# Competitive Equilibrium

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**Abstract**

This report includes a modern account of welfare economics and competitive equilibrium theory. In particular, competitive, or Walrasian, equilibrium is defined. Moreover, existence, optimality, and uniqueness are demonstrated. However, no reliable mechanism for computing equilibrium prices is suggested. At this stage, the problem shifts from the realm of economics to an algorithmic problem in computer science.

# 1 History

In 1776, Adam Smith published *The Wealth of Nations*, and with it, the theory of competitive equilibrium was born. He described a notion which he called the "invisible hand" that refers to the various forces at work in a social system which tend towards a state of balance, despite the individualistic goals pursued by self-interested agents. However, Smith and other classical economists failed to give consideration to the influence of price on supply and demand.

Léon Walras is recognized for introducing the modern concept of competitive equilibrium. Walras envisioned an economic system consisting of consumers who are allocated some proportion of the resources in the economy. Given a price system, consumers sell their resources and earn income. They then spend their earned income on other resources which are both affordable and preferable. In this way, consumer demand is a function of price. An equilibrium set of prices is a set such that supply and demand are equated on all markets. Walras suggested that equilibrium might be achieved via a hypothetical *tâtonnement* process, where prices are adjusted on different markets in turn, until excess demand is zero on all markets.

Although classical economists understood that the allocation of resources at equilibrium was efficient in some sense, the treatment of efficiency was in no way rigorous before Pareto and Edgeworth, who formally defined optimal resource allocation. Edgeworth considered a market game played between two traders who will not trade if there is another that is of greater benefit to both, nor if either will be worse off than in the absence of the trade. These requirements capture a general notion of optimality, and he showed that the set of market allocations satisfying these conditions contains the competitive equilibrium.

The exposition on competitive equilibrium that follows is in the spirit of modern economic theory, which was pioneered by Arrow [1] and DeBreu [2]. A further reference and a more recent publication is Ellickson [3].

# 2 Assumptions

This paper presents an abstract model of an economy formulated under two standard economic assumptions[1]. The first assumption is that the market is perfectly competitive, or equivalently, that economic agents are price-takers: the actions of individual agents do not influence prices. Secondly, economic agents are assumed to be rational; in particular, they act in such a way as to maximize their personal utility.

[1] These assumptions are discussed in further detail and eventually relaxed in the report on game theoretic equilibria.

## 3 Commodities and Prices

Central to an economic system are the dual concepts of commodity and price. A *commodity* is characterized by

1. physical properties
2. date
3. location

*i.e.*, it is specified *physically*, *temporally*, and *spatially*. The two principal types of commodities are *goods* and *services*.

A *price* is an amount to be paid now for the (possibly future) availability of a commodity. Price is dependent on the *date* and the *location* at which commodities are available. In particular, *interest* and *discount rates* arise by comparing prices on different dates, but at the same location. Analogously, *exchange rates* arise by comparing prices on the same date, but at different locations.

Let $\mathcal{N} = \{1, \ldots, N\}$ [2] denote the set of commodities in the economy, and let the *commodity space* $\mathcal{L} = \mathbb{R}^N$. A *price system* is a vector in the commodity space; namely, $p \in \mathcal{L}$, where $p_n$ specifies the price of the $n$th commodity. The non-negative orthant of the commodity space is given by $\mathcal{L}^+ = \{l \in \mathcal{L} \mid l \geq 0\}$. The *total resources* in the economy are given by a vector $w \in \mathcal{L}^+$, where $w_n$ specifies the total amount of the $n$th commodity.

## 4 Economic Agents

In addition to commodities and prices, economic agents are essential components of an economy. The role of an agent is to decide on some quantity of the various commodities to demand or supply. This decision is called a plan of action. The actions of an agent are motivated by goals of maximizing personal utility.

Let $\mathcal{M} = \{1, \ldots, M\}$ be a finite set of economic agents. The set of all possible actions for agent $m$ is denoted by $A_m$. An action taken by an agent is represented by a vector $a_m \in \mathcal{L}$, where $a_{mn}$ specifies the amount of the $n$th commodity that agent $m$ demands or supplies. More specifically, if $a_{mn} \geq 0$ (or $a_{mn} \leq 0$), then agent $m$ demands (or supplies) the given amount of the $n$th commodity.

[2] The variable $n$ ranges over the set $\mathcal{N}$. By convention, lowercase letters (*e.g.*, $n$) range over sets denoted by the corresponding capital, caligraphic letters (*e.g.*, $\mathcal{N}$).

Economic agents are distinguished by their *utility functions*, which reflect their personal preferences, and by limitations on their choice, such as a *wealth constraint*. The utility $u_m$ of agent $m$ is a function $u_m : A_m \to \mathbb{R}$, and the vector $u = (u_1, \ldots, u_m)$ represents the utility of all agents. The wealth of agent $m$ is denoted by $w_m \in W \subset \mathcal{L}$, where $w_{mn}$ specifies the amount of the initial allocation of the $n$th commodity to agent $m$, and $w = \sum_{m \in \mathcal{M}} w_m$ denotes the total resources of the economy.

