SOME RESULTS ON OVERLAPPING SCHWARZ METHODS FOR THE HELMHOLTZ EQUATION EMPLOYING PERFECTLY MATCHED LAYERS

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Abstract. In this paper, we build a class of overlapping Schwarz preconditioners for a finite element approximation of the Helmholtz equation in two dimensions. Perfectly Matched Layers are employed to build the local problems and two kinds of boundary conditions are employed to match the local solutions. Numerical results are presented to compare the different preconditioners.

1. Introduction. In recent years, a considerable effort has been devoted to the the study of preconditioners for scalar and vector Helmholtz equations; see [9, 10, 5, 14]. The standard preconditioners employed for elliptic, positive–definite problems (see [17]) are not effective for propagation problems and new classes of preconditioners have been introduced.

Given a partial differential equation on a bounded domain Ω, a domain decomposition method for its solution can be outlined in the following way:
Partition the domain into subdomains, solve local problems on the subdomains and connect the local solutions by imposing the continuity of suitable quantities defined on the local boundaries, in order to obtain a global solution on Ω. More precisely, an iteration scheme can be obtained, in which, at each step, local problems are solved on the subdomains, in sequence or in parallel, with boundary conditions determined by the solution at the previous steps. The whole procedure can then be employed to build a preconditioner to be used with a Krylov–type accelerator; see [17] for a more detailed discussion.

The characteristics of the method depend on

- the equation solved in the subdomains (either the original or a modified one);
- the way in which the local solutions are matched across the subdomain boundaries;
- the partition of Ω (with or without overlap).

In this paper, we show that, by modifying the equation for the local solvers using Perfectly Matched Layers (PMLs), faster convergence can be achieved.

Our model problem is a scalar Helmholtz problem with a first–order Sommerfeld condition. We consider an overlapping partition of the domain Ω and employ PMLs when building the local problems. PMLs are only employed for the local solvers that form a component of the preconditioner, and a modified equation is, then, solved on the subdomains. In order to connect the local solutions, one imposes the continuity of pointwise values or fluxes, giving rise to Dirichlet and Robin–type local problems, respectively. The use of a coarse solver is also considered.

Our work has been inspired by [5], where some variants of overlapping Schwarz preconditioners are studied.

In the next section, we introduce our model problem. In Section 3, we recall some properties of a class of PMLs, and in Section 4, we build the Schwarz preconditioners. In Section 5, we present some numerical results and compare the different variants.

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2. Model problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded connected polygon. We consider the following Helmholtz problem for the complex–valued function u:

$$P(u) = -\Delta u - k^2 u = f, \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial n} + ik\, u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where the frequency k is positive and the source f has support contained in Ω.

Equation (1) is obtained from the full 3D Maxwell's equations for time-harmonic fields, when considering waves with the electric and magnetic fields, respectively, parallel and perpendicular to the xy–plane (TM waves). Then, (1) is the equation for the z-component of the magnetic field. The time dependence of the field is assumed to be $e^{i\omega t}$, where $i = \sqrt{-1}$, ω is related to k by $\omega = ck$ and c is the speed of light of the medium. The Sommerfeld condition (2) is derived from the first–order Silver–Muller condition; see [15].

We then consider a triangulation $\mathcal{T}_h$ of Ω, made of quadrilaterals, and the standard bilinear finite element (FE) space $V_h \subset H^1(\Omega)$. Triangular linear FE spaces could also be employed; see [16]. The FE approximation of equations (1) and (2) amounts to finding $u \in V_h$, such that

$$b(u,v) = \int_\Omega f v \, dxdy, \quad \forall v \in V_h, \tag{3}$$

where

$$b(u,v) = \int_\Omega (\mathbf{grad} \cdot \mathbf{grad}\, v - k^2 uv)\, dxdy - ik \int_{\partial\Omega} uv\, ds.$$

For a study of the well–posedness, stability and accuracy of Problem (3), see [12, 13, 11]. In particular, we recall that the following stability estimate holds for a domain Ω with unit diameter

$$|u|_{H^1} \leq Ck\, |f|_{H^{-1}},$$

for the FE solution u, if $kh < 1$; see [12]. Additional restrictions on h are required for the accuracy of the FE solution. As is well–known, a restriction on kh requires that there are enough discretization points per wavelength. If the wavelength is defined as $\lambda = 2\pi/k$, the number of points per wavelength, ppw, is

$$ppw = \frac{2\pi}{kh}.$$

3. Perfectly Matched Layers. Electromagnetic scattering problems generally involve one or more objects and an incident electromagnetic wave. The presence of the objects gives rise to a scattered field. In order to find an approximation of the scattered wave, the scatterers are generally enclosed in a bounded computational domain, and Maxwell's equations are approximated inside it. The field outside D must also be modeled suitably. Typically, incident waves on ∂D must be completely transmitted, or reflected as little as possible.

