FETI AND NEUMANN–NEUMANN ITERATIVE SUBSTRUCTURING METHODS: CONNECTIONS AND NEW RESULTS

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Abstract. The FETI and Neumann-Neumann families of algorithms are among the best known and most severely tested domain decomposition methods for elliptic partial differential equations. They are iterative substructuring methods and have many algorithmic components in common but there are also differences. The purpose of this paper is to further unify the theory for these two families of methods and to introduce a new family of FETI algorithms. Bounds on the rate of convergence, which are uniform with respect to the coefficients of a family of elliptic problems with heterogeneous coefficients, are established for these new algorithms. The theory for a variant of the Neumann–Neumann algorithm is also redeveloped stressing similarities to that for the FETI methods.

Key words. domain decomposition, Lagrange multipliers, FETI, Neumann–Neumann, preconditioners, elliptic equations, finite elements, heterogeneous coefficients

AMS subject classifications. 65F10,65N30,65N55

1. Introduction. The FETI and Neumann–Neumann families of algorithms are among the best known and most severely tested domain decomposition methods for elliptic partial differential equations; cf., e.g., [1]. They are iterative substructuring methods and they share many algorithmic components, such as local solvers for both Neumann and Dirichlet problems on the subregions into which the region of the original problem has been partitioned. However, there are also differences and we have seen a need to extend our understanding of the FETI algorithms.

The Finite Element Tearing and Interconnecting (FETI) methods were first introduced by Farhat and Roux [16]. An important advance, making the rate of convergence of the iteration less sensitive to the number of unknowns of the local problems, was made by Farhat, Mandel, and Roux a few years later [14]. Our own work is based on the pioneering work by Mandel and Tezaur [24], who fully analyzed a variant of that algorithm. For a detailed introduction, see [15] or [33].

For early work on the Neumann-Neumann algorithms and their predecessors, see [5, 18, 2, 9, 3, 4, 22]. For a fine introduction, see [21].

The purpose of this paper is to extend, simplify, and unify the theory for the FETI and Neumann–Neumann algorithms. We introduce a new one-parameter family of FETI preconditioners and prove a bound on the rate of convergence which is independent of possible jumps of the coefficients of an elliptic model problem previously considered in the theory of Neumann–Neumann and other iterative substructuring algorithms; see [11, 8, 23, 30, 31]. In fact, we have found it possible to reduce the analytic core of the theory for the new class of FETI methods to a variant of an estimate

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which is central in the Neumann–Neumann theory. We will write an arbitrary element in a product space of traces of finite element functions as the sum of two terms. One of them is central in the FETI theory, the other in the Neumann–Neumann theory. The norm of each of the two terms is bounded by a factor $C(1+\log(H/h))$ times that of the given function. Here, and from now on, C is a generic constant, which may depend on the aspect ratios of the elements and subregions, but which is independent of the mesh parameters h and H and the coefficients of the elliptic problem; h is the diameter of a typical element into which the subregions have been divided and H is the typical subregion diameter. We note that $(H/h)^d, d = 2, 3$, measures the number of degrees of freedom associated with a subregion. We note that our bounds are developed locally for a single subregion and its neighbors and that we therefore can interpret H/h in the logarithmic bound as the maximum value of the diameter of any subregion divided by that of its smallest element.

The results for the new family of FETI algorithms become possible because of two special scalings. One of them, for the preconditioner, is closely related to an important algorithmic idea used in the best of the Neumann–Neumann methods. A proof of one of the two spectral bounds that are required then becomes just as elementary as for the Neumann–Neumann case. We note that our family of scalings of the preconditioner was apparently first introduced by Sarkis [30, 31]; see also [7]. The other scaling affects the choice of the projection which is used in each step of the FETI iteration, whether preconditioned or not.

We will show that, for a certain choice of the two scalings, our preconditioner is the same as one recently tested successfully for very difficult problems; see Bhardwaj et al. [1] and Rixen and Farhat [28] for an important earlier paper. Our algorithms are also defined for the class of problems treated in [1, 28], but in our analysis we have to impose certain restrictions on the coefficients and on the geometry of the subregions. We note that, by now, many variants of the FETI algorithms have been designed and that a number of them have been tested extensively; see in particular the discussion in [29].

This is the second paper on the FETI algorithms by the present authors; cf. [20]. Our present work has also already been extended to Maxwell's equation in two dimensions by Toselli and Klawonn [34].

The remainder of this paper is organized as follows. In section 2, we introduce our elliptic problems and the basic geometry of the decomposition; we have chosen to work only with the more interesting three dimensional case. In section 3, we give a short introduction to the FETI methods. In section 4, we introduce our family of preconditioners and prove our main results; our results should also be extendible to certain other elliptic problems such as those in [20], our recent study of problems of linear elasticity and the use of inexact solvers for the FETI algorithm. A connection between one element of our family of preconditioners and the method recently developed by Rixen and Farhat [28, 1] is established in section 5. In section 6, we summarize the essence of the balancing Neumann–Neumann iterative substructuring method stressing the similarities to the analysis of the FETI algorithms. In an appendix, that concludes our paper, we collect some auxiliary technical results which are needed in the proofs of Lemmas 6 and 8; they are borrowed almost directly from Dryja, Smith, and Widlund [8] and from Dryja and Widlund [11, 10].

2. Elliptic model problem, finite elements, and geometry. Let $\Omega \subset \mathbf{R}^3$, be a bounded, polyhedral region, let $\partial\Omega_D \subset \partial\Omega$ be a closed set of positive measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. We impose Dirichlet and Neumann boundary conditions, respectively, on these two subsets and introduce the standard Sobolev space $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$.

For simplicity, we will only consider a first order, conforming finite element approximation of the following scalar, second order model problem:

Find $u \in H_0^1(\Omega, \partial\Omega_D)$, such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \tag{1}$$

where

$$a(u, v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v dx, \;\; f(v) := \int_\Omega f v dx. \tag{2}$$

Here $\rho(x) > 0$ for $x \in \Omega$. For simplicity, we have chosen zero Neumann boundary data on $\partial\Omega_N$.

We decompose Ω into non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$, also known as substructures, and each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := \overline{\left(\bigcup_{i=1}^N \partial\Omega_i\right)} \setminus \partial\Omega$. We denote the standard finite element space of continuous, piecewise linear functions on Ω_i, which vanish at the nodes of $\partial\Omega_D$, by $W^h(\Omega_i)$. For simplicity, we assume that the triangulation of each subdomain is quasi uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $W_i := W^h(\partial\Omega_i), i = 1, \ldots, N$, and by $W := \prod_{i=1}^N W_i$ the associated product space. We note that we will often consider elements of W which are discontinuous across the interface. The finite element approximation of the elliptic problem is continuous across the interface and we denote the corresponding subspace of W by $\widehat{W}$. We note that all the iterates of the Neumann–Neumann methods belong to $\widehat{W}$ while those of the FETI methods normally do not.

We assume that possible jumps of $\rho(x)$ are aligned with the subdomain boundaries and, for simplicity, that in each subregion Ω_i, $\rho(x)$ has the constant value ρ_i. Our bilinear form and load vector can then be written, in terms of contributions from individual subregions, as

$$a(u, v) = \sum_{i=1}^N \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v dx, \quad f(v) = \sum_{i=1}^N \int_{\Omega_i} f v dx. \tag{3}$$

In our theoretical analysis, we assume that the subregions, Ω_i, are tetrahedra or parallelepipeds and that they are shape regular, i.e., not very thin. We assume that a nonempty intersection of the closure of any pair of subregions is the closure of an entire face, or an entire edge, or just a vertex. We also assume that if a face of a subdomain intersects $\partial\Omega_D$, then the measure of this set is comparable to that of the face. Similarly, if only an edge of a subdomain intersects $\partial\Omega_D$, we assume that the length of this intersection is bounded from below in terms of the length of the edge as a whole. For the FETI methods and the case of arbitrary coefficients, we also have to make a further assumption which is introduced just before Lemma 6.

We next introduce notations related to certain geometrical objects. A face of the substructure Ω_i will be called $\mathcal{F}^{ij}$, $\mathcal{E}^{ik}$ represent an edge, $\mathcal{V}^{i\ell}$ a vertex, and $\mathcal{W}^i$ the wire basket, i.e., the union of the edges and the vertices of the substructure. All the substructures, faces, and edges are regarded as open sets. We note that a face in the interior of the region Ω is common to exactly two substructures, an interior edge is shared by more than two, and a vertex is common to still more substructures. The sets of nodes on $\partial\Omega_i$, Γ, and $\partial\Omega$ are denoted by $\partial\Omega_{i,h}, \Gamma_h$, and $\partial\Omega_h$, respectively.

