of the subdomains, which influence several nonmortar conditions. Therefore, the block diagonal structure of BB' is no longer as dominant, and many non–zero entries of BB' need to be dropped; cf. Figure 5.4.

However, the flop count for the algorithm with $\overline{M}$ is not three times larger than the flop count for the algorithm with $\widehat{M}$. This suggests that the costs of applying $(BB')^{-1}$ are significant in the three dimensional case. In Table 5.5, we present the costs of applying $(BB')^{-1}$ twice during an iteration step, relative to

Table 5.5: Relative complexity study of one iteration step for the new preconditioner

Ns	H/h	MFLOPS due to BB' (A)	MFLOPS per iteration (B)	Ratio (A)/(B)
8	4	3.2e–2	1.5e–1	.22
8	8	5.9e–1	2.7e+0	.22
8	16	6.8e+0	6.9e+1	.10
16	4	1.3e–1	3.9e–1	.34
16	8	2.1e+0	6.6e+0	.32
16	16	2.2e+1	1.5e+2	.15
32	4	6.9e–1	1.3e+0	.55
32	8	1.1e+1	2.1e+1	.55

the total flop count for one iteration step. As expected, this cost is significant, but it decreases when the number of nodes on the edges of the subdomains increases, since, in that case, the costs of multiplying a vector by the Schur complement and the pseudoinverse of the Schur complement grow faster than the cost of applying $(BB')^{-1}$.

Figure 5.5: Partition of the unit cube, 8 subdomains

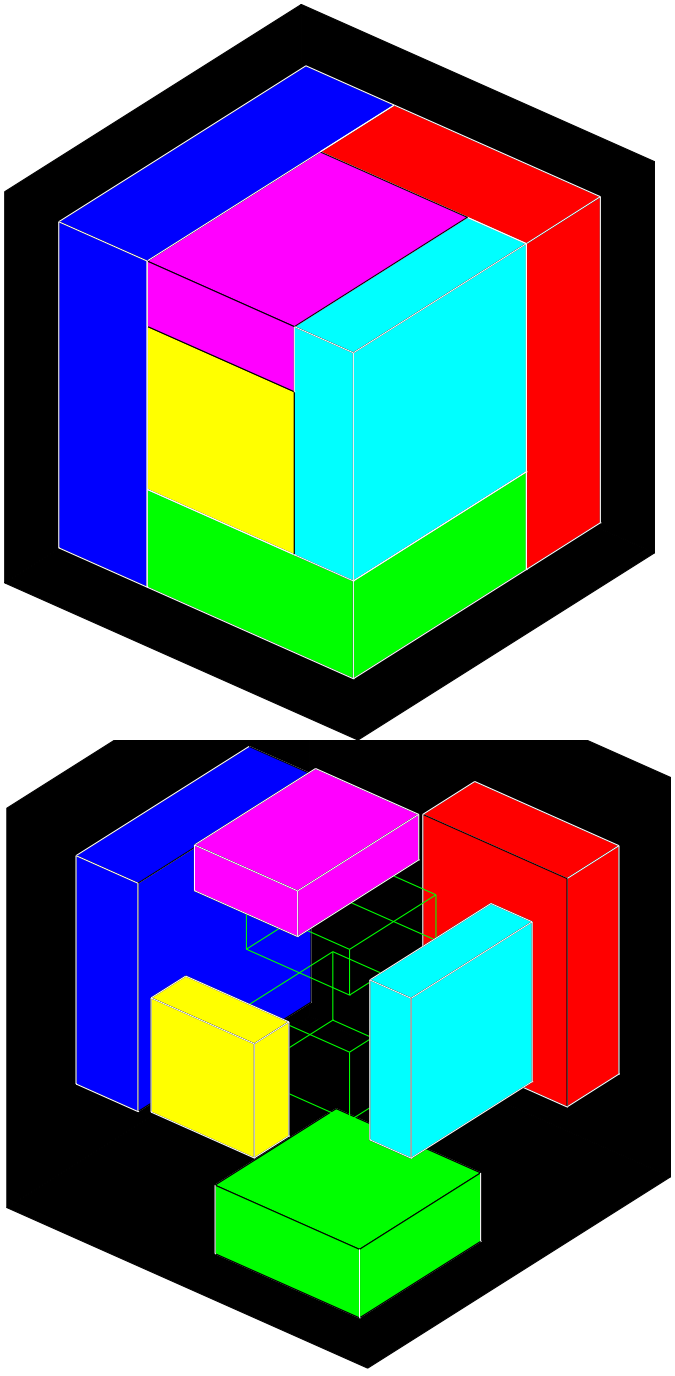
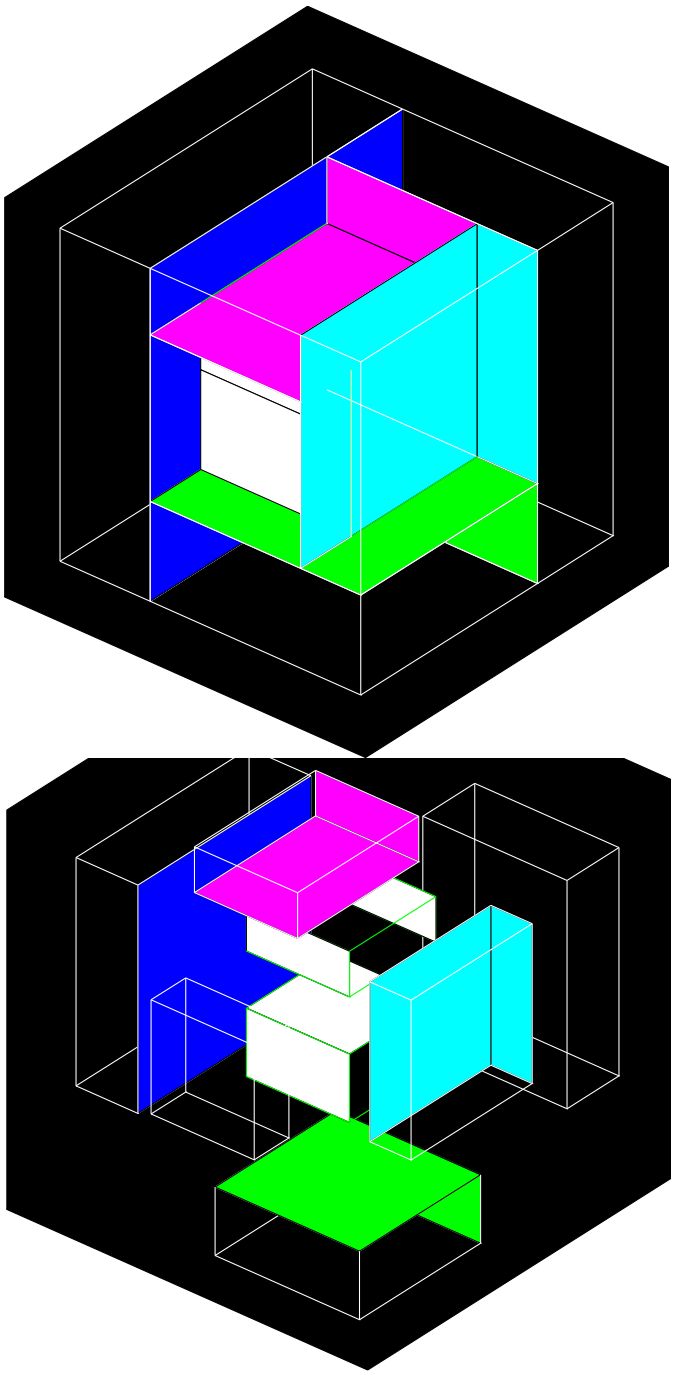


Figure 5.6: Nonmortar faces for the partition the unit cube, 8 subdomains



Chapter 6

A Balancing Algorithm for Mortar Finite Elements

6.1 Introduction

The balancing method is a hybrid nonoverlapping Schwarz domain decomposition method from the Neumann-Neumann family; see Section 3.3.2. It is easy to implement, and uses a natural coarse space of minimal dimension which allows for an unstructured partition of the computational domain. The condition numbers of the resulting algorithms depends only polylogarithmically on the number of degrees of freedom in each subregion.

The balancing method was introduced by Mandel and Brezina [89, 90, 91] for elliptic problems, and was extended to mixed finite elements by Cowsar, Mandel, and Wheeler [42].

The Neumann-Neumann algorithms have been analyzed by Dryja and Widlund [50, 132]; see also [53] for a general study which includes the three dimensional case. Several Neumann-Neumann algorithms for mortars have also been suggested; see, e.g. Dryja [46], Le Tallec [84], and Le Tallec, Sassi, and Vidrascu [85].

In this chapter, we propose an algorithm which can be regarded as the extension of the balancing method for the case when mortar finite elements are used to discretize $H^1(\Omega)$. As in the classical case, every local space is associated with a subregion from the partition of the computational domain. The values of the mortar function on a nonmortar side depend on, but are not equal to, its values on

the mortar sides opposite the nonmortar. To account for this dependence, we chose to work with local spaces defined on extended subregions, instead of using local spaces and local solvers defined on each subregion. In this regard, our algorithm is different from the classical Neumann-Neumann methods.

For each subregion Ω_i, the extended region $\widetilde{\Omega}_i$ is the union of Ω_i and all the neighboring subregions of Ω_i which have a mortar side opposite $\partial\Omega_i$. We note that, in our local solvers, we consider only the nodes on the mortars opposite $\partial\Omega_i$, in addition to the nodes on $\partial\Omega_i$. Therefore, the dimensions of our local problem solved on an extended subregion is at most a fixed multiple (depending on the relative mesh size of the neighboring subregions) of the dimension of the corresponding local problem solved on the individual subregion.

A basis for the coarse space is given by counting functions associated with each local space. The same construction exists in the classical balancing algorithm.

Our algorithm is based on a a similar philosophy to the method suggested by Dryja [46], since for both algorithms the Schwarz framework is used to study their convergence properties. There are major differences between the two algorithms, since our goal was to obtain an algorithm for mortar finite elements which is close to the classical balancing algorithm for conforming finite elements. In the method of Dryja, the local spaces are associated with pairs of opposite nonmortar and mortar sides. Also, a basis for the coarse space is given by the mortar functions which are equal to one at all the genuine degrees of freedom associated with a subregion.

For our algorithm, we obtain an upper bound for the condition number of the form $C(1 + log(H/h))^4$, which is valid for both the geometrically conforming and nonconforming cases. The same bound is valid, in the geometrically nonconforming case, for Dryja's algorithm, as well as for other algorithms, e.g., the iterative substructuring method of Achdou, Maday, and Widlund [3], in the geometrically nonconforming case. We note that, in the geometrically conforming case, which, in some sense, is closer to the conforming finite element case, an upper bound of the form $C(1 + log(H/h))^2$ for the condition number has been established for both algorithms mentioned before. Our algorithm has been designed to be applied in the nonconforming case, and therefore we are satisfied with the quasi–optimal

bound obtained for that case.

The rest of the chapter is structured as follows. In the next section, we present the balancing algorithm for conforming finite elements, and, in Section 6.3, we modify it to obtain an algorithm for mortar finite elements. After we prove several technical results in Section 6.4, we use the abstract Schwarz theory in Section 6.5 to obtain a condition number estimate for our algorithm in the geometrically conforming case. InSection 6.6, we conclude the chapter, and our thesis, by extending the result to the geometrically nonconforming case.

6.2 The Classical Balancing Method

Following Widlund [133], we present the original balancing method of Mandel and Brezina [89], and use abstract Schwarz theory of Section 3.2 to derive a condition number estimate for it. To keep the presentation simple, our model problem will be Poisson's equation with mixed boundary conditions on the boundary of Ω, a bounded polygonal domain in $\mathtt{R}^2$; cf. (5.1).

On Ω, we consider P_1 or Q_1 finite elements with mesh size h. The finite element mesh is partitioned along mesh lines into N non–overlapping subregions Ω_i, $i = 1 : N$. A subdomain Ω_i is floating if $\partial\Omega_i \cap \partial\Omega_D = \emptyset$, and non–floating otherwise. The variational formulation of the problem is given by (1.6); see Section 1.3. As in the FETI method, see Section 5.2, the unknowns interior to the subregions are eliminated, and a Schur complement formulation of the problem is obtained. The reduced system is a Schur complement system, and will be solved in the space $\tilde{V}$ of piecewise harmonic functions on the subregions Ω_i.

Let w_Γ be a piecewise linear function on Γ, the interface of the partition of Ω, and let $\mathcal{H}(w_\Gamma) \in \tilde{V}$ be the discrete harmonic extension of w_Γ. Let $a : \tilde{V} \times \tilde{V} \to \mathtt{R}$ be the inner product on $\tilde{V}$, given by

$$a(w_\Gamma, w_\Gamma) \;=\; a(\mathcal{H}(w_\Gamma), \mathcal{H}(w_\Gamma)), \;=\; |\mathcal{H}(w_\Gamma)|^2_{H^1(\Omega)}.$$

6.2.1 The coarse space and the local spaces

An important role in the balancing algorithm is played by the counting functions associated with the boundary nodes of each subregion. In the classical algorithm, they indicate to how many boundaries of subregions a node on the interface belongs. Let $\nu_i : \Gamma \rightarrow \mathtt{R}$, be the counting function corresponding to Ω_i defined as follows: ν_i is a piecewise linear function with nodal values given by

$$\nu_i(x) = \begin{cases} \text{number of sets } \partial\Omega_j \text{ with } x \in \partial\Omega_j & \text{if} \quad x \in \partial\Omega_i; \\ 0, & \text{if} \quad x \notin \partial\Omega_i; \\ 1, & \text{if} \quad x \in \partial\Omega_i \cap \partial\Omega_D. \end{cases}$$

Let $\nu_i^\dagger$ be the pseudo inverse of ν_i, i.e., $\nu_i^\dagger$ is a piecewise linear function on Γ, with nodal values

$$\nu_i^\dagger(x) = \begin{cases} 1/\nu_i(x), & \text{if } \nu_i(x) \neq 0; \\ 0, & \text{else.} \end{cases}$$

We note that $\nu_i^\dagger$ form a partition of unity, i.e., their sum equals one at each node of $\Gamma \cup \partial\Omega$.

The coarse space V_0 has minimal dimension, i.e., has only one degree of freedom in each subregion. For each floating subregion Ω_i, the harmonic extension of $\nu_i^\dagger$, $\mathcal{H}(\nu_i^\dagger)$, is a basis function for $\tilde{V}_0$. For a non-floating subregion Ω_j, we set the values of $\nu_j^\dagger$ at the nodal points of $\partial\Omega_j \cap \partial\Omega_D$ to zero. Then, the basis function corresponding to Ω_j is the harmonic extension of $\nu_j^\dagger$.