**Definition 4.1** *An* **economy** $\mathcal{E} = (A_m, u_m, w_m)_{m \in \mathcal{M}}$ *is a set of economic agents* $\mathcal{M} = \{1, \ldots, M\}$ *together with, for all* $m \in \mathcal{M}$*,*

- *an action set* $A_m \subset \mathcal{L}$
- *a utility function* $u_m : A_m \to \mathbb{R}$
- *an initial endowment* $w_m \in \mathcal{L}$

A type economy is an economy in which there are distinct types of agents: *e.g.*, consumers, producers, and speculators. The following sections introduce mathematical notation and definitions which facilitate a formal discussion of the concepts of consumption, production, and speculations. Let $I$, $J$, and $K$ be the number of consumers, producers, and speculators, respectively; and, let the total number of economic agents $M = I + J + K$.

# 5 Consumption

*Consumers* are the first example of a class of economic agents. Consumers are characterized by their preferences, which are measured in terms of utility functions, and by constraints on their wealth. The role of a consumer is to choose a plan for consumption. The goal of consumers is to maximize utility subject to a wealth constraint. In what follows, the concepts of consumption set, consumer preference set, utility function, wealth constraint, and utility maximization are formalized.

Let $\mathcal{I} = \{1, \ldots, I\}$ be the set of consumers. The set of possible consumptions for consumer $i$ is called the $i$th *consumption set* and is denoted by $X_i \subset \mathcal{L}$. The *consumption*, or *demand*, of consumer $i$ is given by a vector $x_i \in X_i$. Summing over all consumers, *total consumption* is $x = \sum_{i \in \mathcal{I}} x_i$. Finally, the set $X = \sum_{i \in \mathcal{I}} X_i$ is called the *total consumption set.*

**Assumption 5.1** *The consumption set* $X_i$ *is convex:* i.e., *if* $x_i^1, x_i^2 \in X_i$*, then the weighted average* $t x_i^1 + (1-t) x_i^2 \in X_i$*, for all* $t \in [0, 1]$.

## 5.1 Consumer Preferences

Given two consumptions $x_i, x_i' \in X_i$, exactly one of the following situations holds: $x_i$ is preferred to $x_i'$; or $x_i$ is indifferent to $x_i'$ (more accurately, consumer $i$ is indifferent between $x_i$ and $x_i'$); or $x_i'$ is preferred to $x_i$.

It is convenient to introduce the binary relation on $X_i$ "is not preferred to" (notation $\preceq_i$). Note that $(X_i, \preceq_i)$ is a complete preorder; in particular, it is a total order which is reflexive and transitive:

1. $\preceq_i$ is a total order: for all $x_i$, $x_i' \in X_i$, either $x_i \preceq_i x_i'$ or $x_i' \preceq_i x_i$, since either $x_i$ is not preferred to $x_i'$ or $x_i'$ is not preferred to $x_i$
2. $\preceq_i$ is reflexive: for all $x_i \in X_i$, $x_i \preceq_i x_i$, since $x_i$ is not preferred to $x_i$
3. $\preceq_i$ is transitive: for all $x_i, x_i', x_i'' \in X_i$, if $x_i \preceq_i x_i'$ and $x_i' \preceq_i x_i''$, then $x_i \preceq_i x_i''$, since if $x_i$ is not preferred to $x_i'$ and if $x_i'$ is not preferred to $x_i''$, then $x_i$ is not preferred to $x_i''$

If $x_i \succeq_i x_i'$ but not $x_i' \succeq_i x_i$, write $x_i \succ x_i'$ and read "$x_i$ is preferred to $x_i'$".

**Definition 5.1** *Given a consumption $x_i \in X_i$, the* **preference set** $P_i(x_i)$ *is the set of consumptions that are preferred to the given consumption $x_i$:*

$$P_i(x_i) = \{x_i' \in X_i \mid x_i' \succ_i x_i\}$$

**Assumption 5.2** *The preference preordering is continuous:* i.e., *for $x_i \in X_i$, the preference set $P_i(x_i)$ is closed in $X_i$.*

## 5.2 Utility Functions

A utility function is a measure of the degree of consumer satisfaction with regard to the choice of consumption plan.

**Definition 5.2** *A* **utility function** $u_i$ *is a monotonically increasing, real-valued function on the completely preordered set of preferences:* i.e., $u_i : X_i \to \mathbb{R}$.

Note that the specific choice of utility function is rather arbitrary. If $u_i^1$ is a utility function, and if $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then the function $u_i^2 : X_i \to \mathbb{R}$ defined by $u_i^2(x_i) = f(u_i^1(x_i))$ is also a utility function.

By assumption $(X_i, \preceq_i)$ is a continuous and complete preorder defined on a convex set. These properties imply the existence of a continuous utility function $u_i : X_i \to \mathbb{R}$.

