A Numerical Study of a Class of FETI Preconditioners for Mortar Finite Elements in Two Dimensions

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Abstract

The FETI method is an iterative substructuring method using Lagrange multipliers. It is actively used in industrial–size parallel codes for solving difficult computational mechanics problems, for example the system ANSYS. Mortar finite elements are nonconforming finite elements that also allow for a geometrically nonconforming decomposition of the computational domain and for the optimal coupling of different variational approximations in different subdomains. We present a numerical study of three different FETI preconditioners for two dimensional, self-adjoint, elliptic equations discretized by mortar finite elements.

Key words. FETI preconditioners, mortar finite elements, Lagrange multipliers, domain decomposition

AMS subject classification. 65N12, 65N22, 65N30, 65N55

1 Introduction

The Finite Element Tearing and Interconnecting (FETI) method is a Lagrange multiplier based iterative substructuring method. It was introduced by Farhat and Roux [16]; a detailed presentation is given in [17], a monograph by the same authors. Originally used to solve second order, self-adjoint elliptic equations, it has later been extended to many other problems, e.g. time-dependent problems, cf. Farhat, Chen, and Mandel [11], plate bending problems, cf. Farhat et al [12, 13, 15], and heterogeneous elasticity problems with composite materials, cf. Farhat and Rixen [26, 27].

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The FETI method was designed for conforming finite elements, and is based on the decomposition of the computational domain Ω into non-overlapping subdomains. Continuity across the interface is enforced by using Lagrange multipliers. After eliminating the subdomain variables, the dual problem, given in terms of Lagrange multipliers, is solved by a projected conjugate gradient (PCG) method. Once an accurate approximation for the Lagrange multipliers has been obtained, the values of the primal variables are obtained by solving a local problem for each subdomain.

It was shown experimentally by Farhat, Mandel, and Roux [14] that applying a certain projection operator in the PCG solver plays a role similar to that of a coarse problem for other domain decomposition algorithms, and that certain variants of the FETI algorithm are numerically scalable with respect to both the problem size and the number of subdomains. Mandel and Tezaur later showed that for a FETI method which employs a Dirichlet preconditioner the condition number grows at most in proportion to $(1 + \log(H/h))^3$ where H is the subdomain diameter and h is the mesh size; cf [24].

In this paper, we present a numerical study of the FETI method for two dimensional self-adjoint elliptic equations discretized by low order mortar finite elements. We use geometrically nonconforming mortar finite elements of the second generation, for which no continuity conditions are imposed at the vertices of the subdomains.

We have tested the following three preconditioners: The Dirichlet preconditioner, which has been used successfully for conforming finite elements, see, e.g. Farhat, Mandel, and Roux [12], a block diagonal preconditioner used by Lacour [20, 21], and a new preconditioner introduced by Klawonn and Widlund in [18]. The last preconditioner performs best in terms of both, iteration and flop counts, and has scalability properties similar to those of the algorithm with the Dirichlet preconditioner in a conforming case.

For other work on preconditioners for mortar finite element discretizations, see Achdou, Kuznetsov, and Pironneau [2], Achdou and Kuznetsov [1], Achdou, Maday, and Widlund [3], Casarin and Widlund [6], Kuznetsov [19], and the references therein.

There has been also other work on FETI and Lagrange multiplier based substructuring methods for problems with non–matching grids, see, e.g., Farhat and Géradin [9],[10], and Rixen, Farhat, and Géradin [25] and the references therein.

The rest of the paper is structured as follows. In the next section, we describe the mortar finite element method. In Section 3, we present the classical FETI method and the Dirichlet preconditioner, and in Section 4, we discuss the FETI algorithm for mortars with two different preconditioners. In Section 5, we present numerical results for all preconditioners, and, in Section 6, we draw some conclusions on these results.

2 Mortar finite elements

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [5], and a three dimensional version was developed by Ben Belgacem and Maday in [4]. They are nonconforming finite elements that allow for a nonconforming decomposition of the computational domain and for the optimal coupling of different variational approximations in different subdomains. Here, we mean by optimality that the global error is bounded by the sum of the local approximation errors on each subdomain.

Using mortars instead of conforming finite elements has some significant advantages. The mesh generation is more flexible and can be made quite simple on the individual subregions. It is also possible to move different parts of the mesh relative to each other, which is useful for time dependent problems and in design optimization. The mortar methods also allow for local refinement of finite element models in only certain subdomains of the computational domain, and they are well suited for parallel computing.