The idea underlying a PML, is to surround the domain D by a layer of absorbing material D_d, of thickness d, such that:

- there is no reflection at the boundary between D and D_d, or, equivalently, the two media have the same dielectric and magnetic constants,

- the electromagnetic wave is damped when traveling in the layer.

Homogeneous Dirichlet, Neumann or Sommerfeld–type conditions are enforced on the outside boundary of the layers.

In two dimensions, the domain D is normally a rectangle. In practice, the original form of Maxwell's equations is either changed by splitting one or more field components (see [3, 4]), or is preserved. In the latter case, damping terms are added to the equations and absorption properties are derived either by physical (see [18]), or pure mathematical considerations, (see [2, 1]). We note that PMLs can be extended to curvilinear coordinates; see [8]. See also [8] for a discussion of practical issues of PMLs.

In our analysis, we will choose to work with the PMLs described in [18]. Let D be the unit square. We extend D in each direction, by surrounding it by a layer D_d of thickness d, and obtain a square of side $1+2d$. Layers parallel to the x and y axis will be referred to, respectively, as PML_x and PML_y edges, while the intersections between PML_x and PML_y edges are called PML_{xy} corners.

The scattered field u satisfies the modified Helmholtz equation

$$P_d(u) = F, \quad \text{in } D \cup D_d, \tag{4}$$

where the operator $P_d(u)$ is defined by

$$P_d(u) = \begin{cases} -\Delta u - k^2 u, & \text{in } D \\ -\text{div }(\Lambda\, \mathbf{grad}\, u) - k^2 a_z u, & \text{in } D_d. \end{cases} \tag{5}$$

Here $\Lambda = \text{diag}\,\{a_x(x,y), a_y(x,y)\}$ and $a_z(x,y)$ are suitable functions, and F describes the sources, as well as non-homogeneous conditions on the scatterers. Just as (1), equation (4) gives the z-component of the magnetic field of a TM wave.

The absorption coefficients are defined by

$$a_x = \begin{cases} w_x^{-1}, & \text{in } PML_x \text{ edges}, \\ w_y, & \text{in } PML_y \text{ edges}, \\ w_y/w_x, & \text{at } PML_{xy} \text{ corners}, \end{cases}$$

$$a_y = \begin{cases} w_x, & \text{in } PML_x \text{ edges}, \\ w_y^{-1}, & \text{in } PML_y \text{ edges}, \\ w_x/w_y, & \text{at } PML_{xy} \text{ corners}, \end{cases}$$

$$a_z = \begin{cases} w_x, & \text{in } PML_x \text{ edges}, \\ w_y, & \text{in } PML_y \text{ edges}, \\ w_x\ w_y, & \text{at } PML_{xy} \text{ corners}. \end{cases}$$

The functions w_x and w_y describe the absorption in the PML_x and PML_y edges, respectively. They are complex and equal to one at the boundary between D and D_d. In [18], they are chosen as

$$w_x(x) = 1 - i\frac{\sigma_0}{\omega}\left(\frac{x}{d}\right)^m, \quad w_y(y) = 1 - i\frac{\sigma_0}{\omega}\left(\frac{y}{d}\right)^m. \tag{6}$$

In our experiments, we have also considered absorption coefficients that are independent of the frequency

$$w_x(x) = 1 - i\alpha\left(\frac{x}{d}\right)^m, \quad w_y(y) = 1 - i\alpha\left(\frac{y}{d}\right)^m. \tag{7}$$

In the following, absorption given by (6) and (7) will be called, respectively, *variable* and *constant*, referring to its dependence on k. See [8] for a detailed discussion of the form of the absorption coefficients. We remark that the characteristics of the layer are determined by the thickness d, the coefficients σ_0 or α, and the exponent $m > 0$.

