AN ITERATIVE SUBSTRUCTURING METHOD FOR RAVIART-THOMAS VECTOR FIELDS IN THREE DIMENSIONS

BARBARA I. WOHLMUTH*, ANDREA TOSELLI†, AND OLOF B. WIDLUND‡

Abstract. The iterative substructuring methods, also known as Schur complement methods, form one of two important families of domain decomposition algorithms. They are based on a partitioning of a given region, on which the partial differential equation is defined, into non-overlapping substructures. The preconditioners of these conjugate gradient methods are then defined in terms of local problems defined on individual substructures and pairs of substructures, and, in addition, a global problem of low dimension. An iterative method of this kind is introduced for the lowest order Raviart-Thomas finite elements in three dimensions and it is shown that the condition number of the relevant operator is independent of the number of substructures and grows only as the square of the logarithm of the number of unknowns associated with an individual substructure. The theoretical bounds are confirmed by a series of numerical experiments.

Key words. Raviart-Thomas finite elements, domain decomposition, iterative substructuring methods

AMS subject classifications. 65N22, 65N30, 65N55

1. Introduction. In this paper, we consider a boundary value problem for vector fields, associated with the divergence operator,

$$
\begin{aligned}
Lu := -\mathbf{grad}\,(a\,\mathrm{div}\,\mathbf{u}) + B\,\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\
\mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{1}
$$

Here Ω is a bounded polyhedral domain in $\mathbb{R}^3$ of unit diameter, and $\mathbf{n}$ its outward normal. We assume that $\mathbf{f} \in (L^2(\Omega))^3$, that the coefficient matrix B is a symmetric uniformly positive matrix-valued function with $b_{i,j} \in L^\infty(\Omega)$, $1 \le i,j \le 3$, and that $a \in L^\infty(\Omega)$ is a positive function bounded away from zero.

The weak formulation of problem (1), and the study of the Raviart-Thomas finite elements as well as our iterative method require the introduction of an appropriate Hilbert space $H(\mathrm{div}\,;\Omega)$. It is given by

$$
H(\mathrm{div}\,;\Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^3 | \quad \mathrm{div}\,\mathbf{v} \in L^2(\Omega) \right\},
$$

and equipped with the inner product $(\cdot,\cdot)_{\mathrm{div}}$ and the associated graph norm $||\cdot||_{\mathrm{div}}$, defined by

$$
(\mathbf{u},\mathbf{v})_{\mathrm{div}} := \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\,dx, \qquad ||\mathbf{u}||^2_{\mathrm{div}} := (\mathbf{u},\mathbf{u})_{\mathrm{div}}.
$$

* Math. Institut, Universität Augsburg, Universitätsstr. 14, D-86 159 Augsburg, Germany. E-mail: wohlmuth@math.uni-augsburg.de URL: http://wwwhoppe.math.uni-augsburg.de/wohlmuth
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† Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012. E-mail: toselli@cims.nyu.edu URL: http://www.math.nyu.edu/phd_students/toselli/index.html
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‡ Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012. E-mail: widlund@cs.nyu.edu URL: http://cs.nyu.edu/cs/faculty/widlund/index.html
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The normal component of any vector $\mathbf{u} \in H(\mathrm{div}\,;\Omega)$, on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)$; see [8]. The subspace of vectors in $H(\mathrm{div}\,;\Omega)$ with vanishing normal component on $\partial\Omega$ is denoted by $H_0(\mathrm{div}\,;\Omega)$ and it is the appropriate space for the variational formulation of equation (1):
Find $\mathbf{u} \in H_0(\mathrm{div}\,;\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx, \quad \mathbf{v} \in H_0(\mathrm{div}\,;\Omega), \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega (a\ \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} + B\ \mathbf{u}\cdot\mathbf{v})\ dx, \quad \mathbf{u},\,\mathbf{v} \in H(\mathrm{div}\,;\Omega).$$

We associate an energy norm, defined by $||\cdot||_a^2 := a(\cdot,\cdot)$, with the bilinear form; our assumptions on the coefficients guarantee that this norm is equivalent to the graph norm.

We remark that if there is a g such that $\mathbf{f} = -\nabla(ag)$ then (2) is equivalent to a mixed variational formulation of the following elliptic equation

$$\begin{aligned} -\mathrm{div}\,(B^{-1}\nabla w) + a^{-1}w &= g, \quad \text{in } \Omega,\\ B^{-1}\nabla w\cdot\mathbf{n} &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

To see this, we introduce a flux $\mathbf{q} := -B^{-1}\nabla w$ as an additional unknown. The corresponding mixed variational problem can be written as:
Find $(\mathbf{q},w) \in H_0(\mathrm{div}\,;\Omega)\times L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega B\mathbf{q}\cdot\mathbf{p}\,dx - \int_\Omega w\ \mathrm{div}\,\mathbf{p}\,dx &= 0, \quad \mathbf{p} \in H_0(\mathrm{div}\,;\Omega),\\ \int_\Omega \mathrm{div}\,\mathbf{q}\ v\,dx + \int_\Omega a^{-1}wv\,dx &= (g,v)_0, \quad v \in L^2(\Omega). \end{aligned}$$

The second equation gives

$$w = ag - a\mathrm{div}\,\mathbf{q}$$

and thus, using the first equation, $\mathbf{q} = \mathbf{u}$.

Another application is provided by stabilized mixed formulations of advection-diffusion equations; see [8] and the references therein. Still other applications of the space $H(\mathrm{div}\,;\Omega)$ are given in [2].

In this paper, we will construct a domain decomposition algorithm for the discretization of equation (2) by the lowest order Raviart-Thomas elements. Our algorithm is an iterative substructuring method, based on a decomposition of Ω into nonoverlapping substructures, and it is designed and analyzed in the *Schwarz method framework*; see, e.g. [22]. A Schwarz algorithm is an iteration scheme defined on a finite dimensional space V, in our case that of the Raviart-Thomas finite elements on Ω. It is specified by a family of subspaces $\{V_i,\ i = 0,\cdots,J\}$, and in the simplest case, by projections $P_i : V \to V_i$; these projections are orthogonal with respect to the energy inner product.

An *additive Schwarz method* provides a new operator equation

$$P_{as}\mathbf{u} = \sum_{i=0}^{J} P_i\mathbf{u} = \mathbf{g},$$

which can be much better conditioned than the original discrete elliptic problem; it can often be solved effectively by the conjugate gradient method, without further preconditioning, employing $a(\cdot,\cdot)$ as the inner product. The right hand side $\mathbf{g}$ can be chosen so that the new problem has the same solution $\mathbf{u}$ as the original one; it is possible to compute $P_i\mathbf{u}$ from the data given by the original problem.

A lower bound for the smallest eigenvalue of P_{as} is given by the following well-known lemma; see [22, Section 5.2].

LEMMA 1.1. *If for all* $\mathbf{u} \in V$ *there exists a representation,* $\mathbf{u} = \sum_{i=0}^{J} \mathbf{u}_i$, $\mathbf{u}_i \in V_i$, *such that*

$$\sum_{i=0}^{J} a(\mathbf{u}_i, \mathbf{u}_i) \leq C_0^2 \; a(\mathbf{u}, \mathbf{u}),$$

then the smallest eigenvalue of the additive Schwarz operator P_{as} *is bounded from below by* C_0^{-2}.

As it is often the case, a good bound for the largest eigenvalue P_{as} is routine and can be obtained by a standard coloring argument; see, e.g. [22, p. 165]. We also note that we will only consider the basic case when all the problems defined on the subspaces are solved exactly; this removes the necessity to develop a bound for the norms of certain projection-like operators that otherwise would be required; see [22, Assumption 3]. We note that, as always, the extension of our analysis to other Schwarz methods such as the multiplicative and hybrid variants is completely routine and that we are in no way suggesting that the additive form of the algorithm should be preferred over the others. Good results for all these variants of the Schwarz algorithms will follow from our bound on C_0.

We note that many Schwarz methods have been designed and analyzed for the case of $H^1(\Omega)$ in three dimensions, see, e.g. [11, 22], but that there have been only relatively few studies of the $H(\mathrm{div}\,;\Omega)$ and $H(\mathbf{curl}\,;\Omega)$ cases for three dimensional problems. Among them are [9, 14, 23], on two-level overlapping methods, [3, 13], on multilevel methods, and [1], which is a study of an iterative substructuring method in $H(\mathbf{curl}\,;\Omega)$. We also mention [20, 21] which report on a study of a class of two- and multi-level methods for mixed approximations of Poisson's equation. The present work is a continuation of our recent work in two dimensions; see [5]. See also [2, 7, 12, 15, 16], and the references therein, for some Schwarz methods for problems in $H(\mathrm{div}\,;\Omega)$ in two dimensions.