## 5.3 Wealth Constraint

The goal of consumers is to maximize utility while minimizing expenditure. The *expenditure* of the $i$th consumer relative to price system $p$ is the value of consumption: *i.e.*, $p \cdot x_i$. This expenditure cannot exceed consumer wealth, which is the value of consumer resources: *i.e.*, $p \cdot w_i$. The wealth constraint is expressed mathematically as $p \cdot x_i \leq p \cdot w_i$.

**Definition 5.3** *Given a price system $p$, the $i$th* **budget set** $\hat{B}_i(p) \subset X_i$ *is the set of possible consumptions for the $i$th consumer that satisfy the wealth constraint:*

$$\hat{B}_i(p) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot w_i\}$$

**Definition 5.4** *Given a price system $p$, the set $\hat{X}(p) \subset X$ containing the total consumptions that are consistent with the wealth constraint is given by:*

$$\hat{X}(p) = \sum_{i \in \mathcal{I}} \hat{B}_i(p)$$

*The set $\hat{X}(p)$ is called the* **consumer budget set**.

## 5.4 Utility Maximization

Consumers choose consumptions from their budget sets that optimally satisfy their preferences, thereby maximizing their utility. In other words, consumers choose consumptions from their demand sets.

**Definition 5.5** *Given a price system $p$, the $i$th* **demand set** $X_i(p) \subset X_i$ *is the set of consumptions in the $i$th budget set which are greatest elements in the preference preordering $\preceq_i$, or equivalently, those consumptions which are maximizers of the utility function $u_i$:*

$$X_i(p) = \{x_i \in X_i \mid \arg \sup_{x_i \in \hat{B}_i(p)} u_i(x_i)\}$$

**Definition 5.6** *Given a price system $p$, the set $X(p) \subset \hat{X}(p)$ containing the preferred total consumptions consistent with the wealth constraint is given by:*

$$X(p) = \sum_{i \in \mathcal{I}} X_i(p)$$

*The set $X(p)$ is called the* **total demand set**.

# 6 Production

*Producers* are a second example of a class of agents. The role of a producer is to choose a plan for production. Producers are characterized by technological limitations on their choice. The goal of producers is to maximize profits.

Let $\mathcal{J} = \{1, \ldots, J\}$ be the set of producers. The set of technologically possible productions for producer $j$ is called the $j$th *production set* and is denoted by $Y_j \subset \mathcal{L}$. The *production*, or *supply*, of the $j$th producer is represented by a vector $y_j \in Y_j$. Summing over all producers, *total production* is given by $y = \sum_{j \in \mathcal{J}} y_j$. Finally, the set $Y = \sum_{j \in \mathcal{J}} Y_j$ is called the *total production set*.

## 6.1 Profit Maximization

The goal of producers is to maximize profits. The value of a production by the $j$th producer relative to a price system $p$ is expressed mathematically as $p \cdot y_j$. The supply set of the $j$th producer is the set of possible productions which are of maximal value: *i.e.*, maximize profits.

**Definition 6.1** *Given a price system $p$, the $j$th* **supply set** *$Y_j(p) \subset Y_j$ is the set of productions which maximize profits:*

$$Y_j(p) = \{y_j \in Y_j \mid \arg \sup_{y_j \in Y_j} p \cdot y_j\}$$

**Definition 6.2** *Given a price system $p$, the set $Y(p) \subset Y$ containing the most profitable total productions is given by:*

$$Y(p) = \sum_{j \in \mathcal{J}} Y_j(p)$$

*The set $Y(p)$ is called the* **total supply set**.

# 7 Speculation

*Speculators* are a third example of a class of agents [3]. The role of a speculator is to choose an investment portfolio. Speculators are characterized by their subjective beliefs about potential fluctations in prices. The goal of speculators is to maximize the return on their investments given their subjective beliefs.

[3] The class of agents which are referred to as speculators are not generally distinguished in the literature on mathematical economics. Nonetheless, these agents are of particular interest because of the role they play in computational models of equilibria.

Let $\mathcal{K} = \{1, \ldots, K\}$ be the set of speculators. The set of beliefs of the $k$th speculator is called the $k$th belief set; notation $p_k$. Given the $k$th belief set, the set of possible speculations for the $k$th speculator is called the $k$th *speculation set* and is represented by $V_k \subset \mathcal{L}$. The *speculation*, or *investment portfolio*, of speculator $k$ is denoted by a vector $v_k \in V_k$. Summing over all speculators, *total speculation* is given by $v = \sum_{k \in \mathcal{K}} v_k$. Finally, the set $V = \sum_{k \in \mathcal{K}} V_k$ is called the *total speculation set.*

## 7.1 Wealth Constraint

In addition to acting according to their beliefs, speculators are also constrained by their wealth. The value of the investment of the $k$th speculator relative to price system $p$ is $p \cdot v_k$. This value cannot exceed wealth, which is the value of the resources owned by the speculator: *i.e.*, $p \cdot w_k$.