The bilinear form a_0 is exact, i.e., $a_0(\cdot,\cdot) = a(\cdot,\cdot)$, and therefore the coarse space solver T_0 is a projection. To emphasize this, we write $P_0 = T_0$, and note that

$$a(P_0 u, v_0) \;=\; a(u, v_0), \quad \forall\, v_0 \in V_0.$$

In particular, $a(P_0 u, \nu_i^\dagger) \;=\; a(u, \nu_i^\dagger)$, and therefore,

$$a((I - P_0)u, \nu_i^\dagger) \;=\; 0, \quad \forall\, u \in \tilde{V}. \tag{6.1}$$

Each local space V_i, $i = 1 : N$, is associated with a subregion Ω_i, and is embedded in $\tilde{V}$, i.e., $V_i \subset \tilde{V}$. Thus V_i consists of piecewise harmonic functions which vanish at all the interface nodes on $\Gamma \setminus \partial\Omega_i$. Let $\tilde{I}_h$ be the interpolation

operator onto the piecewise linear finite element space $\tilde{V}$, which preserves the nodal values. The bilinear form $\tilde{a}_i(\cdot,\cdot)$ is given by

$$\tilde{a}_i(v_i, w_i) \;=\; a_{\Omega_i}(\mathcal{H}(\tilde{I}_h(\nu_i v_i)), \mathcal{H}(\tilde{I}_h(\nu_i w_i))). \tag{6.2}$$

The projection-like operator T_i is defined by

$$\tilde{a}_i(T_i u, v_i) \;=\; a(u, v_i), \quad \forall\; v_i \in V_i. \tag{6.3}$$

Since $\mathcal{H}(\nu_i \nu_i^\dagger)$ is equal to 1 on any floating subregion Ω_i, it follows that

$$\tilde{a}_i(T_i(u), \nu_i^\dagger) \;=\; \int_{\Omega_i} \nabla\mathcal{H}(\tilde{I}_h(\nu_i T_i(u))) \cdot \nabla\mathcal{H}(\nu_i \nu_i^\dagger)\; dx \;=\; 0.$$

Therefore, (6.3) is solvable if u satisfies

$$a(u, \nu_i^\dagger) = 0, \tag{6.4}$$

for every $\nu_i^\dagger$ corresponding to a floating subregion. Such functions are called *balanced functions*. Then, from (6.1), we can conclude that any function in Range$(I - P_0)$ is balanced.

Moreover, if Ω_i is a floating subregion, the local problem (6.3) corresponds to a pure Neumann problem, which is not uniquely solvable. We make the solution unique by requiring $T_i u$ to be orthogonal to the null space of $\tilde{a}_i$, i.e.,

$$\int_{\Omega_i} \mathcal{H}(\tilde{I}_h(\nu_i T_i(u)))\; dx \;=\; 0. \tag{6.5}$$

6.2.2 Condition number estimate

The balancing method is a hybrid method, combining features of the additive and multiplicative Schwarz methods. It can be regarded as an additive Schwarz method on the local spaces, after the coarse space component has been projected out. The preconditioned operator is

$$T_{bal} \;=\; P_0 + (I - P_0)(T_1 + \ldots + T_N)(I - P_0), \tag{6.6}$$

and the error propagation operator is

$$(I - P_0)(I - \sum_{i=1}^{N} T_i)(I - P_0).$$

Since the balancing method is a hybrid method, we expect its condition number to be smaller than that of the pure additive Schwarz algorithm, corresponding to the Neumann-Neumann operator

$$T_{N-N} \;=\; P_0 + T_1 + \ldots + T_N, \tag{6.7}$$

and larger than that of the multiplicative algorithm. We note that a Neumann-Neumann algorithm with the spaces and the approximate solvers considered here does not converge.

An important observation is that the convergence analysis for the balancing operator T_{bal} can be obtained from the analysis of the additive operator T_{N-N}, which can be done in the additive Schwarz framework; cf. Section 3.2. This result appears in Mandel and Brezina [89], and will be presented here for the sake of completeness.

Lemma 6.1. *If the coarse space operator P_0 is a projection, then the condition number of the balancing operator T_{bal} is bounded from above by the condition number of the additive Schwarz operator T_{N-N} restricted to the space Range$(I - P_0)$.*

Proof. Since P_0 is a projection, $a(w,w) = a(P_0w, P_0w) + a((I-P_0)w, (I-P_0)w)$, and, from (6.6), it follows that

$$\begin{aligned}
a(T_{bal}w, w) &= a(P_0w, w) \;+\; a((I-P_0)(T_1 + \ldots + T_N)(I-P_0)w, w) \\
&= a(P_0w, w) \;+\; a((T_1 + \ldots + T_N)(I-P_0)w, (I-P_0)w) \\
&= a(P_0w, P_0w) \;+\; a((P_0 + T_1 + \ldots + T_N)(I-P_0)w, (I-P_0)w) \\
&= a(P_0w, P_0w) \;+\; a(T_{N-N}(I-P_0)w, (I-P_0)w).
\end{aligned}$$

If $\lambda_{min}(T_{N-N}) \le 1 \le \lambda_{max}(T_{N-N})$ are the smallest and the largest eigenvalues of T_{N-N} restricted to Range$(I-P_0)$, then

$$\begin{aligned}
a(T_{bal}w, w) &\le a(P_0w, P_0w) \;+\; \lambda_{max}(T_{N-N})a((I-P_0)w, (I-P_0)w) \\
&\le \lambda_{max}(T_{N-N})a(w,w).
\end{aligned}$$

A similar inequality holds for $\lambda_{min}(T_{N-N})$, and we conclude that

$$\kappa(T_{bal}) \;\le\; \kappa(T_{N-N}).$$

□

We now return to the analysis of T_{N-N}. Let $u \in \tilde{V}$, and let $u_i \in V_i$ be the local space components of u, given by

$$u_i \;=\; \mathcal{H}(I_h(\nu_i^\dagger u)). \tag{6.8}$$

Since $\nu_i^\dagger$ form a partition of unity, we obtain

$$\sum_{i=1}^{N} u_i \;=\; \mathcal{H}(I_h(\Big(\sum_i \nu_i^\dagger\Big)u)) \;=\; \mathcal{H}(I_h(u)) \;=\; u, \tag{6.9}$$

since u is a piecewise harmonic function.

From the definition of the approximate bilinear forms, (6.2), we find

$$\tilde{a}_i(u_i, u_i) \;=\; |\mathcal{H}(\tilde{I}_h(\nu_i \nu_i^\dagger u))|^2_{H^1(\Omega_i)} \;=\; |u|^2_{H^1(\Omega_i)},$$

and therefore

$$\sum_{i=1}^{N} \tilde{a}_i(u_i, u_i) \;=\; |u|^2_{H^1(\Omega)}. \tag{6.10}$$

Then, from (6.9) and (6.10), we find that Assumption 1 of the abstract Schwarz theory is satisfied for $C_0 = 1$. We note that, a coarse space component does not exist in the decomposition of u. This will change in the mortar case; see (6.55).

A routine proof using a coloring argument shows that $\rho(\epsilon)$, the spectral radius of Assumption 2, is uniformly bounded by a constant independent of the number of subregions, i.e., $\rho(\epsilon) \leq C$.

In Section 3.2.1, we have noted, following Widlund [133], that it is enough to prove the inequality from Assumption 3 only for the functions from Range(T_i), instead of for every local function in v_i. This observation is crucial for the balancing method, as well as for our mortar balancing algorithm. We conclude this section with the proof of the existence of a constant C such that

$$a(u_i, u_i) \;\leq\; C(1 + log(H/h))^2 \tilde{a}_i(u_i, u_i), \quad \forall\, u_i \in \text{Range}(T_i), \quad \forall i = 1 : N. \tag{6.11}$$

This proofs follows the same steps as in Widlund [133].

Once (6.11) is proven, then Assumption 3 is established, with a parameter ω of order $C(1 + log(H/h))^2$. From Theorem 3.1, it results that

$$\kappa(T_{N-N}) \;\leq\; C(1 + log(H/h))^2,$$

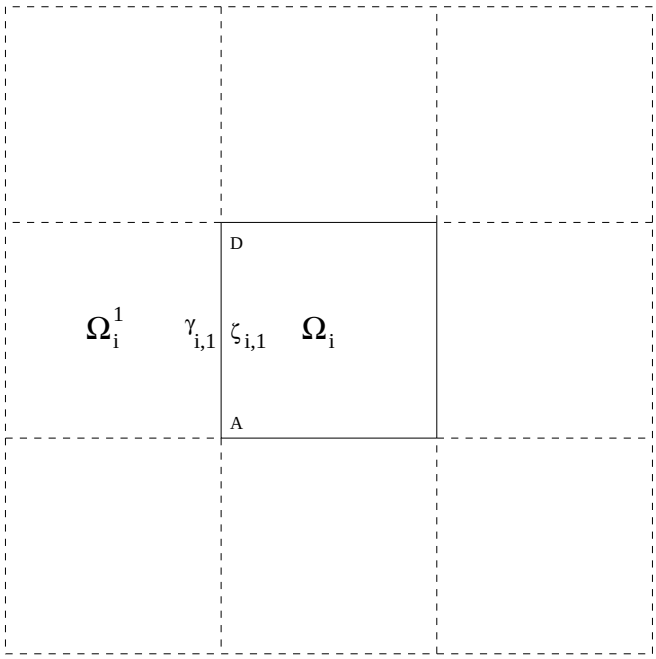


Figure 6.1: Neighboring subregions, conforming finite element case

and, from Lemma 6.1, we obtain that

$$\kappa(T_{bal}) \ \leq \ C(1 + log(H/h))^2.$$

We return to proving (6.11). Let $u_i \in \text{Range}(T_i)$. From (6.2), it follows that (6.11) is equivalent to

$$a(u_i, u_i) \ \leq \ C(1 + log(H/h))^2 |\mathcal{H}(\tilde{I}_h(\nu_i v_i))|^2_{H^1(\Omega)}. \tag{6.12}$$

We note that the support of u_i consists of the union of Ω_i and its neighbors. We only discuss the case when Ω_i and its neighbor have one side in common. The case when Ω_i and one of its neighbors have one vertex in common can be treated similarly.

For clarity, we introduce notations similar to those we will use in the mortar case; see Figure 6.1. Let Ω_i^1 be a neighboring subregion of Ω_i, and let $\gamma_{i,1}$ be a side of Ω_i^1 opposite to the side $\zeta_{i,1}$ of Ω_i. Let A and D be the end points of $\zeta_{i,1}$.

Then $\nu_i(A) = \nu_i(D) = 4$, and ν_i is equal to 2 at all the interior nodes on $\zeta_{i,1}$. Let $\tilde{I}_h(\nu_i u_i)_A$ be the function which is equal to $\nu_i u_i$ at A and vanishes at all the other nodes. The function $\tilde{I}_h(\nu_i u_i)_D$ is defined similarly.

It is easy to see that

$$u_i \mid_{\overline{\gamma}_{i,1}} \;=\; \frac{\tilde{I}_h(\nu_i u_i) \mid_{\overline{\zeta}_{i,1}}}{2} \;-\; \frac{\tilde{I}_h(\nu_i u_i)_A + \tilde{I}_h(\nu_i u_i)_D}{4}.$$

We note that u_i vanishes at all the nodes of $\partial\Omega_i^1$ outside $\overline{\gamma}_{i,1}$ and is harmonic on Ω_i^1. From Theorem 1.5 it follows that

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i^1)} &= |u_i|^2_{H^{1/2}(\partial\Omega_i^1)} \\
&\leq C\Big(|\tilde{I}_h(\nu_i u_i) \mid_{\overline{\zeta}_{i,1}}|^2_{H^{1/2}(\partial\Omega_i)} + |\tilde{I}_h(\nu_i u_i)_A|^2_{H^{1/2}(\partial\Omega_i^1)} \\
&\qquad +|\tilde{I}_h(\nu_i u_i)_D|^2_{H^{1/2}(\partial\Omega_i^1)}\Big) \\
&\leq C(1+log(H/h))^2\Big(|\mathcal{H}(\tilde{I}_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(\tilde{I}_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big).
\end{aligned}$$

Similar estimates can be derived on all the neighboring subregions of Ω_i, as well as on Ω_i. Summing all these estimates, we find

$$a(u_i, u_i) \;\leq\; C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big). \quad (6.13)$$

Since $u_i \in \text{Range}(T_i)$, we find, from (6.5), that

$$\int_{\Omega_i} \mathcal{H}(I_h(\nu_i u_i))\; dx \;=\; 0, \quad \forall\; u_i \in \text{Range}(T_i).$$

Then, from the Poincaré inequality (see Theorem 1.3), it follows that

$$||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)} \;\leq\; CH^2|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)}, \quad (6.14)$$

and the estimate (6.12) results from (6.13) and (6.14).

6.3 The Balancing Algorithm for Mortars

In this section, we modify the classical balancing method for conforming finite elements, discussed in Section 6.2, to obtain an algorithm for mortar finite elements.

To make our presentation clear, we first discuss the geometrically conforming case; we will extend our results to the geometrically nonconforming case in Section 6.6. Our results can also be extended, using the same methods, to any second order self-adjoint elliptic problems with mixed boundary conditions, and to the three dimensional case.

Following Section 2.2, we introduce a mortar finite element V^h on Ω corresponding to the nonoverlapping partition $\{\Omega_i\}_{i=1:N}$. The variational formulation of our Poisson problem is

$$a^\Gamma(u_h, v_h) = f^\Gamma(v_h), \quad \forall\ v_h \in V^h,$$

which has to be solved for $u_h \in V^h$; see Section 2.5 for the definitions of $a^\Gamma(\cdot,\cdot)$ and $f^\Gamma(\cdot)$. For simplicity, we assume that all subregions Ω_i are rectangular.

After that we have eliminated the unknowns in the interior of the subregions, a Schur complement problem results. The variational formulation of this Schur problem was not discussed in Section 2.5, and is different than that of the conforming case; see Section 6.2. Therefore, we give further details.

Let $S^{(i)}$ be the Schur complement matrix corresponding to Ω_i, and let S be the block diagonal Schur complement matrix; see Section 5.2. The reduced system is a Schur complement system,

$$s^\Gamma(u_\Gamma, v_\Gamma) \ = \ \tilde{f}^\Gamma(v_\Gamma), \quad \forall\ v_\Gamma \in V_\Gamma^h. \tag{6.15}$$

We want to solve (6.15) for $u_\Gamma \in V_\Gamma^h$, the restriction of u_h to the interface. Here, $s^\Gamma(u_\Gamma, v_\Gamma) = \underline{v}_\Gamma^T S \underline{u}_\Gamma$, and $\tilde{f}^\Gamma(v_\Gamma)$ is obtained from $f^\Gamma(v_h)$, after eliminating the interior unknowns.