Let us briefly describe the mortar finite element space V^h, restricting our discussion to the two dimensional case. The computational domain Ω is decomposed into a nonoverlapping polygonal partition $\{\Omega_i\}_{i=\overline{1:N}}$. The restriction of the mortar space to any subdomain Ω_i is a conforming P_1 or Q_1 finite element space. Across the interface Γ, i.e. the set of points that belong to the boundaries of at least two subdomains, pointwise continuity is not required. We note that we can work with geometrically nonconforming mortars, i.e. we need not require that the intersection of the boundaries of two different subdomains is empty, a vertex, or an entire edge. We partition Γ into a union of nonoverlapping edges of the subdomains $\{\Omega_i\}_{i=\overline{1:N}}$, called the nonmortars. The edges, across Γ, not chosen to be nonmortars, are called mortars. We note that the partition and the choice of nonmortars are not unique, but any choice can be treated similarly. On the two sides of a nonmortar edge γ, there are two distinct traces of the mortar functions and we only require that their difference is L^2-orthogonal to a space of test functions defined on γ. This space is generally a subspace of codimension two of the restriction of V^h to γ; this allows the values of the mortar function at the end points of any nonmortar to be genuine degrees of freedom.

More formally, if γ is a nonmortar side, let $V^h(\gamma)$ be the continuous piecewise polynomial space which is the restriction of V^h to γ. Let $\widetilde{\gamma}$ be the union of the parts of the mortars opposite γ. Then $w_h \in V^h$ is a mortar function if its restrictions, $w_\gamma = w_{|_\gamma}$ and $w_{\widetilde{\gamma}} = w_{|_{\widetilde{\gamma}}}$, satisfy the following L^2-orthogonality condition for every nonmortar side γ:

$$\int_\gamma (w_\gamma - w_{\widetilde{\gamma}})\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma). \tag{1}$$

Here, $\Psi^h(\gamma)$ is the space of the test functions; it is the subspace of $V^h(\gamma)$ whose restriction to the first and last mesh intervals are polynomials of degree 1 less than the corresponding degree from $V^h(\gamma)$.

3 The classical FETI method

In this section, we review the original FETI method of Farhat and Roux for elliptic problems discretized by conforming finite elements. Only the Dirichlet preconditioner introduced by Farhat, Mandel, and Roux [14] is discussed here. We consider P_1 or Q_1 finite elements with a typical mesh size h.

We first partition the finite element mesh along mesh lines into N non–overlapping subdomains $\Omega_i \subset \Omega, i = 1, \ldots, N$, and assume that the subdomain boundary nodes match across the interface. For each subdomain Ω_i, we construct the local stiffness matrix K_i and local load vector f_i. We denote by K the block–diagonal stiffness matrix with the K_i on the diagonal and by f the vector $[f_1, \ldots, f_N]$. Analogously, we denote by u_i the vector of nodal values on Ω_i and by u the vector $[u_1, \ldots, u_N]$.

Let $B = [B_1, B_2, \ldots, B_N]$ be a matrix which measures the jump of a given vector $u = [u_1, \ldots, u_N]$; $Bu = 0$ means that the values of the degrees of freedom, at all nodes which belong to at least two subdomains, coincide.

Let us consider the following minimization problem with constraints:

$$J(u) := \frac{1}{2} u^t K u - f^t u \to \min \quad \text{subject to } Bu = 0. \tag{2}$$

By introducing Lagrange multipliers λ for the constraint $Bu = 0$, we obtain the saddle point problem

$$\begin{array}{rcrcl} Ku & + & B^t\lambda & = & f, \\ Bu & & & = & 0. \end{array} \tag{3}$$

Let R be a given matrix that spans the null space of K, i.e. $\mathrm{range\,R} = \mathrm{ker\,K}$. The solution of the first equation in (3) exists if and only if $f - B^t\lambda \in \mathrm{range\,K}$. Then, we have

$$u = K^\dagger(f - B^t\lambda) + R\alpha,$$

where $K^\dagger$ is the pseudoinverse of K which provides a solution orthogonal to the null space of K and α has to be determined.