As in previous work on Neumann–Neumann algorithms, a crucial role is played by *the weighted counting functions* $\mu_i \in \widehat{W}$, which are associated with the individual $\partial\Omega_i$; cf. [7, 11, 23, 31]. They are defined, for $\gamma \in [1/2, \infty)$, and for $x \in \Gamma_h \cup \partial\Omega_h$, by a sum of contributions from Ω_i and its relevant next neighbors,

$$\mu_i(x) = \begin{cases} \sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x) & x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}, \\ \rho_i^\gamma(x) & x \in \partial\Omega_{i,h} \cap (\partial\Omega_h \setminus \Gamma_h), \\ 0 & x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h}. \end{cases} \tag{4}$$

Here, $\mathcal{N}_x$ is the set of indices of the subregions which have x on its boundary.

The pseudo inverses $\mu_i^\dagger \in \widehat{W}$ are defined, for $x \in \Gamma_h \cup \partial\Omega_h$, by

$$\mu_i^\dagger(x) = \begin{cases} \mu_i^{-1}(x) & \text{if } \mu_i(x) \neq 0, \\ 0 & \text{if } \mu_i(x) = 0. \end{cases}$$

We note that these functions provide a partition of unity:

$$\sum_i \rho_i^\gamma(x)\mu_i^\dagger(x) \equiv 1 \quad \forall x \in \Gamma_h \cup \partial\Omega_h. \tag{5}$$

3. A review of the FETI method. In this section, we review the original FETI method of Farhat and Roux [16, 17] and the variant with a Dirichlet preconditioner introduced in Farhat, Mandel, and Roux [14]. We will also introduce a general family of projections which was first introduced for heterogeneous problems in [17]. Such methods have recently been tested in very large scale numerical experiments; see [1]. For a more detailed description and extensions beyond scalar elliptic problems, see [12, 13, 25, 27, 33]. Let us point out that there are also other variants of the FETI methods; see, e.g., Park, Justino, and Felippa [26]. The relation of one of them to the FETI method developed by Farhat and Roux is discussed in [29] and a convergence analysis of this method can be found in Tezaur's dissertation [33].

For a chosen finite element method and for each subdomain Ω_i, we first assemble the local stiffness matrix $K^{(i)}$ and the local load vector corresponding to single, appropriate terms in the sums of (3). Any nodal variable, not associated with Γ_h, is called interior and it only belongs to one substructure. The interior variables of any subdomain can be eliminated by a step of block Gaussian elimination. This work can clearly be parallelized across the subdomains. The resulting matrices are the Schur complements

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)}(K_{II}^{(i)})^{-1}K_{I\Gamma}^{(i)}, \quad i = 1, \ldots, N.$$

Here, Γ and I represent the interface and interior, respectively. We note that the $S^{(i)}$ are only needed in terms of matrix-vector products and that therefore the elements of these matrices need not be explicitly computed.

The elimination of the interior variables of a substructure can also be viewed in terms of an orthogonal projection, with respect to the bilinear form $\langle K^{(i)}\cdot, \cdot\rangle$, onto the subspace of vectors with components that vanish at all the nodes of $\overline{\partial\Omega_i \setminus \partial\Omega_N}$. Here $\langle\cdot,\cdot\rangle$ denotes the ℓ_2–inner product. We note that these vectors represent elements of $W^h(\Omega_i) \cap H_0^1(\Omega_i, \overline{\partial\Omega_i \setminus \partial\Omega_N})$. These local subspaces are orthogonal, in this energy inner product, to the space of discrete harmonic vectors which represent discrete harmonic finite element functions. With v_Γ and w_Γ vectors of interface values, such a vector, (w_I, w_Γ), is defined, on the subdomain Ω_i, by

$$\langle K^{(i)} w, v\rangle = 0 \quad \forall v \ \text{ such that } \ v_\Gamma = 0, \tag{6}$$

or, equivalently, by

$$K_{II}^{(i)} w_I + K_{I\Gamma}^{(i)} w_\Gamma = 0. \tag{7}$$

We can regard w_Γ as a vector of Dirichlet data given on $\partial\Omega_{i,h} \cap \Gamma_h$ and note that a piecewise discrete harmonic function is completely defined by its values on the interface.

In what follows, we will almost exclusively work with functions in the trace spaces W_i and, whenever convenient, consider such an element as representing a discrete harmonic function in Ω_i. We will denote the discrete harmonic extension of a function w_i in the trace space W_i to the interior of Ω_i by $\mathcal{H}_i(w_i)$. We can easily show that $|w_i|^2_{S^{(i)}} = |\mathcal{H}_i(w_i)|^2_{K^{(i)}}$.

For $w \in W$, $\mathcal{H}(w)$ denotes the piecewise discrete harmonic extension into all the Ω_i. We also note that it is this piecewise discrete harmonic part of the solution, representing an element of $\widehat{W}$, that is determined by any iterative substructuring method; the other, interior, parts of the solution are computed locally as indicated above.

The values of the right hand vectors also change when the interior variables are eliminated. We denote the resulting vectors, representing the modified load originating in Ω_i, by f_i and the local vectors of interface nodal values by u_i.

We can now reformulate the finite element problem, reduced to the interface Γ, as a minimization problem with constraints given by the requirement of continuity across Γ :

Find $u \in W$, such that

$$\left.\begin{array}{c} J(u) := \frac{1}{2}\langle Su, u\rangle - \langle f, u\rangle \rightarrow \ \min \\ Bu = 0 \end{array}\right\} \tag{8}$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}, \text{ and } S = \begin{bmatrix} S^{(1)} & O & \cdots & O \\ O & S^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & S^{(N)} \end{bmatrix}.$$

The matrix $B = [B^{(1)}, \ldots, B^{(N)}]$ is constructed from $\{0, 1, -1\}$ such that the values of the solution u, associated with more than one subdomain, coincide when $Bu = 0$. We note that the choice of B is far from unique. The local Schur complements $S^{(i)}$ are positive semidefinite and they are singular for any subregion with a boundary which does not intersect $\partial\Omega_D$. The problem (8) is uniquely solvable if and only if $ker\,(S) \cap ker\,(B) = \{0\}$, i.e., if and only if S is invertible on $ker\,(B)$.

By introducing a vector of Lagrange multipliers λ, to enforce the constraints $Bu = 0$, we obtain a saddle point formulation of (8):

Find $(u, \lambda) \in W \times U$, such that

$$\left.\begin{array}{llll} Su & + & B^t\lambda & = & f \\ Bu & & & = & 0 \end{array}\right\}. \tag{9}$$

We note that the solution λ of (9) is unique only up to an additive vector of $ker\,(B^t)$. The space of Lagrange multipliers U is therefore chosen as $range\,(B)$.

We will also use a full column rank matrix built from all of the null space elements of S; these elements are associated with individual subdomains (the rigid body motions in the case of elasticity),

$$R = \begin{bmatrix} R^{(1)} & O & \cdots & O \\ O & R^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & R^{(N)} \end{bmatrix}.$$

Thus, $range\,(R) = ker\,(S)$. We note that there is no contribution to R from any subdomain the boundary of which intersects $\partial\Omega_D$.

A solution of the first equation in (9) exists if and only if $f - B^t\lambda \in range\,(S)$; this constraint will lead to the introduction of a projection P. We obtain,

$$u = S^\dagger(f - B^t\lambda) + R\alpha, \quad \text{if } f - B^t\lambda \perp ker\,(S),$$

where $S^\dagger$ is a pseudoinverse of S. We will see that α can be determined easily once λ has been found.

Substituting u into the second equation of (9) gives

$$BS^\dagger B^t\lambda = BS^\dagger f + BR\alpha. \tag{10}$$

We now introduce a symmetric, positive definite matrix Q which defines an inner product on U; it is defined by $\langle\lambda, \mu\rangle_Q := \langle\lambda, Q\mu\rangle$. By considering the component $Q^{-1}-$orthogonal to $G := BR$, we find that

$$\left.\begin{array}{rcl} P^tF\lambda & = & P^td \\ G^t\lambda & = & e \end{array}\right\} \tag{11}$$

with $F := BS^\dagger B^t, d := BS^\dagger f, P := I - QG(G^tQG)^{-1}G^t$, and $e := R^tf$. We note that P is an orthogonal projection, from U onto $ker\,(G^t)$; the projection is orthogonal in the $Q^{-1}-$inner product, i.e., the inner product defined by $\langle\lambda, Q^{-1}\mu\rangle$.

There are different successful choices for Q. In the case of homogeneous coefficients, it is sufficient to use $Q = I$, while for problems with jumps in the coefficients, we have to make more elaborate choices to make our proofs work satisfactorily. In our analysis, Q will be either a diagonal scaling matrix or the FETI Dirichlet preconditioner; see below, sections 4, 5, and [1, 17]. We note that we could view the introduction of a nontrivial Q in terms of a scaling of the matrix B from the left by the operator $Q^{\frac{1}{2}}$. By multiplying (10) by $(G^tQG)^{-1}G^tQ$, we find that $\alpha := (G^tQG)^{-1}G^tQ(F\lambda - d)$ which then fully determines the primal variables in terms of λ.

