A DOMAIN DECOMPOSITION METHOD WITH LAGRANGE MULTIPLIERS FOR LINEAR ELASTICITY

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Abstract. A new domain decomposition method with Lagrange multipliers for elliptic problems is introduced. It is based on a reformulation of the well–known FETI method as a saddle point problem with both primal and dual variables as unknowns. The resulting linear system is solved with block–structured preconditioners combined with a suitable Krylov subspace method. This approach allows the use of inexact subdomain solvers for the positive definite subproblems. It is shown that the condition number of the preconditioned saddle point problem is bounded independently of the number of subregions and depends only polylogarithmically on the number of degrees of freedom of individual local subproblems. Numerical results are presented for a plane stress cantilever membrane problem.

Key words. domain decomposition, Lagrange multipliers, FETI, preconditioners, elliptic systems, finite elements.

AMS subject classifications. 65F10,65N30,65N55

1. Introduction. In the last decade a great deal of research has been carried out on nonoverlapping domain decomposition methods using Lagrange multipliers. In these methods the original domain is decomposed into nonoverlapping subdomains. The continuity is then enforced by using Lagrange multipliers across the interface defined by the subdomain boundaries. A computationally very efficient member of this class of domain decomposition algorithms is the Finite Element Tearing and Interconnecting (FETI) method introduced by Farhat and Roux [7]. In its original version, a Neumann problem is solved on each subdomain and the method is known to be scalable in the sense that its rate of convergence is independent of the number of subproblems. In a variant of the FETI method introduced in Farhat, Mandel, and Roux [6] an additional Dirichlet problem is solved exactly on each subdomain, in each iteration. This makes the rate of convergence of the iteration even less sensitive to the number of unknowns of the local problems. The use of inexact Dirichlet solvers is possible without a radical change of the FETI method. However, the use of inexact Neumann solvers does require a redesign of these algorithms; this is the topic of the present work.

In this paper, a new domain decomposition method with Lagrange multipliers is introduced by first reformulating the system of the FETI algorithm as a saddle point problem with both primal and dual variables. The resulting system is then solved using block–structured preconditioners and a suitable Krylov subspace method. We can then avoid potentially quite costly direct solvers relying instead on any of a number of well tested preconditioners for positive definite subproblems, such as incomplete

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LU methods, (algebraic) multigrid, etc. The good features of the FETI method such as scalability and efficiency are preserved.

The remainder of this article is organized as follows. In section 2, we present the equations of linear elasticity and a finite element discretization thereof. In section 3, we review the FETI method and we develop our new method in section 4. In subsection 5.1, the convergence analysis of Mandel and Tezaur [15] is extended from scalar, second order elliptic equations to the system of equations of linear elasticity. A convergence analysis and condition number estimates for the block–diagonal preconditioner are given in subsection 5.2. The paper concludes with section 6, in which we report on some of our numerical experiments.

We note that a short conference paper [12] has previously been prepared which describes and discusses a slightly different version of our main algorithm.

2. The elliptic problem. In this section, we introduce our model problem, the elliptic system arising from the displacement formulation of compressible, linear elasticity, and its discretization by conforming finite elements.

2.1. The equations of linear elasticity. The equations of linear elasticity model the displacement of a linear elastic material under the action of external and internal forces. We denote the elastic body by $\Omega \subset \mathbf{R}^d, d = 2, 3$, and its boundary by $\partial\Omega$ and assume that one part of the boundary, Γ_0, is clamped, i.e. with homogeneous Dirichlet boundary conditions, and that the rest, $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, is subject to a surface force $\mathbf{g}$, i.e. a natural boundary condition. We can also introduce an internal volume force $\mathbf{f}$, e.g. gravity. The appropriate space for a variational formulation is the Sobolev space $H^1_{\Gamma_0}(\Omega) := \{\mathbf{v} \in H^1(\Omega)^d \ : \ \mathbf{v}_{|\Gamma_0} = 0\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in H^1_{\Gamma_0}(\Omega)$ of the elastic body Ω, such that

$$(1) \qquad G \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx \ + \ G\beta \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in H^1_{\Gamma_0}(\Omega).$$

Here G and β are material parameters depending on the Poisson ratio ν and Young's modulus E. In the case of plane stress, we have $G = E/(1+\nu), \beta = 1/(1-\nu^2)$ and for plane strain and three dimensional elasticity, we have $G = E/(1+\nu), \beta = \nu/(1-2\nu)$. Furthermore, $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the linearized strain tensor, and

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{d} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v} \rangle := \sum_{i=1}^{d} \int_\Omega f_i v_i \, dx \ + \ \sum_{i=1}^{d} \int_{\Gamma_1} g_i v_i \, d\sigma.$$

The associated bilinear form of linear elasticity is

$$a(\mathbf{u}, \mathbf{v}) = G \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx \ + \ G\beta \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx.$$

In this article, we only consider the case of compressible elasticity. This means that the Poisson ratio ν is bounded away from $1/2$. We will also specialize to the case of $d = 2$ in order to simplify our notations. We note that all our work extends easily to the case of $d = 3$ and to many other elliptic problems.

We will also use the standard Sobolev space norm

$$\|\mathbf{u}\|_{H^1(\Omega)} := \left(|\mathbf{u}|^2_{H^1(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Omega)} \right)^{1/2}$$

with $\|\mathbf{u}\|^2_{L_2(\Omega)} := \int_\Omega |\mathbf{u}|^2 dx$, and $|\mathbf{u}|^2_{H^1(\Omega)} := \|\nabla \mathbf{u}\|^2_{L_2(\Omega)}$. It is obvious that the bilinear form $a(\cdot,\cdot)$ is continuous with respect to $\|\cdot\|_{H^1(\Omega)}$. However, proving ellipticity is far less trivial but it can be established from Korn's first inequality, see, e.g. Ciarlet [3].

LEMMA 1 (KORN'S FIRST INEQUALITY). *Let $\Omega \subset \mathbf{R}^d, d \geq 2$, be a Lipschitz domain. Then, there exists a positive constant $c = c(\Omega, \Gamma_0)$, such that*

$$\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) dx \geq c\, |\mathbf{u}|^2_{H^1(\Omega)} \quad \forall \mathbf{u} \in H^1_{\Gamma_0}(\Omega).$$

The wellposedness of the linear system (1) follows immediately from the continuity and ellipticity of the bilinear form $a(\cdot,\cdot)$.

It follows from Korn's first inequality that $\|\varepsilon(\mathbf{u})\|_{L_2(\Omega)}$ is equivalent to $\|\mathbf{u}\|_{H^1(\Omega)}$ on $H^1_{\Gamma_0}(\Omega)$. This result is not directly valid for the case of pure natural boundary conditions when we are working with the space $(H^1(\Omega))^d$. This case is of interest when considering interior subregions, i.e. those that do not touch Γ_0. However, a Gårding inequality is provided by Korn's second inequality

LEMMA 2 (KORN'S SECOND INEQUALITY). *Let $\Omega \subset \mathbf{R}^d, d \geq 2$, be a Lipschitz domain with diameter one. Then, there exists a positive constant $c = c(\Omega)$, such that*

$$\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) dx + \|\mathbf{u}\|^2_{L_2(\Omega)} \geq c\, \|\mathbf{u}\|^2_{H^1(\Omega)} \quad \forall \mathbf{u} \in (H^1(\Omega))^d.$$

There are several proofs; see, e.g. Nitsche [18].

We can now derive a Korn inequality on the space $\{\mathbf{u} \in (H^1(\Omega))^d : \mathbf{u} \perp \ker(\varepsilon)\}$. The null space $\ker(\varepsilon)$ is the space of rigid body motions. Thus, for $d = 2$, the linearized strain tensor of $\mathbf{u}$ and its divergence vanish only for the elements of the space spanned by the two translations

$$\mathbf{r}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{r}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the single rotation

$$\mathbf{r}_3 := \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}.$$

In three dimensions the corresponding space is spanned by three translations and three rotations.