**Definition 7.1** *Given a price system p, the kth* **budget set** $\hat{B}_k(p)$ *contains the portfolios of the kth speculator that satisfy the wealth constraint:*

$$\hat{B}_k(p) = \{v_k \in V_k \mid p \cdot v_k \leq p \cdot w_k\}$$

*The* **speculator budget set** $\hat{V}(p)$ *is the set of total speculations that satisfy the wealth constraint:*

$$\hat{V}(p) = \sum_{k \in \mathcal{K}} \hat{B}_k(p)$$

## 7.2 Maximization w.r.t. Beliefs

The goal of speculators is to maximize the return on their investments. Their choice of actions is from within their investment portfolio set, which is the set of investments that satisfy the wealth constraint and yield maximal return, given their subjective beliefs about prices.

**Definition 7.2** *Given a price system p, and the beliefs* $p_k$ *of the kth speculator, the kth* **investment portfolio set** $V_k(p)$ *contains the set of investments which maximize return:*

$$V_k(p) = \{v_k \in V_k \mid \arg \sup_{v_k \in \hat{B}_k(p)} p_k \cdot v_k\}$$

*The* **total investment portfolio set** $V(p)$ *contains the most valuable total speculations:*

$$V(p) = \sum_{k \in \mathcal{K}} V_k(p)$$

# 8 Type Economy

In summary, the type economy under consideration consists of:

**Definition 8.1** *A* **type economy** $\mathcal{E}$ *consists of*

- *a set of consumers* $\mathcal{I} = \{1, \ldots, I\}$, *with for all* $i \in \mathcal{I}$, *a consumption set* $X_i \subset \mathcal{L}$, *a utility function* $u_i : X_i \to \mathbb{R}$, *and an initial endowment* $w_i \in \mathcal{L}$
- *a set of producers* $\mathcal{J} = \{1, \ldots, J\}$, *together with for all* $j \in \mathcal{J}$, *a production set* $Y_j \subset \mathcal{L}$
- *a set of speculators* $\mathcal{K} = \{1, \ldots, K\}$, *along with for all* $k \in \mathcal{K}$, *a speculation set* $V_k \subset \mathcal{L}$, *a set of initial beliefs* $p_k$, *and an initial endowment* $w_k \in \mathcal{L}$

# 9 Excess Demand

Total excess demand $z(p)$ is the difference between total demand and total supply. In the type economy under study, total demand is the sum of total consumption and total speculation: $x + v$; and total supply is the sum of total production and total resources: $y + w$. The notation $z(p)$ emphasizes that the total excess demand is a function of price, since the choice of consumption, production, and speculation by the agents is dependent on price.

**Definition 9.1** *Given an economy* $\mathcal{E}$ *and a price system* $p$, *the* **total excess demand** *for the economy is given by*

$$z(p) = x + v - y - w$$

*where* $x$ *is total consumption;* $y$ *is total production;* $v$ *is total speculation.*

The notation $z_m(p)$ denotes the total excess demand for the $m$th agent. The notation $z_n(p)$ denotes the total excess demand for the $n$th commodity. The excess demand vector $z(p)$ is contained in the *total excess demand set*:

$$Z(p) = X(p) + V(p) - Y(p) - W$$

This set is a subset of the *total excess budget set*:

$$\hat{Z}(p) = \hat{X}(p) + \hat{V}(p) - Y(p) - W$$

**Definition 9.2** *Given an economy* $\mathcal{E}$ *and a price system* $p$, *the* **total excess supply** *for the economy is given by* $s(p) = -z(p)$.

## 10 Competitive Equilibrium

**Definition 10.1** *Given an economy $\mathcal{E}$, a specification of actions for all agents $m \in \mathcal{M}$ s.t. $a_m \in A_m$ is called a* **market allocation**, *notation $\langle a_m \rangle$. More specifically, $\langle a_m \rangle = (\langle x_i \rangle, \langle y_j \rangle, \langle v_k \rangle)$, where $x_i \in X_i$, $y_j \in Y_j$, and $v_k \in V_k$. Let*

$$ALL(\mathcal{E}) \equiv \{\langle a_m \rangle \mid a_m \in A_m, \forall m \in \mathcal{M}\}$$

A market allocation is an equilibrium when the various actions of the agents are compatible in the sense that total demand does not exceed total supply. Alternatively, a market is in equilibrium when total excess demand is less than or equal to zero (or when total excess supply is greater than or equal to zero).

**Definition 10.2** *In an economy $\mathcal{E}$, a* **market equilibrium** *is a tuple $(\langle a_m^* \rangle, p^*)$ consisting of a market allocation $\langle a_m^* \rangle = (\langle x_i^* \rangle, \langle y_j^* \rangle, \langle v_k^* \rangle)$ together with a price system $p^*$ s.t. $z(p^*) \leq 0$ (or equivalently, $s(p^*) \geq 0$). Let*

$$ME(\mathcal{E}) \equiv \{\langle a_m^* \rangle \in ALL(\mathcal{E}) \mid \exists p^* \in \mathcal{L} \text{ s.t. } z(p^*) \leq 0\}$$

A competitive equilibrium is a market equilibrium in which all economic agents maximize utility. In particular, at a point of competitive equilibrium, consumers choose plans for consumption which satisfy their preferences (*i.e.*, consumptions in their demand sets); producers choose plans for production which maximize their profits (*i.e.*, productions in their supply sets); speculators choose portfolios which maximize return on investment, given their subjective beliefs (*i.e.*, portfolios in their investment sets).