We introduce the spaces

$$V := \{\lambda \in U \,:\, \langle\lambda, Bz\rangle = 0 \quad \forall z \in ker\,(S)\} = ker\,(G^t) = range\,(P),$$

and

$$V' := \{\mu \in U : \langle \mu, Bz \rangle_Q = 0 \quad \forall z \in ker\,(S)\} = range\,(P^t).$$

It can be easily shown that V' is isomorphic to the dual space of V. Following Farhat, Chen, and Mandel [12], we call V the space of admissible increments. The original FETI method is a conjugate gradient method in the space V applied to

$$P^t F \lambda = P^t d, \qquad \lambda \in \lambda_0 + V, \tag{12}$$

with an initial approximation λ_0 chosen such that $G^t \lambda_0 = e$.

The most basic FETI preconditioner, as introduced in Farhat, Mandel, and Roux [14] for $Q = I$, is of the form

$$M^{-1} := BSB^t.$$

To apply M^{-1} to a vector, N independent Dirichlet problems have to be solved, one on each subregion; it is therefore called the Dirichlet preconditioner.

To keep the search directions of this preconditioned conjugate gradient method in the space V, the application of the preconditioner M^{-1} is followed by an application of the projection P. Hence, the Dirichlet variant of the FETI method is the conjugate gradient algorithm applied to the equation

$$PM^{-1}P^t F\,\lambda = PM^{-1}P^t\,d, \qquad \lambda \in \lambda_0 + V. \tag{13}$$

We note that for $\lambda \in V$, $PM^{-1}P^t F\lambda = PM^{-1}P^t P^t F P\lambda$, and that we can therefore view the operator on the left hand side of (13) as the product of two symmetric matrices.

It is well known that an appropriate norm of the iteration error of the conjugate gradient method will decrease at least by a factor

$$2(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1})^k,$$

in k steps; cf., e.g., [19]. Here κ is the ratio of the largest and smallest eigenvalues of the iteration operator. The main task in the theory is therefore always to obtain a good bound for the condition number κ.

We note that several different possibilities of improving the FETI preconditioner M^{-1} already have been explored. Some interesting variants are discussed by Rixen and Farhat [28], in a framework of mechanically consistent preconditioners, in the case of redundant Lagrange multipliers; see the discussion and analysis in section 5. A new family of improved FETI preconditioners, with non–redundant Lagrange multipliers, is introduced and analyzed in section 4.

4. New FETI methods with non–redundant Lagrange multipliers. In this section, we present and analyze a family of new FETI preconditioners with an improved condition number estimate compared to that of Mandel and Tezaur [24]; the bound in their paper involves three powers of $(1 + \log(H/h))$, in the general case, ours only two. In addition, we obtain a uniform bound for arbitrary positive values of the ρ_i if the operator Q, which enters the definition of P, is chosen carefully. In our proofs, we use a number of arguments developed in [24], but our presentation also differs considerably in several respects. We remark that for the FETI method described in Park, Justino, and Felippa [26] and for the case of continuous coefficients, a bound involving only two powers of $(1 + \log(H/h))$ is given in Tezaur [33].

We now assume, for the rest of this section, that B has full row rank, i.e., the constraints are linearly independent and there are no redundant Lagrange multipliers.

Our new preconditioner is defined, for any diagonal matrix D with positive elements, as

$$\widehat{M}^{-1} := (BD^{-1}B^t)^{-1}BD^{-1}SD^{-1}B^t(BD^{-1}B^t)^{-1}.$$

To obtain a method, which converges at a rate which is independent of the coefficient jumps, we now choose a special family of matrices D; a careful choice of the operator Q will also be required. As in previous work on Neumann–Neumann algorithms, a crucial role is played by the weighted counting functions μ_i, associated with the individual $\partial\Omega_i$, and already introduced in (4) in section 2. The diagonal matrix $D^{(i)}$ has the diagonal entry $\rho_i^\gamma(x)\mu_i^\dagger(x)$ corresponding to the point $x \in \partial\Omega_{i,h}$. Finally, we set

$$D := \begin{bmatrix} D^{(1)} & O & \cdots & O \\ O & D^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & D^{(N)} \end{bmatrix}.$$

We note that this matrix operates on elements in the product space W. This can be regarded as a scaling from the right, by $D^{-\frac{1}{2}}$, of the matrix B.

An important role will be played by $P_D := D^{-1}B^t(BD^{-1}B^t)^{-1}B$; this is a projection which is orthogonal in the scaled ℓ_2-inner product x^tDy, where $x, y \in W$. We note that this operator is invariant if we replace B by $Q^{\frac{1}{2}}B$.

Lemma 1. *For any $\mu \in U$, there exists a $\widehat{w} \in$ range (P_D), such that $\mu = B\widehat{w}$.*

Proof. For any $\mu \in U =$ *range* (B), there exists a $\tilde{w} \in W$, such that $\mu = B\tilde{w}$. We then select $\widehat{w} = P_D\tilde{w} \in W$, since, by a simple computation, $B\widehat{w} = B\tilde{w} = \mu$. □

The next result follows by noting that $BP_D = B$.

Lemma 2. *The projection operator P_D preserves jumps in the sense that*

$$w - P_D w \in \widehat{W},$$

i.e., this function is continuous across Γ, $\forall w \in W$.

To prepare for the analysis of the new preconditioner, we equip V' with the norm

$$\|\mu\|_{V'}^2 := |D^{-1}B^t(BD^{-1}B^t)^{-1}\mu|_S^2 = \langle \widehat{M}^{-1}\mu, \mu\rangle,$$

where $|w|_S := \sqrt{\langle Sw, w\rangle}$ is the semi–norm on the space W induced by the Schur complement S. We have

Lemma 3. *$\|\cdot\|_{V'}$ defines a norm on V'.*

Proof. Since $\|\cdot\|_{V'}$ is clearly a semi–norm, we only need to show that $\|\mu\|_{V'} = 0$ implies $\mu = 0$. Consider any $\mu \in V'$ with $\|\mu\|_{V'} = 0$. By Lemma 1, $\mu = B\widehat{w}$ for some $\widehat{w} \in$ *range* (P_D). Since $P_D\widehat{w} = \widehat{w}$, we obtain,

$$0 = \|\mu\|_{V'}^2 = \|B\widehat{w}\|_{V'}^2 = |D^{-1}B^t(BD^{-1}B^t)^{-1}B\widehat{w}|_S^2 = |\widehat{w}|_S^2.$$

Thus, $\widehat{w} \in$ *ker* (S) and by the definition of V', we find that $\mu = 0$ since

$$\|\mu\|_Q^2 = \langle \mu, Q\mu\rangle = \langle \mu, QB\widehat{w}\rangle = 0.$$

□

We also equip the space of admissible increments V with a norm

$$\|\lambda\|_V := \sup_{\mu \in V'} \frac{\langle \lambda, \mu \rangle}{\|\mu\|_{V'}}.$$

We note that V' is isomorphic to the dual space of V. Since

$$\|\mu\|_{V'}^2 = \langle \widehat{M}^{-1}\mu, \mu \rangle \quad \mu \in V', \tag{14}$$

we find by a simple computation that

$$\langle \widehat{M}\lambda, \lambda \rangle = \|\lambda\|_V^2 \quad \lambda \in V. \tag{15}$$

The next result will be needed in the proof of our main result, Theorem 1; cf. also Lemmas 7 and 8.

LEMMA 4. *For any $w \in W$, there exists a unique $z_w \in ker\,(S)$, such that $\tilde{w} := w + z_w$ with $B\tilde{w} \in V'$. Moreover,*

$$\|Bz_w\|_Q \le \|Bw\|_Q.$$

Proof. We recall that $\tilde{w} := w + z_w$, with $z_w \in ker\,(S)$, and $B\tilde{w} \in V'$, means that $B^tQB\tilde{w} \perp ker\,(S)$; this element can be found by minimizing $\|B(w+z)\|_Q^2$, $z \in ker\,(S)$. The uniqueness of z_w follows from the fact that $ker\,(S) \cap ker\,(B) = \{0\}$. The inequality follows from elementary variational arguments.

□

We now show that $P_D u$ can be computed very easily.

LEMMA 5. *Let $u \in W$. Then,*

$$P_D u = u - E_D u,$$

where $E_D u \in \widehat{W}$, i.e., a function that is continuous across Γ. The value of $E_D u$ at any $x \in \Gamma_h$ equals the $D-$weighted average of the values of u at that point.

Proof. By Lemma 2, $u - P_D u$ is continuous across the interface. Let $e_x \in \widehat{W}$ be equal to 1 at a point $x \in \Gamma_h$ and vanish at all other points of Γ_h. Then, since e_x is continuous across Γ, $Be_x = 0$. We find, by using the definition of P_D, that the $D-$weighted average $P_D u$ at x, which is equal to $e_x^t D P_D u$, vanishes. Thus the $D-$weighted averages of u and $E_D u$ coincide.

□

We will now establish an important stability estimate for P_D, which is at the core of the proof of our main result. It is closely related to a well–known result from the convergence theory of the Neumann–Neumann algorithms. For the choice $Q = \widehat{M}^{-1}$, we are then ready to prove one of our main results, Theorem 1. After its proof, we also consider a choice of a diagonal Q which will require a more detailed analysis. We include a proof of Lemma 6 to make this paper more complete and to prepare for the proof of Lemma 8.