For convenience, we also introduce the notation

$$(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} := \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx,$$

and introduce two inner products on $(H^1(\Omega))^2$, for a region Ω with diameter one,

$$(\mathbf{u}, \mathbf{v})_{E_1} := (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} + (\mathbf{u}, \mathbf{v})_{L_2(\Omega)}$$

and

$$(\mathbf{u}, \mathbf{v})_{E_2} := (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} + \sum_{i=1}^{3} l_i(\mathbf{u}) l_i(\mathbf{v})$$

where the $l_i(\cdot)$ are defined by

$$l_i(\mathbf{u}) := \int_\Omega (\mathbf{r}_i)^t \mathbf{u} dx.$$

By using Lemma 2, $||\cdot||_{E_1}$, given by the inner product $(\cdot,\cdot)_{E_1}$, is a norm and not just a seminorm, and so, by construction, is $||\cdot||_{E_2}$.

These norms are equivalent:

LEMMA 3. *There exist constants $0 < c \leq C < \infty$, such that*

$$c\,||\mathbf{u}||_{E_1} \leq ||\mathbf{u}||_{E_2} \leq C\,||\mathbf{u}||_{E_1} \quad \forall \mathbf{u} \in (H^1(\Omega))^2.$$

Proof: The proof of the right inequality follows immediately from the Cauchy–Schwarz inequality. The left inequality is proven by contradiction and by using Rellich's theorem as in a proof of generalized Poincaré–Friedrichs inequalities, cf., e.g. Nečas [17, Chapt. 2.7].

□

We obviously have

$$||\varepsilon(\mathbf{u})||_{L_2(\Omega)} \leq ||\nabla \mathbf{u}||_{L_2(\Omega)} \quad \forall \mathbf{u} \in (H^1(\Omega))^2. \tag{2}$$

Using (2) and Lemmas 2 and 3, we obtain

LEMMA 4. *There exist constants $0 < c \leq C < \infty$, such that*

$$c\,||\nabla \mathbf{u}||_{L_2(\Omega)} \leq ||\varepsilon(\mathbf{u})||_{L_2(\Omega)} \leq C\,||\nabla \mathbf{u}||_{L_2(\Omega)} \quad \forall \mathbf{u} \in (H^1(\Omega))^2, \mathbf{u} \perp \ker(\varepsilon).$$

2.2. Finite elements and the discrete problem. Since we only consider compressible elastic materials, it follows from Lemma 1 that the bilinear form $a(\cdot,\cdot)$ is uniformly elliptic. We can therefore successfully discretize the system (1) with low–order, conforming finite elements, such as linear or bilinear elements.

We assume that a triangulation τ^h of Ω is given which is shape regular and has a typical element diameter of h. We denote by $\mathbf{W}^h(\Omega) \subset H^1_{\Gamma_0}(\Omega)$ the corresponding conforming space of finite element functions, e.g. piecewise linear or bilinear continuous functions. Thus, it is our goal to solve the discrete problem

$$a(\mathbf{u}_h, \mathbf{v}_h) = \langle \mathbf{F}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{W}^h(\Omega). \tag{3}$$

In what follows, we work exclusively with the discrete problem and we drop the subscript h from now on.

3. A review of the FETI method. In this section, we give a brief review of the original FETI method introduced in Farhat and Roux [7] and the variant with a Dirichlet preconditioner introduced in Farhat, Mandel, and Roux [6]. For more detailed descriptions and proofs, we refer to [4, 5, 16, 21] and the references therein.

Let the domain $\Omega \subset \mathbf{R}^2$ be decomposed into N non-overlapping subdomains $\Omega_i, i = 1,\ldots,N$, each of which is the union of elements and such that the finite element nodes on the boundaries of neighboring subdomains match across the interface $\Gamma := \left(\bigcup_{i=1}^N \partial\Omega_i\right) \setminus \partial\Omega$. Let the corresponding conforming finite element spaces be $\mathbf{W}_i = \mathbf{W}^h(\Omega_i), i = 1,\ldots,N$, and let $\mathbf{W} := \prod_{i=1}^N \mathbf{W}_i$ be the associated product space. When it is necessary to use vectors of nodal values, which define the elements of a finite element space, we underline, e.g. $\underline{\mathbf{W}}$ is the product space that corresponds to $\mathbf{W}$. Analogously, we denote the vector of nodal values associated with the finite element function $\mathbf{u}$ by $\underline{\mathbf{u}}$.

For each subdomain $\Omega_i, i = 1, \ldots, N$, we assemble the local stiffness matrices K_i and local load vectors $\underline{\mathbf{f}}_i$ by integrating the appropriate expressions over individual subdomains. We denote the local vectors of nodal values by $\underline{\mathbf{u}}_i$.

We can now formulate a minimization problem with constraints given by the intersubdomain continuity conditions:

Find $\underline{\mathbf{u}} \in \underline{\mathbf{W}}$, such that

$$\left.\begin{array}{c} J(\underline{\mathbf{u}}) := \frac{1}{2}\underline{\mathbf{u}}^t K \underline{\mathbf{u}} - \underline{\mathbf{f}}^t \underline{\mathbf{u}} \to \min \\ B\underline{\mathbf{u}} = 0 \end{array}\right\} \tag{4}$$

where

$$\underline{\mathbf{u}} = \begin{bmatrix} \underline{\mathbf{u}}_1 \\ \vdots \\ \underline{\mathbf{u}}_N \end{bmatrix}, \underline{\mathbf{f}} = \begin{bmatrix} \underline{\mathbf{f}}_1 \\ \vdots \\ \underline{\mathbf{f}}_N \end{bmatrix}, \text{ and } K = \begin{bmatrix} K_1 & O & \cdots & O \\ O & K_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & K_N \end{bmatrix}.$$

The matrix $B = [B_1, \ldots, B_N]$ is constructed such that the values of the solution $\underline{\mathbf{u}}$, associated with more than one subdomain, coincide when $B\underline{\mathbf{u}} = 0$. The local stiffness matrices K_i are positive semidefinite. The problem (4) is uniquely solvable if and only if $\ker(K) \cap \ker(B) = \{0\}$, i.e. K is invertible on the null space of B. This condition holds since the original finite element model is elliptic.

By introducing a vector of Lagrange multipliers $\underline{\lambda}$ to enforce the constraint $B\underline{\mathbf{u}} = 0$, we obtain a saddle point formulation of (4):

Find $(\underline{\mathbf{u}}, \underline{\lambda}) \in \underline{\mathbf{W}} \times \underline{\mathbf{U}}$, such that

$$\left.\begin{array}{ccccc} K\underline{\mathbf{u}} & + & B^t\underline{\lambda} & = & \underline{\mathbf{f}} \\ B\underline{\mathbf{u}} & & & = & 0 \end{array}\right\} \tag{5}$$

We note that the solution $\underline{\lambda}$ of (5) is in general only unique up to an additive vector from $\ker(B^t)$. The space of Lagrange multipliers $\underline{\mathbf{U}}$ is therefore chosen as the range(B); see also discussion below.

We will also use a full rank matrix, built from the rigid body motions on the interior subdomains,

$$R = \begin{bmatrix} R_1 & O & \cdots & O \\ O & R_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & R_N \end{bmatrix},$$

such that $\text{range}(R) = \ker(K)$.

The solution of the first equation in (5) exists if and only if $\underline{\mathbf{f}} - B^t\underline{\lambda} \in \text{range}(K)$; this constraint will lead to the introduction of a projection P. We obtain

$$\underline{\mathbf{u}} = K^\dagger(\underline{\mathbf{f}} - B^t\underline{\lambda}) + R\underline{\alpha} \text{ if } \underline{\mathbf{f}} - B^t\underline{\lambda} \perp \ker(K),$$

where $K^\dagger$ is the pseudoinverse of K which provides the solution orthogonal to the null space of K and $\underline{\alpha}$ has to be determined; see discussion below.

We assume, for the time being, that B has full rank, i.e., the constraints are linearly independent. Substituting $\underline{\mathbf{u}}$ into the second equation of (5) gives

$$BK^{\dagger}B^t\underline{\lambda} = BK^{\dagger}\underline{\mathbf{f}} + BR\underline{\alpha}.$$

By considering the component orthogonal to BR, we find that

$$\left.\begin{array}{rcl} PF\underline{\lambda} & = & P\underline{\mathbf{d}} \\ G^t\underline{\lambda} & = & \underline{\mathbf{e}} \end{array}\right\} \tag{6}$$

with $G := BR, F := BK^{\dagger}B^t, \underline{\mathbf{d}} := BK^{\dagger}\underline{\mathbf{f}}, P := I - G(G^tG)^{-1}G^t$, and $\underline{\mathbf{e}} := R^t\underline{\mathbf{f}}$. We note that P is an orthogonal projection from $\underline{\mathbf{U}}$ onto $\ker(G^t)$.

