ITERATIVE SUBSTRUCTURING METHODS FOR SPECTRAL ELEMENT DISCRETIZATIONS OF ELLIPTIC SYSTEMS. II: MIXED METHODS FOR LINEAR ELASTICITY AND STOKES FLOW

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Abstract. Iterative substructuring methods are introduced and analyzed for saddle point problems with a penalty term. Two examples of saddle point problems are considered: the mixed formulation of the linear elasticity system and the generalized Stokes system in three dimensions. These problems are discretized with spectral element methods. The resulting stiffness matrices are symmetric and indefinite. The unknowns interior to each element are first implicitly eliminated by using exact local solvers. The resulting saddle point Schur complement is solved with a Krylov space method with block preconditioners. The velocity block can be approximated by a domain decomposition method, e.g., of wire basket type, which is constructed from local solvers for each face of the elements, and a coarse solver related to the wire basket of the elements. The condition number of the preconditioned operator is independent of the number of spectral elements and is bounded from above by the product of the square of the logarithm of the spectral degree and the inverse of the discrete inf-sup constant of the problem.

Key words. linear elasticity, Stokes problem, spectral element methods, mixed methods, preconditioned iterative methods, substructuring, Gauss-Lobatto-Legendre quadrature

AMS(MOS) subject classifications. 65N30, 65N35, 65N55

1. Introduction. In this paper, we continue our study of iterative substructuring methods for elliptic systems of partial differential equations in three dimensions discretized with spectral elements. In part I of this study, [33], we focused on the analysis of symmetric positive definite systems, considering as a model the linear elasticity system for compressible materials, in its pure displacement formulation. In this paper, we instead focus on indefinite systems originating from mixed spectral element discretizations, such as the system of linear elasticity for almost incompressible materials and the Stokes system. In computational elasticity, the mixed formulation provides a well-understood remedy for the problem of locking due to the incompressibility constraint; see, e.g., Babuška and Suri [1] and the references in part I, [33]. We refer to Brezzi and Fortin [10] and Girault and Raviart [21] for a general introduction to mixed finite elements.

Iterative substructuring methods form a main class of domain decomposition methods; see Smith, Bjørstad, and Gropp [40], Chan and Mathew [15], and Dryja, Smith, and Widlund [17]. For a brief review of iterative substructuring methods for linear

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elasticity in pure displacement form, see Section 2; we also refer to part I of this study, [33] and the references therein.

The mixed spectral element discretization considered in this paper has been studied by Maday, Patera, and Rønquist [27] for the Stokes system; see also Canuto, Hussaini, Quarteroni, and Zang [12], and Bernardi and Maday [4, 3] for an introduction to spectral methods. Iterative substructuring methods for spectral and hp discretizations of Stokes and Navier-Stokes problems can be found in Quarteroni [36], Fischer and Rønquist [20], Maday, Meiron, Patera, and Rønquist [26], Rønquist [37], Le Tallec and Patra [25], and Casarin [14]. For h-version finite elements, iterative substructuring methods for Stokes problems can be found in Bramble and Pasciak [6], while multigrid methods for mixed linear elasticity have been studied by Brenner [7, 8, 9].

In this paper, we extend the iterative substructuring methods, previously studied for scalar elliptic problems in [32, 35] and positive definite systems in [33], to saddle point problems with, or without, a penalty term. We will consider two mixed spectral element discretizations, known as the $Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ methods. The unknowns interior to each element are first implicitly eliminated by a substructuring technique as in the positive definite case. The resulting Schur complement corresponds to a saddle point problem, involving the interface unknowns and piecewise constant Lagrange multipliers (pressures in the Stokes case). This saddle point Schur complement system is solved by a Krylov space method such as GMRES or PCR with block-diagonal or block-triangular preconditioners. The velocity block can, e.g., be approximated by using a domain decomposition method of wire basket type, constructed from local solvers for each face of the elements and a coarse solver related to the wire basket of the elements. The main result of this paper is the proof of quasi-optimal bounds on the condition number of the resulting iterative methods. These bounds are independent of N, the number of spectral elements, but depend on the square of the logarithm of the spectral degree n and on the inverse of the discrete inf-sup constant of the mixed discretization. Due to the better stability properties of $Q_n - P_{n-1}$ spectral elements, we will see that the convergence bounds for our algorithm improve when the problem is discretized with $Q_n - P_{n-1}$ instead of $Q_n - Q_{n-2}$ spectral elements. On the other hand, the practical implementation of $Q_n - P_{n-1}$ spectral elements is more complicated.

This paper is organized as follows. In the next section, we introduce the mixed formulation of the linear elasticity system and the analogous generalized Stokes problem. In Section 3, we introduce the mixed spectral element discretization of saddle point problems with a penalty term and we review the inf-sup condition for mixed spectral elements. A relationship between the almost incompressible and the incompressible limit is described in Section 4. In Section 5, we introduce some extension operators from the interface to the interior of each element. In Section 6, we describe the basic iterative substructuring process for saddle point problems resulting in a saddle point Schur complement S_Γ. In Section 7, we show that such a saddle point Schur complement satisfies a uniform inf-sup condition. In Section 8, we introduce some block preconditioners for S_Γ with a wire basket based block. The mixed elasticity and the Stokes case are treated separately. The use of other basic domain decomposition methods, including the Neumann-Neumann algorithm, is also discussed. Section 9 concludes the paper with a report on some of our numerical results.