Let V be the space of piecewise discrete harmonic functions, associated with the partition of Ω. If $w_\Gamma \in V_\Gamma^h$, then $\mathcal{H}(w_\Gamma) \in V$ is constructed as follows. In each subregion Ω_i, let $\mathcal{H}_i(w)$ be the harmonic extension of $w_\Gamma\,|_{\partial\Omega_i}$, with respect to the H^1-seminorm. The restriction of $\mathcal{H}(w_\Gamma)$ to Ω_i is denoted by $\mathcal{H}_i(w)$. It is easy to see that

$$s^\Gamma(w_\Gamma, w_\Gamma) \ = \ a^\Gamma(\mathcal{H}(w_\Gamma), \mathcal{H}(w_\Gamma)), \quad \forall\ w_\Gamma \in V_\Gamma^h, \quad \forall\ \mathcal{H}(w_\Gamma) \in V, \tag{6.16}$$

and that the discrete harmonic extension minimizes the energy of a function for given values on the interface,

$$a^{\Gamma}(\mathcal{H}(w_{\Gamma}),\mathcal{H}(w_{\Gamma})) \;=\; \min_{w\in V^h, w|_{\Gamma}=v_{\Gamma}} a^{\Gamma}(w,w).$$

The major difference between the classical balancing method and our algorithm for mortars is related to the *extended subregions*, which replace the individual subregions in the definition of the local bilinear forms $\tilde{a}_i(\cdot,\cdot)$. In Section 6.5, we will show that defining $\tilde{a}_i(\cdot,\cdot)$ only on Ω_i does not lead to a proof of the convergence of the algorithm.

We are looking for a solution in V_{Γ}^h of the Schur complement system (6.15). From the norm equivalence (6.16), it follows that we can study the convergence properties of the balancing algorithm for piecewise discrete harmonic functions from V, instead of mortar functions restricted to the interface. Therefore, from now on we work in the space V, with inner product $a(\cdot,\cdot)=a^{\Gamma}(\cdot,\cdot)$.

The counting functions are different than in the conforming finite element case, due to the use of the extended subregions.

For each subregion Ω_i, let $\zeta_{i,j}$, $j=1:q(i)$, be the mortar sides of Ω_i, and $\zeta'_{i,j}$, $j=1:q'(i)$, be the mortars opposite $\partial\Omega_i$. Note that, since we are in the geometrically conforming case, opposite a nonmortar there exists exactly one mortar. Let $\mathtt{N}_i$ be the set of the corner nodes of Ω_i, the interior nodes of $\cup_j\zeta_{i,j}$, and all the nodes of $\cup_j\overline{\zeta}'_{i,j}$.

Let $\widetilde{\Omega}_i$ be the extended subregion corresponding to Ω_i, defined as follows: Let $\Omega'_{i,j}$ be the neighboring subregion of Ω_i such that $\zeta'_{i,j}$ is the side of $\Omega'_{i,j}$. Then, $\widetilde{\Omega}_i$ is the union of Ω_i and all the subregions $\Omega'_{i,j}$, with $j=1:q'(i)$.

Let $\nu_i:\Gamma\rightarrow\mathtt{R}$ be the counting function corresponding to Ω_i. By definition, ν_i is a mortar function, piecewise linear on all the mortars, and takes the following values at the genuine degrees of freedom:

$$\nu_i(x)=\begin{cases} \text{number of } N'_j\text{s such that } x\in N_j, & \text{if} \quad x\in\mathtt{N}_i;\\ 0, & \text{if} \quad x\notin\mathtt{N}_i;\\ 1, & \text{if} \quad x\in\partial\Omega_i\cap\partial\Omega_D. \end{cases}$$

In the geometrically conforming case, the value of ν_i at every interior node of the mortar sides where ν_i does not vanish is equal to 2, and $\mathrm{Range}(\nu_i)\subseteq\{1,2,3,4\}$.

Let $\nu_i^\dagger$ be the pseudo inverse of ν_i, i.e., $\nu_i^\dagger$ is a mortar function, piecewise linear on mortars, with nodal values

$$\nu_i^\dagger(x) = \begin{cases} 1/\nu_i(x), & \text{if } \nu_i(x) \neq 0; \\ 0, & \text{else.} \end{cases}$$

As in the conforming case, $\nu_i^\dagger$, as well as $\mathcal{H}(\nu_i^\dagger)$, form a partition of unity:

$$\sum_{i=1}^{N} \nu_i^\dagger \;=\; 1; \quad \sum_{i=1}^{N} \mathcal{H}(\nu_i^\dagger) \;=\; 1. \tag{6.17}$$

The coarse space V_0 is defined as in the conforming case, using the new counting functions. Let I_h be the nodal basis interpolation onto V. A basis function for V_0 corresponding to a floating subregion Ω_i is $\mathcal{H}(\nu_i^\dagger)$. For a nonfloating subregion Ω_j, we set $\nu_j^\dagger$ equal to zero at all the nodes of $\partial\Omega_D$ and $\mathcal{H}(\nu_j^\dagger)$ is a basis function for V_0.

The bilinear form a_0 is exact, i.e., $a_0(\cdot,\cdot) = a(\cdot,\cdot)$, which means that $T_0 = P_0$ is a projection.

Each local space V_i, $i = 1 : N$, is associated with one subregion Ω_i from the partitioning of Ω, is embedded in V, i.e., $V_i \subset V$, and consists of piecewise harmonic functions which vanish at all the genuine degrees of freedom of $\Gamma \setminus \mathtt{N}_i$.

On V_i, we define approximate bilinear forms $\tilde{a}_i(\cdot,\cdot)$ using the extended subregion $\widetilde{\Omega}_i$ as follows:

$$\begin{aligned} \tilde{a}_i(v_i, w_i) &= a_{\widetilde{\Omega}_i}(\mathcal{H}(I_h(\nu_i v_i)), \mathcal{H}(I_h(\nu_i w_i))) \\ &= \sum_{\Omega_j \subset \widetilde{\Omega}_i} \int_{\Omega_j} \nabla\mathcal{H}(I_h(\nu_i v_i)) \cdot \nabla\mathcal{H}(I_h(\nu_i w_i))\; dx. \end{aligned} \tag{6.18}$$

The projection-like operator T_i is defined by

$$\tilde{a}_i(T_i u, v_i) \;=\; a(u, v_i), \quad \forall\; v_i \in V_i. \tag{6.19}$$

If the extended subregion $\widetilde{\Omega}_i$ contains more than one subregion, then any function in V_i vanishes at $\partial\widetilde{\Omega}_i \setminus \partial\Omega_i$. Thus, (6.19) is well–posed since it becomes a Poisson problem on $\widetilde{\Omega}_i$ with intrinsic Dirichlet boundary conditions on $\partial\widetilde{\Omega}_i \setminus \partial\Omega_i$.

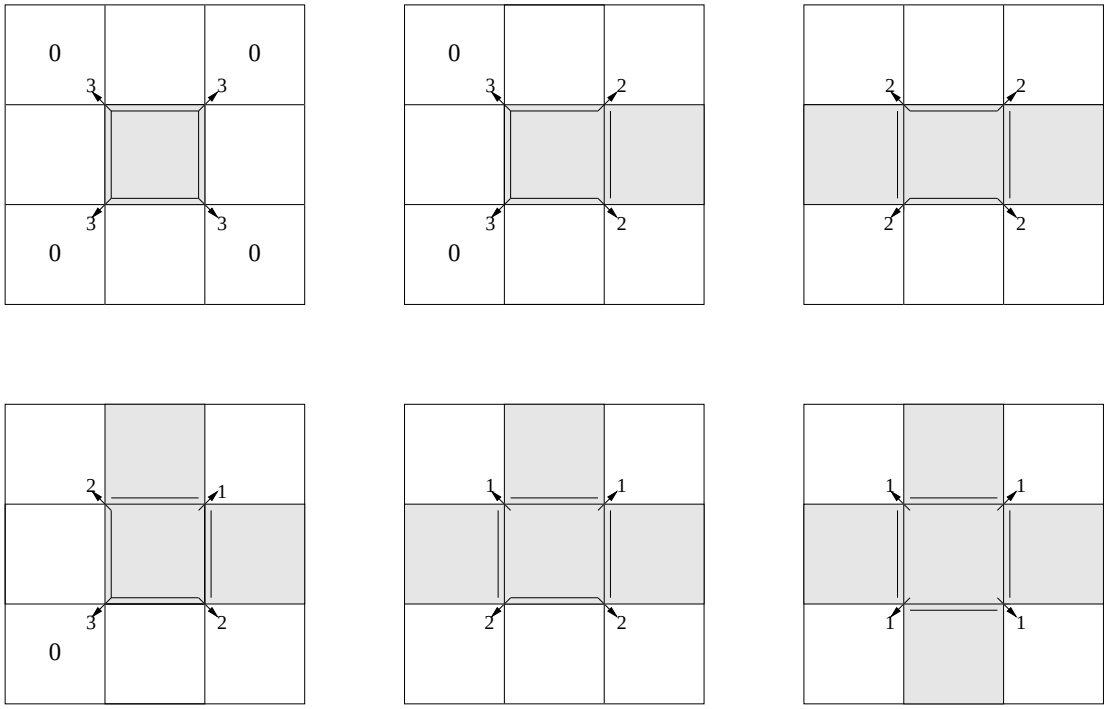


Figure 6.2: Local spaces

If $\widetilde{\Omega}_i = \Omega_i$, then all the sides of Ω_i are mortars; cf. Figure 6.2, upper left picture. In this case, as in the conforming case, we have to work with *balanced functions*; cf. (6.4). Once again, every function from Range$(I - P_0)$ is a balanced function, and we require $T_i u$ to be orthogonal to the null space of $\tilde{a}_i$,

$$\int_{\Omega_i} \mathcal{H}(I_h(\nu_i T_i(u)))\, dx \;=\; 0, \tag{6.20}$$

in order for the solution of (6.19) to be unique.

The balancing algorithm for mortars corresponds to the same preconditioned operator,

$$T_{bal} \;=\; P_0 + (I - P_0)(T_1 + \ldots + T_N)(I - P_0).$$

We note that Lemma 6.1 also holds for the mortar balancing algorithm. Therefore, the analysis of our algorithm will follow the same steps as that of the classical algorithm, and requires an upper bound for $\kappa(T_{N-N})$. This bound will be obtained

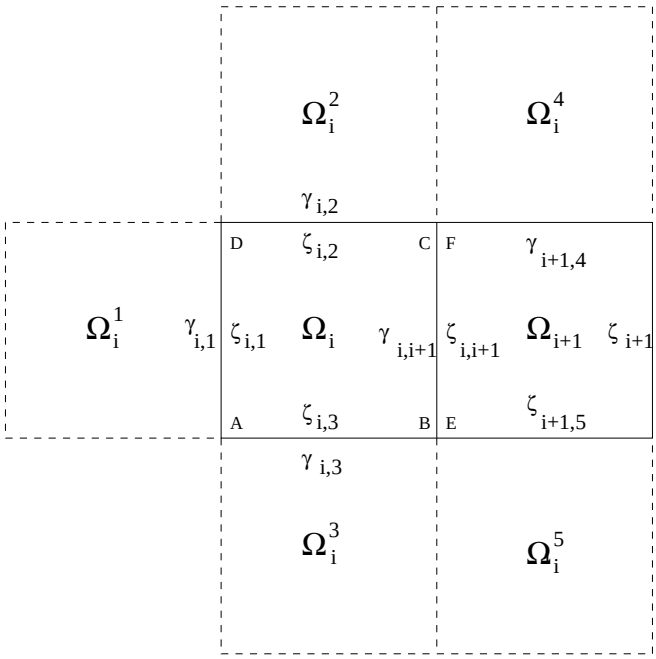


Figure 6.3: Extended subregion $\widetilde{\Omega}_i = \Omega_i \cup \Omega_{i+1}$

using the abstract Schwarz theory, by establishing the validity of Assumptions 1 and 3 from Section 3.2.1, for T_{N-N} restricted to Range$(I - P_0)$; see Section 6.5.

6.4 Technical tools

In this section, we prove two results which will be used in Section 6.5 for the analysis of the case when the extended subregion $\widetilde{\Omega}_i$ contains more than one subregion, i.e., Ω_i has at least one nonmortar side.

Our proofs do not depend on the number of nonmortar sides of Ω_i. To keep the presentation simple, we assume that Ω_i has exactly one nonmortar side, $\gamma_{i,i+1}$; cf. Figure 6.3. Then, $\widetilde{\Omega}_i = \Omega_i \cup \Omega_{i+1}$. Since we are in the geometrically conforming case, ν_i is equal to 2 at all the interior nodes on the mortar sides $\zeta_{i,1}$, $\zeta_{i,2}$, $\zeta_{i,3}$, and $\zeta_{i,i+1}$. Also, $\nu_i(A) = \nu_i(D) = 3$, $\nu_i(B) = \nu_i(C) = 2$, and $\nu_i(E), \nu_i(F) \in \{2, 3\}$, depending on whether or not a mortar and a nonmortar meet at E an F. In

order to present both cases, we assume that $\gamma_{i+1,4}$, the side of Ω_{i+1} opposite Ω_i^4, is a mortar, and that $\zeta_{i+1,5}$, the side of Ω_{i+1} opposite Ω_i^5, is a nonmortar. Then, $\nu_i(E) = 3$ and $\nu_i(F) = 2$. From (6.18), it results that

$$\tilde{a}_i(u_i, u_i) \;=\; |\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} \;+\; |\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})}, \quad \forall\, u_i \in V_i. \tag{6.21}$$

Let $u \in V$ be a piecewise harmonic mortar function, and let $u_i = u\,|_{\widetilde{\Omega}_i}$. Let $v_i = I_h(\nu_i u_i) \in V_i$, and let $[v_i]_{\gamma_{i,i+1}} = v_i\,|_{\zeta_{i,i+1}} - v_i\,|_{\gamma_{i,i+1}}$ be the jump of v_i across $\gamma_{i,i+1}$. From (6.21), we find

$$\tilde{a}_i(u_i, u_i) \;=\; |v_i|^2_{H^1(\Omega_i)} \;+\; |v_i|^2_{H^1(\Omega_{i+1})}. \tag{6.22}$$

In the next lemma, we eliminate the jump of v_i by extending v_i into Ω_{i+1}, in such a way that the energy of the extension is only increased by a constant factor. We use the same construction as in Section 4.3.2, where we have also eliminated the jump of a mortar function in order to estimate the L^2 norm of its jump across a nonmortar side.

Lemma 6.2. *Let $\chi_i \in H^1(\Omega_{i+1})$ satisfy*

$$\chi_i\,|_{\zeta_{i,i+1}} = -[v_i]_{\gamma_{i,i+1}}; \quad \chi_i\,|_{\zeta_{i,i+1}} = 0; \quad |\chi_i|^2_{H^1(\Omega_{i+1})} \;\le\; C\tilde{a}_i(u_i, u_i). \tag{6.23}$$

Let $u_{ext} \in H^1(\widetilde{\Omega}_i)$ be the extension of v_i given by

$$u_{ext} = \begin{cases} v_i & on \quad \Omega_i; \\ v_i + \chi_i & on \quad \Omega_{i+1}. \end{cases}$$

Then,

$$|u_{ext}|^2_{H^1(\widetilde{\Omega}_i)} \;\le\; C\;\tilde{a}_i(u_i, u_i); \tag{6.24}$$

$$||u_{ext}||^2_{L^2(\widetilde{\Omega}_i)} \;\le\; CH^2\;\tilde{a}_i(u_i, u_i), \tag{6.25}$$

where C is a constant that does not depend on Ω_i.