Any solution $\underline{\lambda}$ of (5) and (6) yields the same solution $\underline{\mathbf{u}}$ of (4) and (5) if $\underline{\alpha} := -(G^tG)^{-1}G^t(\underline{\mathbf{d}} - F\lambda)$ is chosen; see Mandel, Tezaur, and Farhat [16, Theorem 2.4].

We define the space of admissible increments by

$$\underline{\mathbf{V}} := \{\underline{\mu} \in \underline{\mathbf{U}} : \underline{\mu} \perp B\underline{\mathbf{w}} \quad \forall \underline{\mathbf{w}} \in \ker(K)\} = \ker(G^t).$$

The original FETI method is a conjugate gradient method in the space $\underline{\mathbf{V}}$ applied to

$$PF\underline{\lambda} = P\underline{\mathbf{d}}, \qquad \underline{\lambda} \in \underline{\lambda}_0 + \underline{\mathbf{V}} \tag{7}$$

with an initial approximation $\underline{\lambda}_0$ chosen such that $G^t\underline{\lambda}_0 = \underline{\mathbf{e}}$. To introduce preconditioned variants, let D be a diagonal matrix. Then, the preconditioner M^{-1}, introduced in Farhat, Mandel, and Roux [6], is of the form

$$M^{-1} = B\begin{bmatrix} O & O \\ O & D^{-1}SD^{-1} \end{bmatrix} B^t$$

with

$$S = \begin{bmatrix} S_1 & O & \cdots & O \\ O & S_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & S_N \end{bmatrix}.$$

The matrix S is the Schur complement of K obtained by eliminating the interior degrees of freedom of each of the subdomains. This computation can clearly be carried out in parallel and results in the block–diagonal matrix S which only operates on the degrees of freedom on the subdomain boundaries. In the application of M^{-1} to a vector, N independent Dirichlet problems have to be solved in each iteration step; it is therefore often called the Dirichlet preconditioner. The simplest choice for D is the identity matrix; this choice is made for the original Dirichlet preconditioner as introduced in Farhat, Mandel, and Roux [6]. Another possibility, which leads to faster convergence, is to choose D as a diagonal matrix where the diagonal elements equal the number of subdomains to which the interface node belongs. This *multiplicity scaling* (MS) is discussed in Rixen and Farhat [21, 22].

To keep the search directions of this preconditioned conjugate gradient method in the space $\underline{\mathbf{V}}$, the application of the preconditioner M^{-1} has to be followed by another application of the projection P. Hence, the Dirichlet variant of the FETI method is the conjugate gradient algorithm applied to

$$PM^{-1}PF\underline{\lambda} = PM^{-1}P\underline{\mathbf{d}}, \qquad \underline{\lambda} \in \underline{\lambda}_0 + \underline{\mathbf{V}}. \tag{8}$$

4. The block–diagonal preconditioner. Using the decomposition $\lambda = \lambda_0 + \mu$, with $\mu \in V$, we can rewrite (7) as

$$PBK^{\dagger}B^t\underline{\mu} = PBK^{\dagger}(\underline{\mathbf{f}} - B^t\underline{\lambda}_0). \tag{9}$$

Since $\underline{\mathbf{u}} = K^{\dagger}(\underline{\mathbf{f}} - B^t\underline{\lambda}) + R\underline{\alpha}$, we see immediately that the solution of (9) can also be obtained by solving

$$\begin{bmatrix} K & B^t \\ PB & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}.$$

Using that $\underline{\mu} \in \underline{\mathbf{V}}$ i.e. $P\underline{\mu} = \underline{\mu}$, we can make the system matrix symmetric

$$\begin{bmatrix} K & (PB)^t \\ PB & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}. \tag{10}$$

We note that in this formulation we are not enforcing $B\underline{\mathbf{u}} = 0$ but only its projected version $PB\underline{\mathbf{u}} = 0$. The addition of an element of $\ker(K)$ does not change the solution $\underline{\mathbf{u}}$ of the first equation in (10). Since $B^t\underline{\mu} \perp \ker(K)$ for $\underline{\mu} \in \underline{\mathbf{V}}$, this is also true for the second equation in (10). We use this fact to post-process $\underline{\mathbf{u}}$, such that $B\underline{\mathbf{u}} = 0$ is finally satisfied. This can be done by setting $\underline{\mathbf{u}}_{cor} := \underline{\mathbf{u}} - R(G^tG)^{-1}G^tB\underline{\mathbf{u}}$; it easily follows that $B\underline{\mathbf{u}}_{cor} = PB\underline{\mathbf{u}} = 0$. Thus, we first compute the component of the solution in $\mathrm{range}(K)$ and then add the correct null space component, such that the solution has no jumps across the interface.

For the solution of the saddle point problem (10), we propose a preconditioned conjugate residual method with a block–diagonal preconditioner. For a detailed description of this algorithm, see Hackbusch [8] or Klawonn [10, 11]. We note that this algorithm will be designed such that the first component of the iterates belong to $\mathrm{range}(K)$.

Our preconditioner has the form

$$\mathcal{B} = \begin{bmatrix} \widehat{K} & O \\ O & \widehat{M} \end{bmatrix}$$

Here $\widehat{K}$ is assumed to be symmetric and a good preconditioner for $K + D_H\,Q$, where

$$Q = \begin{bmatrix} Q_1 & O & \cdots & O \\ O & Q_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & Q_N \end{bmatrix},$$

with Q_i the mass matrices associated with the mesh on Ω_i and $D_H = diag_{i=1}^{N}(H_i^{-2}I_i)$ is a diagonal matrix. Here H_i denotes the diameter of the subdomain Ω_i. We further assume that $\widehat{M}$ is symmetric and a good preconditioner for M, i.e. we assume there exists constants m_0, k_0, k_1, m_0, m_1 with $0 < c \leq m_0 \leq m_1 \leq C < \infty$ and $0 < c \leq k_0 \leq k_1 \leq C < \infty$, such that

$$\begin{array}{rcccll} k_0\,\underline{\mathbf{u}}^t(K + D_HQ)\underline{\mathbf{u}} & \leq & \underline{\mathbf{u}}^t\widehat{K}\underline{\mathbf{u}} & \leq & k_1\,\underline{\mathbf{u}}^t(K + D_HQ)\underline{\mathbf{u}} & \forall\underline{\mathbf{u}} \in \underline{\mathbf{W}}, \\ m_0\,\underline{\lambda}^tM\underline{\lambda} & \leq & \underline{\lambda}^t\widehat{M}\underline{\lambda} & \leq & m_1\,\underline{\lambda}^tM\underline{\lambda} & \forall\underline{\lambda} \in \underline{\mathbf{V}}. \end{array} \tag{11}$$

Here c and C are generic constants independent of, or only weakly dependent on, the mesh size. Because of the block diagonal structure of K and M and the preconditioners, c and C do not depend on the number of subdomains.

A preconditioner is often said to be optimal when the constants c and C can be chosen to be independent of the mesh size and the number of subdomains. We also note that all that is required here are preconditioners for quite benign positive definite problems on individual subregions and that the bounds are independent of the number of subdomains.

From these assumptions it is clear that our preconditioner $\mathcal{B}$ is symmetric positive definite and thus it can be used with the preconditioned conjugate residual method. In order to have a computationally efficient preconditioner, we must also assume that $\widehat{K}^{-1}$ and $\widehat{M}^{-1}$ can be applied to a vector at a low cost.

To guarantee that the iterates belong to range (K), we introduce the projection P_R onto range (K) by

$$P_R := I - R(R^t R)^{-1} R^t.$$

We recall that range $(R) = \ker(K)$ and note that P_R is a block matrix with a 3×3 block for each interior subdomain; the expense of applying P_R to a vector is therefore very modest.