We note that a summary of the results of this paper and those of [33] has been given in a conference paper [34] prepared for the proceedings of a conference held in June 1997 at the IMA, Minneapolis. The first author has also submitted a conference paper

[28] to the proceedings of the Tenth International Conference on Domain Decomposition Methods held in Boulder, Colorado in August 1997. That paper provides a brief discussion of the results of this paper in addition to other methods of solving saddle point problems.

2. The linear elasticity and Stokes systems. The pure displacement form of the linear elasticity problem has been studied in part I, [33]; we briefly review this model here. We refer generally to Ciarlet [16] for a detailed treatment of nonlinear and linear elasticity. Let $\Omega \subset R^3$ be a polyhedral domain and let Γ_0 be a nonempty subset of its boundary. Let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$.

The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, subject to a surface force of density $\mathbf{g}$, along $\Gamma_1 = \partial\Omega - \Gamma_0$, and a body force $\mathbf{f}$:

$$(1) \qquad 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx + \lambda \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v}\ dx \ = \ < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, and the inner products are defined as

$$\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}), \qquad < \mathbf{F}, \mathbf{v} > = \int_\Omega \sum_{i=1}^{3} f_i v_i\ dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i v_i\ ds.$$

When λ approaches infinity, this pure displacement model describes materials that are almost incompressible. In terms of the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$, such materials are characterized by values of ν close to 1/2. It is well known that when low order, h-version finite elements are used in the discretization of (1), *locking* can cause a severe deterioration of the convergence rate as $h \to 0$; see, e.g., Babuška and Suri [1]. If the p-version is used instead, locking in $\mathbf{u}$ is eliminated, but it could still be present in quantities of interest such as $\lambda \mathrm{div}\mathbf{u}$. Moreover, the stiffness matrix obtained by discretizing the pure displacement model (1) has a condition number that goes to infinity when $\nu \to 1/2$. Therefore, we must expect that the convergence rate of any iterative method will deteriorate rapidly as the material becomes almost incompressible.

Locking can be eliminated by introducing the new variable $p = -\lambda \mathrm{div}\mathbf{u} \in L^2(\Omega) = U$ and by replacing the pure displacement problem with a mixed formulation:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(2) \qquad \begin{cases} 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx & - \int_\Omega \mathrm{div}\mathbf{v}\ p\ dx \ = \ < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V} \\ - \int_\Omega \mathrm{div}\mathbf{u}\ q\ dx & - \frac{1}{\lambda}\int_\Omega pq\ dx \ = \ 0 \quad \forall q \in U; \end{cases}$$

see Brezzi and Fortin [10]. Using the notations,

$$e(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx, \qquad b(\mathbf{v}, q) = -\int_\Omega \mathrm{div}\mathbf{v}\ q\ dx, \qquad c(p, q) = \int_\Omega pq\ dx,$$

the problem takes the following form:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(3) \qquad \begin{cases} e(\mathbf{u}, \mathbf{v}) & + \quad b(\mathbf{v}, p) \ = \ < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) & - \quad \frac{1}{\lambda}c(p, q) \ = \ 0 \quad \forall q \in U. \end{cases}$$

When $\lambda \to \infty$ (or, equivalently, $\nu \to 1/2$), we obtain the limiting problem for incompressible linear elasticity; we simply drop the appropriate term in (3).

In case of homogeneous Dirichlet boundary conditions on the whole boundary $\partial\Omega$, problem (2) is equivalent to the following generalized Stokes problem (see Brezzi and Fortin [10]):
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(4) \qquad \begin{cases} s(\mathbf{u},\mathbf{v}) \; + \; b(\mathbf{v},p) \; = \; <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) \; - \; \frac{1}{\lambda+\mu} c(p,q) \; = \; 0 & \forall q \in U. \end{cases}$$

Here,

$$s(\mathbf{u},\mathbf{v}) = \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \; dx,$$

and $U = L_0^2(\Omega)$, since it can be shown that the pressure will have zero mean value due to the homogeneous Dirichlet boundary conditions on $\mathbf{u}$. The penalty term in (4) can also originate from stabilization techniques or penalty formulations for Stokes problems. The classical Stokes system, describing the velocity $\mathbf{u}$ and pressure p of a fluid of viscosity μ, can be obtained from (4) by letting $\lambda \to \infty$; again we simply drop one of the terms in formula (4). We refer to Girault and Raviart [21] for an introduction to the Stokes and Navier-Stokes equations and their finite element discretization. See also Yang [42] for an alternative formulation of saddle point problems.