**Definition 10.3** *In an economy $\mathcal{E}$, a* **competitive equilibrium** *is a market equilibrium $(\langle a_m^* \rangle, p^*) = (\langle x_i^* \rangle, \langle y_j^* \rangle, \langle v_k^* \rangle, p^*)$ s.t. $x_i^* \in X_i(p)$, $y_j^* \in Y_j(p)$, and $v_k^* \in V_k(p)$. Let*

$$CE(\mathcal{E}) \equiv \{(\langle x_i^* \rangle, \langle y_j^* \rangle, \langle v_k^* \rangle) \in ME(\mathcal{E}) \mid \exists p^* \in \mathcal{L} \text{ s.t. } x_i^* \in X_i(p), \; y_j^* \in Y_j(p), \text{ and } v_k^* \in V_k(p)\}$$

This is the definition of the celebrated notion of competitive equilibrium. It remains to demonstrate existence and to prove optimality and uniqueness. This is accomplished in the sections that follow. In order to simplify the exposition, the results are shown in the case of a pure-exchange economy: *i.e.*, an economy without production or speculation, with only consumption[4].

[4] The proofs in the more general case appear in Arrow [1], DeBreu [2], and Ellickson [3].

# 11 Walras' Laws

As mentioned in the historical introduction, Walras is recognized as the first economist to perceive of the concept of competitive equilibrium. In this section, two other insights which are also attributed to Walras are formalized.

## 11.1 Weak Form of Walras' Law

The following lemma is used in the proof of the weak form of Walras' Law and the First Fundamental Theorem of Welfare Economics.

**Lemma 11.1** *Given a price system $p$, the $i$th consumer budget set $\hat{B}_i(p)$ equals:*

$$\hat{B}_i(p) = \{x \in X \mid p \cdot z_i(p) \leq 0\}$$

**Proof 11.1**

$$\begin{aligned} \hat{B}_i(p) &= \{x_i \in X_i \mid p \cdot x_i \leq p \cdot w_i\} \\ &= \{x_i \in X_i \mid p \cdot (x_i - w_i) \leq 0\} \\ &= \{x_i \in X_i \mid p \cdot z_i(p) \leq 0\} \end{aligned}$$

The weak form of Walras' Law states that the value of the total excess demand set is non-positive. In other words, the value of total demand does not exceed the value of total supply in the economy. This result is a direct consequence of the wealth constraint.

**Theorem 11.1 (Weak Form of Walras' Law)** *Given exchange economy $\mathcal{E}$, and given a price system $p$,*

$$p \cdot Z(p) \leq 0$$

**Proof 11.2** *By Lemma 11.1, the consumer excess budget set $\hat{Z}(p)$ is equal to:*

$$\begin{aligned} \hat{Z}(p) &= \hat{X}(p) - W \\ &= \textstyle\sum_{i \in \mathcal{I}} \hat{B}_i(p) - W \\ &= \{x \in X \mid p \cdot z(p) \leq 0\} - W \\ &= \{x - w \in X - W \mid p \cdot z(p) \leq 0\} \\ &= \{z(p) \in Z(p) \mid p \cdot z(p) \leq 0\} \end{aligned}$$

*But the total excess demand set $Z(p) \subset \hat{Z}(p)$. Thus, it follows that $p \cdot Z(p) \leq 0$.*

## 11.2 Strong Form of Walras' Law

By the weak form of Walras' Law, the value of the total excess demand set is non-positive. The strong form of Walras' Law states that in addition this value is non-negative. Thus, the value of the total excess demand set is zero.

This theorem relies on the assumption that consumer demands are insatiable. If consumer demands are not satiated, then consumers spend all of their wealth on consumption. It follows that the value of total demand equals the value of total supply: *i.e.*, the value of total excess demand is zero.

**Definition 11.1** *A* **satiation consumption** *is a consumption $x_i \in X_i$ with no preferred consumptions:* i.e., $P_i(x_i) = \emptyset$.

**Assumption 11.1** *No satiation consumption exists for any consumer.*

**Theorem 11.2 (Strong Form of Walras' Law)** *Given an economy $\mathcal{E}$, and given a price system $p$,*

$$p \cdot Z(p) = 0$$

**Proof 11.3** *Let $z(p) \in Z(p)$. By the weak form of Walras' Law, $p \cdot z(p) \leq 0$. Assume $p \cdot z(p) < 0$:* i.e., *$p \cdot x < p \cdot w$, where $z(p) = x - w$ and $x \in X(p)$. By assumption, consumers are insatiable and their utility functions are continuous. Thus, there exists $x' \in X(p)$* s.t. *$u(x') > u(x)$ and $p \cdot x \leq p \cdot w$. But then the consumption $x$ is not of maximal utility; thus, $x \notin X(p)$. This is a contradiction. Therefore, $p \cdot Z(p) = 0$.*

# 12 Existence

The proof of existence is given under several simplifying assumptions.