To formulate the FETI method, we need the following notations:

$$G := BR, F := BK^\dagger B^t,\ d := BK^\dagger f,\ P := I - G(G^tG)^{-1}G^t,\ e := R^t f.$$

Note that P is an l_2-orthogonal projector onto $V := \mathrm{ker\,G^t}$. Elimination of the primal variables u in (3) gives

$$BK^\dagger B^t\lambda = BK^\dagger f + BR\alpha, \tag{4}$$

which leads to

$$\begin{array}{rcl} PF\lambda & = & Pd, \\ G^t\lambda & = & e. \end{array}$$

The FETI algorithm is the solution of this dual problem with a preconditioned projected conjugate gradient (PCG) method, where all the increments $\lambda_k - \lambda_{k-1}$ are in V.

Given an initial approximation λ_0 with $G^t\lambda_0 = e$, we have to solve

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + V. \tag{5}$$

One possible preconditioner is of the form

$$M := B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t, \tag{6}$$

where S is the Schur complement of K obtained by eliminating the interior degrees of freedom. In the application of M, N independent Dirichlet problems have to be solved in each iteration step. Therefore, M is known as the Dirichlet preconditioner. Note that the Schur complement never has to be computed explicitly, since only the action of S on a vector is needed.

It has been shown by Mandel and Tezaur [24] that the condition number κ of $PMPF$ satisfies

$$\kappa(PMPF) \le C \left(1 + \log \frac{H}{h}\right)^3,$$

where C is a positive constant independent of h, H.

This result is similar to estimates for other non–overlapping domain decomposition methods; see Dryja, Smith, and Widlund [7] for iterative substructuring methods, Dryja and Widlund [8] for Neumann-Neumann algorithms, and Mandel [22] and Mandel and Brezina [23] for balancing algorithms.

4 The FETI method for mortars

The FETI algorithm can also be applied when mortar finite elements are considered on Ω. The price we have to pay for the inherent flexibility of the mortar finite elements is related to the fact that the Lagrange multiplier matrix B is more complicated in this case compared to that arising in the classical FETI method with conforming finite elements. This is due to the fact that we no longer have matching nodes across the interface.

Let us briefly describe the construction of the matrix B. From the mortar conditions (1), we see that the interior nodes of the nonmortar sides are not associated with genuine degrees of freedom in the finite element space V^h. Let w be a mortar finite element function and γ be a nonmortar. Let w_γ^1 be the vector of the values of w at the interior nodes of γ, and let w_γ^2 be the vector of the values of w at the end points of γ and at all the nodes on the edges on the interface opposite to γ, such that the intersection of γ and the support of the corresponding nodal basis functions is not empty. Then w_γ^1 is uniquely determined by w_γ^2, and the mortar conditions (1) for γ can be written in matrix form as

$$M_\gamma w_\gamma^1 - N_\gamma w_\gamma^2 \;=\; 0.$$

Here, M_γ is a tridiagonal matrix and N_γ is a banded matrix. The matrix B will have one block, B_γ, for each nonmortar side, which consists of the columns of the corresponding matrices M_γ and N_γ and zeros in the other columns.

Using mortar finite elements for the FETI method, we do not have to take special care of crosspoints, i.e. the points that belong to the closure of more than two subdomains, since the mortar conditions are only associated with the interior nodes on the nonmortar sides.

The matrix K is again a block–diagonal matrix $diag_{i=1}^{N} K_i$, where the local stiffness matrices $K_i, i = 1, \ldots, N$, are obtained from the finite element discretizations on individual subdomains. As in the case of conforming finite elements, we now have to solve the problem

$$\begin{array}{lclcl} Ku & + & B^t\lambda & = & f, \\ Bu & & & = & 0. \end{array} \tag{7}$$

Note that, in contrast to the conforming finite element case, the K_i can now be build from different discretizations on each subdomain Ω_i. To obtain a dual problem as in (5), we can now proceed completely analogously to sect. 3.