3. Mixed spectral element methods. Let Ω_{ref} be the reference cube $(-1,1)^3$, let $Q_n(\Omega_{\text{ref}})$ be the set of polynomials on Ω_{ref} of degree n in each variable, and let $P_n(\Omega_{\text{ref}})$ be the set of polynomials on Ω_{ref} of total degree n. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an affine image of the reference cube. Thus, $\Omega_i = \phi_i(\Omega_{\text{ref}})$, where ϕ_i is an affine mapping.

a) $Q_n - Q_{n-2}$. This method was proposed by Maday, Patera, and Rønquist [27] for the Stokes system. $\mathbf{V}$ is discretized, component by component, by conforming spectral elements, i.e. by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\Omega_i} \circ \phi_i \in Q_n(\Omega_{\text{ref}}), \; i = 1, \cdots, N, \quad k = 1,2,3\}.$$

The pressure space is discretized by piecewise polynomials of degree $n-2$:

$$U^n = \{q \in L_0^2(\Omega) : q|_{\Omega_i} \circ \phi_i \in Q_{n-2}(\Omega_{\text{ref}}), \; i = 1, \cdots, N\}.$$

We note that the elements of U^n are discontinuous across the boundaries of the Ω_i's. These mixed spectral elements are implemented using Gauss-Lobatto-Legendre (GLL) quadrature, which also allows the construction of very convenient tensor-product bases for $\mathbf{V}^n$ and U^n, described below. Another basis for U^n associated with the Gauss-Legendre (GL) nodes has been studied in [20] and [26]. The $Q_n - Q_{n-2}$ method does not satisfy a uniform inf-sup condition; see Section 3.2.

b) $Q_n - P_{n-1}$. This method uses the same discrete space $\mathbf{V}^n$ as before, together with a different pressure space consisting of piecewise polynomials of total degree $n-1$:

$$\{q \in U : q|_{\Omega_i} \circ \phi_i \in P_{n-1}(\Omega_{ref}), \; i = 1, \cdots, N\}.$$

This choice has been analyzed in Stenberg and Suri [41] and more recently in Bernardi and Maday [5], who proved a uniform inf-sup condition for this and a number of other spaces; see Section 3.2. For P_{n-1} it is not possible to have a tensorial basis, but other standard bases, common in the p-version finite element literature, can be used.

Other interesting choices for U^n have been studied in Canuto [11] and Canuto and Van Kemenade [13] in connection with stabilization techniques for spectral elements using bubble functions.

3.1. GLL quadrature and the discrete problem. Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points of the reference cube $[-1,1]^3$, and by σ_i the quadrature weight associated with ξ_i . Let $l_i(x)$ be the Lagrange interpolating polynomial of degree n which vanishes at all the GLL nodes except ξ_i, where it equals one. The basis functions on the reference cube are then defined by a tensor product as

$$l_i(x)l_j(y)l_k(z), \qquad 0 \le i, j, k \le n.$$

This is a nodal basis, since every element of $Q_n(\Omega_{\text{ref}})$ can be written as

$$u(x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k) l_i(x) l_j(y) l_k(z).$$

If we use the $Q_n - Q_{n-2}$ method, every element of U^n can be written, on the reference cube, using only the internal GLL nodes:

$$p(x,y,z) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} p(\xi_i,\xi_j,\xi_k) \tilde{l}_i(x) \tilde{l}_j(y) \tilde{l}_k(z).$$

Here, $\tilde{l}_i(x)$ is the Lagrange interpolating polynomial of degree $n-2$ vanishing at all the internal GLL nodes except ξ_i, where it equals one. If we use the $Q_n - P_{n-1}$ method, a basis for U^n can be constructed by using integrated Legendre polynomials:

$$p(x,y,z) = \sum_{0 \le i+j+k \le n-1} \alpha_{ijk} \tilde{L}_i(x) \tilde{L}_j(y) \tilde{L}_k(z),$$

where $\tilde{L}_0(x) = 1, \tilde{L}_{i+1}(x) = \int_{-1}^{x} L_i(t)dt$ and L_i is the Legendre polynomial of degree $i,\ i \ge 0$.

We now replace each integral of the continuous models (3)-(4) by GLL quadrature. On Ω_{ref},

$$(u,v)_{n,\Omega_{\text{ref}}} = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k) v(\xi_i,\xi_j,\xi_k) \sigma_i \sigma_j \sigma_k,$$

and in general on Ω,

$$(u,v)_{n,\Omega} = \sum_{s=1}^{N}\sum_{i,j,k=0}^{n} (u \circ \phi_s)(\xi_i,\xi_j,\xi_k)(v \circ \phi_s)(\xi_i,\xi_j,\xi_k) |J_s| \sigma_i \sigma_j \sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. This inner product is uniformly equivalent to the standard L_2-inner product on $Q_n(\Omega_{\text{ref}})$. Thus it is shown in Bernardi and Maday [3, 4] that

$$\|u\|^2_{L_2(\Omega_{\text{ref}})} \le (u,u)_{n,\Omega_{\text{ref}}} \le 27 \|u\|^2_{L_2(\Omega_{\text{ref}})} \qquad \forall u \in Q_n(\Omega_{\text{ref}}). \tag{5}$$

The discrete bilinear forms obtained are

$$e_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega}, \qquad s_n(\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u} : \nabla\mathbf{v})_{n,\Omega},$$

$$b_n(\mathbf{u},p) = -(\mathrm{div}\mathbf{u}, p)_{n,\Omega}, \qquad c_n(p,q) = (p,q)_{n,\Omega}.$$

We note that, since GLL quadrature is exact for all integrands in Q_{2n-1} and we are using affine images of the reference cube, the last two bilinear forms are exact, i.e. $b_n(\mathbf{u},p) = b(\mathbf{u},p)$ and $c_n(p,q) = c(p,q), \quad \forall(\mathbf{u},p) \in \mathbf{V}^n \times U^n$. An analysis of $Q_n - Q_{n-2}$ method for the Stokes case can be found in Bernardi and Maday [3, 4] and Maday, Patera, and Rønquist [27].