Proof. To obtain a uniform C, the constant in (6.23) must have the same property. The extension of the jump of v_i on Ω_{i+1} is done on the reference unit square, and (6.23) is obtained by classical extension and trace theorems for Sobolev spaces; cf. [96]. The details of the construction have been provided in the proof of Lemma 4.7.

An extension χ_i of $[v_i]_{\gamma_{i,i+1}}$ can be obtained such that

$$|\chi_i|_{H^1(\Omega_{i+1})} \ \leq \ C||[v_i]_{\gamma_{i,i+1}}||_{H^{1/2}(\zeta_{i,i+1})}. \tag{6.26}$$

Since v_i is a mortar function, its average on $\gamma_{i,i+1}$ and $\zeta_{i,i+1}$ is the same. Let

$$\overline{v_i} \ = \ \frac{1}{\int_{\gamma_{i,i+1}} v_i dx} \int_{\gamma_{i,i+1}} v_i \psi_i dx \ = \ \frac{1}{\int_{\zeta_{i,i+1}} v_i dx} \int_{\zeta_{i,i+1}} v_i \psi_i dx. \tag{6.27}$$

From (6.26) and (6.27), and using the trace Theorem 1.2 and the Friedrichs' inequality, see Theorem 1.4, we obtain

$$\begin{aligned} |\chi_i|_{H^1(\Omega_{i+1})} &\leq C||v_i - \overline{v_i}||_{H^{1/2}(\gamma_{i,i+1})} \ + \ ||v_i - \overline{v_i}||_{H^{1/2}(\zeta_{i,i+1})} \\ &\leq C||v_i - \overline{v_i}||_{H^1(\Omega_i)} \ + \ ||v_i - \overline{v_i}||_{H^1(\Omega_{i+1})} \\ &\leq C|v_i|_{H^1(\Omega_i)} \ + \ |v_i|_{H^1(\Omega_{i+1})} \\ &= C\tilde{a}_i(u_i, u_i). \end{aligned}$$

By construction, u_{ext} is continuous across $\gamma_{i,i+1}$. Since u_{ext} is also piecewise H^1, it follows from Lemma 1.1 that $u_{ext} \in H^1(\widetilde{\Omega}_i)$. Then, (6.24) follows from (6.22) and (6.23). To obtain (6.25), we apply Friedrichs' inequality to u_{ext} on the extended subregion $\widetilde{\Omega}_i$, and use (6.24) and the fact that u_{ext} vanishes on ζ_{i+1},

$$\begin{aligned} |u_{ext}|^2_{L^2(\tilde{\Omega}_i)} &\leq CH^2 \left(|u_{ext}|^2_{H^1(\tilde{\Omega}_i)} + \frac{1}{H^2} \Big| \int_{\zeta_{i+1}} u_{ext} dx \Big|^2 \right) \\ &= CH^2 |u_{ext}|^2_{H^1(\tilde{\Omega}_i)} \ \leq \ CH^2 \tilde{a}_i(u_i, u_i). \end{aligned}$$

□

The next lemma will be used in Section 6.5 to establish Assumption 1 for T_{N-N}.

Lemma 6.3. *Let $u \in V$ and denote the Lebesgue measure by μ. Let α_i be the average of u over Ω_i, i.e.,*

$$\alpha_i = \frac{1}{\mu(\Omega_i)} \int_{\Omega_i} u \ dx. \tag{6.28}$$

Then

$$||u - \alpha_i||^2_{L^2(\tilde{\Omega}_i)} \ \leq \ CH^2 |u|^2_{H^1(\tilde{\Omega}_i)}, \tag{6.29}$$

and therefore

$$|(u - \alpha_i) \, |_{\zeta_{i,i+1}} \, |^2_{H^{1/2}(\Omega_{i+1})} \ \leq \ C(1 + log(H/h))^2 |u|^2_{H^1(\tilde{\Omega}_i)}. \tag{6.30}$$

Proof. From Friedrichs' inequality, see Theorem 1.4, it follows that

$$\begin{aligned}\frac{1}{H^2}||u-\alpha_i||^2_{L^2(\Omega_{i+1})} &\leq C\left(|u|^2_{H^1(\Omega_{i+1})} + \frac{1}{H^2}\Big|\int_{\zeta_{i,i+1}}(u-\alpha_i)\,d\sigma\Big|^2\right)\\ &= C\left(|u|^2_{H^1(\Omega_{i+1})} + \frac{1}{H^2}\Big|\int_{\delta_{i,i+1}}(u-\alpha_i)\,d\sigma\Big|^2\right). \quad (6.31)\end{aligned}$$

The last equality results from the mortar conditions, since u has the same average over $\zeta_{i,i+1}$ and $\delta_{i,i+1}$, which are opposite mortar and nonmortar sides. From Schwarz's inequality,

$$\frac{1}{H^2}\Big|\int\limits_{\delta_{i,i+1}}(u-\alpha_i)\,d\sigma\Big|^2 \leq \frac{C}{H}\int\limits_{\delta_{i,i+1}}(u-\alpha_i)^2\,d\sigma \leq \frac{C}{H}||u-\alpha_i||^2_{L^2(\partial\Omega_i)}. \quad (6.32)$$

Using a trace inequality, we obtain

$$\frac{1}{H}||u-\alpha_i||^2_{L^2(\partial\Omega_i)} \leq C\Big(|u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}||u-\alpha_i||^2_{L^2(\Omega_i)}\Big), \quad (6.33)$$

and therefore, from (6.32) and (6.33),

$$\frac{1}{H^2}\Big|\int\limits_{\delta_{i,i+1}}(u-\alpha_i)\,d\sigma\Big|^2 \leq C\left(|u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}||u-\alpha_i||^2_{L^2(\Omega_i)}\right). \quad (6.34)$$

From (6.28) it follows that $u-\alpha_i$ has a zero average over Ω_i. Then, from the Poincaré inequality (see Theorem 1.3), we obtain

$$||u-\alpha_i||^2_{L^2(\Omega_i)} \leq CH^2|u|^2_{H^1(\Omega_i)}, \quad (6.35)$$

From (6.34) and (6.35) it follows that

$$\frac{1}{H^2}\Big|\int\limits_{\delta_{i,i+1}}u-\alpha_i\,d\sigma\Big|^2 \leq C|u|^2_{H^1(\Omega_i)}. \quad (6.36)$$

Then, we use (6.36) and (6.31) to conclude that

$$\frac{1}{H^2}||u-\alpha_i||^2_{L^2(\Omega_{i+1})} \leq C\left(|u|^2_{H^1(\Omega_{i+1})} + |u|^2_{H^1(\Omega_i)}\right) = C|u|^2_{H^1(\widetilde{\Omega}_i)}, \quad (6.37)$$

From (6.37) and (6.35), it is easy to see that (6.29) holds:

$$||u-\alpha_i||^2_{L^2(\widetilde{\Omega}_i)} = ||u-\alpha_i||^2_{L^2(\Omega_i)} + ||u-\alpha_i||^2_{L^2(\Omega_{i+1})} \leq CH^2|u|^2_{H^1(\widetilde{\Omega}_i)}.$$

From Theorem 1.5, we obtain

$$|(u-\alpha_i)\,|_{\zeta_{i,i+1}}\,|^2_{H^{1/2}(\Omega_{i+1})} \;\leq\; C(1+log(H/h))^2\Big(|u|^2_{H^1(\Omega_{i+1})}+\frac{||u-\alpha_i||^2_{L^2(\Omega_{i+1})}}{H^2}\Big). \tag{6.38}$$

Thus, (6.30) follows immediately from (6.29) and (6.38). □

6.5 Condition number estimate for the balancing algorithm

In this section, we prove an upper bound for $\kappa(T_{bal})$, the condition number of our mortar balancing algorithm, by using the result of Lemma 6.1.

We begin by establishing Assumption 3 for T_{N-N} restricted to Range$(I-P_0)$.

Lemma 6.4. *There exists a constant C not depending on the local spaces, such that*

$$a(u_i,u_i) \;\leq\; C(1+log(H/h))^2\tilde{a}_i(u_i,u_i), \quad \forall\, u_i\in Range(T_i), \quad \forall i=1:N. \tag{6.39}$$

Proof. The support of a function in V_i can be the union of Ω_i and all its neighboring subregions. Thus, the main difficulty in proving (6.39) comes from the fact that $a(u_i,u_i)$ is computed on a larger domain than $\tilde{a}_i(u_i,u_i)$, which is only computed over the extended subregion $\widetilde{\Omega}_i$; cf. Figure 6.2.

In the proof of (6.39), we discuss separately two cases, which require different tools in their analysis.

Case 1. If the extended subregion $\widetilde{\Omega}_i$ consists only of Ω_i, i.e., if all the sides of Ω_i are mortars, then, from (6.18), it follows that

$$\tilde{a}_i(u_i,u_i) \;=\; |\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)}. \tag{6.40}$$

In this case, the proof of (6.39) is similar to that of the conforming case, see Section 6.2.2. The extra complications due to the mortar spaces are dealt with by using the stability properties of the mortar projection from Theorem 2.1.

Let Ω_i^1 be one of the neighboring subregions of Ω_i, and let $\gamma_{i,1}$ be the nonmortar side of Ω_i^1 opposite the mortar side $\zeta_{i,1}$ of Ω_i. Since $u_i\in V_i$, it is easy to see that

u_i vanishes at all the nodes of $\partial\Omega_i^1$ except at the interior nodes of $\gamma_{i,1}$. Therefore, using the fact that u_i is harmonic on Ω_i^1, we obtain

$$|u_i|^2_{H^1(\Omega_i^1)} \;=\; |u_i|^2_{H^{1/2}(\partial\Omega_i^1)}. \tag{6.41}$$

Let π_γ be the mortar projection corresponding to $\gamma_{i,1}$ with values 0 at the end points of $\gamma_{i,1}$. Then,

$$u_i\mid_{\gamma_{i,1}} \;=\; \pi_\gamma(u_i\mid_{\overline{\zeta}_{i,1}}). \tag{6.42}$$

Let $u_{i,A}$ be the function which is equal to u_i at A and vanishes at all the other nodes, and let $u_{i,int}$ be the function which is equal to u_i at the interior nodes of $\zeta_{i,1}$ and vanishes at all the other nodes. We define $u_{i,D}$, $I_h(\nu_i u_i)_A$, $I_h(\nu_i u_i)_{int}$, and $I_h(\nu_i u_i)_D$ in a similar way. We note that $\nu_i(A) = \nu_i(D) = 3$ and ν_i is equal to 2 at all the interior nodes of $\zeta_{i,1}$. Thus,

$$u_{i,A} = \frac{I_h(\nu_i u_i)_A}{3}; \quad u_{i,D} = \frac{I_h(\nu_i u_i)_D}{3}; \quad u_{i,int} = \frac{I_h(\nu_i u_i)_{int}}{2}.$$

It is easy to see that

$$u_i\mid_{\overline{\zeta}_{i,1}} \;=\; u_{i,A} + u_{i,int} + u_{i,D},$$

and, from (6.42),

$$\begin{aligned} u_i\mid_{\gamma_{i,1}} &= \pi_\gamma(u_{i,A}) + \pi_\gamma(u_{i,int}) + \pi_\gamma(u_{i,D}) \\ &= \frac{\pi_\gamma(I_h(\nu_i u_i)_A)}{3} + \frac{\pi_\gamma(I_h(\nu_i u_i)_{int})}{2} + \frac{\pi_\gamma(I_h(\nu_i u_i)_D)}{3}. \end{aligned}$$

Then, from Theorem 2.1 and Theorem 1.5, it follows that

$$\begin{aligned} |u_i|^2_{H^{1/2}(\partial\Omega_i^1)} &\le C\Big(||\pi_\gamma(I_h(\nu_i u_i)_A)||^2_{H_{00}^{1/2}(\gamma_{i,1})} + ||\pi_\gamma(I_h(\nu_i u_i)_{int})||^2_{H_{00}^{1/2}(\gamma_{i,1})} \\ &\qquad +||\pi_\gamma(I_h(\nu_i u_i)_D)||^2_{H_{00}^{1/2}(\gamma_{i,1})}\Big) \\ &\le C\Big(||I_h(\nu_i u_i)_A||^2_{H^{1/2}(\partial\Omega_i)} + ||I_h(\nu_i u_i)_{int}||^2_{H^{1/2}(\partial\Omega_i)} \\ &\qquad +||I_h(\nu_i u_i)_D||^2_{H^{1/2}(\partial\Omega_i)}\Big) \\ &\le C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big). \end{aligned}$$

Using (6.41) and the last inequality, we obtain an estimate for the energy of u_i on a neighboring subregion of Ω_i in terms of the energy of $\mathcal{H}(I_h(\nu_i u_i))$ on Ω_i, i.e.,

$$|u_i|^2_{H^1(\Omega_i^1)} \leq C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big). \quad (6.43)$$

Similar bounds can be obtained for the H^1 seminorm of u_i on all the other neighboring subregions of Ω_i.

On $\partial\Omega_i$, we note that ν_i is equal to 2 at all the nodes except for the corners, where it equals 3. Therefore,

$$u_i\,|_{\Omega_i} \;=\; \frac{I_h(\nu_i u_i)}{2} \;-\; \frac{I_h(\nu_i u_i)_A + \ldots + I_h(\nu_i u_i)_D}{6}.$$

Since u_i is discretely harmonic on Ω_i, we can use Theorem 1.5 to obtain

$$\begin{aligned} |u_i|^2_{H^1(\Omega_i)} &= |u_i|^2_{H^{1/2}(\partial\Omega_i)} \quad (6.44)\\ &\leq C\Big(|I_h(\nu_i u_i)|^2_{H^{1/2}(\partial\Omega_i)} + |I_h(\nu_i u_i)_A|^2_{H^{1/2}(\partial\Omega_i)} + |I_h(\nu_i u_i)_D|^2_{H^{1/2}(\partial\Omega_i)}\Big)\\ &\leq C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big). \end{aligned}$$

We note that

$$a(u_i,u_i) \;=\; |u_i|^2_{H^1(\Omega_i)} \;+\; \sum_{j=1}^{4} |u_j|^2_{H^1(\Omega_i^j)}$$

From (6.43) and (6.44), it follows that

$$a(u_i,u_i) \;\leq\; C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)}}{H^2}\Big). \quad (6.45)$$

We need to prove (6.39) only for $u_i \in \text{Range}(T_i)$. In Section 6.3, we concluded that, if $\widetilde{\Omega}_i = \Omega_i$, we have to require

$$\int_{\Omega_i} \mathcal{H}(I_h(\nu_i T_i(u)))\; dx \;=\; 0;$$

cf. (6.20). Therefore,

$$\int_{\Omega_i} \mathcal{H}(I_h(\nu_i u_i))\; dx \;=\; 0, \quad \forall\; u_i \in \text{Range}(T_i).$$

Then, from the Poincaré inequality (see Theorem 1.3), we obtain

$$||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_i)} \;\leq\; CH^2|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)}. \tag{6.46}$$

Finally, from (6.40), (6.45) and (6.46), we find

$$a(u_i,u_i) \;\leq\; C(1+log(H/h))^2|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} \;=\; C(1+log(H/h))^2\tilde{a}_i(u_i,u_i).$$

Case 2. We now assume that the extended subregion $\widetilde{\Omega}_i$ contains more than one subregion. For simplicity, we make the same assumption as in Section 6.4, i.e., we assume that Ω_i has exactly one nonmortar side; cf. Figure 6.3 and the notations therein. From (6.18), it follows that

$$\tilde{a}_i(u_i,u_i) \;=\; |\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_i)} \;+\; |\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})}. \tag{6.47}$$

We are going to rely on the construction of $u_{ext} \in H^1(\widetilde{\Omega}_i)$ from Lemma 6.2. We recall that

$$u_{ext} \;=\; \mathcal{H}(I_h(\nu_i u_i)) \quad \text{in } \Omega_i. \tag{6.48}$$

We also need the estimates of the L^2 norm and H^1 seminorm of u_{ext} in terms of $\tilde{a}_i(u_i,u_i)$, which were established in Lemma 6.2; see (6.24) and (6.25).