The choice of the coarse space for a Schur complement method in $H^1(\Omega)$, in three dimensions, is a delicate matter; see, e.g. [11, 25]. Thus, if a standard subspace built on a coarse triangulation is employed in a *vertex-based* algorithm, the condition number of the method cannot be both quasi-optimal and independent on the jumps of the coefficients across the substructures; see [11]. In particular, if only edge, face, and interior spaces are used in addition to a conventional coarse space, the condition number can be made independent of the jumps but it will grow algebraically with the number of unknowns in each subdomain. If, on the other hand, local vertex spaces are added then a logarithmic bound can be found for the condition number of the iteration operator, but this bound will not, in general, be independent of the jumps of the coefficients. The reason is that the standard, vertex based interpolation operator onto the coarse space has a norm that grows algebraically in three dimensions. For this reason, other coarse spaces and iterative substructuring methods have been introduced, among them the *wire-basket based* algorithms; see [11]. (We recall that the wire-basket is the union of the boundaries of the faces which separate the substructures.) In this respect, there is an interesting difference between $H^1(\Omega)$ and $H(\mathrm{div}\,;\Omega)$, since our new method for $H(\mathrm{div}\,;\Omega)$ uses a standard coarse space that is just a smaller instance of the original finite element problem. At the same time, we are able to maintain the same kind of quasi-optimality and independence of the jumps as the best, more complicated algorithms for the $H^1(\Omega)$ case. This is a consequence of a certain stability result, given in Lemma 4.1, for the interpolant for the Raviart-Thomas space, the degrees of freedoms of which are defined by averages of the normal component over the faces of the triangulation.

The rest of this paper is organized as follows. We review some properties of the $H(\mathrm{div}\,;\Omega)$ space in Section 2, and introduce the Raviart-Thomas elements in Section 3. Several important auxiliary results are developed in Section 4 in preparation for the introduction and analysis of our iterative substructuring preconditioner in Section 5. Our main result is a polylogarithmic upper bound for the condition number of the resulting additive Schwarz operator. Section 6 concludes the paper with numerical results which illustrate the performance of our iterative substructuring method.

2. Sobolev and trace spaces. In addition to $H(\mathrm{div}\,;\Omega)$, we will also use some standard Sobolev spaces. Given a bounded open Lipschitz domain $\mathcal{D} \subset \mathbb{R}^3$, with a boundary $\partial\mathcal{D}$ and a diameter $H_{\mathcal{D}}$, let $|\cdot|_{s;\mathcal{D}}$ denote the semi-norm of the Sobolev space $H^s(\mathcal{D})$. In case that $\mathcal{D} = \Omega$, we will drop the reference to the region. Throughout, we will work with scaled norms for the spaces $H^s(\mathcal{D})$, $s > 0$, obtained from the standard definition of the Sobolev norm on a region with diameter one and a dilation. Thus, with $\|\cdot\|_0$ the L_2-norm,

$$||u||_{1;\mathcal{D}}^2 = |u|_{1;\mathcal{D}}^2 + \frac{1}{{H_{\mathcal{D}}}^2}||u||_{0;\mathcal{D}}^2,$$

and

$$||u||_{\frac{1}{2};\partial\mathcal{D}}^2 = |u|_{\frac{1}{2};\partial\mathcal{D}}^2 + \frac{1}{H_{\mathcal{D}}}||u||_{0;\partial\mathcal{D}}^2.$$

As already mentioned, the normal component of any vector field $\mathbf{u} \in H(\mathrm{div}\,;\mathcal{D})$ belongs to $H^{-\frac{1}{2}}(\partial\mathcal{D})$, and the corresponding trace operator is continuous [8]. Here, $H^{-\frac{1}{2}}(\partial\mathcal{D})$ is equipped with the norm

$$||\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2};\partial\mathcal{D}} := \sup_{\substack{\phi\in H^{\frac{1}{2}}(\partial\mathcal{D})\\ \phi\neq 0}} \frac{\langle \mathbf{u}\cdot\mathbf{n}, \phi\rangle}{||\phi||_{\frac{1}{2};\partial\mathcal{D}}}, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\partial\mathcal{D})$ and $H^{\frac{1}{2}}(\partial\mathcal{D})$. Results on the trace are formulated in the following two lemmas; we note that the first bound in the second of these lemmas is quite similar to a lemma given in [4]. From now on, we will denote by C a positive generic constant, uniformly bounded from above, and by c a positive generic constant uniformly bounded away from zero.

LEMMA 2.1. *There exists a constant C, which is independent of the diameter of $\mathcal{D}$, such that*

$$||\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2};\partial\mathcal{D}}^2 \le C\left(||\mathbf{u}||_{0;\mathcal{D}}^2 + H_{\mathcal{D}}^2||\mathrm{div}\,\mathbf{u}||_{0;\mathcal{D}}^2\right). \tag{4}$$

Proof. By using Green's formula on a reference domain $\hat{\mathcal{D}}$ with diameter one, we find

$$\int_{\partial\hat{\mathcal{D}}} \phi\;\mathbf{u}\cdot\mathbf{n}\,d\sigma = \int_{\hat{\mathcal{D}}} \mathbf{u}\cdot\nabla\phi\,dx + \int_{\hat{\mathcal{D}}} \mathrm{div}\,\mathbf{u}\;\phi\,dx,$$

where ϕ is extended harmonically to the interior of $\hat{\mathcal{D}}$. We then arrive at (4) by bounding $\|\phi\|_{1;\hat{\mathcal{D}}}$ in terms of $\|\phi\|_{\frac{1}{2};\partial\hat{\mathcal{D}}}$ and by using a scaling argument. □

We note that the right hand side of inequality (4) is a multiple of a scaled $H(\mathrm{div}\,;\mathcal{D})$ norm but that we will need to work with the unscaled norm in our decomposition of the finite element functions; see also Lemma 4.3

LEMMA 2.2. *There exists a constant c, which is independent of the diameter of $\mathcal{D}$, such that for each $\mathbf{u} \in H(\mathrm{div}\,;\mathcal{D})$ with $\langle \mathbf{u}\cdot\mathbf{n}, 1\rangle = 0$*

$$c \sup_{\substack{\phi\in H^{\frac{1}{2}}(\partial\mathcal{D})\\ \phi\neq\text{const}}} \frac{\langle \mathbf{u}\cdot\mathbf{n}, \phi\rangle}{|\phi|_{\frac{1}{2};\partial\mathcal{D}}} \leq \|\mathbf{u}\cdot\mathbf{n}\|_{-\frac{1}{2};\partial\mathcal{D}} \leq \sup_{\substack{\phi\in H^{\frac{1}{2}}(\partial\mathcal{D})\\ \phi\neq\text{const}}} \frac{\langle \mathbf{u}\cdot\mathbf{n}, \phi\rangle}{|\phi|_{\frac{1}{2};\partial\mathcal{D}}}. \tag{5}$$

Proof. The upper bound in (5) is an immediate consequence of the definition of the $\|\cdot\|_{-\frac{1}{2};\partial\mathcal{D}}$-norm. The proof of the lower bound in (5) is based on the following norm equivalence

$$c\|\phi\|^2_{\frac{1}{2};\partial\mathcal{D}} \leq |\phi|^2_{\frac{1}{2};\partial\mathcal{D}} + \frac{1}{{H_{\mathcal{D}}}^3}\left(\int_{\partial\mathcal{D}} \phi\, d\sigma\right)^2 \leq C\|\phi\|^2_{\frac{1}{2};\partial\mathcal{D}}. \tag{6}$$

This is a Poincaré-type inequality; see, e.g. [17, Chap. 2.7] for a classical introduction to such inequalities. We note that the scale factor results from writing down the result for a region of diameter one and using dilation. Then, the definition of the $H^{-\frac{1}{2}}$-norm and the assumption that $\langle \mathbf{u}\cdot\mathbf{n}, 1\rangle = 0$ yield, for all real α,

$$\|\mathbf{u}\cdot\mathbf{n}\|_{-\frac{1}{2};\partial\mathcal{D}} = \sup_{\substack{\phi\in H^{\frac{1}{2}}(\partial\mathcal{D})\\ \phi\neq\text{const.}}} \frac{\langle \mathbf{u}\cdot\mathbf{n}, \phi\rangle}{\|\phi-\alpha\|_{\frac{1}{2};\partial\mathcal{D}}}.$$