For the dual problem (5) obtained from the mortar system (7), one of the preconditioners to be considered is

$$\widehat{M} \ := \ (BB^t)^{-1} B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t (BB^t)^{-1}, \tag{8}$$

introduced by Klawonn and Widlund in [18]. It will also be shown in [18] that $\widehat{M}$ is an almost optimal preconditioner in the conforming finite element case, in the sense that

$$\kappa(P\widehat{M}PF) \ \leq \ C \left(1 + \log \frac{H}{h}\right)^2 .$$

Another preconditioner $\overline{M}$ has been suggested by Lacour in [20, 21]. It can be obtained from $\widehat{M}$ by taking only the block diagonal part of BB^t, instead of the whole matrix BB^t, i.e.

$$\overline{M} := (diag B_\gamma B_\gamma^t)^{-1} B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t (diag B_\gamma B_\gamma^t)^{-1}, \tag{9}$$

where $diag B_\gamma B_\gamma^t$ has an entry on the block–diagonal for each nonmortar γ.

5 Numerical Results

In this section, we present computational results for the three preconditioners discussed in the previous sections for mortar finite elements. As a model problem we consider the Poisson equation on the unit square $\Omega = [0, 1]^2$ with zero Dirichlet boundary conditions.

We begin by discussing the differences between the new preconditioner $\widehat{M}$, cf. (8), the preconditioner $\overline{M}$, cf. (9), and the Dirichlet preconditioner M, cf. (6), and by presenting some implementation details.

In general, the preconditioners $\widehat{M}$ and $\overline{M}$ require some extra work in comparison to M. In each iteration step, when we multiply a vector by the preconditioner, we have to solve two systems with the matrix BB^t, and $diag B_\gamma B_\gamma^t$,

Figure 1: Geometrically non-conforming decompositions of the unit square into 16, 32, 64, and 128 subdomains.

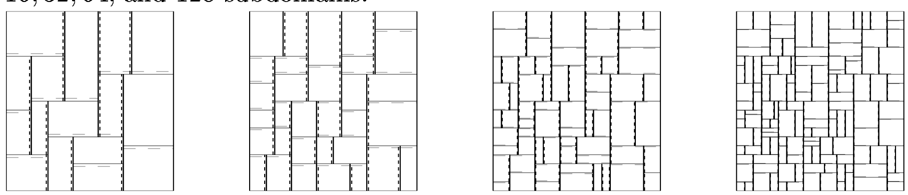
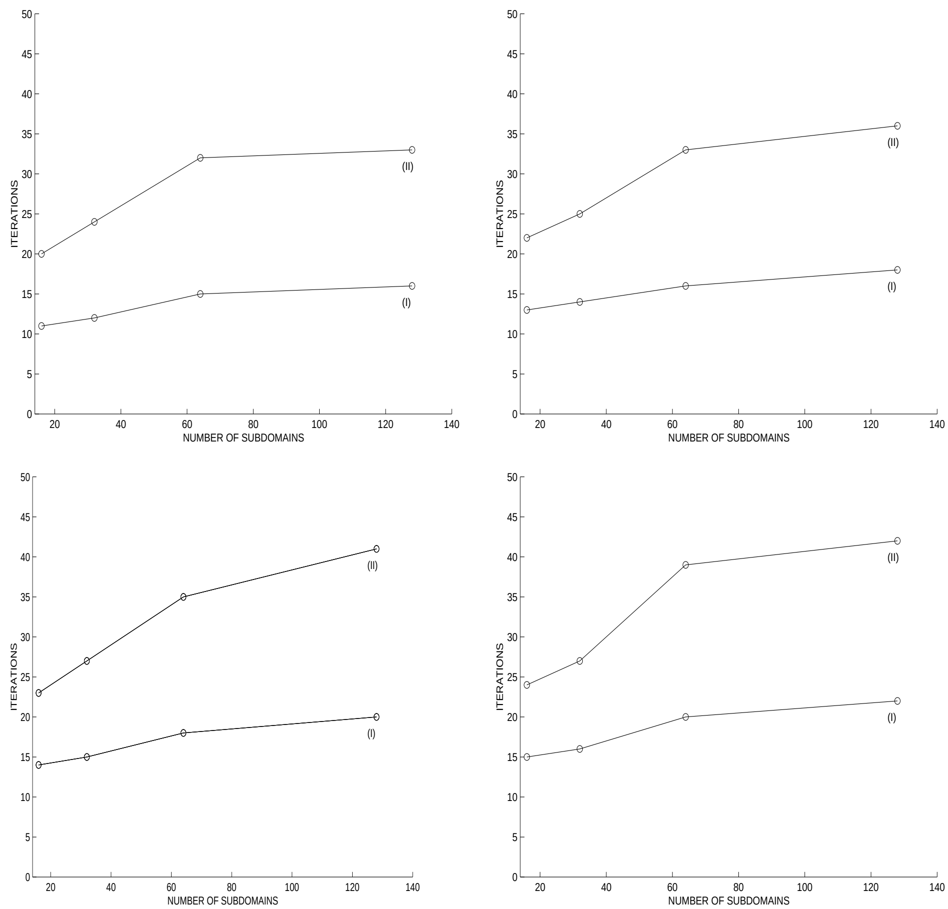