The resulting domain decomposition method is the conjugate residual algorithm applied to the preconditioned system

$$\mathcal{B}^{-1}\mathcal{A}\underline{\mathbf{x}} = \mathcal{B}^{-1}\underline{\mathbf{F}}$$

with

$$\mathcal{A} = \begin{bmatrix} K & (PB)^t \\ PB & O \end{bmatrix}, \mathcal{B}^{-1} = \begin{bmatrix} P_R\widehat{K}^{-1}P_R^t & O \\ O & P\widehat{M}^{-1}P^t \end{bmatrix}, \tag{12}$$

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix}, \underline{\mathbf{F}} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}.$$

We note that it is easy to see that only two matrix-vector products with the projection P and one with the projection P_R are required in each step. We note that the iterates of the conjugate residual method belong to $\underline{\mathbf{W}}_R \times \underline{\mathbf{V}}$ with $\underline{\mathbf{W}}_R :=$ range (K).

5. Analysis. In this section, we will work with both finite element functions and vectors of nodal values representing them. We will make no distinction between operators and their matrix representation.

We will use the spaces $\mathbf{W}$, $\mathbf{U}$, and $\mathbf{V}$, and we begin by defining a norm $||\cdot||_W$ and a semi–norm $|\cdot|_W$ on $\mathbf{W}$ by

$$||\mathbf{v}||_W^2 := \sum_{i=1}^N ||\mathbf{v}_i||^2_{H^1(\Omega_i)}, \quad |\mathbf{v}|_W^2 := \sum_{i=1}^N |\mathbf{v}_i|^2_{H^1(\Omega_i)},$$

where $||\mathbf{v}_i||^2_{H^1(\Omega_i)} := |\mathbf{v}_i|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2}||\mathbf{v}_i||^2_{L_2(\Omega_i)}$ for $\mathbf{v} = [\mathbf{v}_1, \ldots, \mathbf{v}_N] \in \mathbf{W}$. We also need the orthogonal decomposition of $\mathbf{W}$ into $\mathbf{W}_R :=$ range (K) and its orthogonal complement $\mathbf{W}_R^\perp := \ker(K)$, i.e.

$$\mathbf{W} = \mathbf{W}_R \oplus \mathbf{W}_R^\perp.$$

For the analysis we need to consider the trace space $\mathbf{W}_\Gamma$ of $\mathbf{W}$. We equip the space $\mathbf{W}_\Gamma$ with the semi-norm and norm

$$|\mathbf{w}_\Gamma|^2_{W_\Gamma} := \sum_{i=1}^N |\mathbf{w}_{\Gamma,i}|^2_{H^{1/2}(\partial\Omega_i)}, \ ||\mathbf{w}_\Gamma||^2_{W_\Gamma} := |\mathbf{w}_\Gamma|^2_{W_\Gamma} + \sum_{i=1}^N \frac{1}{H_i} ||\mathbf{w}_{\Gamma,i}||^2_{L_2(\partial\Omega_i)}.$$

We also define

$$B_\Gamma \, : \, \mathbf{W}_\Gamma \to \mathbf{U} \quad B_\Gamma := B_{|\mathbf{W}_\Gamma}.$$

From the construction of the jump operator B it is clear that

$$B\mathbf{w} = B_\Gamma \mathbf{w}_\Gamma \text{ for } \mathbf{w} \in \mathbf{W} \text{ where } \mathbf{w}_\Gamma := \mathbf{w}_{|\Gamma},$$

since continuity is only enforced accross the interface. As a consequence, we obtain $\ker(B^t) = \ker(B^t_\Gamma)$. A direct computation yields

$$F = BK^\dagger B^t = B_\Gamma S^\dagger B^t_\Gamma.$$

Let us now introduce a norm on $\mathbf{V}$ by $||\lambda||_V := |B^t_\Gamma \lambda|_{W_\Gamma}$. This defines a norm, and not only a seminorm, since range $(B^t_\Gamma) \perp \ker(S)$. We will also use the dual space $\mathbf{V}'$ with a norm defined by $||\lambda||_{V'} := \sup_{\mu \in V} \frac{(\lambda, \mu)}{||\mu||_V}$. For the sake of simplicity, we will identify the space $\mathbf{V}'$ with $\mathbf{V}$, but we will use both norms.

Let us also define the product space $\mathbf{X} := \mathbf{W} \times \mathbf{V}$ which we equip with the graph norm $||\mathbf{x}||_X := \sqrt{||\mathbf{v}||^2_W + ||\lambda||^2_{V'}}$ for $\mathbf{x} = (\mathbf{v}, \lambda) \in \mathbf{X}$. We also need the subspace $\mathbf{X}_R := \mathbf{W}_R \times \mathbf{V}$ with the same norm as $\mathbf{X}$.

5.1. FETI for linear elasticity. For scalar, second order elliptic equations, it has been shown by Mandel and Tezaur [15] that the condition number of the FETI method with the original Dirichlet preconditioner (D=I) satisfies

$$\kappa(PM^{-1}PF) \leq C\,(1 + \log(H/h))^3.$$

We note that $(H/h)^2$ is proportional to the number of degrees of freedom of a subdomain. In the proof of such results, additional assumptions on the shape of the subregions Ω_i are required; typically it is assumed that the Ω_i are tetrahedra or cubes, or smooth images of such a regular region. In addition, the mesh is assumed to be quasi–uniform on each subregion.

In this section, we extend the results of Mandel and Tezaur [15] from scalar elliptic equations to the system of of linear elasticity. To be able to use the results of [15], we now introduce the vector–valued Laplacian to be used only in our analysis. We denote by

$$K_\Delta := \begin{bmatrix} K_{\Delta,1} & O & \cdots & O \\ O & K_{\Delta,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & K_{\Delta,N} \end{bmatrix},$$

a block–diagonal matrix, where the local stiffness matrices $K_{\Delta,i}, i = 1, \ldots, N$, are obtained from the discretization of the inner product $(\nabla\mathbf{u}, \nabla\mathbf{u})_{H^1(\Omega_i)}$ using the same finite element space as for $a(\cdot,\cdot)$. We denote by S_Δ the Schur complement of K_Δ obtained by eliminating the interior variables of each Ω_i, just as in the case of K and S.

It is well known, that

$$(S_\Delta \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) = \min_{\substack{\mathbf{v}\in\mathbf{W}\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K_\Delta \mathbf{v}, \mathbf{v}) = (K_\Delta \mathbf{v}_{\text{harm}}, \mathbf{v}_{\text{harm}})$$

for $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$. Here $\mathbf{v}_{\text{harm}} \in \mathbf{W}$ is the discrete harmonic extension of $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$ defined as the unique solution of

$$(K_\Delta \mathbf{v}_{\text{harm}}, \mathbf{v}) = 0 \;\; \forall \mathbf{v} \in \mathbf{W}_0 \text{ with } \mathbf{v}_{\text{harm}|\Gamma} = \mathbf{u}_\Gamma.$$

Here $\mathbf{W}_0$ is the subspace of $\mathbf{W}$ with elements that vanish on the interface Γ.

Returning to the elasticity case, it can also easily be seen that

$$(S\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) = \min_{\substack{\mathbf{v}\in\mathbf{W}\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K\mathbf{v}, \mathbf{v}) = (K\mathbf{v}_{\text{elast}}, \mathbf{v}_{\text{elast}})$$

for $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$. Here $\mathbf{v}_{\text{elast}} \in \mathbf{W}$ is the extension of $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$ with the smallest elastic energy defined as the unique solution of

$$(K\mathbf{v}_{\text{elast}}, \mathbf{v}) = 0 \;\; \forall \mathbf{v} \in \mathbf{W}_0 \text{ with } \mathbf{v}_{\text{elast}|\Gamma} = \mathbf{u}_\Gamma.$$

It follows from these formulas, using inequality (2), that for $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$,

$$\begin{aligned}
(S\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) &= \min_{\substack{\mathbf{v}\in\mathbf{W}\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K\mathbf{v}, \mathbf{v}) \\
&\leq (K\mathbf{v}_{\text{harm}}, \mathbf{v}_{\text{harm}}) \\
&\leq C\,(K_\Delta \mathbf{v}_{\text{harm}}, \mathbf{v}_{\text{harm}}) \\
&= C \min_{\substack{\mathbf{v}\in\mathbf{W}\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K_\Delta \mathbf{v}, \mathbf{v}) \\
&= C\,(S_\Delta \mathbf{u}_\Gamma, \mathbf{u}_\Gamma).
\end{aligned}$$