The lower bound is obtained by using (6) and choosing $\alpha = \int_{\partial\mathcal{D}} \phi\, d\sigma$. □

In the next section, we will introduce an alternative formula for the trace norm (3) of Raviart-Thomas elements, based on a finite element discretization $V_h(\partial T)$ of $H^{\frac{1}{2}}(\partial T)$, where T is an element of a coarse triangulation of the region Ω. We begin by introducing two triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$, where the second is a refinement of the first. The coarse triangulation $\mathcal{T}_H$ of Ω consists of shape regular hexahedra the diameters of which can vary across the region Ω. The finite approximation is given on the finer mesh, $\mathcal{T}_h$, which is obtained by quasi-uniform and shape-regular refinements of individual coarse mesh elements, in such a way that $\mathcal{T}_h$ is a conforming triangulation of the whole Ω. A generic element of $\mathcal{T}_h$ or $\mathcal{T}_H$ will be denoted by t and T, respectively. The sets of faces and edges of the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, are denoted by $\mathcal{F}_h$, $\mathcal{F}_H$ and $\mathcal{E}_h$, $\mathcal{E}_H$, respectively. A generic face will be denoted by f and F, and a generic edge by e and E, respectively.

We only consider triangulations based on hexahedra in this paper, but our results are equally valid for finite element spaces built on tetrahedra. Much of the analysis is carried out on a cubic substructure divided into cubic elements but the results remain equally valid if the elements and substructures are images of a reference cube under sufficiently benign mappings, which effectively means that their aspect ratios have to remain uniformly bounded. We remark that our analysis is carried out locally for one substructure at a time. We can therefore interpret the factor H/h, which appears in our the estimates, as

$$\max_{T\in\mathcal{T}_H} \max_{\substack{t\in\mathcal{T}_h\\ t\subset T}} \frac{H_T}{H_t}.$$

Our approximation $V_h(\partial T)$ of $H^{\frac{1}{2}}(\partial T)$, $T \in \mathcal{T}_H$, is given as a direct sum

$$V_h(\partial T) := Q_h(\partial T) + B_h(\partial T).$$

Here $Q_h(\partial T)$ is the space of all continuous piecewise bilinear functions and $B_h(\partial T)$ a space of bubble functions vanishing on the boundary of the elements of $\mathcal{T}_h$:

$$\begin{aligned} Q_h(\partial T) &:= \{\phi \in C^0(\partial T),\ \phi_{|_f} \in Q_1(f), f \subset \partial T,\ f \in \mathcal{F}_h\}, \\ B_h(\partial T) &:= \{\phi \in C^0(\partial T),\ \phi_{|_f} = \alpha_f \varphi_1\ \varphi_2\ \varphi_3\ \varphi_4, f \subset \partial T,\ f \in \mathcal{F}_h, \alpha_f \in \mathbb{R}\}, \end{aligned}$$

where φ_i, $1 \le i \le 4$, are the nodal basis functions that span $Q_1(f)$ on the face f. The support of any bubble basis function is exactly one element. This property is often exploited, e.g. in local a posteriori analysis [24]. The following lemma shows that the $H^{\frac{1}{2}}$-seminorm of an element $\phi = \phi_Q + \phi_B$ in $V_h(\partial T)$, with $\phi_Q \in Q_h(\partial T)$ and $\phi_B \in B_h(\partial T)$, is equivalent to the sum of the seminorms of ϕ_B and ϕ_Q.

LEMMA 2.3. *There exists a constant C, which depends only on the aspect ratio of T, such that for each $\phi = \phi_Q + \phi_B$, with $\phi_Q \in Q_h(\partial T)$ and $\phi_B \in B_h(\partial T)$, the following equivalence holds*

$$|\phi|_{\frac{1}{2};\partial T} \le |\phi_Q|_{\frac{1}{2};\partial T} + |\phi_B|_{\frac{1}{2};\partial T} \le C|\phi|_{\frac{1}{2};\partial T}. \tag{7}$$

Proof. The lower bound follows from the triangle inequality. To prove the upper, we consider one element at a time and note that the restriction of the two subspaces to a face f of an element are of fixed dimension. It then follows immediately from the linear independence of the basis functions that $||\phi_Q||_{0;f} \le C||\phi||_{0;f}$ and that $|\phi_Q|_{1;f} \le C|\phi|_{1;f}$. Squaring these inequalities and adding, gives the same inequalities for ∂T. An interpolation argument then gives a bound in $H^{\frac{1}{2}}$ from which (7) follows directly. □

We conclude this section by introducing an operator $P_h : H^{\frac{1}{2}}(\partial T) \to V_h(\partial T)$, defined by

$$P_h := P_Q + P_B.$$

Here, P_Q is the L^2-projection onto $Q_h(\partial T)$ and P_B a projection onto $B_h(\partial T)$, defined by

$$\int_f P_B\phi \ d\sigma = \int_f (\phi - P_Q\phi)\ d\sigma, \quad f \in \mathcal{F}_h, f \subset \partial T.$$

We note that the operator P_h preserves integrals over each face f.

LEMMA 2.4. *The operator P_h is bounded uniformly in $L_2(\partial T)$ and in $H^{\frac{1}{2}}(\partial T)$.*

Proof. It is well known that P_Q is L^2- and H^1-stable, since $\mathcal{T}_h$ is quasi-uniform; see, e.g. [6]. We then obtain the $H^{\frac{1}{2}}$-stability of P_Q by an interpolation argument. To prove the $H^{\frac{1}{2}}$-stability of P_h, we also have to consider P_B. The proof of its L^2-stability is quite elementary. By using the inverse inequality

$$|\phi_B|^2_{\frac{1}{2};\partial T} \le \frac{C}{h}||\phi_B||^2_{0;\partial T}, \quad \phi_B \in B_h(\partial T),$$

and the approximation property of P_Q, see [6], we find that

$$\begin{aligned}||P_B\phi||^2_{\frac{1}{2};\partial T} &\leq \frac{C}{h}||P_B\phi||^2_{0;\partial T} \leq \frac{C}{h^3}\sum_{\substack{f\subset\partial T\\ f\in\mathcal{F}_h}}\left(\int_f(\phi - P_Q\phi)\,d\sigma\right)^2\\ &\leq \frac{C}{h}||\phi - P_Q\phi||^2_{0;\partial T} \leq C|\phi|^2_{\frac{1}{2};\partial T}.\end{aligned}$$

□

3. Raviart-Thomas finite elements. Our study concerns the lowest order Raviart-Thomas finite elements approximation of (1); see [8]. The Raviart-Thomas element space X_h is defined by

$$X_h := X_h(\Omega) := \left\{\mathbf{u}\in H(\mathrm{div}\,;\Omega)|\quad \mathbf{u}_{|_t}\in\mathcal{RT}(t),\, t\in\mathcal{T}_h\right\},$$

where the local space, for a cube with sides parallel to the coordinate axes, is given by

$$\mathcal{RT}(t) := \begin{pmatrix}\alpha_1+\beta_1 x\\ \alpha_2+\beta_2 y\\ \alpha_3+\beta_3 z\end{pmatrix}.$$

The degrees of freedom of X_h are given by the averages of the normal components over the faces of the triangulation:

$$\lambda_f(\mathbf{u}) := \frac{1}{|f|}\int_f \mathbf{u}\cdot\mathbf{n}\,d\sigma,\quad f\in\mathcal{F}_h. \tag{8}$$

Here $|f|$ is the area of the face f and the direction of the normal can be fixed arbitrarily for each face. This formula also defines the natural interpolation operator onto the space X_h. We note that the normal component of any Raviart-Thomas function is constant on each face. The dimension of the local space $\mathcal{RT}(t)$ is six, and the dimension of the global space X_h equals the number of faces, $f\in\mathcal{F}_h$. We also define the subspaces of vectors with vanishing normal components on the boundary of Ω by

$$X_{0;h} := X_{0;h}(\Omega) := X_h(\Omega)\cap H_0(\mathrm{div}\,;\Omega).$$

We define the coarse spaces X_H and $X_{0;H}$ in exactly the same way, using the coarse triangulation $\mathcal{T}_H$.