Table 1: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

		(I)		(II)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS	Iter	MFLOPS
16	4	11	9.0e–1	20	1.5e+0	108	7.4e+0
16	8	13	1.2e+1	22	1.9e+1	290	2.3e+2
16	16	14	2.2e+2	23	3.4e+2	406	5.6e+3
16	32	15	4.4e+3	24	6.8e+3	486	1.3e+5
32	4	12	2.4e+0	24	4.3e+0	223	3.7e+0
32	8	14	2.7e+1	25	4.6e+1	438	7.4e+2
32	16	15	4.9e+2	27	8.5e+2	620	1.8e+4
32	32	16	1.1e+4	27	1.9e+4	692	4.4e+5
64	4	15	7.2e+0	32	1.3e+1	487	1.9e+2
64	8	16	7.4e+1	33	1.4e+2	1071	4.3e+3
64	16	18	1.3e+3	35	2.4e+3	1725	1.1e+5
64	32	20	3.0e+4	39	5.7e+4	2130	2.9e+6
128	4	16	1.6e+1	33	2.9e+1	1107	9.0e+2
128	8	18	1.7e+2	36	3.2e+2	1413	2.0e+4
128	16	20	3.1e+3	41	6.0e+3	1761	2.5e+5
128	32	22	7.1e+4	42	1.3e+5	–	–

respectively. From the construction of B, it follows that BB^t is an almost block diagonal matrix, with each block corresponding to a nonmortar γ, and of size equal the number of interior nodes on γ. The non-zero entries which do not belong to the diagonal blocks are of two types. Some correspond to Lagrange multipliers associated to the first and last interior points of the nonmortars. Other occur because of nodal basis functions associated to points on the mortar sides, the support of which intersects more than one nonmortar. However, there are relatively few such non-zero entries.

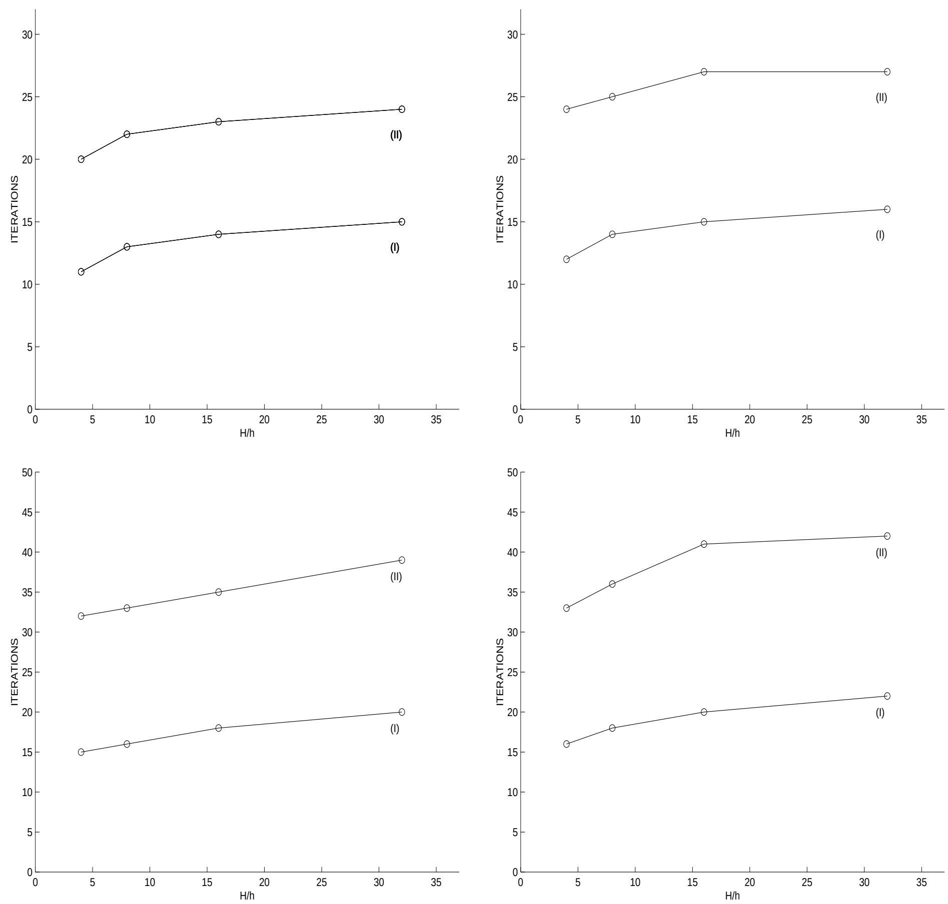
Figure 2: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner. Upper left : $H/h = 4$, Upper right : $H/h = 8$, Lower left : $H/h = 16$, Lower right : $H/h = 32$.