As before, on Ω_i^1, we obtain that

$$u_i\,|_{\gamma_{i,1}} \;=\; \frac{\pi_\gamma(I_h(\nu_i u_i)_A)}{3} \;+\; \frac{\pi_\gamma(I_h(\nu_i u_i)_{int})}{2} \;+\; \frac{\pi_\gamma(I_h(\nu_i u_i)_D)}{3}.$$

We use (6.48) and the stability properties of the mortar projection from Theorem 2.1 to obtain

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i^1)} &= |u_i|^2_{H^{1/2}(\Omega_i^1)} \\
&\leq C\Big(||I_h(\nu_i u_i)_A||^2_{H^{1/2}(\partial\Omega_i)} + ||I_h(\nu_i u_i)||^2_{H^{1/2}(\partial\Omega_i)} + ||I_h(\nu_i u_i)_D||^2_{H^{1/2}(\partial\Omega_i)}\Big) \\
&= C\Big(||u_{ext,A}||^2_{H^{1/2}(\partial\Omega_i)} + ||u_{ext}||^2_{H^{1/2}(\partial\Omega_i)} + ||u_{ext,D}||^2_{H^{1/2}(\partial\Omega_i)}\Big).
\end{aligned}$$

Then, from Theorem 1.5,

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i^1)} &\leq C\Big(|u_{ext}|^2_{H^1((\Omega_i)} + \frac{||u_{ext}||^2_{L^2(\Omega_i)}}{H^2}\Big) \\
&\leq C\Big(|u_{ext}|^2_{H^1(\widetilde{\Omega}_i)} + \frac{||u_{ext}||^2_{L^2(\widetilde{\Omega}_i)}}{H^2}\Big).
\end{aligned}$$

Using the estimates (6.24) and (6.25), we conclude that

$$|u_i|^2_{H^1(\Omega_i^1)} \;\leq\; C(1+log(H/h))^2\;\tilde{a}_i(u_i,u_i). \tag{6.49}$$

Estimates similar to (6.49) can also be derived for the energy of u_i on Ω_i^2 and Ω_i^3.

On Ω_{i+1}, we note that u_i is equal to the mortar projection of $u_{i,F}$ at the interior nodes of $\gamma_{i+1,4}$, the nonmortar side adjacent to the vertex F, i.e.,

$$u_i\mid_{\gamma_{i+1,4}} \;=\; \pi_\gamma(u_{i,F}).$$

Also, u_i vanishes on $\zeta_{i+1,5}$, since all the nodes on the mortar side $\zeta_{i+1,5}$ are genuine degrees of freedom which do not belong to $\mathtt{N}_i$. Then,

$$u_i\mid_{\partial\Omega_{i+1}} \;=\; u_{i,F} \;+\; \pi_\gamma(u_{i,F}) \;+\; u_i\mid_{\zeta_{i,i+1}} \;+\; u_{i,E}.$$

Using the values of ν_i at the boundary nodes of Ω_{i+1}, we obtain that

$$u_i\mid_{\partial\Omega_{i+1}} \;=\; \frac{I_h(\nu_i u_i)_F}{2}+\frac{\pi_\gamma(I_h(\nu_i u_i)_F}{2}+\frac{I_h(\nu_i u_i)\mid_{\zeta_{i,i+1}}}{2}+\frac{I_h(\nu_i u_i)_E}{3}.$$

From Theorem 2.1, we obtain that

$$||\pi_\gamma(I_h(\nu_i u_i)_F||^2_{H^{1/2}_{00}(\gamma_{i+1,4})} \;\leq\; C||I_h(\nu_i u_i)_F||^2_{H^{1/2}(\partial\Omega_{i+1})},$$

and therefore, from Theorem 1.5,

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_{i+1})} &= |u_i|^2_{H^{1/2}(\partial\Omega_{i+1})}\\
&\leq C\Big(||I_h(\nu_i u_i)_F||^2_{H^{1/2}(\partial\Omega_{i+1})}+||I_h(\nu_i u_i)||^2_{H^{1/2}_{00}(\zeta_{i,i+1})}\\
&\qquad +||I_h(\nu_i u_i)_E||^2_{H^{1/2}(\partial\Omega_{i+1})}\Big)\\
&\leq C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})}+\frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_{i+1})}}{H^2}\Big).
\end{aligned}$$

Since ν_i, and therefore $\mathcal{H}(I_h(\nu_i u_i))$ vanish on ζ_{i+1}, it follows from Friedrichs' inequality (see Theorem 1.4) that

$$||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_{i+1})} \;\leq\; C|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})}. \tag{6.50}$$

Thus,

$$|u_i|^2_{H^1(\Omega_{i+1})} \;\leq\; C(1+log(H/h))^2|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})},$$

and, from (6.47), we conclude that

$$|u_i|^2_{H^1(\Omega_{i+1})} \;\leq\; C(1+log(H/h))^2\tilde{a}_i(u_i,u_i). \tag{6.51}$$

We note that u_i vanishes on Ω_i^4, and that it is equal to the mortar projection of $u_{i,E}$ on the nonmortar side of Ω_i^5 opposite $\zeta_{i+1,5}$. Using the same arguments as before, it is easy to see that

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i^5)} &\leq C||I_h(\nu_i u_i)_E||^2_{H^{1/2}(\partial\Omega_{i+1})} \\
&\leq C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_{i+1})}}{H^2}\Big). \\
&\leq C(1+log(H/h))^2|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})},
\end{aligned}$$

and therefore, from (6.50),

$$|u_i|^2_{H^1(\Omega_i^5)} \;\leq\; C(1+log(H/h))^2\tilde{a}_i(u_i,u_i). \tag{6.52}$$

The last step of the proof of this case consists of estimating the H^1 seminorm of u_i on Ω_i. We will use the fact that $I_h(\nu_i u_i)$ is equal to twice u_i at almost all the genuine degrees of freedom. More precisely, let π_γ be the mortar projection corresponding to $\gamma_{i,i+1}$ which vanishes at the end points of $\gamma_{i,i+1}$. Then, on $\partial\Omega_i$,

$$\begin{aligned}
u_i &= u_{i,A} + \ldots + u_{i,D} + u_i\,|_{\zeta_{i,1}} + \ldots + u_i\,|_{\zeta_{i,3}} \\
&\quad + \pi_\gamma(u_{i,B}) + \pi_\gamma(u_{i,C}) + \pi_\gamma(u_{i,E}) + \pi_\gamma(u_{i,F}) + \pi_\gamma(u_i\,|_{\zeta_{i,i+1}}).
\end{aligned}$$

Using once again the nodal values of ν_i, the construction of u_{ext}, and (6.48), it follows that

$$\begin{aligned}
u_i &= \frac{u_{ext,A}+u_{ext,D}}{3} + \frac{u_{ext}\,|_{\zeta_{i,1}} + \ldots \;+\; u_{ext}\,|_{\zeta_{i,3}}}{2} \;+\; \frac{u_{ext,B}+u_{ext,C}}{2} \\
&\quad + \frac{\pi_\gamma(u_{ext,B})+\pi_\gamma(u_{ext,C})}{2} \;+\; \frac{\pi_\gamma(I_h(\nu_i u_i)_E)}{3} \;+\; \frac{\pi_\gamma(I_h(\nu_i u_i)_F)}{2} \\
&\quad + \frac{\pi_\gamma(I_h(\nu_i u_i)\,|_{\zeta_{i,i+1}})}{2}.
\end{aligned}$$

From Theorem 2.1, it results that

$$\begin{aligned}
||\pi_\gamma(u_{ext,B})||^2_{H^{1/2}_{00}(\gamma_{i,i+1})} &\leq ||u_{ext,B}||^2_{H^{1/2}(\partial\Omega_i)} \\
||\pi_\gamma(I_h(\nu_i u_i)_E)||^2_{H^{1/2}_{00}(\gamma_{i,i+1})} &\leq ||I_h(\nu_i u_i)_E||^2_{H^{1/2}(\partial\Omega_{i+1})} \\
||\pi_\gamma(I_h(\nu_i u_i)\,|_{\zeta_{i,i+1}})||^2_{H^{1/2}_{00}(\gamma_{i,i+1})} &\leq ||I_h(\nu_i u_i)\,|_{\zeta_{i,i+1}}\,||^2_{H^{1/2}_{00}(\zeta_{i,i+1})}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i)} &= |u_i|^2_{H^{1/2}(\partial\Omega_i)} \\
&\leq C\Big(||u_{ext,A}||^2_{H^{1/2}(\partial\Omega_i)} + \ldots + ||u_{ext,D}||^2_{H^{1/2}(\partial\Omega_i)} \\
&\quad + ||u_{ext}\,|_{\zeta_{i,1}}\,||^2_{H^{1/2}_{00}(\zeta_{i,1})} + \ldots + ||u_{ext}\,|_{\zeta_{i,3}}\,||^2_{H^{1/2}_{00}(\zeta_{i,i+1})} \\
&\quad + ||I_h(\nu_i u_i)_E||^2_{H^{1/2}(\partial\Omega_{i+1})} + ||I_h(\nu_i u_i)_F||^2_{H^{1/2}(\partial\Omega_{i+1})} \\
&\quad + ||I_h(\nu_i u_i)\,|_{\zeta_{i,i+1}})||^2_{H^{1/2}_{00}(\gamma_{i,i+1})}\Big)
\end{aligned}$$

From Theorem 1.5, we obtain

$$\begin{aligned}
|u_i|^2_{H^1(\Omega_i)} &\leq C(1+log(H/h))^2\Big(|u_{ext}|^2_{H^1(\Omega_i)} + \frac{1}{H^2}||u_{ext}||^2_{L^2(\Omega_i)}\Big) \\
&\quad + C(1+log(H/h))^2\Big(|\mathcal{H}(I_h(\nu_i u_i))|^2_{H^1(\Omega_{i+1})} + \frac{||\mathcal{H}(I_h(\nu_i u_i))||^2_{L^2(\Omega_{i+1})}}{H^2}\Big).
\end{aligned}$$

Finally, from (6.24) and (6.25), and from (6.50) and (6.47), we obtain

$$|u_i|^2_{H^1(\Omega_i)} \leq C(1+log(H/h))^2\ \tilde{a}_i(u_i,u_i). \tag{6.53}$$

Summing up the energy estimates for u_i on all the neighboring subregions of Ω_i, i.e., (6.49), (6.51), (6.52), and (6.53), we obtain

$$a(u_i,u_i) \leq C(1+log(H/h))^2\ \tilde{a}_i(u_i,u_i).$$

□

We now establish Assumption 1 for T_{N-N}. An important ingredient is Lemma 6.3.

Lemma 6.5. *For every $u \in V$ there exist $u_0 \in V_0$ and $u_i \in V_i$, $i = 1:N$, such that $u = u_0 + \sum_{i=1}^N u_i$ and*

$$a(u_0,u_0) + \sum_{i=1}^N \tilde{a}_i(u_i,u_i) \leq C(1+log(H/h))^2 a(u,u). \tag{6.54}$$

Proof. For simplicity, we assume once again that we are in the case when Ω_i has exactly one nonmortar side; cf. Figure 6.3 and the notations therein.

Let α_i be the weighted averages of u over Ω_i, given by (6.28). Let u_0, the coarse space component in the splitting of u, be given by

$$u_0 \;=\; \sum_{i=1}^{N} \alpha_i \mathcal{H}(\nu_i^\dagger), \tag{6.55}$$

where $\mathcal{H}(\nu_i^\dagger)$ are the basis functions of the coarse space V_0. The local space components are given by

$$u_i \;=\; \mathcal{H}(I_h(\nu_i^\dagger(u-\alpha_i))). \tag{6.56}$$

Since $\nu_i^\dagger$ form a partition of unity, see (6.17), it is easy to see that

$$\sum_{i=1}^{N} u_i \;=\; \sum_{i=1}^{N} \mathcal{H}(I_h(\nu_i^\dagger u)) - \sum_{i=1}^{N} \alpha_i \mathcal{H}(I_h(\nu_i^\dagger) \;=\; u - u_0,$$

and therefore $u = u_0 + \sum_{i=1}^{N} u_i$.