## 12.1 Assumptions

The following lemma shows that equilibrium price vectors are not unique; but equilibrium prices are relative.

**Lemma 12.1** *In an exchange economy $\mathcal{E}$, if the tuple $(\langle x_i \rangle, p)$ is a competitive equilibrium, then $(\langle x_i \rangle, \lambda p)$ is also a competitive equilibrium, for all $\lambda > 0$.*

**Proof 12.1** *First note the following: if $\lambda > 0$ and $p \in \mathcal{L}^*$, then for all $i \in \mathcal{I}$,*

$$\begin{aligned} \hat{B}_i(\lambda p) &= \{x_i \in X_i \mid \lambda p \cdot x_i \leq \lambda p \cdot w_i\} \\ &= \{x_i \in X_i \mid p \cdot x_i \leq p \cdot w_i\} \\ &= \hat{B}_i(p) \end{aligned}$$

*Moreover,*

$$\begin{aligned} X_i(\lambda p) &= \{x_i \in X_i \mid \arg\sup_{x_i \in \hat{B}_i(\lambda p)} u_i(x_i)\} \\ &= \{x_i \in X_i \mid \arg\sup_{x_i \in \hat{B}_i(p)} u_i(x_i)\} \\ &= X_i(p) \end{aligned}$$

*By definition, if $(\langle x_i \rangle, p)$ is a competitive equilibrium, then $x_i \in X_i(p)$ for all $i \in \mathcal{I}$. But then $x_i \in X_i(\lambda p)$, for all $i \in \mathcal{I}$. Thus, $(\langle x_i \rangle, \lambda p)$ is a competitive equilibrium.*

In light of this observation, it is convenient to normalize prices. In the existence proof, prices are restricted to the n-dimensional unit simplex $S_n$:

$$S_n = \{p \in \mathcal{L} \mid \sum_{n \in \mathcal{N}} p_n = 1 \text{ and } p_n \geq 0, \text{ for all } n \in \mathcal{N}\}$$

Note that this excludes the possibility that $p \leq 0$. This is consistent with the assumptions of free disposal and consumer insatiability.

**Assumption 12.1** *Free disposal of commodities.*

If there is free disposal in the economy, then no economic agent would supply a commodity with a negative price, as it is free to dispose of it. Thus, in accordance with free disposal, assume $p \geq 0$. It follows from the next lemma and consumer insatiability that $p \neq 0$.

**Lemma 12.2** *Let $p^* = (p_1^*, \ldots, p_N^*)$ be a competitive equilibrium in exchange economy $\mathcal{E}$. If $p_n^* > 0$, for some $n \in \mathcal{N}$, then $Z_n(p^*) = S_n(p^*) = 0$.*

**Proof 12.2** *For all $n \in \mathcal{N}$, $p_n^* \geq 0$ and $Z_n(p^*) \leq 0$. This implies that the product $p_n^* \cdot Z_n(p^*) \leq 0$. But in fact, by the strong form of Walras' Law,*

$$p^* \cdot Z(p^*) = \sum_{n \in \mathcal{N}} p_n^* \cdot Z_n(p^*) = 0$$

*It follows that for all $n \in \mathcal{N}$, $p_n^* \cdot Z_n(p^*) = 0$. Thus, if $p_n^* > 0$, for some $n \in \mathcal{N}$, then $Z_n(p^*) = 0$.*

In words, this lemma states that if the equilibrium price of a commodity is positive, then the excess demand for that commodity is zero. Equivalently, if the excess demand for a commodity is negative (*i.e.*, the excess supply is positive), then the equilibrium price of the commodity is zero (*i.e.*, the commodity is free). This implies that consumer demand for free goods is satiable. But consumers are insatiable, by assumption. Thus, there exists at least one good that is not free: *i.e.*, $p \neq 0$.

There is a final simplifying assumption, followed by the existence proof. This assumption implies that the total excess demand $z(p)$ is a function, rather than a relation: *i.e.*, it is single-valued. Moreover, $z(p)$ is a continuous function, since the utility function $u$ is continuous.

**Assumption 12.2** *The total excess demand set $Z(p)$ is a singleton.*

## 12.2 Existence Proof

The proof of existence of competitive equilibrium utilizes a fundamental result in topology; namely, Brouwer's Fixed Point Theorem [5]. In fact, the existence of a competitive equilibrium in an economy is equivalent to the existence of a fixed point in a topological space [**?**].