Throughout the rest of this section, and in section 5, we will add an extra assumption on the geometry of the subregions; this extra assumption is not necessary in the Neumann–Neumann theory; cf. Dryja and Widlund [11] and section 6.

ASSUMPTION 1. *There are no subregion Ω_i with a boundary that intersects $\partial\Omega_{D,h}$ in only one or a few points.*

LEMMA 6. *For any $w \in$ range (S), we have*

$$|P_D w|_S^2 \leq C\,(1+\log(H/h))^2 |w|_S^2.$$

Here C is independent of h, H, γ, and the values of the ρ_i.

Proof. We will work with the $H^{1/2}(\partial\Omega_i)-$semi–norm, which is equivalent to the $S^{(i)}-$semi–norm; see the appendix for a short discussion and references to the literature. We begin our proof by using Lemma 5 to obtain a formula for $P_D w$ for an arbitrary element $w \in W$. We find, by using formula (5), that

$$P_D w(x) \;=\; \sum_{j \in \mathcal{N}_x} \rho_j^{\gamma} (\mu_j^{\dagger} (w_i(x) - w_j(x))),\; x \in \partial\Omega_{i,h}. \tag{16}$$

Here, $\mathcal{N}_x$ is again the set of indices of the subregions which have x on its boundary. We note that the coefficients in this expression are constant on each face and on each edge of $\partial\Omega_i$, and that they are independent of the particular choice of B.

On a face $\mathcal{F}^{ij}$, we have $P_D w = \rho_j^{\gamma}(\mu_j^{\dagger}(w_i - w_j))$, at any node, and we note, for future reference, that on this face Bw equals $w_i - w_j$ or $w_j - w_i$.

We cut the function $P_D w$ using the functions $\theta_{\mathcal{F}^{ij}}$, described in the appendix, and write $P_D w(x)$ as a sum of terms which vanish at all the interface nodes outside individual faces, edges, and vertices; cf., e.g., [8, 11, 10]. Using notations from the appendix, we find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}} \rho_j^{\gamma} \mu_j^{\dagger} (w_i - w_j))$$

and we have to estimate its $H_{00}^{1/2}(\mathcal{F}^{ij})-$norm.

With $\gamma \geq 1/2$, we can easily prove that

$$\rho_i (\rho_j^{\gamma} \mu_j^{\dagger})^2 \leq \min(\rho_i, \rho_j). \tag{17}$$

Using this inequality and Lemma 11, we obtain

$$\begin{aligned}
&\rho_i |I^h(\theta_{\mathcal{F}^{ij}} \rho_j^{\gamma} \mu_j^{\dagger}(w_i - w_j))|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\
&\leq C\,(1+\log(\frac{H_i}{h_i}))^2 \min(\rho_i,\rho_j) \left[|w_i - w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i} \|w_i - w_j\|^2_{L^2(\mathcal{F}^{ij})} \right] \\
&\leq C\,(1+\log(\frac{H_i}{h_i}))^2 \left[2\rho_i (|w_i|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H_i} ||w_i||^2_{L^2(\partial\Omega_i)}) \; + \right. \\
&\qquad \left. + \; 2\rho_j (|w_j|^2_{H^{1/2}(\partial\Omega_j)} + \frac{1}{H_j} ||w_j||^2_{L^2(\partial\Omega_j)}) \right].
\end{aligned}$$

We note that H_j and H_i are comparable since our subdomains, Ω_i and Ω_j, by assumption, are shape regular and share an entire face.

By using Lemma 12, we can estimate the contributions of the edges of Ω_i to the energy in terms of L_2-norms over the edges. These L_2-norms are then estimated by using Lemma 13. If four subdomains, e.g., $\Omega_i, \Omega_j, \Omega_k$, and Ω_ℓ, have an edge $\mathcal{E}^{ik}$ in common, then there are three contributions to the estimate of the contribution of Ω_i to $|P_D w|_S^2$, namely

$$\rho_i ||\rho_j^{\gamma} \mu_j^{\dagger}(w_i - w_j)\|^2_{L^2(\mathcal{E}^{ik})} + \rho_i ||\rho_k^{\gamma} \mu_k^{\dagger}(w_i - w_k)\|^2_{L^2(\mathcal{E}^{ik})} + \rho_i ||\rho_\ell^{\gamma} \mu_\ell^{\dagger}(w_i - w_\ell)\|^2_{L^2(\mathcal{E}^{ik})}.$$

We first consider the second term in detail assuming that Ω_i shares a face with each of Ω_j and Ω_ℓ, but that it shares only an edge with Ω_k as in Figure 1. We apply formula (17) and Lemma 13 and obtain,

$$\begin{aligned} & \rho_i \, ||\rho_k^\gamma \mu_k^\dagger (w_i - w_k)\|^2_{L^2(\mathcal{E}^{ik})} \\ \leq \;& 2 \left(\rho_i \|w_i\|^2_{L^2(\mathcal{E}^{ik})} + \rho_k \|w_k\|^2_{L^2(\mathcal{E}^{ik})} \right) \\ \leq \;& C(1+\log(H/h)) \Big[\rho_i \left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \tfrac{1}{H_i} \|w_i\|^2_{L^2(\mathcal{F}^{ij})} \right) + \\ & \qquad + \rho_k \left(|w_k|^2_{H^{1/2}(\mathcal{F}^{kj})} + \tfrac{1}{H_k} \|w_k\|^2_{L^2(\mathcal{F}^{kj})} \right) \Big] , \end{aligned}$$

since $\mathcal{F}^{ij}$ is a face of Ω_i and $\mathcal{F}^{kj}$ a face of Ω_k, which have the edge $\mathcal{E}^{ik}$ in common. We have now obtained a bound which, in fact, is better than that given above for the face contributions since there is only one logarithmic factor. Since Ω_i and Ω_j, as well as Ω_i and Ω_ℓ, have a face in common, the argument given above can be simplified for the first and third edge contributions.

As for the contributions from a vertex $\mathcal{V}^{i\ell}$ of a substructure, we can use an elementary argument to show that the square of the $H^{1/2}-$semi–norm of a finite element function w, which vanishes at all points of $\partial\Omega_{i,h}$ except at that vertex, is bounded by $C\,h|w(\mathcal{V}^{i\ell})|^2$. Such a term can trivially be estimated by the square of the L_2-norm over any edge which has the vertex as an end point. The rest of the argument is then the same as for the edge contributions.

Before we discuss a final case of special boundary subregions, we show how the the L_2-terms in our estimates can be eliminated. We use Poincaré's inequality, for the interior subregions, and the fact that $w \in \mathit{range}\,(S)$. For the subregions which have at least a substantial part of a face in common with $\partial\Omega_D$, we use the standard Friedrichs inequality.

By Assumption 1, there are no subregions which touch $\partial\Omega_D$ in just isolated nodes. The final case, consistent with our assumptions, is therefore that of a subregion Ω_i with only an edge intersecting $\partial\Omega_D$; the argument can easily be extended to the case when the measure of the intersection of an edge with $\partial\Omega_D$ is bounded from below in terms the subdomain diameter. The arguments above then have to be modified and we have to use a variant of Friedrichs' inequality given as Lemma 14 of our appendix; this type of work was done already in [11, Lemma 7]. We first note that the terms attributable to edges and vertices of such a subdomain create no problems since we only obtain one logarithmic factor from the basic estimates given above and then only one additional logarithmic factor from Lemma 14. For a face $\mathcal{F}^{ij}$, we define the average $\overline{w}$ of a finite element function w by

$$\overline{w} := \frac{1}{m_{ij}} \int_{\mathcal{F}^{ij}} w(x)dx \text{ with } m_{ij} := \int_{\mathcal{F}^{ij}} 1dx.$$

For the face term related to $\mathcal{F}^{ij}$, we write it as a sum of $(w_i - w_j) - \overline{(w_i - w_j)}$ and $\overline{(w_i - w_j)}$. We can now apply Lemma 11 directly to the first term and obtain an estimate of the form $C\,(1+\log(H/h))^2 |w_i - w_j|^2_{H^{1/2}(\mathcal{F}^{ij})}$, since $(w_i - w_j) - (\overline{w_i - w_j})$ has a zero average; a Friedrichs inequality is not required. The second term can be estimated by using Lemma 10, which gives only one logarithmic factor and a factor H_i, and the elementary inequality

$$\overline{(w_i - w_j)}^2 \leq C\, H_i^{-2}\, \|w_i - w_j\|^2_{L^2(\mathcal{F}^{ij})},$$

which is a direct consequence of the Cauchy–Schwarz inequality. Finally, Lemma 14 contributes one more logarithmic factor and the overall estimate therefore only involves two logarithmic factors.

□

In preparation for our first theorem, we combine the results of Lemmas 4 and 6.

LEMMA 7. *For any* $w \in range\,(S)$, *and the unique* $z_w \in ker\,(S)$ *given in Lemma 4, and for* $Q = \widehat{M}^{-1}$, *we have*

$$|P_{D} z_w|_S^2 \leq C\,(1 + \log(H/h))^2 |w|_S^2.$$

Here C *is independent of* h, H, γ, *and the values of the* ρ_i.