As our results show, the costs for applying $(BB^t)^{-1}$ and $(diagB_\gamma B_\gamma^t)^{-1}$ are relatively small compared to the costs for other operations performed during one iteration, e.g. multipling a vector by the Schur complement, or by its pseudoinverse. These costs further decrease if the LU factorizations of BB^t and $diagB_\gamma B_\gamma^t$ are computed only once, and the factors stored. Then, solving systems with BB^t or $diagB_\gamma B_\gamma^t$ only amounts to one back and one forward solve. Moreover, the improvement of the iteration count offsets this extra cost easily.

We note that we do not compute the Schur complements explicitly, nor their

Figure 3: Geometrically nonconforming partition, non–matching grids across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner. Upper left : $Ns = 16$, Upper right : $Ns = 32$, Lower left : $Ns = 64$, Lower right : $Ns = 128$, Ns = Number of subdomains.



pseudoinverses, but only the stiffness matrices for each subdomain.

In the geometrically nonconforming case, the domain Ω is partitioned into $N \in \{16, 32, 64, 128\}$ rectangles of average diameter H, cf. Figure 1. We use Q_1 elements with mesh size h. The local meshes on the subdomains do not match across the interface and the mortar conditions are enforced by using Lagrange multipliers. The PCG iteration is stopped when the residual norm has decreased by a factor of 10^{-6}.

In Table 1, we report the iteration and flop counts for the new preconditioner

Table 2: Geometrically conforming partition, Matching grids and continuity constraints across the interface : (I) = Dirichlet Preconditioner, (II) = New Preconditioner, Ns = Number of Subdomains

		(I)		(II)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS
16	4	18	1.3e+0	7	5.9e–1
16	8	19	1.5e+1	9	7.7e+0
16	16	20	3.0e+2	10	1.6e+2
16	32	21	6.6e+3	11	3.7e+3
36	4	23	5.2e+0	9	2.3e+0
36	8	24	5.0e+1	10	2.3e+1
36	16	26	9.1e+2	11	4.2e+2
36	32	28	2.2e+4	13	1.1e+4
64	4	25	1.2e+1	9	4.6e+0
64	8	25	1.1e+2	10	4.7e+1
64	16	27	2.2e+3	11	1.0e+3
64	32	28	4.6e+4	13	2.1e+4
121	4	25	2.5e+1	9	9.7e+0
121	8	25	1.8e+2	10	8.1e+1
121	16	27	3.1e+3	11	1.4e+3
121	32	28	1.0e+5	13	5.1e+4

$\widehat{M}$, cf. (8), the preconditioner $\overline{M}$, cf. (9), and the Dirichlet preconditioner M, cf. (6). Note that in order to ensure matching grids across the interface, in the geometrically conforming case, the number of nodes per edge of subdomain (i.e. H/h) is the same for every edge, while in the nonconforming case the mesh ratio can be quite arbitrary.

The Dirichlet preconditioner M does not yield a numerically scalable method and converges only in hundreds of iterations. The iteration count appears to be linear in H/h, and the computational costs are one to two orders of magnitude greater than for the other preconditioners. The new preconditioner $\widehat{M}$ has scalability properties similar to those of M in the conforming case. When the number of nodes per subdomain edge (i.e. H/h) is fixed and the number of subdomains, N, is increased, the iteration count shows only a slight growth, cf. also Figure 2. When H/h is increased while the partition is kept unchanged, the increase in the number of iterations is quite satisfactory and very similar to that of the conforming case, cf. also Figure 3. Note that the number of iterations and the computational cost for $\widehat{M}$ are about half of those for $\overline{M}$.