As in the case of Lagrangian finite elements, the L^2-norm of these discrete vector fields can be bounded from above and below by means of the values of their degrees of freedom. This is a simple matter; we can, e.g. easily adapt the proof given for Lagrangian elements in [19, Proposition 6.3.1].

LEMMA 3.1. *Let t be an element of $\mathcal{T}_h$. There exist constants, c and C, which only depend on the aspect ratio of the element t, such that*

$$c\sum_{f\subset\partial t}\left(H_f^{3/2}\,\lambda_f(\mathbf{u})\right)^2 \leq ||\mathbf{u}||^2_{0;t} \leq C\sum_{f\subset\partial t}\left(H_f^{3/2}\,\lambda_f(\mathbf{u})\right)^2,\quad \mathbf{u}\in\mathcal{RT}(t). \tag{9}$$

Moreover, the following inverse estimate holds:

$$||\mathrm{div}\,\mathbf{u}||_{0;t} \leq C\frac{1}{H_t}||\mathbf{u}||_{0;t},\quad \mathbf{u}\in\mathcal{RT}(t). \tag{10}$$

The same bounds hold for a coarse element $T \in \mathcal{T}_H$.

We will also need some trace spaces associated with the substructure $T \in \mathcal{T}_H$. We define $S_H(\partial T)$ as the space of functions which are constant on each coarse face $F \subset \partial T$; its dimension is six. We also define $S_{0;h}(\partial T)$ as the space of functions that are constant on each fine face $f \in \mathcal{F}_h$, $f \subset \partial T$, and that have mean value zero on ∂T.

4. Stability estimates. In this section, we will study two operators and analyze their stability properties; they will play essential roles in the proof of the bound for our iterative substructuring method. The first operator is the standard interpolant onto the global coarse subspace and the second is an extension operator. Finally, we will prove a decomposition lemma for the traces of Raviart-Thomas functions on a substructure, which is analogous to similar results for conforming finite elements in H^1; see [11]. However, the bounds are now given in terms of the norm of $H^{-1/2}$, the trace space of vectors in $H(\mathrm{div}\,;\Omega)$. We note that just as in the H^1 case, some of our bounds will not be uniform in the mesh size.

We first consider the interpolation operator ρ_H onto X_H, which is defined in terms of the degrees of freedom of X_H, i.e.

$$\lambda_F(\rho_H \mathbf{u}) := \frac{1}{|F|} \int_F \mathbf{u} \cdot \mathbf{n} \, d\sigma, \quad F \in \mathcal{F}_H.$$

Our next lemma establishes the stability of the interpolant ρ_H. As previously remarked, ρ_H is logarithmically stable in the $\| \cdot \|_{\mathrm{div}}$-norm, in three dimensions, in contrast to the nodal interpolant on continuous finite element spaces, which has a norm which grows algebraically with H/h; see [11]. We also note that for the case at hand the best bound for the L^2-norm alone involves a factor of H/h; this can easily be seen by considering an element $\mathbf{u}$, for which all the interior degrees of freedom vanish.

LEMMA 4.1. *There exists a constant C, which depends only on the aspect ratios of $T \in \mathcal{T}_H$ and the elements of $\mathcal{T}_h$, such that for all $\mathbf{u} \in X_h$,*

$$\|\mathrm{div}\,(\rho_H \mathbf{u})\|_{0;T}^2 \leq \|\mathrm{div}\,\mathbf{u}\|_{0;T}^2, \tag{11}$$

$$\|\rho_H \mathbf{u}\|_{0;T}^2 \leq C \left(\left(1 + \log\left(\frac{H}{h}\right) \right) \|\mathbf{u}\|_{0;T}^2 + H_T^2 \|\mathrm{div}\,\mathbf{u}\|_{0;T}^2 \right). \tag{12}$$

Proof. By a simple computation and the use of Green's formula, we find that

$$(\mathrm{div}\,(\rho_H \mathbf{u}))\,|_T = (\Pi_H \mathrm{div}\,(\mathbf{u}))\,|_T, \tag{13}$$

where Π_H is the L^2-projection operator onto the space of constants on $T \in \mathcal{T}_H$; see [8, Sect. III.3.4]. Inequality (11) follows immediately.

The proof of (12) uses Green's formula, Lemma 3.1, and a partition of unity very similar to one given in [11] for the simplex case. Consider a face $F \subset \partial T$, and note that it is partitioned into N_F non-overlapping faces $f \in \mathcal{F}_h$; see Figure 1 depicting, for simplicity, just a very regular case. Number these faces so that f_i, $1 \leq i \leq n_F$ have at least one vertex on an edge of F, (see Fig 1), and let $\{f_1, f_2, f_3, f_4\}$ be the faces that contain a corner point. Let $t_i \subset T$, be the associated elements. We note that since, by assumption, the triangulation of the face is quasi-uniform, $n_F \leq C(H/h)$.

Let ϑ_F be a continuous, piecewise trilinear function defined on T. It vanishes on $\partial T \setminus F$ and is equal to 1 at all interior mesh points of F. The extension of ϑ_F to

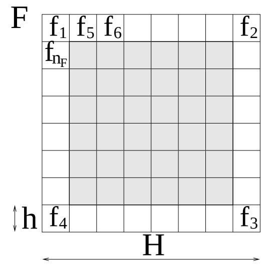

FIG. 1. *Decomposition of F*

the interior of T has values between 0 and 1, and the absolute value of its gradient is bounded by $C/\max(r,h)$ where r denotes the distance to the wire-basket of T. We refer to [11] for an explicit construction of such a function for a simplex; this construction can easily be adapted to the cubic case. It is established in [11] that

$$|\vartheta_F|_{1;T}^2 \leq C(1+\log H/h)H, \quad ||\vartheta_F||_{0;T}^2 \leq CH^3. \tag{14}$$

Using (9), it is sufficient to bound $\lambda_F(\rho_H \mathbf{u})$, for each face $F \subset \partial T$. Applying Green's formula, we obtain

$$\begin{aligned} |F|\,\lambda_F(\rho_H\mathbf{u}) &= \int_F (\mathbf{u}\cdot\mathbf{n})\,d\sigma \\ &= \int_{\partial T} \vartheta_F(\mathbf{u}\cdot\mathbf{n})\,d\sigma + \frac{3}{4}\sum_{i=1}^{4}|f_i|\left(\mathbf{u}\cdot\mathbf{n}_{|f_i}\right) + \frac{1}{2}\sum_{i=5}^{n_F}|f_i|\left(\mathbf{u}\cdot\mathbf{n}_{|f_i}\right) \\ &= \int_T (\vartheta_F \mathrm{div}\,\mathbf{u} + \mathbf{grad}\,\vartheta_F\cdot\mathbf{u})\,dx + \frac{3}{4}\sum_{i=1}^{4}|f_i|\lambda_{f_i}(\mathbf{u}) + \frac{1}{2}\sum_{i=5}^{n_F}|f_i|\lambda_{f_i}(\mathbf{u}). \end{aligned}$$

Thanks to (9), the absolute values of the last two terms can be bounded by

$$C\sum_{i=1}^{n_F} h^{1/2}||\mathbf{u}||_{0;t_i} \leq C n_F^{1/2} h^{1/2}\left(\sum_{i=1}^{n_F}||\mathbf{u}||_{0;t_i}^2\right)^{1/2} \leq CH^{1/2}||\mathbf{u}||_{0;T},$$

and, by using (14), we find the following bound for $|F|\ |\lambda_F(\rho_H\mathbf{u})|$

$$C\left(H^{3/2}||\mathrm{div}\,\mathbf{u}||_{0;T} + (H(1+\log H/h))^{1/2}||\mathbf{u}||_{0;T} + H^{1/2}||\mathbf{u}||_{0;T}\right). \tag{15}$$

Summing over all $F \subset \partial T$, (15) finally gives

$$||\rho_H\mathbf{u}||_{0;T}^2 \leq CH^2||\mathrm{div}\,\mathbf{u}||_{0;T}^2 + C\left(1+\log\left(\frac{H}{h}\right)\right)||\mathbf{u}||_{0;T}^2.$$

□

Remark: *We can obtain a similar estimate for the energy norm on each substructure T*

$$\begin{aligned} \int_T B\ (\rho_H\mathbf{u})\cdot(\rho_H\mathbf{u})\,dx \quad &\leq \quad C\tfrac{\gamma_T}{\beta_T}\left(1+\log\left(\tfrac{H}{h}\right)\right)\int_T B\ \mathbf{u}\cdot\mathbf{u}\,dx \\ &\quad + C\tfrac{H_T^2\,\gamma_T}{a_T}\int_T a\ \mathrm{div}\,\mathbf{u}\ \ \mathrm{div}\,\mathbf{u}\,dx. \end{aligned}$$

Here, a_T is the minimum of $a(x)$ on T, and β_T and γ_T satisfy

$$\beta_T\,\eta^T\eta \le \eta^T B(x)\eta \le \gamma_T\,\eta^T\eta, \quad \forall \eta \in \mathbb{R}^3,\ \forall x \in T.$$

The constant in the corresponding global estimate depends on the ratio of the coefficients B and a on individual substructures, and is independent of the jumps of the coefficients between the substructures.