Proof. For any $u \in W$, we have

$$|P_{D} u|_S^2 = \langle S P_{D} u, P_{D} u \rangle = \langle \widehat{M}^{-1} B u, B u \rangle = \|Bu\|_Q^2.$$

According to Lemma 4, for any $w \in range\,(S)$, the unique $z_w \in ker\,(S)$, such that $w + z_w \in V'$, satisfies

$$\|B z_w\|_Q \leq \|Bw\|_Q.$$

The proof is completed by combining these results with Lemma 6.

□

We are now ready to prove a condition number estimate for the preconditioned FETI operator $P\widehat{M}^{-1}P^tF$.

THEOREM 1. *The condition number of the FETI method, with the new preconditioner* $\widehat{M}$, *and with* $Q = \widehat{M}^{-1}$, *satisfies*

$$\kappa(P\widehat{M}^{-1}P^tF) \leq C\,(1 + \log(H/h))^2.$$

Here, C *is independent of* h, H, γ, *and the values of the* ρ_i.

Proof. We have to estimate the smallest eigenvalue $\lambda_{min}(P\widehat{M}^{-1}P^tF)$ from below and the largest eigenvalue $\lambda_{max}(P\widehat{M}^{-1}P^tF)$ from above. We will show that

$$\langle \widehat{M}\lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq C\,(1 + \log(H/h))^2 \langle \widehat{M}\lambda, \lambda \rangle \quad \forall \lambda \in V. \tag{18}$$

Lower bound: We note that this bound is optimal in the sense that it is independent of h and H and possible coefficient jumps. It is derived using purely algebraic arguments.

Following Mandel and Tezaur [24, Proof of Lemma 3.11], we will use the formula

$$\langle F\lambda, \lambda \rangle = \sup_{w \in range\,(S)} \frac{\langle \lambda, Bw \rangle^2}{|w|_S^2}, \quad \lambda \in V. \tag{19}$$

For completeness, we provide a short proof of (19). We first note that $S^{-1/2}B^t\lambda \in range(S)$ has a good meaning since $\lambda \in V$ means that $B^t\lambda \in range\,(S)$. We find that

$$\begin{aligned} \langle F\lambda, \lambda \rangle &= \langle S^{\dagger} B^t \lambda, B^t \lambda \rangle &&= \|S^{-1/2} B^t \lambda\|^2 \\ &= \sup_{v \in range\,(S)} \frac{\langle S^{-1/2} B^t \lambda, v \rangle^2}{\|v\|^2} &&= \sup_{w \in range\,(S)} \frac{\langle B^t \lambda, w \rangle^2}{|w|_S^2}. \end{aligned}$$

Let $\mu \in V'$ be arbitrary. It follows from Lemma 1, that there exists a $\widehat{w} \in W$ such that $\mu = B\widehat{w}$ with $\widehat{w} \in range\,(P_{D})$. We denote by $\widehat{w}_\perp$ the component of $\widehat{w}$ which is orthogonal to $ker\,(S)$. Clearly, we have

$$\sup_{w \in range\,(S)} \frac{\langle \lambda, Bw \rangle^2}{|w|_S^2} \geq \frac{\langle \lambda, B\widehat{w}_\perp \rangle^2}{|\widehat{w}_\perp|_S^2}.$$

We also observe that, $\forall \widehat{w}$,

$$\langle S\widehat{w}_\perp, \widehat{w}_\perp \rangle = \langle S\widehat{w}, \widehat{w} \rangle, \tag{20}$$

and it also follows, from the definition of V, that

$$\langle \lambda, Bw_\perp \rangle = \langle \lambda, Bw \rangle, \quad \lambda \in V. \tag{21}$$

Using (20) and (21), we obtain, since $\widehat{w} = P_{D}\widehat{w}$,

$$\frac{\langle \lambda, B\widehat{w}_\perp \rangle^2}{|\widehat{w}_\perp|_S^2} = \frac{\langle \lambda, B\widehat{w} \rangle^2}{|\widehat{w}|_S^2} = \frac{\langle \lambda, \mu \rangle^2}{|D^{-1}B^t(BD^{-1}B^t)^{-1}\mu|_S^2} = \frac{\langle \lambda, \mu \rangle^2}{\|\mu\|_{V'}^2}.$$

The proof of the left inequality of (18) concludes by using the definition of the norm $\|\cdot\|_V$ and formula (15).

Upper bound: We will derive an upper bound for $\langle F\lambda, \lambda \rangle$ which depends only polylogarithmically on H/h and is independent of possible coefficient jumps.

Let $w \in \mathit{range}\,(S)$ be arbitrary. By Lemma 4, there exists a unique $z_w \in \mathit{ker}\,(S)$ such that $B(w + z_w) \in V'$. By using Lemmas 6 and 7, we obtain

$$|P_{D}(w + z_w)|_S^2 \leq C\,(1 + \log(H/h))^2 |w|_S^2.$$

Combining this formula with (19), we obtain, $\forall \lambda \in V$,

$$\begin{aligned}
\langle F\lambda, \lambda \rangle &= \sup_{w \in range\,(S)} \frac{\langle \lambda, Bw \rangle^2}{|w|_S^2} \\
&\leq C\,(1 + \log(H/h))^2 \sup_{w \in range\,(S)} \frac{\langle \lambda, Bw \rangle^2}{|P_{D}(w + z_w)|_S^2} \\
&= C\,(1 + \log(H/h))^2 \sup_{w \in range\,(S)} \frac{\langle \lambda, B(w + z_w) \rangle^2}{\|B(w + z_w)\|_{V'}^2} \\
&= C\,(1 + \log(H/h))^2 \sup_{\substack{\tilde{w} \in W \\ B\tilde{w} \in V'}} \frac{\langle \lambda, B\tilde{w} \rangle^2}{\|B\tilde{w}\|_{V'}^2} \\
&= C\,(1 + \log(H/h))^2 \sup_{\mu \in V'} \frac{\langle \lambda, \mu \rangle^2}{\|\mu\|_{V'}^2} \\
&= C\,(1 + \log(H/h))^2\, \|\lambda\|_V^2.
\end{aligned}$$

The proof of the right inequality of (18) concludes by using (15).

□

We conclude this section by showing that a quite different, diagonal choice of the operator Q can lead to an equally strong result. We observe that the proof of Theorem 1 remains valid for a different choice of Q if we can replace Lemma 7 by a bound of the same quality of $|P_{D} z_w|_S^2$ in terms of $|w|_S^2$; we will do so in Lemma 8. We recall that the operator Q was introduced in section 3 in the definition of the projection P, and that we also then noted that $Q^{\frac{1}{2}}$ provides a scaling of B from the left, i.e., a scaling of the rows of B. We also recall that the norm defined by Q is used in Lemma 4.

The following recipe for Q is successful for arbitrary values of the coefficients ρ_i, provided that the operator B is chosen in the particular way illustrated in Figure 1. This figure shows, without loss of generality, an edge and four subregions, Ω_i,

Ω_j, Ω_k, and Ω_ℓ, which have that edge in common. The subregion with the largest coefficient ρ_k plays a special role as indicated in the figure. For a vertex, we select the constraints, i.e., the rows of B, in the same way.

The elements of the diagonal matrix Q can be chosen as follows for the case of arbitrary coefficients:

$$\begin{array}{rcl} q_{\mathcal{F}^{ij}} & = & \min(\rho_i,\rho_j)\,(1+\log(H_i/h_i))\,\frac{h_i^2}{H_i} \\ q_{\mathcal{E}^{ik}} & = & \min(\rho_i,\rho_k)\,h_i \\ q_{\mathcal{V}^{i\ell}} & = & \min(\rho_i,\rho_\ell)\,h_i \end{array} \tag{22}$$

We note that we use the same scaling for all the pairs of points on the face $\mathcal{F}^{ij}$, and similarly, that the scale factor is also the same for all the constraints that force the nodal values of w_i and w_k, on the common edge, to match. We note that for the edge shown in Figure 1 there are three different edge weights, $q_{\mathcal{E}^{ik}}, q_{\mathcal{E}^{jk}}$, and $q_{\mathcal{E}^{\ell k}}$ corresponding to the three sets of constraints across that edge.

LEMMA 8. *For any $w \in$ range (S) and the unique $z_w \in$ ker (S), given in Lemma 4, and the diagonal scaling matrix Q given by (22), we have*

$$|P_D z_w|_S^2 \le C\,(1+\log(H/h))^2 |w|_S^2.$$

Here C is independent of h, H, γ, and the values of the ρ_i.

Proof. We recall that any element of *ker* (S), in particular z_w as constructed in Lemma 4, is constant in each subdomain; we denote the value of z_w associated with $\partial\Omega_i$ by z_i.

As in the proof of Lemma 6, we will focus on the contribution to $|P_D z_w|_S^2$ from one subdomain Ω_i. We note that for any nodal point on a face $\mathcal{F}^{ij}$ the value of Bz is $z_i - z_j$ or $z_j - z_i$. For the choice of B as indicated in Figure 1, there will be three components of Bz associated with any node on the common edge, namely, $z_i - z_k$, $z_j - z_k$, and $z_\ell - z_k$.