To bound $(S\mathbf{u}_\Gamma, \mathbf{u}_\Gamma)$ from below, we restrict ourselves to a subspace. Using the left inequality in Lemma 4, we obtain for $\mathbf{u}_\Gamma \in \text{range}\,(S)$,

$$\begin{aligned}
(S_\Delta \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) &= \min_{\substack{\mathbf{v}\in\mathbf{W}\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K_\Delta \mathbf{v}, \mathbf{v}) \\
&\leq (K_\Delta \mathbf{v}_{\text{elast}}, \mathbf{v}_{\text{elast}}) \\
&\leq C\,(K\mathbf{v}_{\text{elast}}, \mathbf{v}_{\text{elast}}) \\
&= C \min_{\substack{\mathbf{v}\in\text{range}\,(K)\\ \mathbf{v}_{|\Gamma}=\mathbf{u}_\Gamma}} (K\mathbf{v}, \mathbf{v}) \\
&= C(S\mathbf{u}_\Gamma, \mathbf{u}_\Gamma).
\end{aligned}$$

Since $(S_\Delta \mathbf{u}_\Gamma, \mathbf{u}_\Gamma)$ and $|\mathbf{u}_\Gamma|^2_{W_\Gamma}$ are equivalent for $\mathbf{u}_\Gamma \in \text{range}\,(S)$, we have proven the following

LEMMA 5. *There exist constants $0 < c \leq C < \infty$, independent of the mesh size and the subdomain diameters, such that*

$$c\,|\mathbf{u}_\Gamma|^2_{W_\Gamma} \leq (S\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \leq C\,|\mathbf{u}_\Gamma|^2_{W_\Gamma} \quad \forall \mathbf{u}_\Gamma \;\in \text{range}\,(S).$$

The next lemma follows from Mandel and Tezaur [15, Proof of Lemma 3.11].

LEMMA 6.

$$\inf_{\lambda\in\mathbf{V}} \sup_{\mathbf{w}_\Gamma\in\mathbf{W}_\Gamma} \frac{(\lambda, B_\Gamma \mathbf{w}_\Gamma)}{||\lambda||_{V'}||\mathbf{w}_\Gamma||_{W_\Gamma}} \geq C\,(1+\log(H/h))^{-1/2},$$
$$\sup_{\lambda\in\mathbf{V}} \sup_{\mathbf{w}_\Gamma\in\mathbf{W}_\Gamma} \frac{(\lambda, B_\Gamma \mathbf{w}_\Gamma)}{||\lambda||_{V'}||\mathbf{w}_\Gamma||_{W_\Gamma}} \leq C\,(1+\log(H/h)).$$

The next result is essentially an extension of that lemma to the system of equations of linear elasticity.

LEMMA 7. *There exist constants $0 < c \leq C < \infty$, independent of the mesh size and the number of subdomains, such that*

$$c(1+\log(H/h))^{-1}||\lambda||^2_{V'} \leq (F\lambda,\lambda) \leq C(1+\log(H/h))^2||\lambda||^2_{V'} \quad \forall \lambda\in V.$$

Proof. As shown in [15, Proof of Lemma 3.11], we have for $\lambda \in V$

$$(F\lambda,\lambda) = \sup_{\mathbf{w}_\Gamma\in \mathrm{range}\,(S)} \frac{(\lambda, B_\Gamma\mathbf{w}_\Gamma)^2}{(S\mathbf{w}_\Gamma, \mathbf{w}_\Gamma)}.$$

Using Lemma 5 and that $B_\Gamma^t\lambda \perp \ker(S)$ for $\lambda\in\mathbf{V}$, we obtain

$$\begin{aligned}(F\lambda,\lambda) &\leq C \sup_{\mathbf{w}_\Gamma\in \mathrm{range}\,(S)} \frac{(\lambda, B_\Gamma\mathbf{w}_\Gamma)^2}{|\mathbf{w}_\Gamma|^2_{W_\Gamma}} \leq C \sup_{\mathbf{w}_\Gamma\in \mathrm{range}\,(S)} \frac{(\lambda, B_\Gamma\mathbf{w}_\Gamma)}{\inf_{\mathbf{z}_\Gamma\in\ker(S)} ||\mathbf{w}_\Gamma+\mathbf{z}_\Gamma||^2_{W_\Gamma}} \\ &= C \sup_{\substack{\tilde{\mathbf{w}}_\Gamma=\mathbf{w}_\Gamma+\mathbf{z}_\Gamma\in\mathbf{W}_\Gamma \\ \mathbf{w}_\Gamma\in\mathrm{range}\,(S),\mathbf{z}_\Gamma\in\ker(S)}} \frac{(\lambda, B_\Gamma\mathbf{w}_\Gamma)}{||\mathbf{w}_\Gamma+\mathbf{z}_\Gamma||^2_{W_\Gamma}} = C \sup_{\tilde{\mathbf{w}}_\Gamma\in\mathbf{W}_\Gamma} \frac{(\lambda, B_\Gamma\tilde{\mathbf{w}}_\Gamma)^2}{||\tilde{\mathbf{w}}_\Gamma||^2_{W_\Gamma}}.\end{aligned}$$

Analogously, we obtain

$$(F\lambda,\lambda) \geq c \sup_{\mathbf{w}_\Gamma\in\mathbf{W}_\Gamma} \frac{(\lambda, B_\Gamma\mathbf{w}_\Gamma)^2}{||\mathbf{w}_\Gamma||^2_{W_\Gamma}}.$$

The bounds of $(F\lambda,\lambda)$ now follow from Lemma 6.

□

Combining the definitions of the (exact) Dirichlet preconditioner M^{-1} and of the norm $||\cdot||_V$ with Lemma 5, we obtain

LEMMA 8. *There exist constants $0 < c \leq C < \infty$, independent of the mesh size and the number of subdomains, such that*

$$c\,||\lambda||^2_V \leq (M^{-1}\lambda,\lambda) \leq C\,||\lambda||^2_V \quad \forall\lambda\in V.$$

A condition number estimate of $PM^{-1}PF$ follows easily from these estimates; cf. also [15]. The proof for the algorithm using an inexact Dirichlet solver proceeds along very similar lines.

THEOREM 1. *There exists a positive constant C, independent of the mesh size and the number of subdomains, such that*

$$\kappa(PM^{-1}PF) \leq C\,(1+\log(H/h))^3,$$

where $\kappa(PM^{-1}PF) := \sigma_{max}/\sigma_{min}$ is the spectral condition number defined by the ratio of the largest and the smallest eigenvalue σ_{max} and σ_{min} of $PM^{-1}PF$.

Similarly, there exists a positive constant C, independent of the mesh size and the number of subdomains, such that

$$\kappa(P\widehat{M}^{-1}PF) \leq C\,(1 + \log(H/h))^3,$$

for any preconditioner $\widehat{M}^{-1}$ that is spectrally equivalent to the exact Dirichlet preconditioner.

In Tezaur [26], a condition number estimate of $C(1 + \log(H/h))^2$ is given for an algebraic FETI method developed by Park et al. [19, 20]; see also the discussion in Rixen et al. [23]. A modification of the FETI preconditioner with almost no extra cost, is introduced by the authors in [13]. For the case when B has full rank, it is of the form

$$\widehat{M}^{-1}_{BB^t} = (BB^t)^{-1}B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t(BB^t)^{-1}.$$

and it is shown that the condition number is bounded by $C\,(1+\log(H/h))^2$. It is easy to see that all that is required, in addition to the previous algorithm, is the matrix–vector product with the matrix $B^t(BB^t)^{-1}B$. Such a product can be computed very inexpensively since only local operations are required.

5.2. Analysis of the block–diagonal preconditioner. In this section, we give a condition number estimate for the block–diagonal preconditioner for the systems of equations arising from linear elasticity. This results in a convergence estimate for the preconditioned conjugate residual method.

As shown in section 4, the system of equations (10) involves an operator from $\mathbf{W} \times \mathbf{V}$ onto itself. The component of $\mathbf{u}$ in $ker(K)$ is determined after the completion of the iteration. It is therefore appropriate to consider the restriction of the operator $\mathcal{A}$ to the subspace $\mathbf{X}_R = \mathbf{W}_R \times \mathbf{V}$. Similarly, we can view the preconditioner $\mathcal{B}^{-1}$ as a mapping from $\mathbf{X}_R$ onto itself; see (12).