The discrete elasticity problem obtained by spectral element discretization is: Find $(\mathbf{u},p) \in \mathbf{V}^n \times U^n$ such that

$$(6) \qquad \begin{cases} e_n(\mathbf{u},\mathbf{v}) & + & b_n(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}>_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}^n \\ b_n(\mathbf{u},q) & - & \frac{1}{\lambda}c_n(p,q) & = & 0 & \forall q \in U^n. \end{cases}$$

In the incompressible case, we remove the $c_n(\cdot,\cdot)$ term, since $1/\lambda = 0$. The discretization of the generalized Stokes problem (4) leads to similar saddle point problems, with $s_n(\cdot,\cdot)$ in place of $e_n(\cdot,\cdot)$ and the penalty parameter equal to $1/(\lambda+\mu)$.

These are all saddle point problems, with a penalty term in the elasticity and generalized Stokes case. Using, for simplicity, the same notation for functions and their coefficient vectors, we can write the matrix form of (6) as

$$(7) \qquad K\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -t^2C \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix},$$

where A, B, and C are the matrices associated with $s_n(\cdot,\cdot)$ or $e_n(\cdot,\cdot)$, and with $b_n(\cdot,\cdot)$, and $c_n(\cdot,\cdot)$, respectively. The penalty parameter is $t^2 = \frac{1}{\lambda}$ for elasticity problems and $t^2 = \frac{1}{\lambda+\mu}$ for generalized Stokes problems. The stiffness matrix K is symmetric and indefinite. It is less sparse than the stiffness matrices obtained by low-order finite elements, but still well-structured in particular in the $Q_n - Q_{n-2}$ case, and the corresponding matrix-vector multiplication is then relatively inexpensive if advantage is taken of the tensor product structure; see, e.g., Bernadi and Maday [3].

In the following, we will also use $c > 0$ and $C < +\infty$ to denote generic constants in our inequalities; it will be clear from the context if we are referring to generic constants or to the bilinear form $c(\cdot,\cdot)$ and the associated matrix C.

Block-diagonal and block-triangular preconditioners for saddle point problems with a penalty parameter have been studied in Klawonn [24, 22, 23] for low-order finite elements and by Pavarino [30, 31] for spectral element methods. The resulting preconditioned operators have a convergence rate which is independent of the penalty parameter t, the number of spectral elements N, and which depends only mildly on the spectral degree n. Domain decomposition techniques can be applied to each diagonal block of these preconditioners. In contrast to this approach, we will in this paper apply iterative substructuring techniques directly to the saddle point problem (7). The resulting Schur complement problem is itself of saddle point form and of reduced dimension, and can be solved in an iteration using a block preconditioner, based again on domain decomposition techniques.

3.2. The inf-sup condition for spectral elements. The convergence of mixed methods depends not only on the approximation properties of the discrete spaces $\mathbf{V}^n$ and U^n, but also on a stability condition known as the inf-sup (or LBB) condition; see, e.g., Brezzi and Fortin [10]. For numerical studies of the inf-sup constant of various h-version finite elements, see Bathe and Chapelle [2]. While many important h-version finite elements for Stokes problems satisfy the inf-sup condition with a constant independent of h, several important spectral elements proposed for Stokes problems, such as the $Q_n - Q_{n-2}$ method, satisfy the following inf-sup condition:

$$\sup_{\mathbf{v}\in\mathbf{V}^n} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}} \geq C n^{-(\frac{d-1}{2})} \|q\|_{L^2} \quad \forall q \in U^n, \tag{8}$$

where $d = 2, 3$ and the constant C is independent of n and q. This result has been proven for the $Q_n - Q_{n-2}$ method by Maday, Patera, and Rønquist [27] and by Stenberg and Suri [41] for more general discrete mixed spaces. For the $Q_n - Q_{n-2}$ method, an example is also given in [27] showing that the estimate is sharp, i.e. the inf-sup constant approaches zero as $n^{-(d-1)/2}$ ($d = 2, 3$). However, numerical experiments by Maday, Meiron, Patera, and Rønquist [26] and [27], have also shown that for practical values of n (e.g. $n \leq 16$), the inf-sup constant β_n of the $Q_n - Q_{n-2}$ method decays much slower than could be expected from the theoretical bound.

Very recently, Bernardi and Maday [5] improved the bound in (8) by proving a uniform inf-sup condition for $Q_n - P_{n-1}$. Indeed, our numerical experiments reported in [30, 31] and in Section 9 indicated that in fact the $Q_n - P_{n-1}$ method might be uniformly stable. On the other hand, the loss of a tensorial basis for the pressures makes the implementation and use of $Q_n - P_{n-1}$ more complicated.