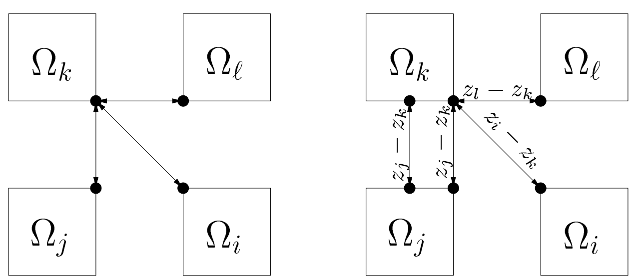

FIG. 1. *Left figure: Four subdomains meeting at an edge and a "fork" choice of the Lagrange multipliers. Right figure: Displacement differences for this choice.*

The strategy is now to estimate the contributions to $|P_D z_w|_S^2$ from individual faces, edges, and vertices of the substructure Ω_i in terms of jumps of z_w across the interface. We then interpret the jumps as elements of Bz_w and use the inequality given in Lemma 4, and the choice of Q given in (22), to obtain a bound in terms of $|w|_S^2$. Many of our arguments will be quite similar to those of the proof of Lemma 6.

We first consider the contribution to $|P_D z_w|_S^2$ from the face $\mathcal{F}^{ij}$ of Ω_i. By using Lemma 10 and formula (16), we can easily see that it can be bounded by

$$C\min(\rho_i,\rho_j)(1+\log(H_i/h_i))H_i(z_i-z_j)^2.$$

We then replace $z_i - z_j$ by the components of Bz_w corresponding to the nodal points on $\mathcal{F}^{ij}$. We will soon see that for the chosen scaling this expression can be bounded by appropriate terms in $\|Bz_w\|_Q^2$.

As in the proof of Lemma 6, we can use Lemma 12 to reduce the estimate of the $H^{1/2}(\partial\Omega_i)-$semi–norm of the three relevant edge terms of $P_D z_w$ to L_2-estimates. We obtain an upper bound of the form

$$CH_i\big(\min(\rho_i,\rho_j)(z_i - z_j)^2 + \min(\rho_i,\rho_k)(z_i - z_k)^2 + \min(\rho_i,\rho_\ell)(z_i - z_\ell)^2\big). \tag{23}$$

We can now absorb the first and third terms into the expressions for the faces $\mathcal{F}^{ij}$ and $\mathcal{F}^{i\ell}$, respectively. We also note that $z_i - z_k$ is an element of Bz_w.

Since the choice of B introduces a certain nonsymmetry, we will also examine the contributions from Ω_k and Ω_ℓ. We note that the edge terms related to Ω_k all contain factors that can be found among the elements of Bz_w; this is an easy case. The subdomain Ω_ℓ gives rise to the expression

$$CH_\ell\big(\min(\rho_\ell,\rho_i)(z_\ell - z_i)^2 + \min(\rho_\ell,\rho_j)(z_\ell - z_j)^2 + \min(\rho_\ell,\rho_k)(z_\ell - z_k)^2\big).$$

Of these, the first and third terms can be absorbed into face terms related to $\mathcal{F}^{\ell i}$ and $\mathcal{F}^{\ell k}$, respectively, but the second requires special attention since $z_\ell - z_j$ is neither an element of Bz_w, nor do Ω_j and Ω_ℓ share a face. Here we can instead use our assumption that ρ_k is at least as large as ρ_j and ρ_ℓ. The second term can then be bounded from above by

$$CH_\ell\big(2\min(\rho_\ell,\rho_k)(z_\ell - z_k)^2 + 2\min(\rho_j,\rho_k)(z_j - z_k)^2\big);$$

this alternative expression contains elements of Bz_w only.

The vertex contributions to the norm of $P_D z_w$ can be handled as those from the edges without introducing any new ideas.

We are now ready to use our scaling coefficients and Lemma 4. We first examine the contribution to $\|Bz_w\|_Q$ from the face $\mathcal{F}^{ij}$. The constant weight of $q_{\mathcal{F}^{ij}}$ is assigned to all the nodes of $\mathcal{F}^{ij}$, and since there are on the order of $(H_i/h_i)^2$ nodes on the face, we find that the contribution is proportional to $q_{\mathcal{F}^{ij}}(H_i/h_i)^2(z_i - z_j)^2$. The corresponding expression for one of the edge contributions can similarly be shown to be on the order of $q_{\mathcal{E}^{ik}}(H_i/h_i)(z_i - z_k)^2$.

By comparing coefficients, and using elementary estimates of different contributions to $\|Bw\|_Q$, we find that the contribution from the face $\mathcal{F}^{ij}$ can be bounded from above by

$$C\min(\rho_i,\rho_j)(1+\log(H/h))1/H_i\|w_i - w_j\|^2_{L_2(\mathcal{F}^{ij})},$$

and that of one of the edge contributions by

$$C\min(\rho_i,\rho_k)\|w_i - w_k\|^2_{L_2(\mathcal{E}^{ik})}.$$

By using Lemma 13, which introduces a logarithmic factor, we can estimate this edge contribution in terms of the full $H^{1/2}(\partial\Omega_i)-$norm. Except for possible special boundary subregions the boundaries of which only have an edge in common with Γ_D, we can now use standard Poincaré and Friedrichs inequalities to eliminate the $L_2(\mathcal{F}^{ij})-$terms. For the special subregions, we instead use Lemma 14, which gives rise to an additional, second logarithmic term. The contributions from the vertices can be handled very similarly; cf. the discussion in the proof of Lemma 6.

□

We have now completed all the work necessary for the proof of the following result.

THEOREM 2. *The condition number of the FETI method, with the preconditioner $\widehat{M}$, with Q as given in (22), and with B chosen as in Figure 1, satisfies*

$$\kappa(P\widehat{M}^{-1}P^tF) \le C\,(1+\log(H/h))^2.$$

Here, C is independent of h, H, γ, and the values of the ρ_i.

We conclude this section by considering simplifications possible if the collection of coefficients is less general. We first note that if the ρ_i are constant, or all of the same order of magnitude, then we can choose B arbitrarily and write any difference $z_i - z_m$ as the sum of such terms corresponding to faces; these terms can then be absorbed into face contributions to Bz_w. The matrix Q can then be chosen as a multiple of the identity matrix. In a more general case, we see that the special choice of B, used above in the discussion of the contributions of Ω_ℓ, is not necessary, if between any pair of subdomains, with an edge or vertex in common, there is a path through faces of neighboring subdomains, such that the coefficients are monotonically non–increasing or non–decreasing along the path. We note that this condition resembles, but is different from, the concept of quasi–monotone coefficients introduced in [7]. While the elements of Q corresponding to the faces still generally must depend on the coefficients, we note that in such a case we can decrease the values of the scale factors corresponding to edges and vertices quite arbitrarily.

5. FETI with redundant Lagrange multipliers. In this section, we extend our analysis to the case of redundant Lagrange multipliers. For a detailed algorithmic description of FETI preconditioners in this case, with $\gamma = 1$, together with an analysis based on mechanics, see Rixen and Farhat [27, 28]. To distinguish the redundant from the non–redundant case, we will denote Q by Q_r in this section. We will first choose Q_r to be the Dirichlet preconditioner and note that the resulting algorithm has proven successful for industrial problems; cf. Bhardwaj et al. [1]. At the end of this section, we also consider a diagonal Q_r constructed using the recipe given in (22).

Following Rixen and Farhat, we consider the case where a maximum number of redundant Lagrange multipliers are introduced, i.e., when all possible pair of degrees of freedom of the primal variables u, that belongs to the same nodal point $x \in \Gamma_h$, are connected by a Lagrange multiplier. Any edge or vertex node, where at least three subregions meet, will then contribute at least one additional Lagrange multiplier in comparison with the non–redundant case. An illustration of an edge common to four subregions is given in Figure 2.

We denote the new vector of Lagrange multipliers by λ_r. Similarly, we obtain a jump operator B_r with additional rows. We also introduce scaling matrices $D_r^{(i)}$, that operate on the Lagrange multiplier space, as follows: Consider, for a point $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$, the Lagrange multiplier connecting the corresponding two degrees of freedom in W_i and W_j, respectively. Then the diagonal entry of $D_r^{(i)}$ for that point is chosen as $\rho_j^\gamma(x)\mu_j^\dagger(x)$.