The existence proof is based on a procedure that adjusts a given price system $p$ in the appropriate direction such that excess demand tends toward zero. The rules are as follows:

1. if there is excess demand in the economy, then increase $p$:
   *i.e.*, $\Delta p > 0$ iff $z(p) > 0$
2. if there is excess supply in the economy, then decrease $p$ $s.t.$ $p \geq 0$:
   *i.e.*, $\Delta p \leq 0$ iff $z(p) < 0$
3. if there is no excess demand in the economy, then $p$ is constant:
   *i.e.*, $\Delta p = 0$ iff $z(p) = 0$

In accordance with these specifications, define the function $g : S_n \to \mathcal{L}$

$$g(p) = p + z(p)$$

This function has a fixed point $p^*$ iff $p^* = g(p^*)$ iff $p^* = p^* + z(p^*)$ iff $z(p^*) = 0$. Unfortunately, Brouwer's Fixed Point Theorem is not applicable to the function $g$ since $\mathcal{L}$ is not compact. But if $g$ is normalized such that the image of $g$ is $S_n$ (*i.e.*, $g : S_n \to S_n$), then this proof technique is valid.

**Theorem 12.1** *There exists* $p^* \in S_n$ s.t. $z(p^*) \leq 0$.

**Proof 12.3** *Consider the function* $f : S_n \to S_n$ s.t. *for all* $n \in \mathcal{N}$,

$$f_n(p) = \frac{p + [z_n(p)]^+}{F(p)}$$

*where*

$$[z_n(p)]^+ = \max[0, z_n(p)]$$

*and*

$$F(p) = \sum_{n \in \mathcal{N}} (p + [z_n(p)]^+) = 1 + \sum_{n \in \mathcal{N}} [z_n(p)]^+$$

*Note that $f$ is a continuous function, since $z(p)$ is continuous and $F(p)$ does not vanish. In particular, $F(p) \geq 1$. Thus, by Brouwer's Fixed Point Theorem, $f$ has a fixed point, namely $p^*$* s.t. *$p^* = f(p^*)$. In particular, for all $n \in \mathcal{N}$,*

$$p^* = \frac{p^* + [z_n(p^*)]^+}{F(p^*)}$$

[5] Every continuous mapping of a compact, topological space into itself has a fixed point.

*It remains to show that $F(p^*) = 1$. Rewriting the above equation:*

$$(F(p^*) - 1)p^* = [z_n(p^*)]^+$$

*Since $p^* \in S_n$, it follows that $p^* > 0$; thus, $p^*_n \neq 0$ for some $n \in \mathcal{N}$. So if $F(p^*) > 1$, then $[z_n(p^*)]^+ = \max[0, z_n(p^*)] > 0$, which implies that $z_n(p^*) > 0$, and moreover $p^*_n \cdot z_n(p^*) > 0$. But then $\sum_{n \in \mathcal{N}} (p^*_n \cdot z_n(p^*)) = p^* \cdot z(p^*) > 0$. This violates the strong form of Walras' Law. Thus, $F(p^*) = 1$.*

*Finally, $p^* = f(p^*)$ iff for all $n \in \mathcal{N}$,*

$$\begin{aligned} & p^* = p^* + [z_n(p^*)]^+ \\ \Leftrightarrow \quad & [z_n(p^*)]^+ = 0 \\ \Leftrightarrow \quad & \max[0, z_n(p^*)] = 0 \\ \Leftrightarrow \quad & z_n(p^*) \leq 0 \end{aligned}$$

*Thus, $z(p^*) = \sum_{n \in \mathcal{N}} z_n(p^*) \leq 0$. This completes the proof of existence.*

This establishes the existence of competitive equilibria. However, the proof of the theorem is highly non-constructive, as it is equivalent to Brouwer's Fixed Point Theorem[6]. Consequently, it is not satisfactory from the point of view of computer science. At best, there exist numerical algorithms which compute approximations to this fixed point as the solution to some nonlinear system of equations [4].

# 13 Pareto Optimality

An optimal allocation is a market equilibrium in which the utility of no consumer can be increased without decreasing the utility of at least one other consumer. The sets of Pareto-dominated and Pareto-optimal allocations partition the set of market allocations.

**Definition 13.1** *Given an exchange economy $\mathcal{E}$, a market allocation $\langle x_i \rangle$ is said to be is* **Pareto-dominated** *iff there exists another market allocation $\langle x'_i \rangle$* s.t. *$\langle x'_i \rangle$ is a market equilibrium and $x'_i \succ_i x_i$, for all $i \in \mathcal{I}$. Let*

$$DOM(\mathcal{E}) \equiv \{\langle x_i \rangle \in ALL(\mathcal{E}) \mid \exists \langle x'_i \rangle \in ME(\mathcal{E}) \text{ s.t. } x'_i \succ_i x_i, \forall i \in \mathcal{I}\}$$

**Definition 13.2** *Given an exchange economy $\mathcal{E}$, a market allocation $\langle x_i \rangle$ is said to be* **Pareto-optimal** *iff it is not Pareto-dominated. Let*

$$PO(\mathcal{E}) \equiv ALL(\mathcal{E}) \backslash DOM(\mathcal{E})$$

According to the above definition, a Pareto-optimal allocation is a market equilibrium. Conversely, by the following fundamental theorem, a competitive equilibrium is a Pareto-optimal market allocation.