4. Schwarz methods. In the following, we suppose, for simplicity, that the domain Ω is a rectangle and the triangulation $\mathcal{T}_h$ is uniform. We want to build a preconditioner for Equation (3), and consider an overlapping decomposition, built in the following way:

We first consider a decomposition of the rectangle Ω into M nonoverlapping rectangles. We, then, extend each rectangle and let $\delta_0 h$ be the thickness of the extended part. We further extend the subdomains, by putting PMLs around them, with thickness $d = \delta_L h$, and obtain a family of overlapping subdomains $\{\Omega'_i\}$. The decomposition is thus

$$\Omega = \cup_{i=1}^{M} \Omega'_i.$$

Each local subproblem is thus a Helmholtz problem with PMLs, given by the operator P_d defined in (5). The total overlap *ovl* is given by

$$ovl = (\delta_0 + \delta_L)h = \delta_0 h + d. \tag{8}$$

We obtain different preconditioners, by choosing different boundary conditions for the local problems. We consider Dirichlet conditions (*Algorithm 1L*) and Sommerfeld conditions (*Algorithm 2L*). We define the following linear iterations. We start with an initial vector u^0. A full iteration step is performed through M fractional steps, where $u^{n+\frac{j}{M}}$ is the solution of the following problem on the subdomain Ω'_j, $j = 1, \cdots, M$:

- **Algorithm 1L** (Dirichlet + Layers)

$$\begin{cases} P_d\left(u_j^{n+\frac{j}{M}} - u^{n+\frac{j-1}{M}}\right) = f - P\left(u^{n+\frac{j-1}{M}}\right), & \text{in } \Omega'_j, \\ u_j^{n+\frac{j}{M}} = u_{out}^{n+\frac{j-1}{M}}, & \text{on } \partial\Omega'_j. \end{cases} \tag{9}$$

- **Algorithm 2L** (Sommerfeld + Layers)

$$\begin{cases} P_d\left(u_j^{n+\frac{j}{M}} - u^{n+\frac{j-1}{M}}\right) = f - P\left(u^{n+\frac{j-1}{M}}\right), & \text{in } \Omega'_j, \\ \frac{\partial u_j^{n+\frac{j}{M}}}{\partial n_{int}} - ik\, u_j^{n+\frac{j}{M}} = -\frac{\partial u_{out}^{n+\frac{j-1}{M}}}{\partial n_{out}} - ik\, u_{out}^{n+\frac{j-1}{M}}, & \text{on } \partial\Omega'_j. \end{cases} \tag{10}$$

Here n_{int} and n_{out} are the outward and inward normal vectors to $\partial\Omega'_j$, respectively. The function $u_{out}^{n+\frac{j-1}{M}}$ is the iterate at step $n + \frac{j-1}{M}$, defined in $\Omega \setminus \Omega'_j$.

In the definition of the fractional steps (9) and (10), we have chosen to solve the local problems in sequence and obtained multiplicative algorithms, but additive algorithms can also be considered. We have also chosen to update the solution at step $n + j/M$ on the whole $\overline{\Omega}'_j$, but restricted algorithms can also be considered. In practice, it may be convenient to color the subregions $\{\Omega'_i\}$ using different colors for subregions that intersect. The original decomposition is then partitioned into sets of subregions with the same color, reducing the number of subregions, and consequently the fractional steps. These basic iterations can be employed to build preconditioners to be combined with a Krylov–type accelerator. A coarse solver can also be added in a standard way, by using the FE discretization of Equation (3) on a coarse mesh $\mathcal{T}_H$, $H > h$. See [17] for a general discussion of these issues and [6] for an introduction to restricted Schwarz preconditioners.

We note that the corresponding algorithms with no absorption ($\alpha = 0$) have already been studied: see Algorithms 1 and 2 in [5].

5. Numerical results. In this section, we compare the performance of the two algorithms introduced in the previous section, when varying the overlap, the number of subregions and the diameter of the coarse mesh.

We can make the following preliminary remarks; the supporting results are not shown here.

- Multiplicative algorithms give far better performances than standard and restricted additive ones. Therefore, we will only present results for multiplicative preconditioners.
- For a fixed value of the overlap (see (8)), the best performance is achieved when $\delta_0 = 0$, i.e., when the whole overlapping region is a PML:

  $$ovl = \delta_L h = \delta h.$$

  Therefore, in the following, we will always assume $\delta_0 = 0$.
- As far as the convergence of the preconditioner is concerned, constant absorption gives faster convergence than the variable one (see Section 3). Therefore, in the following, we will only consider absorption coefficients given by (7), with $m = 2$.

In the following, we will denote by *overlap* both the integer δ and the length $ovl = \delta h$, where there is no ambiguity.