The following lemma is an easy consequence of the stability of the operator P_h; cf. Lemma 2.4. It ensures that an equivalent discrete norm can be found for the trace spaces of the Raviart-Thomas finite elements. It will be employed in the proof of Lemma 4.3. We note that we use the stability of P_h in $H^{\frac{1}{2}}$ and the fact that this operator preserves the integrals over the faces f. This latter property is not satisfied by P_Q alone.

LEMMA 4.2. *There exist constants, c and C, such that*

$$c \sup_{\substack{\phi\in V_h(\partial T)\\ \phi\neq 0}} \frac{\langle \mathbf{u}\cdot\mathbf{n},\phi\rangle}{||\phi||_{\frac{1}{2};\partial T}} \le ||\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2};\partial T} \le C \sup_{\substack{\phi\in V_h(\partial T)\\ \phi\neq 0}} \frac{\langle \mathbf{u}\cdot\mathbf{n},\phi\rangle}{||\phi||_{\frac{1}{2};\partial T}}. \tag{16}$$

Furthermore, if $\langle \mathbf{u}\cdot\mathbf{n}, 1\rangle = 0$, the $||\cdot||_{\frac{1}{2};\partial T}$-norm in (16) can be replaced by the seminorm and the supremum can be taken over the non-constant functions ϕ.

Proof. The lower bound follows directly from the definition of the $||\cdot||_{-\frac{1}{2};\partial T}$-norm. For the upper bound, let $\mathbf{u} \in X_{0;h}(T)$. There then exists a $\phi_\mathbf{u} \in H^{\frac{1}{2}}(\partial T)$ such that

$$||\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2};\partial T} \le 2\frac{\langle \mathbf{u}\cdot\mathbf{n},\phi_\mathbf{u}\rangle}{||\phi_\mathbf{u}||_{\frac{1}{2};\partial T}}.$$

Recalling the definition of P_h and the fact that $\mathbf{u}\cdot\mathbf{n}$ is constant on each element, and using Lemma 2.4, we find that

$$||\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2};\partial T} \le 2\frac{\langle \mathbf{u}\cdot\mathbf{n},P_h\phi_\mathbf{u}\rangle}{||\phi_\mathbf{u}||_{\frac{1}{2};\partial T}} \le C\frac{\langle \mathbf{u}\cdot\mathbf{n},P_h\phi_\mathbf{u}\rangle}{||P_h\phi_\mathbf{u}||_{\frac{1}{2};\partial T}}.$$

The proof is now completed by proceeding as in the proof of Lemma 2.2. □

An important step in finding a stable decomposition of $X_h(\Omega)$ involves a discrete extension operator from the boundary of a substructure to its interior. The stable extension operator, defined in the next lemma, provides a divergence-free extension of the boundary data given on ∂T. This fact ensures that the stability constant will be independent of the diameter of T. This will not be true for some other extension procedures.

LEMMA 4.3. *There exists an extension operator $\mathcal{H}_T : S_{0;h}(\partial T) \to X_h(T)$, such that, for any $\mu \in S_{0;h}(\partial T)$,*

$$\operatorname{div} \mathcal{H}_T\mu = 0,$$

and

$$||\mathcal{H}_T\mu||_{0;T} \le C||\mu||_{-\frac{1}{2};\partial T}. \tag{17}$$

Here C is independent of h, H, and μ.

Proof. The proof is similar to one given in [13, Lemma 2.47]. We will first prove the result for a substructure T of unit diameter. Consider a Neumann problem

$$\begin{aligned} \Delta\phi &= 0, \quad \text{in } T,\\ \frac{\partial\phi}{\partial n} &= \mu, \quad \text{on } \partial T. \end{aligned}$$

Here, $\partial/\partial n$ is the derivative in the direction of the outward normal of ∂T. This problem is solvable, since the boundary data μ has mean value zero on ∂T. We can select any solution, e.g. that with mean value zero on T. Our extension operator $\mathcal{H}_T$ is defined by $\mathcal{H}_T\mu := \rho_h \mathbf{u}$, where $\mathbf{u} = \mathbf{grad}\,\phi$, and ρ_h is the interpolant onto the Raviart-Thomas space $X_h(T)$, defined by the degrees of freedom given in (8); we will show below that the $\{\lambda_f(\mathbf{u})\}$ are well defined.

An elementary variational argument shows that $\|\mathbf{u}\|_{0;T} = |\phi|_{1;T} \leq C\|\mu\|_{-\frac{1}{2};\partial T}$. In order to estimate $\|\rho_h \mathbf{u}\|_{0;T}$, we will now estimate $\|\mathbf{u} - \rho_h \mathbf{u}\|_{0;T}$. This requires the use of a regularity result and a finite element error bound. Since μ is piecewise constant on ∂T, it belongs to $H^s(\partial T)$, for all $s < 1/2$. Using the surjectivity of the map $\phi \mapsto \partial\phi/\partial n$ from $H^{3/2+s}(T)$ onto $H^s(\partial T)$ and a regularity result given in [10, Corollary 23.5], we deduce

$$\|\phi\|_{\frac{3}{2}+s;T} \leq C\|\mu\|_{s;\partial T}, \quad s < \epsilon_T. \tag{18}$$

Here ϵ_T is strictly positive and depends on T.

The $\{\lambda_f(\mathbf{u}),\ f \in \mathcal{F}_h\}$ are well-defined, since $\mathbf{u} \in H^{\frac{1}{2}+s}(T)$, with $s > 0$, and, as in (13),

$$\mathrm{div}\ (\rho_h \mathbf{u}) = \Pi_h\,(\mathrm{div}\,\mathbf{u}) = 0,$$

where Π_h is the L^2 projection onto the space of constant functions on each fine element $t \subset T$.

We now use the following error estimate for the interpolating operator:

$$\|\mathbf{u} - \rho_h \mathbf{u}\|_{0;T} \leq Ch^r |\mathbf{u}|_{r;T}, \quad \frac{1}{2} < r \leq 1, \tag{19}$$

where C depends only on the aspect ratios of the elements of $\mathcal{T}_h$ and the exponent r; inequality (19) can be proven using standard arguments as in [19, Sect. 3.4.2]. Employing (19) with $r = 1/2 + s$, (18) and an inverse inequality, we find that for $s < \epsilon_T$,

$$\|\mathbf{u} - \rho_h \mathbf{u}\|_{0;T} \leq Ch^{\frac{1}{2}+s}\|\phi\|_{\frac{3}{2}+s;T} \leq C\|\mu\|_{-\frac{1}{2};\partial T}. \tag{20}$$

The bound for the L_2-norm of $\rho_h \mathbf{u}$ is then obtained by applying the triangle inequality.

We now consider a substructure T of diameter H, obtained by dilation from the substructure of unit diameter. Using the previous result and a scaling argument, we obtain

$$\begin{aligned} \mathrm{div}\,\mathcal{H}_T\mu &= 0, \\ \|\mathcal{H}_T\mu\|_{0;T} &\leq C\|\mu\|_{-\frac{1}{2};\partial T}, \end{aligned}$$

where C is independent of the diameter of T. □

We conclude this section by proving a decomposition lemma for the traces of Raviart-Thomas functions on a substructure.