An upper bound for the convergence rate of the conjugate residual method can be given in terms of the condition number $\kappa(\mathcal{B}^{-1}\mathcal{A})$ of the preconditioned system. A theory of block–diagonal preconditioners for saddle point problems of different origins has been developed by several authors; see Rusten and Winther [24], Silvester and Wathen [25], Kuznetsov [14], and Klawonn [11]. To the best of our knowledge, the first proof for block–preconditioners applied to saddle point problems with a singular block K is given in Klawonn [9, 10] and independently in Arnold, Falk, and Winther [1]. In order to obtain a condition number estimate for $\mathcal{B}^{-1}\mathcal{A}$, we follow the short argument given in [1] which is in the same spirit as the argument given by Mandel and Tezaur [15, Lemma 3.1] for the positive definite case. For completeness, we include the short proof here using our notations and the norms of X_R and its dual. Denoting by $\rho(\cdot)$ the spectral radius of a matrix, we find

$$\begin{aligned}
\kappa(\mathcal{B}^{-1}\mathcal{A}) &= \rho(\mathcal{B}^{-1}\mathcal{A})\rho((\mathcal{B}^{-1}\mathcal{A})^{-1}) \\
&\leq ||\mathcal{B}^{-1}\mathcal{A}||_{X_R \to X_R}||(\mathcal{B}^{-1}\mathcal{A})^{-1}||_{X_R \to X_R} \\
&\leq ||\mathcal{B}^{-1}||_{X'_R \to X_R}||\mathcal{A}||_{X_R \to X'_R}||\mathcal{B}||_{X_R \to X'_R}||\mathcal{A}^{-1}||_{X'_R \to X_R}.
\end{aligned}$$

Hence, we need estimates of the norms of the operators $\mathcal{B}, \mathcal{A}$ and their inverses. The next lemma is given in Brezzi [2].

LEMMA 9. *Let $B : \mathbf{W}_R \to \mathbf{V}$, satisfy an inf–sup and a sup–sup condition, i.e.*

$$
\begin{aligned}
\inf_{\lambda \in \mathbf{V}} \sup_{\mathbf{w} \in \mathbf{W}_R} \frac{(\lambda, B\mathbf{w})}{\|\lambda\|_{V'} \|\mathbf{w}\|_W} &\geq \beta_0, \\
\sup_{\lambda \in \mathbf{V}} \sup_{\mathbf{w} \in \mathbf{W}_R} \frac{(\lambda, B\mathbf{w})}{\|\lambda\|_{V'} \|\mathbf{w}\|_W} &\leq \beta_1,
\end{aligned} \tag{13}
$$

where $\beta_0, \beta_1 > 0$.

Furthermore, let $K : \mathbf{W}_R \to \mathbf{W}_R$ be a symmetric operator satisfying

$$
\begin{aligned}
|(K\mathbf{w}, \mathbf{v})| &\leq \alpha_1 \|\mathbf{w}\|_W \|\mathbf{v}\|_W \quad && \forall \mathbf{w}, \mathbf{v} \in \mathbf{W}_R, \\
(K\mathbf{w}, \mathbf{w}) &\geq \alpha_0 \|\mathbf{w}\|_W^2 && \forall \mathbf{w} \in \mathbf{W}_R,
\end{aligned} \tag{14}
$$

where $\alpha_0, \alpha_1 > 0$.

Then, $\mathcal{A} : \mathbf{X}_R \to \mathbf{X}_R$ is an isomorphism with $\|\mathcal{A}\|_{X_R \to X_R'} \leq \bar{C}(\alpha_1, \beta_1)$ and $\|\mathcal{A}^{-1}\|_{X_R' \to X_R} \leq \underline{C}(\alpha_0, \alpha_1, \beta_0)$, where $\bar{C}(\alpha_1, \beta_1) := \alpha_1 + \beta_1$ and $\underline{C}(\alpha_0, \alpha_1, \beta_0) := \max\{(\alpha_0^{-1} + \beta_0^{-1}(1 + \alpha_1/\alpha_0)), (\beta_0^{-1} + \alpha_1 \beta_0^{-2})(1 + \alpha_1/\alpha_0)\}$.

The uniform boundedness and ellipticity of K on $\mathbf{W}_R$ follows directly from the definition of the norm $\|\cdot\|_W$. Thus, we obtain constants $\alpha_0, \alpha_1 > 0$ which are independent of h, H.

We are left with showing the inf–sup and sup–sup conditions for B.

LEMMA 10.

$$
\begin{aligned}
\inf_{\lambda \in \mathbf{V}} \sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{\|\lambda\|_{V'} \|\mathbf{w}\|_W} &\geq C\,(1 + \log(H/h))^{-1/2} =: \beta_0 \\
\sup_{\lambda \in \mathbf{V}} \sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{\|\lambda\|_{V'} \|\mathbf{w}\|_W} &\leq C\,(1 + \log(H/h)) =: \beta_1
\end{aligned}
$$

Proof. For $\lambda \in \mathbf{V}$, let us now consider

$$
\begin{aligned}
\sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{\|\mathbf{w}\|_W} &\leq C \sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{|\mathbf{w}|_W} \\
&\leq C \sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{(K\mathbf{w}, \mathbf{w})^{1/2}} \\
&= C \sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B_\Gamma \mathbf{w}_\Gamma)}{\inf\limits_{\substack{\mathbf{v} \in \mathbf{W} \\ \mathbf{v}|_\Gamma = \mathbf{w}_\Gamma}} (K\mathbf{v}, \mathbf{v})^{1/2}} \\
&= C \sup_{\mathbf{w}_\Gamma \in \mathrm{range}(S)} \frac{(\lambda, B_\Gamma \mathbf{w}_\Gamma)}{(S\mathbf{w}_\Gamma, \mathbf{w}_\Gamma)^{1/2}} \\
&\leq C \sup_{\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma} \frac{(\lambda, B_\Gamma \mathbf{w}_\Gamma)}{\|\mathbf{w}_\Gamma\|_{W_\Gamma}} \\
&\leq \beta_1 \|\lambda\|_{V'},
\end{aligned}
$$

with $\beta_1 := C\,(1 + \log(H/h))$. The last inequality follows from Lemma 6. Analogously, we obtain

$$
\sup_{\mathbf{w} \in \mathbf{W_R}} \frac{(\lambda, B\mathbf{w})}{\|\mathbf{w}\|_W} \geq \beta_0 \|\lambda\|_{V'},
$$

with $\beta_0 := C\,(1 + \log(H/h))^{-1/2}$.

□

Combining these results with Lemma 9, we obtain estimates of the norm of $\mathcal{A}$ and $\mathcal{A}^{-1}$:

LEMMA 11. *The operator $\mathcal{A} : \mathbf{X}_R \to \mathbf{X}_R$ is an isomorphism and satisfies*

$$\begin{aligned} \|\mathcal{A}\|_{X_R \to X'_R} &\leq C\,(1+\log(H/h)), \\ \|\mathcal{A}^{-1}\|_{X'_R \to X_R} &\leq C\,(1+\log(H/h)), \end{aligned}$$

where $C > 0$ is a generic constant independent of the mesh size and the subdomain diameters.

The next lemma provides bounds for $\|\mathcal{B}\|_{X_R \to X'_R}$ and $\|\mathcal{B}^{-1}\|_{X'_R \to X_R}$.

LEMMA 12. *There exist constants $0 < c \leq C < \infty$, which depend only on k_0, k_1, m_0, and m_1, such that*

$$\|\mathcal{B}\|_{X_R \to X'_R} \leq C, \quad \|\mathcal{B}^{-1}\|_{X'_R \to X_R} \leq \frac{1}{c}.$$

These bounds are uniform with respect to the mesh size and the number of subdomains if $\mathcal{B}$ is an optimal preconditioner, i.e. if the constants k_0, k_1, m_0, m_1 are uniformly bounded.