[6] In fact, Brouwer, the father of the intuitionistic school of thought at the turn of the century, rejected his own proof on these grounds.

**Theorem 13.1 ( First Fundamental Theorem of Welfare Economics )** *In an exchange economy $\mathcal{E}$, if $(\langle x_i \rangle, p)$ is a competitive equilibrium, then $\langle x_i \rangle$ is a Pareto-optimal market allocation. In short,*

$$CE(\mathcal{E}) \subset PO(\mathcal{E})$$

**Proof 13.1** *Suppose not: suppose the market allocation $\langle x_i \rangle$ is not Pareto-optimal. Then $\langle x_i \rangle$ is Pareto-dominated:* i.e., *there exists a market equilibrium* $\langle x_i' \rangle$ s.t. $x_i' \succ_i x_i$*, for all* $i \in \mathcal{I}$*. In words $x_i'$ is preferred to $x_i$ by all consumers. Now $(\langle x_i \rangle, p)$ is a competitive equilibrium; thus, $x_i$ is in the ith demand set:* i.e., $x_i \in X_i(p)$*. This implies that there is no consumption in the ith budget set that is preferred to $x_i$. But $x_i'$ is in fact preferred to $x_i$; so $x_i'$ must not be in the ith budget set:* i.e., $x_i' \notin \hat{B}_i(p)$*. Thus, by Lemma 11.1, $p \cdot z_i(p) > 0$, for all $i \in \mathcal{I}$. It follows that*

$$\sum_{i \in \mathcal{I}} p \cdot z_i(p) = p \cdot \sum_{i \in \mathcal{I}} z_i(p) = p \cdot Z(p) > 0$$

*But this contradicts the weak form of Walras' Law. Therefore, the allocation $\langle x_i \rangle$ is indeed Pareto-optimal.*

# 14 Uniqueness

In general, a competitive equilibrium price vector is not unique. Nonetheless, competitive equilibrium is unique under the assumption of gross substitutability. It is convenient to describe gross substitutability in terms of excess supply.

**Definition 14.1** *Two commodities $n, n' \in \mathcal{N}$ are called* **gross substitutes** *iff*

1. *if the price of commodity $n$ (or $n'$) decreases, then the excess supply of commodity $n'$ (or $n$) does not increase*
2. *if the price of commodity $n$ (or $n'$) increases, then the excess supply of commodity $n'$ (or $n$) does not decrease*

**Theorem 14.1** *In an economy $\mathcal{E}$, if all goods are gross substitutes, then the competitive equilibrium price vector $p^*$ is unique.*

**Proof 14.1** *Suppose not:* i.e., *let $p \neq p^*$ be a competitive equilibrium. For all $n \in \mathcal{N}$, consider the ratio $\lambda_n = p_n / p_n^*$. Let $\bar{n}$ be the commodity which maximizes this ratio. Note that $p_{\bar{n}} > 0$, since $p, p^* > 0$ and $p_{\bar{n}}$ is a maximizer. Moreover, $\lambda_{\bar{n}} > 0$. By Lemma 12.1, $\lambda_{\bar{n}} p^*$ is a competitive equilibrium:* i.e., $s_{\bar{n}}(\lambda_{\bar{n}} p^*) = s_{\bar{n}}(p^*) \leq 0$*. And in fact, by lemma 12.2, $s_{\bar{n}}(p^*) = 0$.*

*Now consider a sequence of price vectors from $\lambda_{\bar{n}} p^*$ to $p$ s.t. the price of the $n$th commodity never changes, but the prices of all the other commodities decreases. By gross substitutability, $s_{\bar{n}}$ increases until $s_{\bar{n}}(p) > 0$. But then by Lemma 12.2, $p_{\bar{n}} = 0$. This is a contradiction. Therefore, $p^*$ is the unique competitive equilibrium.*

# 15 Remarks

This overview of the theory of competitive equilibrium described a particular type economy consisting of consumers, producers, and speculators. Existence, optimality, and uniqueness of competitive equilibrium were established, but no mechanisms for discovering equilibrium prices were suggested. At this stage, the problem shifts from the realm of economics to an algorithmic problem in computer science. Perhaps computer simulation of market behavior can offer new insights on the convergence of the price adjustment process to competitive equilibrium.

# References


[1] Kenneth Arrow and Frank Hahn. *General Competitive Analysis.* North-Holland, Amsterdam, 1971.

[2] Gerard DeBreu. *Theory of Value.* Wiley, New York, 1959.

[3] Bryan Ellickson. *Competitive Equilibrium.* Cambridge University Press, Cambridge, England, 1993.

[4] Louis H. Scarf. *The Computation of Economic Equilibria.* Yale University Press, New Haven and London, 1973.