As a comparison, we also present iteration counts for a geometrically conforming case. The computational domain Ω is partitioned in a geometrically conforming fashion into 16, 36, 64, and 121 squares.

Table 3: Geometrically conforming partition, matching grids and mortar conditions across the interface : (I) = New Preconditioner, (II) = Block–diagonal Preconditioner, (III) = Dirichlet Preconditioner, Ns = Number of Subdomains

		(I)		(II)		(III)	
Ns	H/h	Iter	MFLOPS	Iter	MFLOPS	Iter	MFLOPS
16	4	6	5.4e–1	6	5.4e–1	6	1.0e+0
16	8	6	5.6e+0	7	6.4e+0	18	2.0e+1
16	16	6	1.1e+2	8	1.4e+2	38	6.8e+2
16	32	7	2.6e+3	8	2.9e+3	47	1.7e+4
36	4	8	2.1e+0	9	2.3e+0	14	5.6e+0
36	8	8	1.9e+1	10	2.3e+1	33	9.1e+1
36	16	9	3.6e+2	10	3.9e+2	49	2.0e+3
36	32	11	9.5e+3	12	1.0e+4	56	5.3e+4
64	4	8	4.4e+0	10	5.2e+0	20	1.5e+1
64	8	9	4.4e+1	11	5.2e+1	25	1.1e+2
64	16	11	1.0e+3	11	1.0e+3	52	5.2e+3
64	32	12	1.9e+4	13	2.1e+4	61	1.1e+5
121	4	10	1.1e+1	13	1.4e+1	48	5.9e+1
121	8	11	8.9e+1	13	1.0e+2	78	6.5e+2
121	16	12	1.5e+3	15	1.8e+3	116	1.4e+4
121	32	14	5.4e+4	17	6.4e+4	121	4.2e+5

Across the interface Γ, we can use continuity conditions, as in the classical FETI method, and the preconditioners $\widehat{M}$ and M, or we can use mortar conditions and the preconditioners $\widehat{M}$ and $\overline{M}$.

When continuity constraints are enforced, the new preconditioner $\widehat{M}$ converges in less than half the number of iterations required for the Dirichlet preconditioner. In this case, BB^t is very close to twice the identity matrix, and therefore almost no extra work is required when a system with the matrix BB^t is solved. This observation is supported by a comparison of the flop counts; cf. Table 2.

When mortar conditions are used, $\widehat{M}$ and $\overline{M}$ behave similarly. There is also little difference in terms of iteration count and computational costs between the use of $\widehat{M}$ when continuity constraints or mortar conditions are used; cf. Table 3. Also, computing the matrix B is simple for matching nodes, in particular no computations of integrals are necessary.

6 Conclusions

In this paper, we have considered a new FETI preconditioner for the mortar finite element method with Lagrange multipliers and compared it to two other FETI preconditioners.

Our experiments show that the new preconditioner $\widehat{M}$ performs very well, both for mortar and conforming finite elements. The Dirichlet preconditioner M is no longer scalable, for mortar finite elements.

We are currently working on a study of the three dimensional case. Implementing mortars is more difficult in that case, and the computational cost of applying $(BB^t)^{-1}$ might also be more significant than in the two dimensional case.

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References

[1] Yves Achdou and Yuri A. Kuznetsov. Substructuring preconditioners for finite element methods on nonmatching grids. *East-West J. Numer. Math.*, 3(1):1–28, 1995.

[2] Yves Achdou, Yuri A. Kuznetsov, and Olivier Pironneau. Substructuring preconditioners for the Q_1 mortar element method. *Numer. Math.*, 71(4):419–449, 1995.

[3] Yves Achdou, Yvon Maday, and Olof B. Widlund. Iterative substructuring preconditioners for mortar element methods in two dimensions. Technical Report 735, Department of Computer Science, Courant Institute, April 1997. To appear in SIAM J. Numer. Anal.

[4] Faker Ben Belgacem and Yvon Maday. The mortar element method for three dimensional finite elements. *RAIRO Modél. Math. Anal. Numér.*, 31:289–302, 1997.

[5] Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar*. Pitman, 1994.