Proof: From the first inequalities of (11) and Lemma 2, we obtain by a standard scaling argument that $(\widehat{K}\mathbf{u}, \mathbf{u})$ and $\|\mathbf{u}\|_W^2$ are spectrally equivalent for $\mathbf{u} \in \mathbf{W}_R$. From the second inequalities in (11) and Lemma 5, it follows that for $\mu \in \mathbf{V}$

$$(\widehat{M}^{-1}\mu, \mu) \leq C\,(M^{-1}\mu, \mu) = C\,(B_\Gamma S B_\Gamma^t \mu, \mu) \leq C\,|B_\Gamma^t \mu|_{W_\Gamma}^2 = C\,\|\mu\|_V^2.$$

In the last inequality, we have used that $\mu \in \mathbf{V}$ implies $B_\Gamma^t \mu \in \text{range}\,(S)$. Analogously, we obtain

$$(\widehat{M}^{-1}\mu, \mu) \geq c\,\|\mu\|_V^2.$$

By using these inequalities, we get for $\lambda \in \mathbf{V}$

$$\|\lambda\|_{V'}^2 = \sup_{\mu \in V} \frac{(\lambda, \mu)^2}{\|\mu\|_V^2} \leq C \sup_{\mu \in V} \frac{(\lambda, \mu)^2}{(\widehat{M}^{-1}\mu, \mu)} = C \sup_{\nu \in V} \frac{(\lambda, \widehat{M}^{1/2}\nu)^2}{(\nu, \nu)} = C\,(\widehat{M}\lambda, \lambda),$$

and analogously

$$\|\lambda\|_{V'}^2 \geq c\,(\widehat{M}\lambda, \lambda).$$

From the definition of $\mathcal{B}$ follows with $\mathbf{x} = (\mathbf{u}, \lambda) \in \mathbf{X_R}$

$$(\mathcal{B}\mathbf{x}, \mathbf{x}) = (\widehat{K}\mathbf{u}, \mathbf{u}) + (\widehat{M}\lambda, \lambda) \leq C(\|\mathbf{u}\|_W^2 + \|\lambda\|_{V'}^2) = C\,\|\mathbf{x}\|_X^2$$

and

$$(\mathcal{B}\mathbf{x}, \mathbf{x}) \geq c\,\|\mathbf{x}\|_X^2.$$

The boundedness of $\mathcal{B}$ and $\mathcal{B}^{-1}$ follows by using the following formulas

$$\begin{aligned} \|\mathcal{B}\|_{X_R \to X'_R} &= \sup_{\mathbf{x} \in \mathbf{X}_R} \frac{(\mathcal{B}\mathbf{x}, \mathbf{x})}{\|\mathbf{x}\|_X^2} \\ \|\mathcal{B}^{-1}\|_{X'_R \to X_R}^{-1} &= \inf_{\mathbf{x} \in \mathbf{X}_R} \frac{(\mathcal{B}\mathbf{x}, \mathbf{x})}{\|\mathbf{x}\|_X^2}. \end{aligned}$$

□

From Lemmas 11 and 12 follows
THEOREM 2.

$$\kappa(\mathcal{B}^{-1}\mathcal{A}) \leq C\,(1 + \log(H/h))^2,$$

with a constant C independent of H, h.

6. Numerical Results. We have applied our domain decomposition method to a plane stress problem described in section 2. The Poisson ratio is $\nu = 0.3$ and the elasticity modulus $E = 2.1 \cdot 10^{11} N/m^2$, which models steel. The domain Ω is the unit square fixed on the left hand side and free on the other three edges except for the upper right corner, where we impose a non-homogeneous natural boundary condition in the form of a point force that has components in the positive x and y directions both equal to $10^5 N$; cf. Figure 1.

All our computations have been performed in MATLAB 5.0. Our Krylov subspace method is the preconditioned conjugate residual method with a zero initial guess. The stopping criterion is $||\underline{\mathbf{r}}_n||_2/||\underline{\mathbf{r}}_0||_2 < 10^{-6}$, where $\underline{\mathbf{r}}_n$ and $\underline{\mathbf{r}}_0$ are the $n-$th and initial residual, respectively.

Our domain Ω is decomposed into $N \times N$ square subdomains with $H := 1/N$; see Figure 1. In our implementation, we use the maximal possible number of constraints, pairwise connecting all degrees of freedom, which physically belong to the same location, using a Lagrange multiplier for each possible pair. There are therefore redundancies in the compatibility constraints at the crosspoints. This is known to yield a smaller number of iterations. An explanation from a mechanical viewpoint is given in Rixen and Farhat [22]; see also our forthcoming paper [13].

We have carried out three different types of experiments for different combinations of preconditioners $\widehat{K}$ and $\widehat{M}$, in order to analyze the numerical scalability of the method. In our first set of runs, we have kept the dimension of the subproblems, and H/h, fixed and have increased the number of subdomains and thus the overall problem size. In a second set of experiments, we have considered a fixed mesh size h and again increased the number of subdomains. This results in decreasing dimensions of the subproblems and in decreasing values of H/h. Our last series of experiments is carried out with a fixed number of subdomains and increasing values of H/h resulting in an increased $1/h$.

In order to see how our method behaves in the best possible case, we first report on results for $\widehat{K} = K + \frac{1}{H^2}Q$ and $\widehat{M} = M$; cf. Tables 1,2, and 3 and Figures 2, and 4. For all three cases, we present results for $\widehat{M}$ constructed using $D = I$ as well as with the multiplicity scaling $D = MS$; cf. section 3. As in the original FETI algorithm, the convergence is considerably faster with multiplicity scaling; using this scaling the asymptotic convergence rate is also reached much earlier than for $D = I$. For both choices of D, we obtain scalable domain decomposition methods, in all three set of experiments.

To gain insight into the convergence behavior with inexact blocks $\widehat{K}$ and $\widehat{M}$, we have used preconditioners based on an incomplete Cholesky factorization (ILU). In the following, ILU(0) stands for an incomplete Cholesky factorization with no fill in while ILU(*tol*) is a threshold ILU factorization, with a threshold of *tol*, as provided in MATLAB 5.0; any entry in a column of the Cholesky factor L is dropped if its magnitude is smaller than the drop tolerance *tol* times the norm of its column. We denote by $\hat{S}$ the matrix that replaces S when ILU(0) is used to solve the Dirichlet

FIG. 1. *Sample domain decomposition of the cantilever with 16 subdomains.*

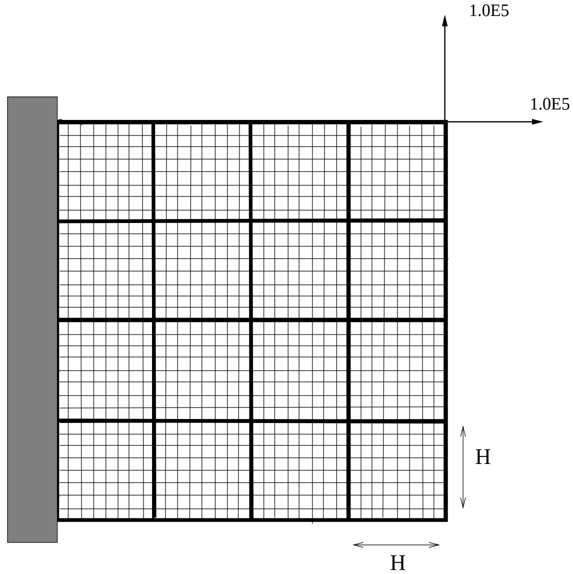

TABLE 1

(I) : $\widehat{K} = K + 1/H^2 M_Q$ *and* $\widehat{M} = M$*, (II) :* $\widehat{K} = K + 1/H^2 M_Q$ *and* $\widehat{M}$ *using ILU(0), (III) :* $\widehat{K}$ *ILU(*10^{-3}*) of* $K + 1/H^2 M_Q$ *and* $\widehat{M} = M$*, (IV) :* $\widehat{K}$ *using ILU(*10^{-3}*) of* $K + 1/H^2 M_Q$ *and* $\widehat{M}$ *using ILU(0). MS : D= multiplicity scaling, I : D=Identity.*

H/h = 8		Iter (I)		Iter (II)		Iter (III)		Iter (IV)	
$1/h$	$1/H$	MS	I	MS	I	MS	I	MS	I
16	2	11	19	11	19	23	37	25	38
32	4	17	27	17	27	33	69	37	73
64	8	21	35	25	35	41	85	45	89
96	12	21	41	25	41	47	91	51	97
128	16	21	41	25	43	51	93	57	101

problems in each subdomain. Three different combinations are considered: 1) $\widehat{K} = K + \frac{1}{H^2}Q$ and

$$\widehat{M}^{-1} = B \begin{bmatrix} O & O \\ O & D^{-1}\hat{S}D^{-1} \end{bmatrix} B^t;$$

2) $\widehat{K}$ is built with ILU(10^{-3}) applied to $K + \frac{1}{H^2}Q$ and $\widehat{M}^{-1} = M^{-1}$; 3) $\widehat{K}$ is built with ILU(10^{-3}) applied to $K + \frac{1}{H^2}Q$ and $\widehat{M}^{-1}$ again, as in 1), uses the inexact Schur complement $\hat{S}$. The computational results are given in Tables 1, 2, and 3; cf. also Figures 2, 3, and 4. We also present results with a more accurate ILU decomposition based on a threshold tolerance of $tol = 10^{-6}$ in Table 4. We see that a more accurate preconditioner $\widehat{K}$ improves the overall rate of convergence significantly.