To estimate $\tilde{a}_i(u_i, u_i)$, we use (6.21) and (6.56), and obtain

$$\begin{aligned} \tilde{a}_i(u_i,u_i) &= |\mathcal{H}(I_h(\nu_i\nu_i^\dagger(u-\alpha_i)))|^2_{H^1(\Omega_i)} \;+\; |\mathcal{H}(I_h(\nu_i\nu_i^\dagger(u-\alpha_i)))|^2_{H^1(\Omega_{i+1})} \\ &= |I_h(\nu_i\nu_i^\dagger(u-\alpha_i))|^2_{H^{1/2}(\partial\Omega_i)} \;+\; |I_h(\nu_i\nu_i^\dagger(u-\alpha_i))|^2_{H^{1/2}(\partial\Omega_{i+1})}. \end{aligned} \tag{6.57}$$

From the definition of ν_i we obtain that $\nu_i\nu_i^\dagger(u-\alpha_i)$ is equal to $u-\alpha_i$ at all the nodes of $\mathtt{N}_i$, and vanishes at all other nodes representing genuine degrees of freedom. Thus

$$\nu_i\nu_i^\dagger(u-\alpha_i) = u-\alpha_i, \quad \text{on } \; (\bigcup_{j=1}^{3} \overline{\zeta}_{i,j}) \cup \overline{\zeta}_{i,i+1}.$$

Since $u \in V$ is a mortar function, $I_h(\nu_i\nu_i^\dagger(u-\alpha_i)) = u-\alpha_i$ on $\partial\Omega_i$. Then, from the trace Theorem 1.2 and Lemma 6.3, it follows that

$$\begin{aligned} |I_h(\nu_i\nu_i^\dagger(u-\alpha_i))|^2_{H^{1/2}(\partial\Omega_i)} &= |u-\alpha_i|^2_{H^{1/2}(\partial\Omega_i)} \\ &\leq C\Big(|u|^2_{H^1(\Omega_i)} \;+\; \frac{1}{H^2}||u-\alpha_i||^2_{L^2(\Omega_i)}\Big) \\ &\leq C|u|^2_{H^1(\tilde{\Omega}_i)}. \end{aligned} \tag{6.58}$$

If both sides of Ω_{i+1} adjacent to $\zeta_{i,i+1}$ are mortar sides, then $\nu_i\nu_i^\dagger(u-\alpha_i)$ vanishes at all the nodes of $\partial\Omega_{i+1} \setminus \overline{\zeta}_{i,i+1}$, and therefore

$$||I_h(\nu_i\nu_i^\dagger(u-\alpha_i))||^2_{H^{1/2}(\partial\Omega_{i+1})} \;=\; ||(u-\alpha_i)\mid_{\zeta_{i,i+1}}||^2_{H^{1/2}(\partial\Omega_{i+1})} \tag{6.59}$$

We now assume there exists at least one nonmortar side of Ω_{i+1} adjacent to $\zeta_{i,i+1}$. Let $\gamma_{i+1,4}$, adjacent to the vertex F and opposite $\partial\Omega_i^4$ be that side. As before, let $(u-\alpha_i)_F$ be the function which is equal to $u-\alpha_i$ at F and vanishes at all the other nodes on $\partial\Omega_{i+1}$. The restriction of $I_h(\nu_i\nu_i^\dagger(u-\alpha_i))$ to the interior of $\gamma_{i+1,4}$ is the mortar projection of $(u-\alpha_i)_F$, i.e.,

$$I_h(\nu_i\nu_i^\dagger(u-\alpha_i))\mid_{\gamma_{i+1,4}} \;=\; \pi_\gamma((u-\alpha_i)_F),$$

where π_γ is the mortar projection on $\gamma_{i+1,4}$ with values 0 at the end points of $\gamma_{i+1,4}$. From the stability properties of the mortar projection, see Theorem 2.1, it follows that

$$||I_h(\nu_i\nu_i^\dagger(u-\alpha_i))||^2_{H^{1/2}_{00}(\gamma_{i+1,4})} \;\leq\; C||(u-\alpha_i)_F||^2_{H^{1/2}(\partial\Omega_{i+1})}. \tag{6.60}$$

From (1.3), it results that

$$||(u-\alpha_i)_F||^2_{H^{1/2}(\partial\Omega_{i+1})} \;\leq\; C||(u-\alpha_i)\mid_{\zeta_{i,i+1}}||^2_{H^{1/2}(\partial\Omega_{i+1})}, \tag{6.61}$$

and therefore, from (6.60) and (6.61),

$$||I_h(\nu_i\nu_i^\dagger(u-\alpha_i))||^2_{H^{1/2}_{00}(\gamma_{i+1,4})} \;\leq\; C||(u-\alpha_i)\mid_{\zeta_{i,i+1}}||^2_{H^{1/2}(\partial\Omega_{i+1})}. \tag{6.62}$$

A similar bound would result if the side of Ω_{i+1} adjacent to $\zeta_{i,i+1}$ at the vertex E is a nonmortar.

We note that $I_h(\nu_i\nu_i^\dagger(u-\alpha_i)) = u-\alpha_i$ on $\overline{\zeta}_{i,i+1}$. Then, using (6.62), we conclude that

$$|I_h(\nu_i\nu_i^\dagger(u-\alpha_i))|^2_{H^{1/2}(\partial\Omega_{i+1})} \;\leq\; C||(u-\alpha_i)\mid_{\zeta_{i,i+1}}||^2_{H^{1/2}(\partial\Omega_{i+1})}. \tag{6.63}$$

From (6.59) and (6.63), it follows that, regardless of whether the sides of Ω_{i+1} adjacent to $\zeta_{i,i+1}$ are mortar or nonmortar sides,

$$\begin{aligned} |I_h(\nu_i\nu_i^\dagger(u-\alpha_i))|^2_{H^{1/2}(\partial\Omega_{i+1})} &\leq C||(u-\alpha_i)\mid_{\zeta_{i,i+1}}||^2_{H^{1/2}(\partial\Omega_{i+1})}. \\ &\leq C(1+log(H/h))^2|u|^2_{H^1(\tilde{\Omega}_i)}, \end{aligned} \tag{6.64}$$

where the last inequality comes from Lemma 6.3.

From (6.57), (6.58), and (6.64), it follows that

$$\tilde{a}_i(u_i, u_i) \ \leq \ C(1 + log(H/h))^2 |u|^2_{H^1(\widetilde{\Omega}_i)}.$$

Summing over $i = 1 : N$, we obtain

$$\begin{aligned} \sum_{i=1}^{N} \tilde{a}_i(u_i, u_i) \ &\leq \ C(1 + log(H/h))^2 \sum_{i=1}^{N} |u|^2_{H^1(\widetilde{\Omega}_i)} \\ &\leq \ C(1 + log(H/h))^2 a(u, u), \end{aligned}$$

where C is a constant which does not depend on the number of subregions, since any subregion belongs to at most five extended subregions.

We have thus reduced the proof of (6.54) to obtaining a bound for the energy of u_0, i.e.,

$$a(u_0, u_0) \ \leq \ C(1 + log(H/h))^2 a(u, u). \tag{6.65}$$

Since $u_0 = u - \sum_{i=1}^{N} u_i$, we can use Schwarz's inequality to estimate $a(u_0, u_0)$, and obtain

$$a(u_0, u_0) \ \leq \ 2a(u, u) \ + \ 2 \sum_{i=1}^{N} a(u_i, u_i). \tag{6.66}$$

A proof along these lines for the conforming case appears in [53]. For the mortar case, extra work is necessary due to the presence of the extended regions. Here, we only give a short proof, since the details can be worked out as in the proof of the second case of Lemma 6.4.

Since $u_i \in V_i$, the support of u_i is embedded in the union of Ω_i and its neighboring subregions. By the definition (6.56),

$$a(u_i, u_i) \ = \ |\mathcal{H}(I_h(\nu_i^{\dagger}(u - \alpha_i)))|^2_{H^1(\Omega)}.$$

Once again, we have two different cases according to whether the extended subregion $\widetilde{\Omega}_i$ consists only of Ω_i, or contains other subregions as well.

Case 1. If $\widetilde{\Omega}_i = \Omega_i$, then all the sides of Ω_i are mortars, and

$$a(u_i, u_i) \ \leq \ |\mathcal{H}(I_h(\nu_i^{\dagger}(u - \alpha_i)))|^2_{H^1(\Omega_i)} \ + \ \sum_{j=1}^{4} ||I_h(\nu_i^{\dagger}(u - \alpha_i))||^2_{H_{00}^{1/2}(\gamma_{i,j})} \tag{6.67}$$

On Ω_i, we use the fact that $u - \alpha_i = \nu_i(\nu_i^\dagger(u - \alpha_i))$, and obtain as in (6.43) that

$$|\mathcal{H}(I_h(\nu_i^\dagger(u-\alpha_i)))|^2_{H^1(\Omega_i)} \;\leq\; C(1+log(H/h))^2\Big(|u-\alpha_i|^2_{H^1(\Omega_i)} + \frac{||u-\alpha_i||^2_{L^2(\Omega_i)}}{H^2}\Big)$$

Then, from the Poincaré inequality (see Thereom 1.3), we find

$$|\mathcal{H}(I_h(\nu_i^\dagger(u-\alpha_i)))|^2_{H^1(\Omega_i)} \;\leq\; C(1+log(H/h))^2|u|^2_{H^1(\Omega_i)}. \tag{6.68}$$

On Ω_i^1,

$$\begin{aligned}
||I_h(\nu_i^\dagger(u-\alpha_i))||^2_{H^{1/2}_{00}(\gamma_{i,1})} &= C||\pi_\gamma(\nu_i^\dagger(u-\alpha_i))||^2_{H^{1/2}_{00}(\gamma_{i,1})} \\
&\leq C||(\nu_i^\dagger(u-\alpha_i))\mid_{\zeta_{i,1}}||^2_{H^{1/2}(\partial\Omega_i)} \\
&\leq C(1+log(H/h))^2\Big(|u-\alpha_i|^2_{H^1(\Omega_i)} + \frac{||u-\alpha_i||^2_{L^2(\Omega_i)}}{H^2}\Big) \\
&\leq C(1+log(H/h))^2|u|^2_{H^1(\Omega_i)}. \qquad (6.69)
\end{aligned}$$

Similar bounds can be obtained for the other neighboring subregions of Ω_i.

From (6.67), (6.68), and (6.69), we conclude that

$$a(u_i,u_i) \;\leq\; C(1+log(H/h))^2|u|^2_{H^1(\Omega_i)} \;=\; C(1+log(H/h))^2|u|^2_{H^1(\widetilde{\Omega}_i)}. \tag{6.70}$$

Case 2. If $\widetilde{\Omega}_i$ contains more than one subregion, we assume once again that Ω_i has exactly one nonmortar side; cf. Figure 6.3.

We can prove once again that

$$a(u_i,u_i) \;=\; C(1+log(H/h))^2|u|^2_{H^1(\widetilde{\Omega}_i)},$$

by using the fact that $u-\alpha_i = \nu_i(\nu_i^\dagger(u-\alpha_i))$, if we note that u_{ext} and $u-\alpha_i$ have similar L^2 and H^1 upper bounds; cf. Lemma 6.2 and Lemma 6.3.

In both cases, we conclude that an upper bound of the form (6.70) is satisfied. When summing (6.70) for $i = 1 : N$, we obtain, using a simple coloring argument as before, that

$$\begin{aligned}
\sum_{i=1}^N a(u_i,u_i) &\leq C(1+log(H/h))^2\sum_{i=1}^N |u|^2_{H^1(\widetilde{\Omega}_i)} \\
&\leq C(1+log(H/h))^2 a(u,u). \qquad (6.71)
\end{aligned}$$

Finally, from (6.66) and (6.71), it results that

$$a(u_0, u_0) \ \leq \ C(1 + log(H/h))^2 a(u, u).$$

Thus, (6.65), and therefore (6.54), are proven. □

We are now able to derive an estimate for the condition number of the balancing algorithm for mortars introduced in Section 6.3.

Theorem 6.1. *The condition number of the balancing algorithm grows only polylogarithmically with the number of nodes in each subregion, and is independent of the number of subregions, N. More precisely,*

$$\kappa(T_{bal}) \ \leq \ C(1 + log(H/h))^4,$$

where C is a constant that does not depend on the properties of the partition.

Proof. From Lemma 6.1, we find $\kappa(T_{bal}) \leq \kappa(T_{N-N})$, where T_{N-N} is the Neumann-Neumann operator (6.7) restricted to Range$(I - P_0)$.

Since T_{N-N} is an additive Schwarz operator, we use the abstract Schwarz theory of Section 3.2 to obtain a bound on $\kappa(T_{N-N})$. From Lemma 6.5, it follows that the constant C_0 from Assumption 1 satisfies

$$C_0^2 \ \leq \ C(1 + log(H/h))^2.$$

Assumption 2 follows from strengthened Cauchy-Schwarz inequalities. For each local space there is a finite fixed number of local spaces which are not orthogonal to it. Then the inequality $\rho(\epsilon) \leq C$ can be established as in [52]. Since the coarse space solver is exact, and using Lemma 6.4 for the local spaces, the following estimate for the parameter ω of Assumption 3 can be established,

$$\omega \ \leq \ C(1 + log(H/h))^2.$$

We have now estimates for all the parameters measuring the convergence of an additive Schwarz method. From Theorem 3.1,

$$\kappa(T_{N-N}) \ \leq \ C(1 + log(H/h))^4,$$

and therefore, from Lemma 6.1 we obtain that

$$\kappa(T_{bal}) \ \leq \ C(1 + log(H/h))^4.$$

□

We conclude this section by placing Theorem 6.1 in a general context.

The condition number estimate of our balancing algorithm for mortar finite elements has two extra logarithmic factors compared to that of the balancing algorithm for conforming finite elements [89, 91]. In the classical algorithm, the constant C_0 of Assumption 1 is equal to 1. The splitting we have used for our algorithm is similar to the classical one, but fails to preserve $C_0 = 1$ because the extended subregion are not mutually disjoint. Thus, when establishing (6.54), we have to use Theorem 1.5 and we therefore can only prove that C_0 is of order $(1 + log(H/h))^2$.

We now discuss a possible version of the algorithm for mortars, the purpose of which is to make $C_0 = 1$ in Assumption 1. Two conditions must be satisfied for C_0 to equal 1: the approximate bilinear forms should be computed over Ω_i instead of $\widetilde{\Omega}_i$, and the local space component u_i multiplied with the counting function should equal u on $\partial\Omega_i$. Since the values of u on the nonmortar sides of Ω_i depend on its values at the opposite mortar sides, the local space must include degrees of freedom of the nodes of those mortars, including the vertex values. Thus, we obtain the V_i space we have introduced.

If, instead of integrating over $\widetilde{\Omega}_i$ in (6.18), the definition of $\tilde{a}_i(\cdot,\cdot)$, we integrate only over Ω_i, the Assumption 1 is satisfied with $C_0 = 1$. However, Assumption 3 will not hold. Consider the case from Figure 6.3. Assume there are twice as many nodes on the mortar side $\zeta_{i,i+1}$ than on the nonmortar side $\gamma_{i,i+1}$. Then, there exists a mortar function $v_i \in V_i$ which vanishes on $\overline{\gamma}_{i,i+1}$, and is nonzero on $\zeta_{i,i+1}$. If we set the values of v_i to zero at all the other nodes of $\partial\Omega_i$, then

$$\tilde{a}_i(v_i, v_i) \ = \ |\mathcal{H}(I_h(\nu_i^\dagger v_i))|^2_{H^1(\Omega_i)} \ = \ 0,$$

while

$$a(v_i, v_i) \geq |v_i|^2_{H^1(\Omega_{i+1})} > 0.$$

Therefore Assumption 3 is not satisfied.

6.6 The geometrically nonconforming case

The proof for the geometrically nonconforming case follows along the same lines as for the geometrically conforming case. In this section, we show how to overcome the technical difficulties inherent to a nonconforming partition.

The main difference between the two cases arises in Lemma 6.2, since opposite the nonmortar side γ there might be more than one mortar side. Therefore, a simple extension of the jump which involves only one subregion is no longer possible. A remedy can be found using the techniques introduced in Section 4.3 to estimate the L^2 norm of the jumps of mortar functions across nonmortars.