We can rewrite the inf-sup condition in matrix form as

$$q^t B A^{-1} B^t q \geq \beta_n^2 q^t C q \quad \forall q \in U^n, \tag{9}$$

where β_n is the inf-sup constant of the method; see Brezzi and Fortin [10]. Therefore β_n^2 scales as $\lambda_{min}(C^{-1}BA^{-1}B^t)$. Similarly, if $\tilde{\beta}$ is the continuity constant of the bilinear form $b(\cdot,\cdot)$, we have

$$\mathbf{v}^t B^t q \leq \tilde{\beta}(q^t C q)^{1/2} (\mathbf{v}^t A \mathbf{v})^{1/2} \quad \forall \mathbf{v} \in \mathbf{V}^n, \forall q \in U^n. \tag{10}$$

From (9) and (10), it follows that

$$\beta_n^2 \leq \frac{q^t B A^{-1} B^t q}{q^t C q} \leq \tilde{\beta}^2 \quad \forall q \in U^n.$$

We remark that the dependence on n of the inf-sup constant implies only a loss (of order $n^{-(d-1)/2}$) in the order of convergence for the pressure p, but not for the velocity $\mathbf{u}$; see the classical error estimates as given in Bernardi and Maday [3, Theorems 2.5 and 7.7] and Stenberg and Suri [41, Theorem 5.2 and Remark 5.3]. For problems with regular solutions (for which spectral methods are most appropriate), we still have spectral convergence for both components of the discrete solution.

4. The incompressible limit. The almost incompressible case (*Problem* $\mathcal{P}_t$ with t small) can be seen as a regularized version (by penalty) of the incompressible case (*Problem* $\mathcal{P}_0$). In fact, the following result concerning an abstract saddle point problem with penalty parameter t and right hand-side $\mathbf{F}$, can be found in Girault and Raviart [21, Theorem II.1.3. p.121].

THEOREM 4.1. *Assume that*

i) $b(\cdot,\cdot)$ *satisfies an inf-sup condition, i.e. there exists a constant* $\beta > 0$ *such that*

$$\sup_{\mathbf{v}\in\mathbf{V}^n} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{V}^n}} \geq \beta \|q\|_{U^n} \qquad \forall q \in U^n; \tag{11}$$

ii) $c(\cdot,\cdot)$ *is* U^n*-elliptic, i.e. there exists a constant* $\gamma > 0$ *such that*

$$c(q,q) \geq \gamma \|q\|^2_{U^n} \qquad \forall q \in U^n; \tag{12}$$

iii) there exists a constant $\alpha > 0$ *such that*

$$s(\mathbf{v},\mathbf{v}) + < \mathcal{C}^{-1}\mathcal{B}\mathbf{v}, \mathcal{B}\mathbf{v} > \geq \alpha \|\mathbf{v}\|^2_{\mathbf{V}} \qquad \forall \mathbf{v} \in \mathbf{V}^n, \tag{13}$$

where the operators $\mathcal{B} \in \mathcal{L}(\mathbf{V}^n, (U^n)')$ *and* $\mathcal{C} \in \mathcal{L}(U^n, (U^n)')$ *are defined by*

$$\begin{aligned} < \mathcal{B}\mathbf{v}, q > &= b(\mathbf{v},q) \qquad \forall q \in U^n, \qquad \forall \mathbf{v} \in \mathbf{V}^n; \\ < \mathcal{C}p, q > &= c(p,q) \qquad \forall p,q \in U^n. \end{aligned}$$

Then the Problem $\mathcal{P}_t$*, for* $t^2 \leq 1$*, and Problem* $\mathcal{P}_0$*, have unique solutions* $(\mathbf{u}_t, p_t)$ *and* $(\mathbf{u}, p)$ *in* $\mathbf{V}^n \times U^n$*, respectively. Moreover, if* $t^2 \leq t_0^2$ *and* t_0 *is small enough, we have the error bound:*

$$\|\mathbf{u}_t - \mathbf{u}\|_{\mathbf{V}} + \|p_t - p\|_U \leq Ct^2 \|\mathbf{F}\|_{\mathbf{V}'}, \tag{14}$$

where the constant C *depends only on* $\alpha, \beta, \|a\|, \|b\|$ *and* $\|c\|$.

This result shows that *Problem* $\mathcal{P}_t$ can be used as a preconditioner for *Problem* $\mathcal{P}_0$ and vice versa. In fact

$$\|\mathbf{u}_t - \mathbf{u}\|_{\mathbf{V}} + \|p_t - p\|_U \leq Ct^2 \|\mathbf{F}\|_{\mathbf{V}'} \leq Ct^2 (\|\mathbf{u}_t\|_{\mathbf{V}} + \|p_t\|_U).$$

Therefore

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_U \leq (1 + Ct^2)(\|\mathbf{u}_t\|_{\mathbf{V}} + \|p_t\|_U),$$

i.e.

$$\| \begin{bmatrix} A & B^T \\ B & -t^2 C \end{bmatrix}^{-1} \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \|_{\mathcal{L}(\mathbf{V}\times U, \mathbf{V}\times U)} \leq (1 + Ct^2).$$

We can therefore concentrate our analysis on the incompressible case and build a preconditioner for *Problem* $\mathcal{P}_0$ in the Schwarz framework by splitting the discrete spaces $\mathbf{V}^n$ and U^n into subspaces. By Theorem 4.1 such a preconditioner will also be a good preconditioner for *Problem* $\mathcal{P}_t$ when t^2 is small.