Experience in overlapping preconditioners for Helmholtz problems, has shown that an important parameter that determines the convergence rate is the *wavelap*; see [5]. It is defined as the fraction of a wavelength that is covered by the overlap:

$$wlp = \frac{\delta h}{\lambda} = \frac{\delta}{ppw}.$$

We summarize the parameters we use:

- n Number of discretization nodes in each direction.
- h Step-size, equal to $1/(n-1)$.
- $nsub$ Number of subdomains in the x and y direction (the total number of subdomains is $nsub^2$).
- nc Number of discretization steps in each direction for the coarse mesh.
- δ Number of layers of elements that are added to the nonoverlapping subdomains, to produce the overlapping ones ($ovl = \delta h$).
- ppw Number of mesh points per wavelength.
- wlp Wavelap.
- α Absorption coefficient; see Equation (7).

Our numerical results are obtained with Matlab. GMRES acceleration and right preconditioning is employed, with restart equal to 40, a maximum number of iterations equal to 70 and a reduction of the relative residual of the preconditioned system, by a factor of 10^{-6}. We have taken $\Omega = (0,1)^2$.

The first set of tables (Table 1 for Algorithm 1L and Table 2 for Algorithm 2L) shows the dependence of the number of iterations on the absorption coefficient and on the overlap δ, when ppw and $nsub$ are fixed and no coarse space is used.

TABLE 1

Algorithm 1L: number of GMRES iterations, versus δ and α; $n = 33$, $ppw = 10$, $k = 20.7$, $nsub = 2$, $nc = 0$.

δ	1	2	3	4	5	6
$\alpha = 0$	19	20	26	18	23	21
$\alpha = 0.5$	17	14	14	12	11	11
$\alpha = 1$	15	12	11	10	9	9
$\alpha = 2$	13	11	10	9	9	8
$\alpha = 3$	13	11	10	9	9	8
$\alpha = 4$	13	11	10	10	9	9
$\alpha = 5$	13	11	11	11	10	10

TABLE 2

Algorithm 2L: number of GMRES iterations, versus δ and α; $n = 33$, $ppw = 10$, $k = 20.7$, $nsub = 2$, $nc = 0$.

δ	1	2	3	4	5	6
$\alpha = 0$	12	10	10	9	9	9
$\alpha = 0.25$	11	10	9	9	8	9
$\alpha = 0.5$	11	9	8	8	7	7
$\alpha = 0.75$	11	9	8	9	8	8
$\alpha = 1$	11	8	9	8	8	7
$\alpha = 2$	10	9	9	8	8	8
$\alpha = 4$	11	10	10	10	9	8

For a fixed value of the overlap, the number of iterations initially decreases, reaches a minimum and then starts increasing with the absorption coefficient, consistent with the fact that too much absorption makes the local problems unstable (see [8]) and that the standard theory for Schwarz preconditioners for indefinite problems requires that the local problems be stable to ensure fast convergence (see [7]). We also remark that in Algorithm 1L, without absorption, the local problems may not be solvable or unstable in practice.

For the two methods, an optimal range of values of the absorption coefficient α has been found, which is fairly insensitive to the frequency, the number of points per wavelength, the number of subregions and the diameter of the coarse triangulation. We have chosen $\alpha = 2.0$ for Algorithm 1L and $\alpha = 0.75$ for Algorithm 2L.

As expected, the number of iterations decreases if the overlap increases. Algorithm 2L gives better performances than Algorithm 1L for moderate values of the overlap. When δ is large, the two methods give comparable numbers of iterations.

The second set of tables shows the dependence on ppw (or n, equivalently), the overlap and the number of subregions, for a fixed value of the frequency and no coarse space; see Table 3 for $ppw = 10.1$ and Table 4 for $ppw = 13.5$. The tables show results for Algorithm 1L with $\alpha = 2.0$, Algorithm 2L with $\alpha = 0.75$ and Algorithm 2L with $\alpha = 0$.

As expected, without a coarse space and for a fixed δ, the number of iterations increases with the number of subregions. By comparing the results for Algorithm 2L, one can see that the increase is larger if no absorption is present. We also remark that Sommerfeld boundary conditions for the local problems ensure faster convergence; see also [5]. In this case, results for $\alpha > 0$ are somewhat better than those for $\alpha = 0$ for

TABLE 3

Number of GMRES iterations, versus δ (wlp) and nsub; $n = 121$, $ppw = 10.1$, $k = 75$, $nc = 0$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ wavelap	1 0.10	2 0.20	3 0.30	4 0.39	5 0.49	6 0.59
nsub = 4	>70	37	29	27	26	25
	21	20	18	18	17	17
	24	19	19	20	18	21
nsub = 5	>70	44	32	29	28	26
	25	22	21	20	19	18
	28	22	23	23	23	26
nsub = 8	>70	61	43	36	33	31
	35	31	28	27	26	23
	41	34	38	48	44	>70