We use the notation of Section 6.4 and of Section 4.3. Let $\Gamma(\gamma_{i,i+1})$ be the union of the parts of mortars opposite the nonmortar side $\gamma_{i,i+1}$. As in Section 4.3.2, we consider a rectangle, Ω_{new}, opposite $\gamma_{i,i+1}$, such that $\Gamma(\gamma_{i,i+1})$ is a side of Ω_{new}. As before, it is possible to construct a function v_{new}, such that $v_{new} \in H^1(\Omega_{new})$ and v_{new} has the same average as v_i on $\Gamma(\gamma_{i,i+1})$. From (4.45), we also obtain

$$|v_{new}|^2_{H^1(\Omega_{new})} \;\leq\; \sum_{j=1}^{q(i)} |v_i|^2_{H^1(\Omega_i^j)}, \tag{6.72}$$

where Ω_i^j, $j = 1 : q(i)$, are the subregions opposite γ.

As in the proof of Lemma 6.2, we construct $\chi_i \in H^1(\Omega_{new})$, an extension of

$$v_{new}\,|_{\Gamma(\gamma_{i,i+1})} - v_i\,|_{\gamma_{i,i+1}}$$

from $\Gamma(\gamma_{i,i+1})$ to Ω_{new}, such that χ_i vanishes on the side of Ω_{new} opposite $\Gamma(\gamma_{i,i+1})$, and

$$|\chi_i|^2_{H^1(\Omega_{new})} \;\leq\; C|v_i|^2_{H^1(\Omega_i)} \;+\; |v_{new}|^2_{H^1(\Omega_{new})}. \tag{6.73}$$

Then, $u_{ext} \in H^1(\Omega_i \cup \Omega_{new})$ is given by

$$u_{ext} = \begin{cases} v_i & \text{on } \Omega_i \\ v_{new} + \chi_i & \text{on } \Omega_{new}. \end{cases}$$

From (6.72) and (6.73), we find

$$|u_{ext}|^2_{H^1(\Omega_i \cup \Omega_{new})} \;\leq\; C\Big(|v_i|^2_{H^1(\Omega_i)} \;+\; |v_{new}|^2_{H^1(\Omega_{new})} \;+\; |\chi_i|^2_{H^1(\Omega_{new})}\Big)$$

$$
\begin{aligned}
&\leq C\Big(|v_i|^2_{H^1(\Omega_i)} + |v_{new}|^2_{H^1(\Omega_{new})}\Big) \\
&\leq C\Big(|v_i|^2_{H^1(\Omega_i)} + \sum_{j=1}^{q(i)} |v_i|^2_{H^1(\Omega_i^j)}\Big) \\
&= C\ \tilde{a}_i(u_i, u_i).
\end{aligned}
$$

An estimate for the L^2 norm of u_{ext} follows immediately from the Poincaré inequality (see Theorem 1.3) applied on $\Omega_i \cup \Omega_{new}$.

After this step, the rest of the proofs of Assumptions 1 and 3, see Lemma 6.4 and Lemma 6.5 follow as in Section 6.5, if the new function u_{ext} is used.

Bibliography

[1] Yves Achdou and Yuri A. Kuznetsov. Substructuring preconditioners for finite element methods on nonmatching grids. *East-West J. Numer. Math.*, 3(1):1–28, 1995.

[2] Yves Achdou, Yuri A. Kuznetsov, and Olivier Pironneau. Substructuring preconditioners for the Q_1 mortar element method. *Numer. Math.*, 71(4):419–449, 1995.

[3] Yves Achdou, Yvon Maday, and Olof B. Widlund. Iterative substructuring preconditioners for mortar element methods in two dimensions. *SIAM J. Numer. Anal.*, 36:551–580, 1999.

[4] Robert Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[5] Faker Ben Belgacem, Yvon Maday, and Annalisa Buffa. The mortar finite element method for 3D Maxwell equations: First results. Technical report, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie – Centre National de la Recherche Scientifique, 1999.

[6] Faker Ben Belgacem. *Discretisations 3D Non Conformes pour la Méthode de Decomposition de Domaine des Elément avec Joints: Analyse Mathématique et Mise en Œvre pour le Probleme de Poisson.* PhD thesis, Université Pierre et Marie Curie, Paris, France, January 1993. Tech. Rep. HI-72/93017, Electricité de France.

[7] Faker Ben Belgacem. The mortar element method with Lagrange multipliers. Technical report, Université Paul Sabatier, Toulouse, France, 1994. To appear in Numer. Math.

[8] Faker Ben Belgacem and Yvon Maday. Non-conforming spectral element method for second order elliptic problem in 3D. Technical Report R93039, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie – Centre National de la Recherche Scientifique, 1993.

[9] Faker Ben Belgacem and Yvon Maday. The mortar element method for three dimensional finite elements. *RAIRO Modél. Math. Anal. Numér.*, 31:289–302, 1997.

[10] Christine Bernardi and Yvon Maday. Mesh adaptivity in finite elements by the mortar method. Technical Report R94029, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie – Centre National de la Recherche Scientifique, January 1995.

[11] Christine Bernardi and Yvon Maday. Raffinement de maillage en elements finis par la methode des joints. *C. R. Acad. Sci. Paris*, 320:373–377, 1995.

[12] Christine Bernardi and Yvon Maday. Spectral methods. In *Handbook of Numerical Analysis, Volume V: Techniques of Scientific Computing (Part 2)*, pages 209–485. North-Holland, 1997.

[13] Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[14] Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In H. Brezzi et al., editor, *Nonlinear partial differential equations and their applications*, pages 13–51. Paris, 1994.

[15] Ion Bică. *Iterative Substructuring Algorithms for the p-version Finite Element Method for Elliptic Problems*. PhD thesis, Courant Institute of Mathematical Sciences, September 1997. Tech. Rep. 743, Department of Computer Science, Courant Institute.

[16] Petter E. Bjørstad, Magne Espedal, and David E. Keyes, editors. *Ninth International Conference of Domain Decomposition Methods*. John Wiley & Sons, Ltd, Strasbourg, France., 1997.

[17] Dietrich Braess and Wolfgang Dahmen. Stability estimates for the mortar finite element method for 3–dimensional problems. *East-West J. Numer. Math.*, 6(4):249–263, 1998.

[18] Dietrich Braess, Wolfgang Dahmen, and Christian Wieners. A multigrid algorithm for the mortar finite element method. Technical Report 153, Institut für Geometrie und Praktische Mathematik, Aachen, March 1998. To appear in SIAM J. Numer. Anal.

[19] Dietrich Braess, Maksymilian Dryja, and Wolfgang Hackbusch. Multigrid method for nonconforming FE–discretisations with application to nonmatching grids. *Computing*, 63:1–25, 1999.

[20] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, I. *Math. Comp.*, 47(175):103–134, 1986.

[21] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, II. *Math. Comp.*, 49:1–16, 1987.

[22] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, III. *Math. Comp.*, 51:415–430, 1988.

[23] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, IV. *Math. Comp.*, 53:1–24, 1989.

[24] James H. Bramble, Joseph E. Pasciak, Junping Wang, and Jinchao Xu. Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.*, 57(195):1–21, 1991.

[25] Susanne C. Brenner. A two-level Schwarz preconditioner for nonconforming plate elements. *Numer. Math.*, 72(4):419–447, 1996.

[26] Susanne C. Brenner. Lower bounds for two-level additive Schwarz preconditioners with small overlap. Technical Report 98:01, Industrial Mathematics Institute, 1998. To appear in SIAM J. Sci. Comput.

[27] Susanne C. Brenner and L. Ridgway Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[28] Susanne C. Brenner and Li-Yeng Sung. Lower bounds for nonoverlapping domain decomposition preconditioners in two dimensions. Technical Report 98:04, Industrial Mathematics Institute, 1998. To appear in Math. Comp.

[29] Haim Brezis. *Analyse fonctionnelle: théorie et applications*. Masson, Paris, 1983.

[30] Annalisa Buffa, Yvon Maday, and Francesca Rapetti. A sliding mesh-mortar method for a two-dimensional eddy currents model of electric engines. Technical Report R99002, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie – Centre National de la Recherche Scientifique, 1999.

[31] Xiao-Chuan Cai. An optimal two-level overlapping domain decomposition method for elliptic problems in two and three dimensions. *SIAM J. Sci. Comp.*, 14:239–247, January 1993.

[32] Xiao-Chuan Cai. The use of pointwise interpolation in domain decomposition methods with non-nested meshes. *SIAM J. Sci. Comput.*, 16(1):250–256, 1995.

[33] Xiao-Chuan Cai, Maksymilian Dryja, and Marcus Sarkis. Overlapping nonmatching grid mortar element methods for elliptic problems. *SIAM J. Numer. Anal.*, 36:581–606, 1999.

[34] Mario A. Casarin. *Schwarz Preconditioners for Spectral and Mortar Finite Element Methods with Applications to Incompressible Fluids*. PhD thesis, Courant Institute of Mathematical Sciences, March 1996. Tech. Rep. 717, Department of Computer Science, Courant Institute.

[35] Mario A. Casarin. Quasi-optimal Schwarz methods for the conforming spectral element discretization. *SIAM J. Numer. Anal.*, 34(6):2482–2502, 1997.

[36] Mario A. Casarin and Olof B. Widlund. A hierarchical preconditioner for the mortar finite element method. *ETNA*, 4:75–88, June 1996.

[37] Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors. *Domain Decomposition Methods*, Philadelphia, PA, 1989. SIAM. Proceedings of the Second International Symposium on Domain Decomposition Methods, Los Angeles, California, January 14 - 16, 1988.

[38] Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors. *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1990. SIAM.

[39] Tony F. Chan, Barry F. Smith, and Jun Zou. Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. *Numer. Math.*, 73(2):149–167, 1996.

[40] Tony F. Chan and Jun Zou. Additive Schwarz domain decomposition methods for elliptic problems on unstructured meshes. *Numer. Algorithms*, 8(2–4):329–346, 1994.

[41] Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[42] Lawrence C. Cowsar, Jan Mandel, and Mary F. Wheeler. Balancing domain decomposition for mixed finite elements. *Math. Comp.*, 64(211):989–1015, July 1995.

[43] M. Crouzeix and V. Thomée. The stability in L^p and W_p^1 of the L^2-projection onto finite element spaces. *Math. Comp.*, 48(178):521–532, 1987.

[44] Maksymilian Dryja. Additive Schwarz methods for elliptic mortar finite element problems. In K. Malanowski, Z. Nahorski, and M. Peszynska, editors, *Modeling and Optimization of Distributed Parameter Systems with Applications to Engineering*. IFIP, Chapman & Hall, London, 1996.

[45] Maksymilian Dryja. An additive Schwarz method for elliptic mortar finite element problems in three dimensions. In Petter E. Bjørstad, Magne Espedal, and David E. Keyes, editors, *Ninth International Conference of Domain Decomposition Methods*, 1997.

[46] Maksymilian Dryja. An iterative substructuring method for elliptic mortar finite element problems with a new coarse space. *East-West J. Numer. Math.*, 5(2):79–98, 1997.

[47] Maksymilian Dryja, Marcus V. Sarkis, and Olof B. Widlund. Multilevel Schwarz methods for elliptic problems with discontinuous coefficients in three dimensions. *Numer. Math.*, 72(3):313–348, 1996.

[48] Maksymilian Dryja, Barry F. Smith, and Olof B. Widlund. Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. *SIAM J. Numer. Anal.*, 31(6):1662–1694, December 1994.

[49] Maksymilian Dryja and Olof B. Widlund. Some domain decomposition algorithms for elliptic problems. In Linda Hayes and David Kincaid, editors, *Iterative Methods for Large Linear Systems*, pages 273–291, San Diego, California, 1989. Academic Press. Proceeding of the Conference on Iterative Methods for Large Linear Systems held in Austin, Texas, October 19 - 21, 1988, to celebrate the sixty-fifth birthday of David M. Young, Jr.

[50] Maksymilian Dryja and Olof B. Widlund. Towards a unified theory of domain decomposition algorithms for elliptic problems. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 3–21. SIAM, Philadelphia, PA, 1990.

[51] Maksymilian Dryja and Olof B. Widlund. Multilevel additive methods for elliptic finite element problems. In Wolfgang Hackbusch, editor, *Parallel Algorithms for Partial Differential Equations, Proceedings of the Sixth GAMM-Seminar, Kiel, January 19–21, 1990*, pages 58–69, Braunschweig, Germany, 1991. Vieweg & Son.

[52] Maksymilian Dryja and Olof B. Widlund. Domain decomposition algorithms with small overlap. *SIAM J. Sci. Comput.*, 15(3):604–620, May 1994.

[53] Maksymilian Dryja and Olof B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, February 1995.

[54] Charbel Farhat, Po-Shu Chen, and Jan Mandel. A scalable Lagrange multiplier based domain decomposition method for time-dependent problems. *Int. J. Numer. Meth. Eng.*, 38:3831–3853, 1995.

[55] Charbel Farhat, Po-Shu Chen, and François-Xavier Roux. The two-level FETI method - part II: Extensions to shell problems, parallel implementation, and performance results. *Comput. Methods Appl. Mech. Eng.*, 155:153–180, 1998.

[56] Charbel Farhat and Michel Géradin. Using a reduced number of Lagrange multipliers for assembling a parallel incomplete field finite element approximation. *Comput. Methods Appl. Mech. Eng.*, 97:333–354, 1992.

[57] Charbel Farhat, Antonini P. Macedo, and Michel Lesoinne. A two-level domain decomposition method for the iterative solution of high frequency exterior Helmholtz problems. *Numer. Math.*, 1999. To appear.

[58] Charbel Farhat, Antonini P. Macedo, Michel Lesoinne, François-Xavier Roux, Frédéric Magoulès, and Armel de La Bourdonnaie. A non-overlapping domain decomposition method for the exterior Helmholtz problem. In Jan Mandel, Charbel Farhat, and Xiao-Chuan Cai, editors, *Tenth International Conference of Domain Decomposition Methods*, volume 218 of *Contemporary Mathematics*, pages 42–66. AMS, 1998.

[59] Charbel Farhat and Jan Mandel. The two-level FETI method for static and dynamic plate problems - part I: An optimal iterative solver for biharmonic systems. *Comput. Methods Appl. Mech. Eng.*, 155:129–152, 1998.

[60] Charbel Farhat, Jan Mandel, and François-Xavier Roux. Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Eng.*, 115:367–388, 1994.

[61] Charbel Farhat and François-Xavier Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Meth. Eng.*, 32:1205–1227, 1991.

[62] Charbel Farhat and François-Xavier Roux. Implicit parallel processing in structural mechanics. In J. Tinsley Oden, editor, *Computational Mechanics Advances*, volume 2 (1), pages 1–124. North-Holland, 1994.

[63] Gerald B. Folland. *Introduction to Partial Differential Equations*. Princeton Univ. Pr., 1996. 2nd edition.

[64] Leopoldo Franca, Charbel Farhat, Antonini P. Macedo, and Michel Lesoinne. Residual-free bubbles for the Helmholtz equation. *Internat. J. Numer. Meth. Eng.*, 40:4003–4009, 1997.

[65] Leopoldo Franca and Antonini P. Macedo. A two-level finite element method and its application to the Helmholtz equation. *Internat. J. Numer. Meth. Eng.*, 43:23–32, 1998.