We note that $D_r := diag_{i=1}^N(D_r^{(i)})$ is a mapping from the Lagrange multiplier space onto itself, in contrast to the matrix D of the non–redundant case, discussed in section 4, which maps the space of primal variables W onto itself. We note that in the special case of continuous coefficients, we obtain the multiplicity scaling described

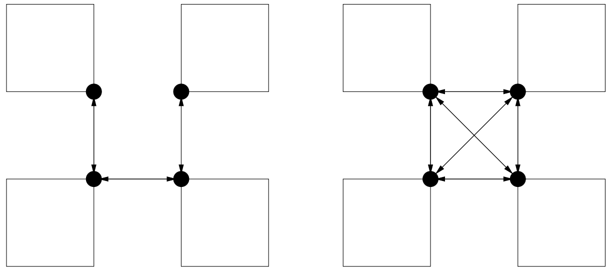

FIG. 2. *Left: U-shaped distribution of Lagrange multipliers for an edge in a non–redundant case. Right: Distribution of Lagrange multipliers for an edge in the fully redundant case.*

in [28, section 3]. Finally, we define a scaled jump operator by

$$B_{D_r} := [D_r^{(1)} B_r^{(1)}, \ldots, D_r^{(N)} B_r^{(N)}],$$

and the FETI preconditioner by

$$\widehat{M}_r^{-1} := \sum_i D_r^{(i)} B_r^{(i)} S^{(i)} B_r^{(i)t} D_r^{(i)} = B_{D_r} S B_{D_r}^t.$$

This preconditioner, with $\gamma = 1$, was introduced in Rixen and Farhat [28, section 5] in a framework of mechanically consistent preconditioners.

The matrix of the reduced linear system can be written as

$$F_r := B_r S^\dagger B_r^t.$$

Thus, we now have to solve the preconditioned system

$$P_r \widehat{M}_r^{-1} P_r^t F_r \lambda_r = P_r \widehat{M}_r^{-1} P_r^t d_r, \tag{24}$$

with $P_r := I - Q_r G_r (G_r^t Q_r G_r)^{-1} G_r^t$, $G_r := B_r R$, and $d_r := B_r S^\dagger f$. Here, Q_r can be either chosen as $\widehat{M}_r^{-1}$ or as the diagonal matrix defined in (22). We denote the inner product induced by Q_r by $\langle \lambda_r, \mu_r \rangle_{Q_r}$.

The next lemma shows that the redundant and the non–redundant implementations of the Lagrange multiplier methods yield the same corrections of the primal variables, in each iteration step. Using our notations and Lemma 5 in combination with formulae (28), (61), and (68) in [28, section 5], we can prove

LEMMA 9. *The operator $B_{D_r}^t B_r$, with its two factors just defined in this section, and $P_{\scriptscriptstyle D}$, defined in section 4, are the same:*

$$B_{D_r}^t B_r = P_{\scriptscriptstyle D}.$$

Informally, one can say that the Lagrange multipliers, of the two variants, could be viewed as temporary variables that can be hidden in two otherwise identical iterative methods, both written in terms of the primal variables.

A formal analysis of this FETI variant, with redundant Lagrange multipliers, can now be carried out using Lemma 9, adapting the arguments of section 4 to the current context, step by step. To start this process, we define the space of Lagrange multipliers as $U_r := \mathit{range}\,(B_r)$. This guarantees uniqueness of the Lagrange multiplier solution;

otherwise the solution of (24) would only be unique up to an additive term from $ker\,(B_r^t)$. Another possibility would be to work in the space of Lagrange multipliers modulo $ker\,(B_r^t)$.

As before, we then define a space of admissible increments

$$V_r := \{\lambda_r \in U_r : \langle \lambda_r, B_r z\rangle = 0 \quad \forall z \in ker\,(S)\} = range\,(P_r),$$

and the space

$$V_r' := \{\mu_r \in U_r : \langle \mu_r, B_r z\rangle_{Q_r} = 0 \quad \forall z \in ker\,(S)\} = range\,(P_r^t).$$

We equip V_r' with the norm

$$\|\mu_r\|_{V_r'} := |B_{D_r}^t \mu_r|_S, \quad \mu_r \in V_r',$$

and V_r with the norm

$$\|\lambda_r\|_{V_r} := \sup_{\mu_r \in V_r'} \frac{\langle \lambda_r, \mu_r\rangle}{\|\mu_r\|_{V_r'}}.$$

The fact that $\|\cdot\|_{V_r'}$ is a norm is established exactly as in the non–redundant case by using Lemma 9. Continuing completely as in the non–redundant case, we obtain

THEOREM 3. *The condition number of the FETI method defined by $\widehat{M}_r$ and $Q_r = \widehat{M}_r^{-1}$ satisfies*

$$\kappa(P_r \widehat{M}_r^{-1} P_r^t F_r) \leq C\,(1 + \log(H/h))^2.$$

Here C is independent of h, H, γ, and the values of the ρ_i.

In the rest of this section, we consider a diagonal Q_r given by the same recipe as in (22). As in section 4, we only have to prove a result equivalent to Lemma 8 for the fully redundant case. Examining the proof of that lemma, we see that we need only reexamine the edge and vertex contributions since there are no redundant Lagrange multipliers associated with the faces. The estimates of the vertex contributions can be reduced to those for the edge contributions, and it is therefore sufficient to consider only the latter.

In the fully redundant case, we have all possible Lagrange multipliers available and any formula such as (23) therefore already contains only elements of $B_r z_w$. The arguments in the proof of Lemma 8 can be simplified and we readily obtain a result analogous to Theorem 2.

THEOREM 4. *The condition number of the FETI method using the Dirichlet preconditioner $\widehat{M}_r$ and the diagonal matrix Q_r defined as in (22), satisfies*

$$\kappa(P_r \widehat{M}_r^{-1} P_r^t F_r) \leq C\,(1 + \log(H/h))^2.$$

Here C is independent of h, H, γ, and the values of the ρ_i.

Finally, as in section 4, we see that Theorem 4 still holds with Q_r chosen as a multiple of the identity matrix, when the ρ_i are constant or all of the same order of magnitude. As in section 4, we can also obtain other results for other special coefficient patterns.

6. The Neumann–Neumann balancing method. The purpose of this section is to review the theory for the balancing variant of the Neumann–Neumann methods. We note that there has been a number of previous studies of iterative substructuring methods in which condition number bounds are established which are independent of the ρ_i and grow only polylogarithmically with H/h; see, e.g., [32, 8, 11, 21, 23].

In this section, we will work exclusively with finite element functions that are continuous across the interface Γ, i.e., belong to the space $\widehat{W}$; the Neumann–Neumann iterates are all continuous functions. We recall that all elements of W and $\widehat{W}$ are piecewise discrete harmonic functions. There is no need to consider any finite element function not in this class; cf. discussion in section 3.

The Neumann–Neumann balancing method is analyzed in Mandel and Brezina [23]. It is a two level method with a coarse global space, $\widehat{W}_0$. Each interior substructure, i.e., one that does not intersect $\partial\Omega_D$, contributes one basis function, $\mu_i^\dagger$, to $\widehat{W}_0$. In addition, any substructure which touches $\partial\Omega_D$ in only one or a few points, also contribute a basis function. A detailed discussion of this matter is given in [11]; we believe these details are of no real importance to our discussion here.

We will solve one coarse problem in each iteration and will use an exact solver for this relatively small subspace, which has at most one degree of freedom per substructure. We denote the projection onto $\widehat{W}_0$, defined in terms of the bilinear form $a(\cdot,\cdot)$, by P_0.

In addition, just as for the FETI algorithms, there are local problems which are associated with the trace spaces W_i already introduced. We will only discuss those of the interior substructures in detail. We associate a subspace $\widehat{W}_i$ of continuous finite element functions with W_i. An element of $\widehat{W}_i$ takes on arbitrary values on $\partial\Omega_{i,h}$ and vanishes at all points of $\Gamma_h \setminus \partial\Omega_{i,h}$.

The local part of the balancing preconditioner is built from Neumann and Dirichlet solvers on the individual substructures. A bilinear form $\tilde{a}_i(u,v)$ is defined for the subspace $\widehat{W}_i$ by

$$\tilde{a}_i(u,v) \; = \; \rho_i^{1-2\gamma} \int_{\Omega_i} \nabla \mathcal{H}_i(\mu_i u) \cdot \nabla \mathcal{H}_i(\mu_i v) dx,$$

where, as before, $\mathcal{H}_i$ represents the discrete harmonic extension into the interior of Ω_i. This form is used to define a projection-like operator T_i onto $\widehat{W}_i$, given by

$$\tilde{a}_i(T_i u, v) = a(u,v) \quad \forall v \in \widehat{W}_i. \tag{25}$$

This operator is well defined only for finite element functions u for which $a(u,v)=0$ for all v for which $\mathcal{H}_i(\mu_i v)$ is constant on Ω_i. This condition is satisfied if $a(u,\mu_i^\dagger)=0$; the right hand side of (25) is then said to be balanced. We recall that this test function is a basis function for $\widehat{W}_0$.

We make the solution $T_i u$ of (25) unique by imposing the constraint

$$\int_{\Omega_i} \mathcal{H}_i(\mu_i T_i u) dx = 0, \tag{26}$$

which just means that we select the solution orthogonal to the null space of the Neumann problem.