LEMMA 4.4. *Let T be in $\mathcal{T}_H$ and let $\{\mu_F,\ F \subset \partial T\}$ be functions in $S_{0;h}(\partial T)$, which vanish on $\partial T \setminus F$. Let $\mu := \sum_{F\subset\partial T} \mu_F$. Then there exists a constant C, independent of h and μ_H, such that, for all $\mu_H \in S_H(\partial T)$,*

$$\|\mu_F\|^2_{-\frac{1}{2};\partial T} \leq C(1 + \log H/h)\Big((1 + \log H/h)\|\mu + \mu_H\|^2_{-\frac{1}{2};\partial T} + \|\mu\|^2_{-\frac{1}{2};\partial T}\Big). \tag{21}$$

Proof. Since $\mu_F \in S_{0;h}(\partial T)$, we obtain

$$||\mu_F||_{-\frac{1}{2};\partial T} \leq C \sup_{\substack{\phi \in V_h(\partial T)\\ \phi \neq \text{const.}}} \frac{\langle \mu_F, \phi \rangle}{|\phi|_{\frac{1}{2};\partial T}},$$

by applying Lemma 4.2.

Now, $\phi \in V_h(\partial T)$ can be split uniquely into $\phi_Q + \phi_B$, $\phi_Q \in Q_h(\partial T)$, $\phi_B \in B_h(\partial T)$ and by using Lemma 2.3, we find that

$$||\mu_F||_{-\frac{1}{2};\partial T} \leq C \left(\sup_{\substack{\phi_Q \in Q_h(\partial T)\\ \phi_Q \neq \text{const.}}} \frac{\langle \mu_F, \phi_Q \rangle}{|\phi_Q|_{\frac{1}{2};\partial T}} + \sup_{\substack{\phi_B \in B_h(\partial T)\\ \phi_B \neq 0}} \frac{\langle \mu_F, \phi_B \rangle}{|\phi_B|_{\frac{1}{2};\partial T}} \right). \tag{22}$$

For any $\phi_Q \in Q_h(\partial T)$, we now define a weighted average c_{ϕ_Q} by

$$c_{\phi_Q} \int_F \vartheta_F \, d\sigma = \int_F I_h(\vartheta_F \phi_Q) \, d\sigma.$$

Here, ϑ_F is given in the proof of Lemma 4.1 and I_h is the nodal interpolation operator onto $Q_h(\partial T)$. Then, the supremum in the first term on the right in (22) can be replaced by

$$\sup_{\substack{\phi_Q \in Q_h(\partial T)\\ \phi_Q \neq \text{const.}}} \frac{\langle \mu_F, \phi_Q \rangle}{|\phi_Q|_{\frac{1}{2};\partial T}} = \sup_{\substack{\phi_Q \in Q_h(\partial T)\\ \phi_Q \neq \text{const.}}} \frac{\langle \mu_F, \phi_Q - c_{\phi_Q} \rangle}{|\phi_Q - c_{\phi_Q}|_{\frac{1}{2};\partial T}} = \sup_{\substack{\phi_Q \in Q_h(\partial T)\\ \phi_Q \neq 0,\, c_{\phi_Q} = 0}} \frac{\langle \mu_F, \phi_Q \rangle}{|\phi_Q|_{\frac{1}{2};\partial T}}, \tag{23}$$

i.e., we need only consider functions ϕ_Q which have a zero weighted average. The following norm equivalence is similar to (6) and can be proved by the same standard techniques

$$c(|\phi_Q|^2_{\frac{1}{2};\partial T} + H c^2_{\phi_Q}) \leq \|\phi_Q\|^2_{\frac{1}{2};\partial T} \leq C(|\phi_Q|^2_{\frac{1}{2};\partial T} + H c^2_{\phi_Q}). \tag{24}$$

We remark that, because of (24), in the last term of (23), the $H^{\frac{1}{2}}$-seminorm can be replaced by the full norm.

We next decompose ϕ_B into the sum of terms $\phi_{B;F}$ supported on individual faces $F \subset \partial T$

$$\phi_B = \sum_{\substack{F \in \mathcal{F}_H\\ F \subset \partial T}} \phi_{B;F}. \tag{25}$$

Similarly we decompose ϕ_Q into a sum of contributions $\phi_{Q;F}$ supported on individual faces $F \subset \partial T$ and $\phi_{Q;w}$ supported in a neighborhood of the wire-basket which is one element wide; see Figure 2. Thus,

$$\phi_Q = \sum_{\substack{F \in \mathcal{F}_H\\ F \subset \partial T}} \phi_{Q;F} + \phi_{Q;w}. \tag{26}$$

Local inverse estimates combined with interpolation arguments easily give

$$|\phi_{B;F}|^2_{\frac{1}{2};\partial T} \quad \leq \quad C|\phi_B|^2_{\frac{1}{2};\partial T}. \tag{27}$$

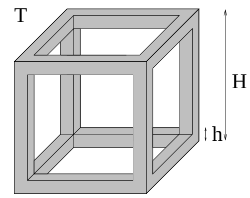

FIG. 2. *Neighborhood of the wire-basket*

Similar arguments give

$$|\phi_{Q;w}|^2_{\frac{1}{2};\partial T} \leq C\frac{1}{h}||\phi_{Q;w}||^2_{0;\partial T} \leq C||\phi_{Q;w}||^2_{0;W}, \tag{28}$$

where W is the wire-basket of T. We note that $||\phi_{Q;w}||^2_{0;W}$ is defined by a line integral. By using the inequality $||u^h||_{1;T} \leq C||u^h||^2_{1/2;\partial T}$, which is valid for discrete harmonic functions, and [11, Lemma 4.3 and Lemma 4.5], we can prove

$$||\phi_{Q;w}||^2_{0;W} \leq C(1+\log H/h)\ ||\phi_Q||^2_{\frac{1}{2};\partial T}, \tag{29}$$

$$||\phi_{Q;F}||^2_{\frac{1}{2};\partial T} \leq C(1+\log H/h)^2||\phi_Q||^2_{\frac{1}{2};\partial T}. \tag{30}$$

The proofs in [11] are for the simplicial case; they can be carried out in exactly the same way for the rectangular case and are therefore omitted. Combining (28) and (29), we find

$$||\phi_{Q;w}||^2_{\frac{1}{2};\partial T} \leq C(1+\log H/h)\ ||\phi_Q||^2_{\frac{1}{2};\partial T}. \tag{31}$$

We find, by using the splitting (26), that

$$\begin{aligned} \langle \mu_F, \phi_Q\rangle &= \sum_{\hat{F}\subset\partial T} \langle \mu_F, \phi_{Q;\hat{F}}\rangle + \langle \mu_F, \phi_{Q;w}\rangle \\ &= \langle \mu, \phi_{Q;F}\rangle + \langle \mu_F, \phi_{Q;w}\rangle. \end{aligned} \tag{32}$$

Since $I_h(\vartheta_F\phi_Q) = \phi_{Q;F} = I_h(\vartheta_F\phi_{Q;F})$ and since we can always assume that $c_{\phi_Q} = 0$, we obtain

$$\langle \mu_H, \phi_{Q;F}\rangle = 0, \quad \forall \mu_H \in S_H(\partial T),$$

and $c_{\phi_{Q;F}} = 0$. The first term on the right side of (32) can be bounded by means of (30)

$$|\langle \mu, \phi_{Q;F}\rangle| = |\langle \mu+\mu_H, \phi_{Q;F}\rangle| \leq C(1+\log H/h)||\phi_Q||_{\frac{1}{2};\partial T}\ ||\mu+\mu_H||_{-\frac{1}{2};\partial T}. \tag{33}$$

For each $\phi_{Q;w}$ there is a unique $\tilde{\phi}_{B;F} \in B_h(\partial T)$ such that

$$\int_f \phi_{Q;w}\ d\sigma = \int_f \tilde{\phi}_{B;F}\ d\sigma, \quad f \in \mathcal{F}_h,\ f \subset F,$$

with $\tilde{\phi}_{B;F} = 0$ on $\partial T \setminus F$. Moreover, this mapping is continuous

$$||\tilde{\phi}_{B;F}||^2_{\frac{1}{2};\partial T} \leq C\frac{1}{h}||\tilde{\phi}_{B;F}||^2_{0;\partial T} \leq C\frac{1}{h}||\phi_{Q;w}||^2_{0;\partial T} \leq C||\phi_{Q;w}||^2_{\frac{1}{2};\partial T}.$$

By means of this bound and (31), we finally obtain

$$
\begin{aligned}
|\langle \mu_F, \phi_{Q;w} \rangle| &= \left| \langle \mu_F, \tilde{\phi}_{B;F} \rangle \right| = \left| \langle \mu, \tilde{\phi}_{B;F} \rangle \right| \leq C ||\mu||_{-\frac{1}{2};\partial T} ||\phi_{Q;w}||_{\frac{1}{2};\partial T} \\
&\leq C(1+\log H/h)^{1/2} ||\mu||_{-\frac{1}{2};\partial T} ||\phi_Q||_{\frac{1}{2};\partial T}.
\end{aligned} \tag{34}
$$