TABLE 4

Number of GMRES iterations, versus δ (wlp) and nsub; $n = 161$, $ppw = 13.5$, $k = 75$, $nc = 0$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ wavelap	1 0.07	2 0.15	3 0.22	4 0.30	5 0.37	6 0.44
nsub = 4	>70	48	33	28	28	27
	26	21	20	19	19	19
	56	21	20	21	22	22
nsub = 5	>70	64	40	32	30	28
	29	24	22	22	21	20
	>70	25	24	25	27	27
nsub = 8	>70	>70	53	43	39	36
	42	33	32	31	30	29
	>70	38	37	42	65	65

a few subregions, and considerably better for many subregions. In particular, some absorption ensures a steady decrease of the number of iterations when the overlap is increased.

By comparing Tables 3 and 4, one can see that, for a fixed number of subregions, a constant value of the wavelap gives comparable numbers of iterations. This shows the importance of this parameter in the analysis of overlapping methods for Helmholtz problems. This has already been pointed out in [5, 14] for other Schwarz algorithms.

Tables 5 and 6 show the results when a coarse space is added. For a fixed value of *ppw* and *nsub*, they show the number of iterations when varying the wavelap and the size of the coarse space, for different values of the frequency. Results are given for Algorithm 1L with $\alpha = 2.0$, Algorithm 2L with $\alpha = 0.75$ and Algorithm 2L with $\alpha = 0$.

We observe an initial deterioration of the performances when a very coarse space is added, but note a considerable improvement, when the number of coarse points per wavelength (*cppw*) is sufficiently large (greater than or equal to 4).

As for Tables 3 and 4, we remark that for Algorithm 2L, some absorption ensures

TABLE 5

Number of GMRES iterations, versus δ (wlp) and nc; ppw = 20, nsub = 8, n = 121, k = 38; first rows for Algorithm 1L with α = 2.0, second rows for Algorithm 2L with α = 0.75 and third rows for Algorithm 2L with α = 0.

δ wavelap	1 0.05	2 0.10	3 0.15	4 0.20	5 0.25	6 0.30
nc = 0	>70	>70	>70	58	51	49
cppw = 0.0	68	55	43	39	41	47
	>70	>70	52	47	52	>70
nc = 8	>70	>70	>70	61	52	45
cppw = 1.3	>70	56	42	38	38	42
	>70	>70	51	45	52	85
nc = 16	>70	>70	61	44	42	40
cppw = 2.6	>70	63	44	46	52	68
	>70	>70	51	55	>70	>70
nc = 24	30	31	28	27	25	24
cppw = 4.0	51	28	24	22	22	22
	>70	>70	27	24	25	53
nc = 32	25	25	24	23	23	21
cppw = 5.3	43	21	18	19	19	21
	>70	43	18	18	27	33

better performances. We also remark that, for a fixed value of the wavelap and the number of coarse points per wavelength, the number of iterations increases with the frequency.

We conclude with some remarks.

We do not show any results for Algorithm 1L with $\alpha = 0$. This case was considered in [5] and it performs very poorly. From the numerical results, we can deduce that, in general, adding PMLs to the local problems, improves the performance of Schwarz methods for Helmholtz equations.

The key parameters of the algorithms are the wavelap for the one–level algorithms and the wavelap, the number of coarse points per wavelengths, and the frequency, for the two–level algorithms.

The methods developed in this paper can be easily generalized to the full Maxwell's equations, using the theory of PMLs developed in [18] for the three–dimensional case and results are forthcoming.

Acknowledgments. The author is grateful to Olof Widlund for his help and enlightening discussions of my work and to Jean David Benamou for suggesting the problem.

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TABLE 6

Number of GMRES iterations, versus δ (wlp) and nc; $ppw = 20$, $nsub = 8$, $n = 161$, $k = 50.6$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ wavelap	1 0.05	2 0.10	3 0.15	4 0.20	5 0.25	6 0.30
nc = 0	>70	>70	>70	58	48	45
cppw = 0.0	69	46	39	37	35	37
	>70	>70	47	43	47	>70
nc = 8	>70	>70	>70	55	47	44
cppw = 1.3	>70	47	38	36	36	35
	-	-	-	-	-	-
nc = 16	>70	>70	>70	>70	>70	>70
cppw = 2.6	>70	>70	>70	>70	>70	>70
	-	-	-	-	-	-
nc = 24	>70	63	52	41	40	44
cppw = 4.0	59	47	37	48	52	40
	>70	-	49	60	>70	>70
nc = 32	69	36	30	33	33	31
cppw = 5.3	46	30	36	30	25	24
	>70	>70	43	31	28	>70

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