[6] Mario A. Casarin and Olof B. Widlund. A hierarchical preconditioner for the mortar finite element method. *ETNA*, 4:75–88, June 1996.

[7] Maksymilian Dryja, Barry F. Smith, and Olof B. Widlund. Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. *SIAM J. Numer. Anal.*, 31(6):1662–1694, December 1994.

[8] Maksymilian Dryja and Olof B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, February 1995.

[9] C. Farhat and M. Géradin. A hybrid formulation of a component mode synthesis method. In *Structural Dynamics and Material Conference*, Dallas USA, April 13–15 1992. 33rd AIAA/ASME/AHS/ASC Structures.

[10] C. Farhat and M. Géradin. Using a reduced number of Lagrange multipliers for assembling a parallel incomplete field finite element approximation. *Comp. Meth. Appl. Mech. Eng.*, 97:333–354, 1992.

[11] Charbel Farhat, Po-Shu Chen, and Jan Mandel. A scalable Lagrange multiplier based domain decomposition method for time-dependent problems. *Int. J. Numer. Meth. Eng.*, 38:3831–3853, 1995.

[12] Charbel Farhat, Po-Shu Chen, and François-Xavier Roux. The two-level FETI method - part II: Extensions to shell problems, parallel implementation, and performance results. *Comput. Methods Appl. Mech. Eng.*, 155:153–180, 1998.

[13] Charbel Farhat and Jan Mandel. The two-level FETI method for static and dynamic plate problems - part I: An optimal iterative solver for biharmonic systems. *Comput. Methods Appl. Mech. Eng.*, 155:129–152, 1998.

[14] Charbel Farhat, Jan Mandel, and François-Xavier Roux. Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Eng.*, 115:367–388, 1994.

[15] Charbel Farhat, Jan Mandel, and Radek Tezaur. A scalable substructuring method by Lagrange multipliers for plate bending problems. Technical report, Center for Computational Mathematics, University of Colorado at Denver, 1996. To appear in SIAM J. Num. Anal.

[16] Charbel Farhat and François-Xavier Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Meth. Eng.*, 32:1205–1227, 1991.

[17] Charbel Farhat and François-Xavier Roux. Implicit parallel processing in structural mechanics. In J. Tinsley Oden, editor, *Computational Mechanics Advances*, volume 2 (1), pages 1–124. North-Holland, 1994.

[18] Axel Klawonn and Olof B. Widlund. A modified FETI preconditioner for elliptic problems. Technical report, Department of Computer Science, Courant Institute, 1998. In preparation.

[19] Yuri A. Kuznetsov. Efficient iterative solvers for elliptic finite element problems on nonmatching grids. *Russ. J. Numer. Anal. Math. Modelling*, 10:187–211, 1995.

[20] Catherine Lacour. *Analyse et Resolution Numérique de Méthodes de Sous-Domaines Non Conformes pour des Problèmes de Plaques*. PhD thesis, Université Pierre et Marie Curie, Paris, 1997.

[21] Catherine Lacour. Iterative substructuring preconditioners for the mortar finite element method. In Petter Bjørstad, Magne Espedal, and David Keyes, editors, *Ninth International Conference of Domain Decomposition Methods*, 1997. URL = `http://www.ddm.org/DD9/`.

[22] Jan Mandel. Balancing domain decomposition. *Comm. Numer. Meth. Engrg.*, 9:233–241, 1993.

[23] Jan Mandel and Marian Brezina. Balancing domain decomposition for problems with large jumps in coefficients. *Math. Comp.*, 65:1387–1401, 1996.

[24] Jan Mandel and Radek Tezaur. Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.*, 73:473–487, 1996.

[25] D. Rixen, C. Farhat, and M. Géradin. A two-step, two-field hybrid method for the static and dynamic analysis of substructure problems with conforming and non-conforming interfaces. *Comp. Meth. Appl. Mech. Eng.*, 154:229–264, 1998.

[26] Daniel Rixen and Charbel Farhat. A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. *Computers and Structures*. To appear.

[27] Daniel Rixen and Charbel Farhat. Preconditioning the FETI method for problems with intra- and inter-subdomain coefficient jumps. In Petter Bjørstad, Magne Espedal, and David Keyes, editors, *Ninth International Conference of Domain Decomposition Methods*, 1997. URL = `http://www.ddm.org/DD9/`.