5. Extensions from the interface. In the construction and analysis of our algorithms, we will need to consider a number of subspaces of the space $\mathbf{V}^n$. Many of them involve extensions into the interior of the elements of the interface values of elements of the spectral finite element space $\mathbf{V}^n$. The interface Γ of the decomposition $\{\Omega_i\}$ of Ω is defined by

$$\Gamma = (\cup_{i=1}^{N} \partial\Omega_i) \setminus \partial\Omega.$$

The space of restrictions to the interface is defined by

$$\mathbf{V}^n(\Gamma) = \{\mathbf{v}|_\Gamma : \quad \mathbf{v} \in \mathbf{V}^n\}.$$

Γ is composed of N_F faces F_k (open sets) of the elements and the wire basket W, defined as the union of the edges and vertices of the elements, i.e.

$$\Gamma = \cup_{k=1}^{N_F} F_k \cup W.$$

We first define local subspaces consisting of elements of V^n with support in the interior of individual elements,

$$\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3, \qquad i = 1, \cdots, N. \tag{15}$$

We will often also use related local subspaces of pressures, with support and zero mean value in individual elements, defined by

$$U_i^n = U^n \cap L_0^2(\Omega_i), \qquad i = 1, \cdots, N. \tag{16}$$

We will now examine several useful ways of extending elements of $\mathbf{V}^n(\Gamma)$. These extensions are all constructed locally, i.e. element by element.

5.1. The discrete harmonic extension. The discrete harmonic extension $\mathcal{H}^n$: $\mathbf{V}^n(\Gamma) \longrightarrow \mathbf{V}^n$, is defined as the operator that maps a piecewise polynomial $\mathbf{u} \in \mathbf{V}^n(\Gamma)$ into the unique solution $\mathcal{H}^n\mathbf{u} \in \mathbf{V}^n$ of

$$s_n(\mathcal{H}^n\mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_i^n, \qquad \mathcal{H}^n\mathbf{u} = \mathbf{u} \ \text{ on } \ \partial\Omega_i, \quad i = 1, \cdots, N.$$

This is just an application, component by component, of the well-known scalar discrete harmonic extension. As in the scalar case, the discrete harmonic extension satisfies the minimization property

$$s_n(\mathcal{H}^n\mathbf{u}, \mathcal{H}^n\mathbf{u}) = \min_{\mathbf{v}\in\mathbf{V}^n,\ \mathbf{v}|_\Gamma=\mathbf{u}} s_n(\mathbf{v}, \mathbf{v})$$

5.2. The discrete Stokes extension. We can extend a piecewise polynomial from Γ to the interior of each element by solving a Stokes problem in each element. The discrete Stokes extension $(\mathcal{S}^n, \mathcal{S}_p^n) : \mathbf{V}^n(\Gamma) \longrightarrow \mathbf{V}^n \times U^n$, is the operator that maps a piecewise polynomial $\mathbf{u} \in \mathbf{V}^n(\Gamma)$ into the solution of the following Stokes problem on each element:
Find $\mathcal{S}^n\mathbf{u} \in \mathbf{V}^n$ and $\mathcal{S}_p^n\mathbf{u} = p \in \sum_{i=1}^N U_i^n$ such that on each Ω_i

$$\begin{cases} s_n(\mathcal{S}^n\mathbf{u}, \mathbf{v}) + b_n(\mathbf{v}, \mathcal{S}_p^n\mathbf{u}) = 0 & \forall \mathbf{v} \in \mathbf{V}_i^n \\ b_n(\mathcal{S}^n\mathbf{u}, q) = 0 & \forall q \in U_i^n \\ \mathcal{S}^n\mathbf{u} = \mathbf{u} \quad \text{on} \quad \partial\Omega_i \end{cases} \tag{17}$$

In our applications to Stokes problems, we will choose the range of this extension operator

$$\mathbf{V}^n_{\mathcal{S}} = \mathcal{S}^n(\mathbf{V}^n(\Gamma)) \tag{18}$$

as the subspace of interface velocities. As with the discrete harmonic extension, the velocities in this subspace are completely determined by their values on Γ.

The discrete Stokes extension satisfies the minimization property

$$s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) = \min_{\mathbf{v}|_\Gamma = \mathbf{u}} s_n(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in \{\mathbf{v} \in \mathbf{V}^n \; : \; b_n(\mathbf{v}, q) = 0 \;\; \forall q \in \sum_{i=1}^{N} U_i^n\}.$$

The following comparison of the energy of the discrete Stokes and harmonic extensions can be found in [21], [6], [25], and [14].

LEMMA 5.1.

$$c\beta_n s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) \le s_n(\mathcal{H}^n\mathbf{u}, \mathcal{H}^n\mathbf{u}) \le s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) \qquad \forall \mathbf{u} \in \mathbf{V}^n(\Gamma).$$

5.3. The discrete mixed elastic extension. We can also extend a piecewise polynomial from Γ to the interior of each element by solving an incompressible linear elasticity problem (in mixed form) in each element. The discrete elastic extension $(\mathcal{M}^n, \mathcal{M}^n_p) : \mathbf{V}^n(\Gamma) \longrightarrow \mathbf{V}^n \times U^n$, is the operator that maps a piecewise polynomial $\mathbf{u} \in \mathbf{V}^n(\Gamma)$ into the solution of the following incompressible elasticity problem:
Find $\mathcal{M}^n\mathbf{u} \in \mathbf{V}^n$ and $\mathcal{M}^n_p\mathbf{u} = p \in \sum_{i=1}^N U_i^n$ such that on each Ω_i

$$\begin{cases} e_n(\mathcal{M}^n\mathbf{u}, \mathbf{v}) \;\; + \;\; b_n(\mathbf{v}, \mathcal{M}^n_p\mathbf{u}) \;\; = \;\; 0 \quad \forall \mathbf{v} \in \mathbf{V}^n_i \\ b_n(\mathcal{M}^n\mathbf{u}, q) \;\; = \;\; 0 \quad \forall q \in U_i^n \\ \mathcal{M}^n\mathbf{u} = \mathbf{u} \quad \text{on} \quad \partial\Omega_i \end{cases} \tag{19}$$

In our applications to elasticity problems, we will choose the range of this extension operator

$$\mathbf{V}^n_{\mathcal{M}} = \mathcal{M}^n(\mathbf{V}^n(\Gamma)). \tag{20}$$

as the subspace of interface displacements. As with the other extensions, the displacements in this subspace are completely determined by their values on Γ.