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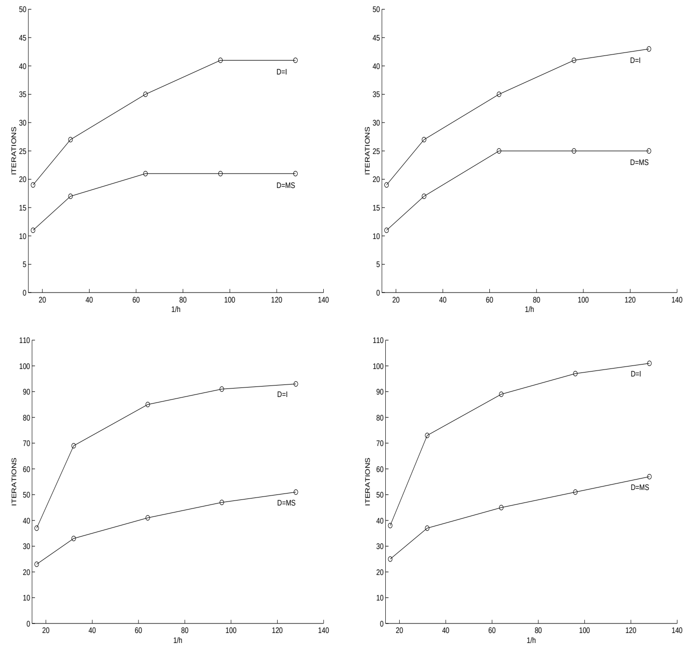

FIG. 2. *$H/h = 8$, Upper left : $\widehat{K} = K + \frac{1}{H^2}Q, \widehat{M} = M$, Upper right : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), Lower left : $\widehat{K}$ ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, Lower right : $\widehat{K}$ using ILU(10^{-3}) of $K + 1/H^2 M_Q$*
and $\widehat{M}$ using ILU(0).

TABLE 2

(I) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M} = M$, (II) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), (III) : $\widehat{K}$ ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, (IV) : $\widehat{K}$ using ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0). MS : D= multiplicity scaling, I : D=Identity.

h=1/96		Iter (I)		Iter (II)		Iter (III)		Iter (IV)	
H/h	$1/H$	MS	I	MS	I	MS	I	MS	I
24	4	19	33	49	79	56	111	101	165
16	6	21	33	27	39	55	113	75	131
12	8	25	35	29	39	56	114	73	129
8	12	21	41	25	41	47	91	51	97
6	16	19	37	23	39	48	87	52	89

TABLE 3

(I) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M} = M$, (II) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), (III) : $\widehat{K}$ ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, (IV) : $\widehat{K}$ using ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0). MS : D= multiplicity scaling, I : D=Identity.

H=1/4		Iter (I)		Iter (II)		Iter (III)		Iter (IV)	
$1/h$	H/h	MS	I	MS	I	MS	I	MS	I
16	4	15	25	15	25	41	84	41	84
32	8	17	27	17	27	33	69	37	73
64	16	19	29	27	49	49	97	63	114
128	32	19	35	51	69	72	142	118	209

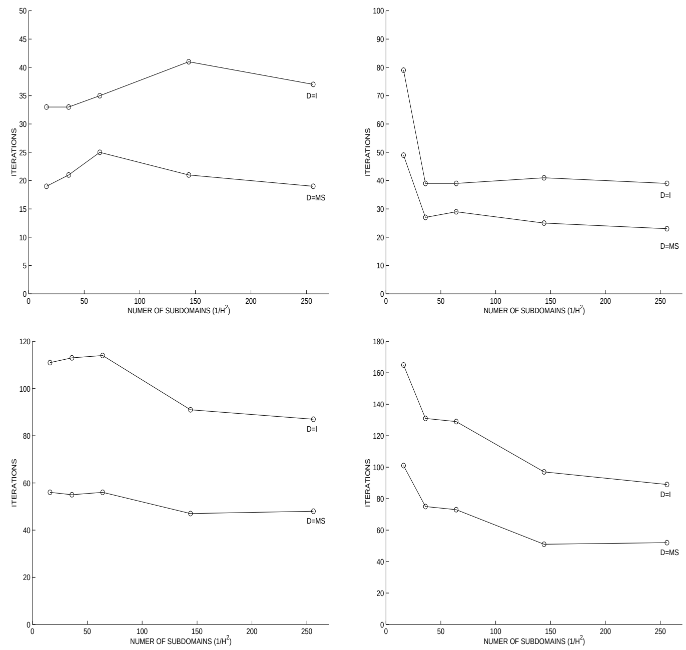

FIG. 3. *$h = 1/96$, Upper left : $\widehat{K} = K + \frac{1}{H^2}Q, \widehat{M} = M$, Upper right : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), Lower left : $\widehat{K}$ ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, Lower right : $\widehat{K}$ using ILU(10^{-3}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0).*

TABLE 4

(I) : $\widehat{K}$ based on ILU(10^{-6}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, (II) : $\widehat{K}$ based on ILU(10^{-6}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0). MS : D = multiplicity scaling, I : D= Identity.

H/h = 8		Iter (I)		Iter (II)	
$1/h$	$1/H$	MS	I	MS	I
16	2	18	30	20	32
32	4	17	29	19	29
64	8	21	35	25	35
128	16	23	41	25	43

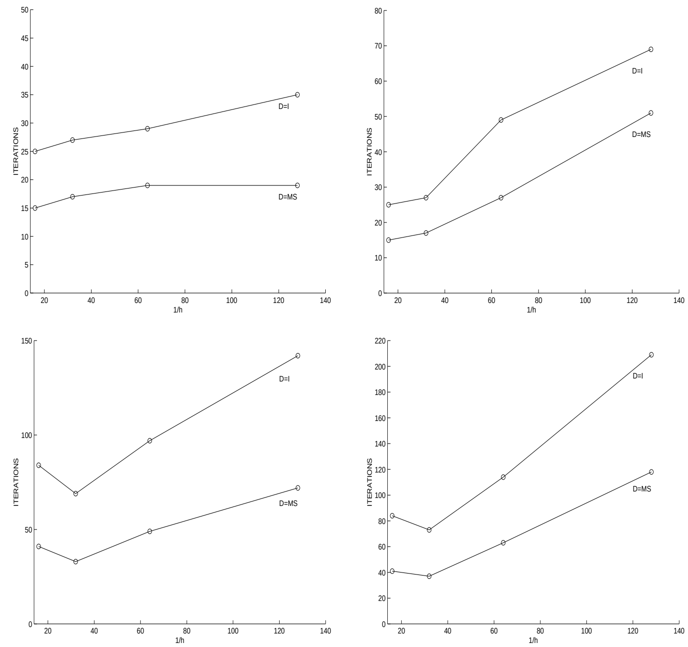

FIG. 4. $H = 1/4$, *Upper left :* $\widehat{K} = K + \frac{1}{H^2}Q, \widehat{M} = M$, *Upper right :* $\widehat{K} = K + 1/H^2 M_Q$ *and* $\widehat{M}$ *using ILU(0), Lower left :* $\widehat{K}$ *ILU*(10^{-3}) *of* $K + 1/H^2 M_Q$ *and* $\widehat{M} = M$, *Lower right :* $\widehat{K}$ *using ILU*(10^{-3}) *of* $K + 1/H^2 M_Q$ *and* $\widehat{M}$ *using ILU(0).*

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