[66] Alan George and Joseph Liu. *Computer Solution of Large Sparse Positive Definite Systems*. Prentice-Hall, Englewood Cliffs, NJ, 1981.

[67] Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors. *Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1988. SIAM. Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Paris, France, January 1987.

[68] Roland Glowinski, Yuri A. Kuznetsov, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors. *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1991. SIAM. Held in Moscow, USSR, May 21–25, 1990.

[69] Roland Glowinski, Jacques Périaux, Zhong-Ci Shi, and Olof B. Widlund, editors. *Eight International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1997. Wiley. Held in Beijing, PR China, May 16-20, 1995.

[70] Michael Griebel and Peter Oswald. On the abstract theory of additive and multiplicative Schwarz algorithms. *Numer. Math.*, 70:163–180, 1995.

[71] R. H. W. Hoppe, Y. Iliash, Y. Kuznetsov, Y. Vassilevski, and B.I. Wohlmuth. Analysis and parallel implementation of adaptive mortar finite element methods. *East-West J. Numer. Math.*, 6(4):223–248, 1998.

[72] Ronald H.W. Hoppe. Mortar edge element methods in R^3. Technical report, Universität Augsburg, Germany, 1998.

[73] David E. Keyes, Tony F. Chan, Gérard A. Meurant, Jeffrey S. Scroggs, and Robert G. Voigt, editors. *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1992. SIAM. Held in Norfolk, VA, May 6–8, 1991.

[74] David E. Keyes and Jinchao Xu, editors. *Domain Decomposition Methods in Science and Engineering*, volume 180 of *Contemporary Mathematics*,

Providence, R.I., 1994. AMS. Proceedings of the Seventh International Conference on Domain Decomposition, October 27-30, 1993, The Pennsylvania State University.

[75] Axel Klawonn and Luca F. Pavarino. An overlapping additive Schwarz method for a saddle point problem from linear elasticity. In *Lecture Notes in Computational and Applied Mathematics*. IMACS, 1997. Submitted.

[76] Axel Klawonn and Luca F. Pavarino. Overlapping Schwarz methods for mixed linear elasticity and Stokes problems. *Comput. Methods Appl. Mech. Engrg.*, 165(1–4):233–245, 1998.

[77] Axel Klawonn and Olof B. Widlund. A domain decomposition method with Lagrange multipliers for linear elasticity. Technical Report 780, Department of Computer Science, Courant Institute, 1999.

[78] Axel Klawonn and Olof B. Widlund. FETI and Neumann–Neumann iterative substructuring methods: Connections and new results. Technical report, Department of Computer Science, Courant Institute, 1999. In preparation.

[79] Catherine Lacour. *Analyse et Resolution Numérique de Méthodes de Sous-Domaines Non Conformes pour des Problèmes de Plaques*. PhD thesis, Université Pierre et Marie Curie, Paris, 1997.

[80] Catherine Lacour. Iterative substructuring preconditioners for the mortar finite element method. In Petter Bjørstad, Magne Espedal, and David Keyes, editors, *Ninth International Conference of Domain Decomposition Methods*, 1997. URL = `http://www.ddm.org/DD9/`.

[81] Catherine Lacour and Yvon Maday. Two different approaches for matching nonconforming grids: the mortar element method and the FETI method. *BIT*, 37:720–738, 1997.

[82] Choi-Hong Lai, Petter E. Bjørstad, Mark Cross, and Olof B. Widlund, editors. *Eleventh International Conference of Domain Decomposition Methods*. DDM.org, 1999. To appear.

[83] Peter Lax and Arthur Milgram. Parabolic equations. In *Contributions to the theory of partial differential equations*, volume 33 of *Annals of Mathematics Studies*, pages 167–190. Princeton University Press, 1954.

[84] Patrick Le Tallec. Neumann-Neumann domain decomposition algorithms for solving 2D elliptic problems with nonmatching grids. *East-West J. Numer. Math.*, 1(2):129–146, 1993.

[85] Patrick Le Tallec, Taoufik Sassi, and Marina Vidrascu. Three-dimensional domain decomposition methods with nonmatching grids and unstructured grid solvers. In David E. Keyes and Jinchao Xu, editors, *Seventh International Conference of Domain Decomposition Methods in Scientific and Engineering Computing*, volume 180 of *Contemporary Mathematics*, pages 61–74. AMS, 1994. Held at Penn State University, October 27-30, 1993.

[86] Jacques-Louis Lions and Enrico Magenes. *Nonhomogeneous Boundary Value Problems and Applications*, volume I. Springer, New York, Heidelberg, Berlin, 1972.

[87] Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42, Philadelphia, PA, 1988. SIAM.

[88] Pierre Louis Lions. On the Schwarz alternating method. II. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Domain Decomposition Methods*, pages 47–70, Philadelphia, PA, 1989. SIAM.

[89] Jan Mandel. Balancing domain decomposition. *Comm. Numer. Meth. Engrg.*, 9:233–241, 1993.

[90] Jan Mandel and Marian Brezina. Balancing domain decomposition: Theory and computations in two and three dimensions. Technical Report UCD/CCM 2, Center for Computational Mathematics, University of Colorado at Denver, 1993.

[91] Jan Mandel and Marian Brezina. Balancing domain decomposition for problems with large jumps in coefficients. *Math. Comp.*, 65:1387–1401, 1996.

[92] Jan Mandel, Charbel Farhat, and Xiao-Chuan Cai, editors. *Tenth International Conference of Domain Decomposition Methods*, volume 218 of *Contemporary Mathematics*. AMS, 1998.

[93] Jan Mandel and Radek Tezaur. Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.*, 73:473–487, 1996.

[94] Jan Mandel, Radek Tezaur, and Charbel Farhat. A scalable substructuring method by Lagrange multipliers for plate bending problems. *SIAM J. Numer. Anal.*, 1999.

[95] Vladimir G. Mazia. *Sobolev spaces*. Springer, New York, Heidelberg, Berlin, 1985.

[96] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Academia, Prague, 1967.

[97] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Academia, Prague, 1967.

[98] Shannon S. Pahl. Schwarz type domain decomposition methods for spectral element discretizations. Master's thesis, Department of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa, December 1993.

[99] K. C. Park, M. R. Justino, and C. A. Felippa. An algebraically partitioned FETI method for parallel structural analysis: Algorithm description. *Int. J. Numer. Meth. Eng.*, 40:2717–2737, 1997.

[100] K. C. Park, M. R. Justino, and C. A. Felippa. An algebraically partitioned FETI method for parallel structural analysis: Performance evaluation. *Int. J. Numer. Meth. Eng.*, 40:2739–2758, 1997.

[101] Luca F. Pavarino. *Domain Decomposition Algorithms for the p-version Finite Element Method for Elliptic Problems*. PhD thesis, Courant Institute, New York University, September 1992.

[102] Luca F. Pavarino. Additive Schwarz methods for the p-version finite element method. *Numer. Math.*, 66(4):493–515, 1994.

[103] Luca F. Pavarino and Olof B. Widlund. Iterative substructuring methods for spectral elements in three dimensions. In M. Křižek, P. Neittaanmäki, and R. Stenberg, editors, *The Finite Element Method: Fifty Years of the Courant Element*, pages 345–355, New York, Basel, Hong Kong, 1994. Marcel Dekker.

[104] Luca F. Pavarino and Olof B. Widlund. A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions. *SIAM J. Numer. Anal.*, 33(4):1303–1335, August 1996.

[105] Luca F. Pavarino and Olof B. Widlund. Iterative substructuring methods for spectral element discretizations of elliptic systems. I: Compressible linear elasticity. Technical Report 744, Dept. of Computer Science, Courant Institute of Mathematical Sciences, October 1997. To appear in SIAM J. Numer. Anal.

[106] Luca F. Pavarino and Olof B. Widlund. Iterative substructuring methods for spectral element discretizations of elliptic systems. II: Mixed methods for linear elasticity and Stokes flow. Technical Report 755, Dept. of Computer Science, Courant Institute of Mathematical Sciences, 1997. To appear in SIAM J. Numer. Anal.

[107] Alfio Quarteroni, Yuri A. Kuznetsov, Jacques Périaux, and Olof B. Widlund, editors. *Domain Decomposition Methods in Science and Engineering: The Sixth International Conference on Domain Decomposition*, volume 157 of *Contemporary Mathematics*. AMS, 1994. Held in Como, Italy, June 15–19,1992.

[108] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, 1999.

[109] Daniel Rixen and Charbel Farhat. Preconditioning the FETI method for problems with intra- and inter-subdomain coefficient jumps. In Petter Bjørstad, Magne Espedal, and David Keyes, editors, *Ninth International Conference of Domain Decomposition Methods*, 1997. URL = `http://www.ddm.org/DD9/`.

[110] Daniel Rixen and Charbel Farhat. A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. *Int. J. Numer. Meth. Engng.*, 44:489–516, 1999.

[111] Daniel Rixen, Charbel Farhat, and Michel Géradin. A two-step, two-field hybrid method for the static and dynamic analysis of substructure problems with conforming and non-conforming interfaces. *Comput. Methods Appl. Mech. Eng.*, 154:229–264, 1998.

[112] Daniel Rixen, Charbel Farhat, Radek Tezaur, and Jan Mandel. Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver. Technical report, Center for Computational Mathematics, University of Colorado at Denver, 1998. To appear in Int. J. Num. Meth. Eng.

[113] Y. Saad and M. H. Schultz. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comp.*, 7:856–869, 1986.

[114] Marcus V. Sarkis. *Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Conforming and Non-Conforming Elements*. PhD thesis, Courant Institute, New York University, September 1994.

[115] H. A. Schwarz. *Gesammelte Mathematische Abhandlungen*, volume 2, pages 133–143. Springer, Berlin, 1890. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, volume 15, 1870, pp. 272–286.

[116] Padmanabhan Seshaiyer. *Non-conforming hp Finite Element Methods*. PhD thesis, University of Maryland, Baltimore County, 1998.

[117] Padmanabhan Seshaiyer and Manil Suri. Uniform *hp* convergence results for the mortar finite element method. Technical report, University of Maryland, Baltimore County, July 1997. To appear in Math. Comp.

[118] Padmanabhan Seshaiyer and Manil Suri. *hp* submeshing via non-conforming finite element methods. Technical report, University of Maryland, Baltimore County, July 1998.

[119] Barry F. Smith. *Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity.* PhD thesis, Courant Institute of Mathematical Sciences, September 1990. Tech. Rep. 517, Department of Computer Science, Courant Institute.

[120] Barry F. Smith. A domain decomposition algorithm for elliptic problems in three dimensions. *Numer. Math.*, 60(2):219–234, 1991.

[121] Barry F. Smith. An optimal domain decomposition preconditioner for the finite element solution of linear elasticity problems. *SIAM J. Sci. Stat. Comput.*, 13(1):364–378, January 1992.

[122] Barry F. Smith, Petter Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge University Press, 1996.

[123] Dan Stefanica. On the L^2 stability of the 1-D mortar projection. Technical Report 767, Department of Computer Science, Courant Institute, 1998.

[124] Dan Stefanica. Poincaré and Friedrichs inequalities for mortar finite element methods. Technical Report 774, Department of Computer Science, Courant Institute, 1998.

[125] Dan Stefanica and Axel Klawonn. A numerical study of a class of FETI preconditioners for mortar finite elements in two dimensions. Technical Report 773, Department of Computer Science, Courant Institute, 1998.

[126] Dan Stefanica and Axel Klawonn. The FETI method for mortar finite elements. In Choi-Hong Lai, Petter E. Bjørstad, Mark Cross, and Olof B. Widlund, editors, *Eleventh International Conference on Domain Decomposition Methods*. DDM.org, 1999. To appear.

[127] Gilbert Strang. Variational crimes in the finite element method. In A. K. Aziz, editor, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, pages 689–710. Academic Press, New Tork, 1972.

[128] Daniel B. Szyld and Olof B. Widlund. Variational analysis of some conjugate gradient methods. *East-West J. Numer. Math.*, 1(1):51–74, 1993.

[129] Radek Tezaur. *Analysis of Lagrange multiplier based Domain Decomposition Methods*. PhD thesis, University of Colorado at Denver, 1998.

[130] Lloyd N. Trefethen and III David Bau. *Numerical Linear Algebra*. SIAM, Philadelphia, 1997.

[131] Hans Triebel. *Interpolation theory, function spaces, differential operators*. North-Holland, Amsterdam, New York, 1978.

[132] Olof B. Widlund. Some Schwarz methods for symmetric and nonsymmetric elliptic problems. In David E. Keyes, Tony F. Chan, Gérard A. Meurant, Jeffrey S. Scroggs, and Robert G. Voigt, editors, *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 19–36, Philadelphia, PA, 1992. SIAM.

[133] Olof B. Widlund. Domain decomposition methods for elliptic partial differential equations. In H. Bulgak and C. Zenger, editors, *Error control and Adaptivity in Scientific Computing*, pages 325–354. Kluwer Academic Publishers, 1999. Proceedings of the NATO Academic Study Institute, Antalya, 1999.

[134] Olof B. Widlund. Two-level Schwarz algorithms, using overlapping subregions, for mortar finite element methods. Technical report, Department of Computer Science, Courant Institute, 1999. In preparation.

[135] Christian Wieners and Barbara Wohlmuth. The coupling of mixed and conforming finite element discretizations. In Jan Mandel, Charbel Farhat, and Xiao-Chuan Cai, editors, *Proceedings of the 10th International Conference on Domain Decomposition*, pages 546–553. AMS, Contemporary Mathematics series, 1998.

[136] Christian Wieners and Barbara Wohlmuth. A general framework for multigrid methods for mortar finite elements. Technical Report 415, Math.-Nat. Fakultät, Universität Augsburg, 1999.

[137] Barbara Wohlmuth. Hierarchical a posteriori error estimators for mortar finite element methods with Lagrange multipliers. Technical Report 749, Department of Computer Science, Courant Institute, 1997. To appear in SIAM J. Numer. Anal.

[138] Barbara Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. Technical Report 407, Math.-Nat. Fakultät, Universität Augsburg, 1998.

[139] Barbara Wohlmuth. Multigrid methods for saddlepoint problems arising from mortar finite element discretizations. Technical Report 413, Math.-Nat. Fakultät, Universität Augsburg, 1999.

[140] Barbara Wohlmuth and Rolf Krause. Multigrid methods on the unconstrained product space arising from mortar finite element discretizations. Technical Report A18-99, Institut für Mathematik I, Freie Universität Berlin, 1999.

[141] Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.

[142] Jinchao Xu and Jun Zou. Some nonoverlapping domain decomposition methods. *SIAM Review*, 40(4):857–914, December 1998.

[143] Xuejun Zhang. *Studies in Domain Decomposition: Multilevel Methods and the Biharmonic Dirichlet Problem*. PhD thesis, Courant Institute, New York University, September 1991.

[144] Xuejun Zhang. Multilevel Schwarz methods. *Numer. Math.*, 63(4):521–539, 1992.