Thus, the bilinear form of the left hand side of (25) is defined in terms of a diagonal scaling of the values on $\partial\Omega_{i,h}$, and by the $S^{(i)}-$inner product. This scaling has the real advantage that there is a convenient decomposition of any $u \in \widehat{W}$:

$$u = \sum_{i=1}^{N} u_i, \text{ with } u_i(x) = \rho_i^\gamma \mathcal{H}(\mu_i^\dagger u)(x). \tag{27}$$

We also note that this can equally well be expressed by saying that

$$E_D u = u, \quad u \in \widehat{W};$$

see Lemma 5. This is easily seen by using formula (5) and a simple computation. One can also show, straightforwardly, that

$$\sum_{i=1}^{N} \tilde{a}_i(u_i, u_i) = a(u, u). \tag{28}$$

We can now use P_0 and the T_i to construct a special hybrid Schwarz operator, see [32, Chapter 5.1], with the error propagation operator

$$(I - \sum_{i=1}^{N} T_i)(I - P_0),$$

or after an additional coarse solve,

$$(I - P_0)(I - \sum_{i=1}^{N} T_i)(I - P_0). \tag{29}$$

This is an operator symmetric with respect to $a(\cdot, \cdot)$, with which we can work without any extra real computational cost, since $(I - P_0)^2 = (I - P_0)$.

We note that the condition on the right hand side of (25) is satisfied for all elements in *range* $(I - P_0)$. The Schwarz operator is therefore well defined. We also note that for any given right hand side, we can use any solution of (25) in our computations since any two such solutions will differ only by an element in *ker* $(I - P_0)$.

Subtracting the operator (29) from I, we obtain the operator

$$T_{hyb} = P_0 + (I - P_0)(\sum_{i=1}^{N} T_i)(I - P_0). \tag{30}$$

It represents the preconditioned operator and is the operator relevant for the conjugate gradient iteration; see, e.g., [32, Chapter 5.1]. A condition number estimate for T_{hyb} is given in the next theorem.

THEOREM 5. *The condition number of T_{hyb} of the balancing Neumann–Neumann method satisfies*

$$\kappa(T_{hyb}) \leq C \, (1 + \log(H/h))^2.$$

Here, C is independent of h, H, γ, and the values of ρ_i.

Proof. It is easy to see that all that is required to estimate the condition number of T_{hyb} are upper and lower bounds of $T = \sum_{i=1}^{N} T_i$ restricted to *range* $(I - P_0)$. We choose to prove these bounds directly rather than in the framework of the abstract Schwarz theory as developed in [11, 8, 32] and in a number of other papers.

Lower bound: This bound is obtained quite easily using (27), (28), the definition given in (25), and Cauchy–Schwarz's inequality:

$$\begin{aligned} a(u,u) &= \sum_i a(u,u_i) = \sum_i \tilde{a}_i(T_i u, u_i) \\ &\leq \left(\sum_i \tilde{a}_i(T_i u, T_i u)\right)^{1/2} \left(\sum_i \tilde{a}_i(u_i,u_i)\right)^{1/2} \\ &= \left(\sum_i a(u, T_i u)\right)^{1/2} a(u,u)^{1/2} = a(Tu,u)^{1/2} a(u,u)^{1/2}. \end{aligned}$$

Therefore, squaring and cancelling a common factor, we find that $a(u,u) \leq a(Tu,u)$. *Upper bound:* For this bound, we introduce the notation $w_i = \rho_i^{-\gamma} \mathcal{H}_i(\mu_i T_i u)$ and note that this function is a multiple of the solution of the Neumann problem defined by (25). The w_i define an element $w = (w_i)_{i=1,\ldots,N}$ in the product space W and we also see that

$$Tu = \sum_i T_i u = \sum_i \rho_i^{\gamma} \mathcal{H}({\mu_i}^{\dagger} w_i) = E_D w.$$

We note that the computation of Tu requires the solution of a Dirichlet problem for each substructure since we have to find the discrete harmonic extension of the boundary data provided by the weighted averages computed at the points of Γ_h.

We can now essentially use the bound on the energy of $P_D w$ developed in the proof of Lemma 6. We can show, by a simple computation, that $\forall w \in W$ which satisfy the constraints (26), we have

$$a(E_D w, E_D w) \leq C(1+\log(H/h))^2 \sum_i \rho_i \int_{\Omega_i} |\nabla w_i|^2 dx. \tag{31}$$

Therefore, by using the definitions of the T_i, $\tilde{a}_i(\cdot,\cdot)$, and w_i, and selecting $v = T_i u$ as a test function in (25), we obtain,

$$\begin{aligned} a(Tu,Tu) &= a(E_D w, E_D w) \\ &\leq C(1+\log(H/h))^2 \sum_i \rho_i \int_{\Omega_i} |\nabla w_i|^2 dx \\ &= C(1+\log(H/h))^2 a(u,Tu). \end{aligned}$$

The upper bound

$$a(Tu,Tu)^{1/2} \leq C(1+\log(H/h))^2 a(u,u)^{1/2},$$

now follows immediately, by using Cauchy–Schwarz's inequality.

□

Appendix. Some auxiliary results. The purpose of this appendix is to provide, without proofs, the few auxiliary results that are required for complete proofs of Lemmas 6 and 8. These results are all borrowed from [8, 11, 10]. Here, we formulate them using trace spaces on the subdomain boundaries, i.e., $H^{1/2}(\partial\Omega_i)$, instead of the spaces $H^1(\Omega_i)$ and discrete harmonic extensions; given the well–known equivalence of the norms, nothing essentially new needs to be proven. The equivalence of the $S^{(i)}-$ and the $H^{1/2}(\partial\Omega_i)-$semi–norm of elements of W_i was established already in [2] for the case of piecewise linear elements and two dimensions. The tools necessary to extend this result to more general finite elements were provided in [35]; in our case, we of course have to multiply $|w_i|^2_{H^{1/2}(\partial\Omega_i)}$ by the factor ρ_i.

We also recall that we can define the $H^{1/2}_{00}(\tilde{\Gamma})-$norm of an element of W_i, $\tilde{\Gamma} \subset \partial\Omega_i$, as the $H^{1/2}(\partial\Omega_i)-$norm of the function extended by zero onto the rest of $\partial\Omega_i$.

The next lemma can, essentially, be found in Dryja, Smith, and Widlund [8, Lemma 4.4].

LEMMA 10. *Let $\theta_{\mathcal{F}^{ij}}$ be the finite element function that is equal to* 1 *at the nodal points on the face $\mathcal{F}^{ij}$, which is common to two subregions Ω_i and Ω_j, and that vanishes on $(\partial\Omega_{i,h} \cup \partial\Omega_{j,h}) \setminus \mathcal{F}^{ij}_h$. Then,*

$$|\theta_{\mathcal{F}^{ij}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C(1 + \log(H_i/h_i))H_i.$$

The same bounds also hold for the other subregion Ω_j.

We remark that the proof of Lemma 10 involves the explicit construction of a partion of unity constructed from functions $\vartheta_{\mathcal{F}^{ij}}$, with the same boundary conditions as the $\theta_{\mathcal{F}^{ij}}$, and which satisfies the bound of the lemma. This set of functions are well defined in the interior of the substructure where they form a partition of unity. The discrete harmonic function $\theta_{\mathcal{F}^{ij}}$ will have a smaller energy than $\vartheta_{\mathcal{F}^{ij}}$. Further details are not provided here; see, e.g., [8], [32, Chapter 5.3.2]. The following result can, essentially, be found in Dryja, Smith, and Widlund [8, Lemma 4.5] or in Dryja [6, Lemma 3]

LEMMA 11. *Let $\theta_{\mathcal{F}^{ij}}(x)$ be the function introduced in Lemma 10 and let I^h denote the interpolation operator onto the finite element space $W^h(\Omega_i)$. Then, $\forall u \in W_i$,*

$$|I^h(\theta_{\mathcal{F}^{ij}}u)|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \leq C(1 + \log(H_i/h_i))^2\big(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\big).$$

We will also need two additional results which are used to estimate the contributions to our bounds from the values on the wire basket. For the next lemma, see Dryja, Smith, and Widlund [8, Lemma 4.7].

LEMMA 12. *Consider all finite element functions $u \in W_i$ that vanish at all the nodal points on the faces of Ω_i. Then,*

$$|u|^2_{H^{1/2}(\partial\Omega_i)} \leq C\|u\|^2_{L^2(\mathcal{W}^i)}.$$

This result follows by estimating the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy.

We will also need a Sobolev-type inequality for finite element functions, see Dryja and Widlund [10, Lemma 3.3] or Dryja [6, Lemma 1].

LEMMA 13. *Let $\mathcal{E}^{ik}$ be any edge of Ω_i which forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then, $\forall u \in W_i$,*

$$\|u\|^2_{L^2(\mathcal{E}^{ik})} \leq C(1 + \log(H_i/h_i))\big(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\big).$$

Finally, we state a nonstandard version of Friedrichs' inequality that is given in a somewhat different form in [11, Lemma 6].

LEMMA 14. *Consider all finite element functions $u \in W_i$ that vanish on an edge $\mathcal{E}^{ik}$ of $\mathcal{F}^{ij}$. Then,*

$$\|u\|^2_{L^2(\mathcal{F}^{ij})} \leq C H_i (1 + \log(H_i/h_i)) |u|^2_{H^{1/2}(\mathcal{F}^{ij})}.$$

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