The discrete elastic extension satisfies the minimization property

$$e_n(\mathcal{M}^n\mathbf{u}, \mathcal{M}^n\mathbf{u}) = \min_{\mathbf{v}|_\Gamma = \mathbf{u}} e_n(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in \{\mathbf{v} \in \mathbf{V}^n \; : \; b_n(\mathbf{v}, q) = 0 \;\; \forall q \in \sum_{i=1}^{N} U_i^n\}.$$

6. Iterative substructuring for saddle point problems. In this section, we describe how to eliminate the interior unknowns in our saddle point problems. The remaining interface unknowns and constant pressures in each spectral element satisfy a reduced saddle point problem, analogous to the Schur complement in the positive definite case. This process is the starting point of several substructuring methods for Stokes problems; see Bramble and Pasciak [6] for h-version finite elements, Le Tallec and Patra [25] for hp-version finite elements, and Casarin [14] for spectral elements. The following description applies to both Stokes and elasticity problems, but for simplicity we adopt the Stokes terminology (velocity and pressure).

The velocity space $\mathbf{V}^n$ is decomposed as

$$\mathbf{V}^n = \mathbf{V}_1^n + \mathbf{V}_2^n + \cdots + \mathbf{V}_N^n + \mathbf{V}_\Gamma^n,$$

where the local spaces $\mathbf{V}_i^n$ have been defined in (15) and $\mathbf{V}_\Gamma^n = \mathbf{V}_{\mathcal{S}}^n$ in the Stokes case or $\mathbf{V}_\Gamma^n = \mathbf{V}_{\mathcal{M}}^n$ in the elasticity case. The pressure space U^n is decomposed as

$$U^n = U_1^n + U_2^n + \cdots + U_N^n + U_0,$$

where the local spaces U_i^n have been defined in (16) and

$$U_0 = \{q \in U^n : q|_{\Omega_i} = \text{ constant}, \; i = 1, \cdots, N\}$$

consists of piecewise constant pressures in each element. The vector of unknowns is now reordered placing first the interior unknowns, element by element, and then the interface velocities and the piecewise constant pressures in each element:

$$(\mathbf{u}, p)^T = (\mathbf{u}_1 \; p_1, \mathbf{u}_2 \; p_2, \; \cdots, \; \mathbf{u}_N \; p_N, \; \mathbf{u}_\Gamma \; p_0)^T.$$

With this reordering, our saddle point problem $\mathcal{P}_0 : \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$ has the following matrix structure:

$$\begin{bmatrix} A_{11} & B_{11}^T & \cdots & 0 & 0 & A_{1\Gamma} & 0 \\ B_{11} & 0 & \cdots & 0 & 0 & B_{1\Gamma} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{NN} & B_{NN}^T & A_{N\Gamma} & 0 \\ 0 & 0 & \cdots & B_{NN} & 0 & B_{N\Gamma} & 0 \\ A_{\Gamma 1} & B_{1\Gamma}^T & \cdots & A_{\Gamma N} & B_{N\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & \cdots & 0 & 0 & B_0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ p_1 \\ \vdots \\ \mathbf{u}_N \\ p_N \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ 0 \\ \vdots \\ \mathbf{b}_N \\ 0 \\ \mathbf{b}_\Gamma \\ 0 \end{bmatrix} \tag{21}$$

The leading block of this matrix is the direct sum of N local saddle point problems for the interior velocities and pressures $(\mathbf{u}_i, p_i)$. In addition there is a saddle point problem for the interface velocities and piecewise constant pressures $(\mathbf{u}_\Gamma, p_0)$. These subsystems are given by

$$\left\{ \begin{array}{lcl} A_{ii}\mathbf{u}_i + B_{ii}^T p_i & = & \mathbf{b}_i - A_{i\Gamma}\mathbf{u}_\Gamma \\ B_{ii}\mathbf{u}_i & = & -B_{i\Gamma}\mathbf{u}_\Gamma \end{array} \right. , \qquad i = 1, 2, \cdots, N, \tag{22}$$

$$\left\{ \begin{array}{lcl} A_{\Gamma\Gamma}\mathbf{u}_\Gamma + A_{\Gamma 1}\mathbf{u}_1 + \cdots + A_{\Gamma N}\mathbf{u}_N + B_{1\Gamma}^T p_1 + \cdots + B_{N\Gamma}^T p_N + B_0^T p_0 & = & \mathbf{b}_\Gamma \\ B_0 \mathbf{u}_\Gamma & = & 0 \end{array} \right. \tag{23}$$