Using (27), we find for the second term on the right hand side of (22)

$$
\begin{aligned}
\frac{|\langle \mu_F, \phi_B \rangle|}{|\phi_B|_{\frac{1}{2};\partial T}} &= \frac{|\langle \mu, \phi_{B;F} \rangle|}{|\phi_B|_{\frac{1}{2};\partial T}} \leq \frac{||\mu||_{-\frac{1}{2};\partial T} \, ||\phi_{B;F}||_{\frac{1}{2};\partial T}}{|\phi_B|_{\frac{1}{2};\partial T}} \\
&\leq C \frac{||\mu||_{-\frac{1}{2};\partial T} \, |\phi_{B;F}|_{\frac{1}{2};\partial T}}{|\phi_B|_{\frac{1}{2};\partial T}} \leq C ||\mu||_{-\frac{1}{2};\partial T}.
\end{aligned} \tag{35}
$$

The proof is completed by combining (22), (23), (24), (26), (33), (34), and (35). □

5. The iterative substructuring method. It follows from Lemma 1.1 that the first step towards the introduction of our Schwarz method is to define a decomposition of the space $X_{0;h}$. For each interior face $F \in \mathcal{F}_H$ there are two elements T_i, $T_j \in \mathcal{T}_H$ such that $\bar{F} := \partial T_i \cap \partial T_j$, and we set $\bar{T}_F := \bar{T}_i \cup \bar{T}_j$. We will now decompose $X_{0;h}$ into the coarse space $X_{0;H}$, the face spaces X_F, $F \in \mathcal{F}_H$, and the interior spaces $X_T := X_{0;h}(T)$, $T \in \mathcal{T}_H$. The face spaces are defined by

$$
X_F := \{ \mathbf{v} \in X_{0;h} \mid a(\mathbf{v}, \mathbf{w}) = 0, \ \mathbf{w} \in X_{T_i} \cup X_{T_j}, \ \operatorname{supp} \mathbf{v} \subset \overline{T}_F \}.
$$

We note that an element $\mathbf{v} \in X_F$ is defined uniquely by the value of $\mathbf{v} \cdot \mathbf{n}$ on F and that the coarse space $X_{0;H}$ is contained in the union of the face and interior spaces. Thus, the decomposition

$$
X_{0;h} = X_{0;H} + \sum_{T \in \mathcal{T}_H} X_T + \sum_{F \in \mathcal{F}_H} X_F, \tag{36}
$$

is not a direct sum.

In our proof, we will also use $\{\widetilde{X}_F\}$ which are divergence free subspaces of $X_{0;h}$, and are built in the following way:
Consider any function μ on F, that is piecewise constant and has mean-value zero on F. Then, μ can be extended by zero to all of ∂T_i, to obtain a function of $S_{0;h}(\partial T_i)$, still denoted by μ. Let $\mathbf{u}_i := \mathcal{H}_{T_i} \mu$. In a similar way, we can extend $-\mu$ by zero on $\partial T_j \setminus F$, and construct a function $\mathbf{u}_j = \mathcal{H}_{T_j}(-\mu)$, on T_j. The minus sign has to be chosen, since the elements T_i and T_j have outward normals in opposite directions. We define $\widetilde{X}_F$ as the space of functions $\mathbf{u}$, the restriction of which to T_i and T_j are equal to $\mathbf{u}_i$ and $\mathbf{u}_j$, respectively, and that are zero outside $\bar{T}_F$. Thus, each element in $\widetilde{X}_F$ is uniquely defined by its normal component on F, and its dimension is equal to the number of fine faces in F minus one.

THEOREM 5.1. *For each $\mathbf{u} \in X_{0;h}$, there exists a decomposition*

$$\mathbf{u} = \mathbf{u}_H + \sum_{T \in \mathcal{T}_H} \mathbf{u}_T + \sum_{F \in \mathcal{F}_H} \mathbf{u}_F,$$

corresponding to (36), such that

$$a(\mathbf{u}_H, \mathbf{u}_H) + \sum_{T \in \mathcal{T}_H} a(\mathbf{u}_T, \mathbf{u}_T) + \sum_{F \in \mathcal{F}_H} a(\mathbf{u}_F, \mathbf{u}_F) \leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 a(\mathbf{u}, \mathbf{u}),$$

with a constant C, independent of h, H and $\mathbf{u}$.

Proof. We remark that, because of the equivalence of the graph and energy norms, we only have to prove the stability of the decomposition (36) with respect to the graph norm.

We will first prove the stability of the decomposition

$$X_{0;h} = X_{0;H} + \sum_{T \in \mathcal{T}_H} X_T + \sum_{F \in \mathcal{F}_H} \widetilde{X}_F, \tag{37}$$

and we will then employ the energy-minimizing property of the harmonic extensions $\{\mathbf{u}_F\}$. We also consider one subdomain T at a time; the global result is obtained by summing over all subdomains.

Counting the degrees of freedom shows that (37) is a direct sum, and consequently for each $\mathbf{u}$, (37) defines a unique $\mathbf{u}_H$. In particular, $\rho_H(\mathbf{u} - \mathbf{u}_H) = 0$ yields $\rho_H \mathbf{u} = \mathbf{u}_H$. Using Lemma 4.1, we immediately obtain an upper bound for the first term

$$||\mathbf{u}_H||^2_{\mathrm{div}\,;T} \leq C \left(1 + \log\frac{H}{h}\right) ||\mathbf{u}||^2_{\mathrm{div}\,;T}.$$

For each face $F \subset \partial T$ there is a unique $\mu_F \in S_{0;h}(\partial T)$ which is zero on $\partial T \setminus F$, such that $(\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{n}_{|_F} = \mu_F$. By means of the definition of $\widetilde{X}_F$, we obtain $\widetilde{\mathbf{u}}_F = \mathcal{H}_T\, \mu_F$, in T. Combining (17) and Lemma 4.4, we obtain, for any $\mu_H \in S_H(\partial T)$,

$$\begin{aligned} ||\widetilde{\mathbf{u}}_F||^2_{\mathrm{div}\,;T} \quad &\leq \quad C(1 + \log H/h)||(\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{n}||^2_{-\frac{1}{2};\partial T} \\ &\quad + C(1 + \log H/h)^2 ||(\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{n} + \mu_H||^2_{-\frac{1}{2};\partial T}. \end{aligned} \tag{38}$$

Lemma 2.1, the triangle inequality, and Lemma 4.1 yield

$$||(\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{n}||^2_{-\frac{1}{2};\partial T} \leq C(1 + \log H/h)||\mathbf{u}||^2_{\mathrm{div}\,;T},$$

and the choice $\mu_H = \mathbf{u}_H \cdot \mathbf{n}$, finally, gives

$$||\widetilde{\mathbf{u}}_F||^2_{\mathrm{div}\,;T} \leq C(1 + \log H/h)^2 ||\mathbf{u}||^2_{\mathrm{div}\,;T}.$$

An upper bound for $||\mathbf{u}_T||^2_{\mathrm{div}\,;T}$ is now an easy consequence of the triangle inequality.

The stability of (37) with respect to the energy norm $||\cdot||_a$ is a consequence of the norm equivalence of the graph norm $||\cdot||_{\mathrm{div}}$ and the energy norm. More precisely, the constant C in (37) is proportional to

$$\max_{T \in \mathcal{T}_H} \max\left(\frac{\gamma_T}{\beta_T}, \frac{h_T^2\, \gamma_T}{a_T}\right);$$

see the Remark to Lemma 4.1.

In order to prove the stability of the decomposition (36), we set $\mathbf{u}_H := \rho_H \mathbf{u}$ and extend the trace $\mu_F = (\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{n}_{|_F}$, harmonically in T_i and T_j to obtain a function $\mathbf{u}_F \in X_F$. The energy-minimizing property of the harmonic extension yields

$$a(\mathbf{u}_F, \mathbf{u}_F) \leq a(\tilde{\mathbf{u}}_F, \tilde{\mathbf{u}}_F) \leq C \left(1 + \log \frac{H}{h}\right) a(\mathbf{u}, \mathbf{u}).$$

The remainder $\mathbf{u} - \mathbf{u}_H - \sum_{F \in \mathcal{F}_H} \mathbf{u}_F$ is a sum of elements belonging to the interior spaces, the contributions of which can be bounded using the triangle inequality. □

Finally, we consider the splitting (36) for the limit case $a = 0$. In this case, the bilinear form $a(\cdot, \cdot)$ is just a weighted L^2-scalar product

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega B\mathbf{v} \cdot \mathbf{w} \, dx.$$

Let us first decompose $\mathbf{u}$ as follows

$$\mathbf{u} = \sum_{T \in \mathcal{T}_H} \bar{\mathbf{u}}_T + \sum_{F \in \mathcal{F}_H} \bar{\mathbf{u}}_F,$$

where $\bar{\mathbf{u}}_F \in X_{0;h}$ with $\lambda_f(\bar{\mathbf{u}}_F) := \lambda_f(\mathbf{u})$, $f \subset F$, $f \in \mathcal{F}_h$ and $\lambda_f(\bar{\mathbf{u}}_F) = 0$ elsewhere, and $\mathbf{u}_T \in X_T$. Then, Lemma 3.1 guarantees that

$$\sum_{F \in \mathcal{F}_H} ||\bar{\mathbf{u}}_F||_0^2 \leq C ||\mathbf{u}||_0^2.$$

We remark that $\bar{\mathbf{u}}_F$ is an extension by zero to the interior of the substructures and, therefore, in general not contained in X_F. Let us now consider the unique decomposition

$$\mathbf{u} = \sum_{T \in \mathcal{T}_H} \mathbf{u}_T + \sum_{F \in \mathcal{F}_H} \mathbf{u}_F,$$

where $\mathbf{u}_T \in X_T$ and $\mathbf{u}_F \in X_F$. By using the minimization property of $\mathbf{u}_F$ and the fact that $\mathbf{u}_F \cdot \mathbf{n}_{|_F} = \bar{\mathbf{u}}_F \cdot \mathbf{n}_{|_F}$, we obtain

$$\sum_{F \in \mathcal{F}_H} ||\mathbf{u}_F||_0^2 \leq \sum_{F \in \mathcal{F}_H} ||\bar{\mathbf{u}}_F||_0^2 \leq C ||\mathbf{u}||_0^2.$$

This proves the stability of the decomposition of $\mathbf{u}$ with respect to the L^2-norm. Thus, as the ratio between the coefficients B and a becomes large, we get an upper bound for the condition number which is independent of H/h. We remark that this result cannot be obtained with the splitting (37).

In the second limit case, $B = 0$, the bilinear form $a(\cdot, \cdot)$ is no longer positive definite. However, we can still work with the preconditioned conjugate gradient in a subspace, if the right hand side $\mathbf{f}$ is consistent. Then, the stability of ρ_H with respect to the L^2-norm of the divergence, (11), gives us an optimal result, i.e., we obtain a condition number which is independent of H/h.

Remark: *In the multilevel context, we can immediately get an additive Schwarz method by using a decomposition of $X_{0;h}$ in terms of the hierarchical surplus spaces associated with the different levels and a vertical splitting into divergence-free and* *complementary spaces. Using Lemma 4.1, we get a lower bound for the minimal eigenvalue that is proportional to l^{-2}, where l is the number of refinement levels. A strengthened Cauchy-Schwarz inequality, similar to the one established in [26] for the two dimensional case, proves that the largest eigenvalue is bounded independently of the number of refinement levels. Altogether a multilevel preconditioner is obtained, where the number of conjugate gradient steps to obtain a fixed reduction of the residual norm grows linear with the number of refinement steps.*

6. Numerical results. In this section, we present some numerical results on the performance of the iterative substructuring method based on the decomposition (36), for varying coarse and fine mesh sizes, and varying coefficients a and B. We refer to [22], for a general discussion of practical issues concerning Schwarz methods.

We consider the domain $\Omega = (0,1)^3$ and uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$. The fine triangulation $\mathcal{T}_h$ consists of n^3 cubical elements, with $h = 1/n$. The matrix B is given by

$$B = \text{diag}\{b, b, b\}.$$

TABLE 1

Estimated condition number and number of conjugate gradient iterations for a residual norm reduction of 10^{-6} *(in parentheses), versus* H/h *and* n*. Case of* $a = 1$, $b = 1$.

H/h	8	4	2
n=8	-	13.28 (14)	15.15 (22)
n=16	19.46 (16)	23.26 (24)	17.37 (24)
n=24	32.78 (27)	25.55 (26)	17.43 (21)
n=32	33.48 (27)	26.01 (26)	17.42 (21)
n=40	35.50 (27)	26.08 (25)	-
n=48	36.47 (28)	25.91 (22)	-

Table 1 shows the estimated condition number and the number of iterations to obtain a reduction of the residual norm by a factor 10^{-6}, as a function of the dimensions of the fine and coarse meshes. The estimate of the condition number is obtained from the parameters calculated during the conjugate gradient iteration, as described in [18]. For a fixed H/h, the condition number appears to remain bounded independently of the number of fine mesh points n. The number of iterations varies slowly with H/h and n.

We remark that the supports of the face spaces, consisting of the union of two substructures, can be colored in such a way that spaces with the same color do not intersect. Therefore, the largest eigenvalue of the additive Schwarz operator P_{as} is bounded by the number of colors plus one; see [22, p. 165]. The largest eigenvalue is 7 in all the cases in Table 1, except for ($n = 8$, $H/h = 4$) and ($n = 16$, $H/h = 8$); the latter cases correspond to a partition into 2 by 2 by 2 subregions and, consequently, the largest eigenvalue is bounded by 4.

In Figure 3, we plot the results of Table 1, together with the best least-square fit second order logarithmic polynomial. Our numerical results are in good agreement with the theoretical bound obtained in the previous section and they suggest that our bound is sharp.

In Table 2, we show some results when the ratio of the coefficients b and a is changed. For a fixed value of $n = 24$ and $a = 1$, the estimated condition number and the number of iterations are shown as functions of H/h and b. These numerical

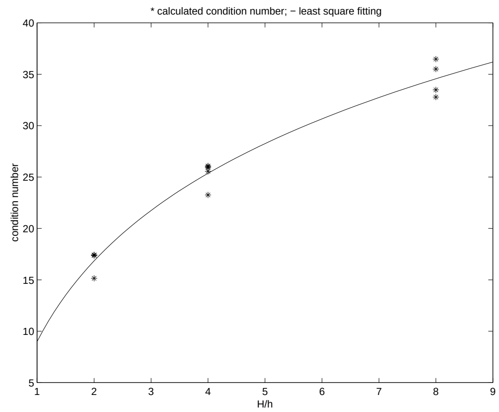

FIG. 3. *Estimated condition number from Table 1 (asterisk) and least-square second order logarithmic polynomial (solid line), versus H/h; the relative fitting error is about* 4.5 *per cent.*

TABLE 2
Estimated condition number and number of conjugate gradient iterations for a residual norm reduction of 10^{-6} *(in parentheses), versus H/h and b. Case of $n = 24$ and $a = 1$.*

H/h	8	4	2
b=1e-09	4.00 (10)	5.81 (16)	6.29 (15)
b=1e-08	4.00 (10)	5.81 (16)	6.29 (15)
b=1e-07	4.00 (10)	5.82 (16)	6.29 (15)
b=1e-06	4.00 (10)	21.0 (18)	6.29 (15)
b=1e-05	17.5 (11)	25.0 (19)	16.1 (18)
b=0.0001	29.5 (12)	25.0 (19)	17.1 (18)
b=0.001	30.9 (15)	25.3 (21)	17.2 (18)
b=0.01	32.3 (20)	25.4 (22)	17.2 (18)
b=0.1	32.6 (22)	25.5 (25)	17.4 (20)
b= 1	32.8 (27)	25.6 (26)	17.4 (21)
b= 10	30.0 (29)	23.4 (26)	17.1 (23)
b=1e+02	23.6 (26)	20.4 (25)	15.1 (22)
b=1e+03	14.4 (21)	14.1 (22)	12.6 (19)
b=1e+04	8.42 (16)	8.57 (17)	9.43 (17)
b=1e+05	6.75 (14)	6.98 (15)	7.92 (17)
b=1e+06	6.72 (14)	6.91 (15)	7.80 (16)

results also confirm the theoretical results in the limit cases $b = 0$ and $b = \infty$, as given in the previous section, since we observe that the condition number appears to be bounded independently of the ratio H/h when the ratio b/a is very small or very large.

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