The local saddle point problems (22) are uniquely solvable because the local pressures are constrained to have zero mean value. The reduced saddle point problem (23) can be written more clearly by introducing linear operators R_i^b, R_i^Γ and P_i^b, P_i^Γ representing the solutions of the $i-$th local saddle point problem,

$$\mathbf{u}_i = R_i^b \mathbf{b}_i + R_i^\Gamma \mathbf{u}_\Gamma, \qquad p_i = P_i^b \mathbf{b}_i + P_i^\Gamma \mathbf{u}_\Gamma, \qquad i = 1, 2, \cdots, N.$$

Then (23) can be rewritten as

$$(24)\qquad \begin{cases} S_\Gamma \mathbf{u}_\Gamma & + & B_0^T p_0 & = & \tilde{\mathbf{b}}_\Gamma \\ B_0 \mathbf{u}_\Gamma & & & = & 0, \end{cases}$$

where

$$S_\Gamma = A_{\Gamma\Gamma} + \sum_{i=1}^{N} A_{\Gamma i} R_i^\Gamma + \sum_{i=1}^{N} B_{i\Gamma}^T P_i^\Gamma, \qquad \tilde{\mathbf{b}}_\Gamma = \mathbf{b}_\Gamma - \sum_{i=1}^{N} A_{\Gamma i} R_i^b \mathbf{b}_i - \sum_{i=1}^{N} B_{i\Gamma}^T P_i^b \mathbf{b}_i.$$

Of course, the matrices R_i^b, R_i^Γ and P_i^b, P_i^Γ need not be assembled explicitly; their action on a given vector is computed by solving the corresponding local saddle point problem. Analogously, S_Γ need not be assembled, since its action on a given vector can be computed by solving the N local saddle point problems (22) with $\mathbf{b}_i = 0$. The right-hand side $\tilde{\mathbf{b}}_\Gamma$ is formed from an additional set of solutions of the N local saddle point problems (22) with $\mathbf{u}_\Gamma = 0$.
We solve the saddle point Schur complement system (24) by a preconditioned Krylov space method such as PCR if we use a symmetric positive definite preconditioner or GMRES if we use a more general preconditioner.

7. Stability of the saddle point Schur complement.

7.1. The Stokes problem. In this section, we will prove that problem (24) is uniformly stable, i.e. that it satisfies an inf-sup condition with a constant β_Γ bounded away from zero independently of n and N. We remark that Bramble and Pasciak [6] have proven a stability result for (24) for h-version finite elements. However, their proof bounds β_Γ in terms of the inf-sup constant of the original system (in our case β_n), which leads to a nonuniform bound in the spectral element case, since β_n can approach zero when n increases. In order to establish a uniform bound on β_Γ, we first give a variational formulation of the saddle point Schur complement (24).

LEMMA 7.1. *The variational form of the saddle point Schur complement (24) is: Find $\mathcal{S}^n\mathbf{u} \in \mathcal{S}^n(\mathbf{V}^n)$ and $p_0 \in U_0$ such that*

$$(25)\qquad \begin{cases} s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{v}) & + & b_n(\mathcal{S}^n\mathbf{v}, p_0) & = & <\tilde{\mathbf{F}}, \mathbf{v}>_{n,\Omega} & \forall \mathcal{S}^n\mathbf{v} \in \mathcal{S}^n(\mathbf{V}^n) \\ b_n(\mathcal{S}^n\mathbf{u}, q_0) & & & = & 0 & \forall q_0 \in U_0. \end{cases}$$

Proof. Let $\mathbf{u} \in \mathbf{V}^n$ and $p \in U^n$ be the solution of the discrete Stokes problem and let $\mathbf{u}_i \in \mathbf{V}_i^n$ and $p_i \in U_i^n$ be the solutions of the local Stokes problems

$$(26)\qquad \begin{cases} s_n(\mathbf{u}_i, \mathbf{v}) & + & b_n(\mathbf{v}, p_i) & = & <\mathbf{F}, \mathbf{v}>_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}_i^n \\ b_n(\mathbf{u}_i, q) & & & = & 0 & \forall q \in U_i^n. \end{cases}$$

Then $\mathbf{u}_\Gamma = \mathbf{u} - \sum_{i=1}^N \mathbf{u}_i$ and $p_\Gamma = p - \sum_{i=1}^N p_i$ satisfy the saddle point problem

$$(27)\qquad \begin{cases} s_n(\mathbf{u}_\Gamma, \mathbf{v}) & + & b_n(\mathbf{v}, p_\Gamma) & = & <\tilde{\mathbf{F}}, \mathbf{v}>_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}^n \\ b_n(\mathbf{u}_\Gamma, q) & & & = & -\sum_{i=1}^N b_n(\mathbf{u}_i, q) & \forall q \in U^n, \end{cases}$$

where $< \tilde{\mathbf{F}}, \mathbf{v} >_{n,\Omega} = < \mathbf{F}, \mathbf{v} >_{n,\Omega} - \sum_{i=1}^{N} s_n(\mathbf{u}_i, \mathbf{v}) - b_n(\mathbf{v}, p_i)$. The right-hand sides of both equations in (27) are zero when $\mathbf{v} \in \mathbf{V}_i^n$ and $q \in U_i^n$. This implies that

$$\mathbf{u}_\Gamma = \mathcal{S}^n \mathbf{u} \qquad \text{and} \qquad p_\Gamma = \mathcal{S}_p^n \mathbf{u} + p_0,$$

where p_0 is the piecewise constant polynomial that equals the mean value of p on each element. In fact, $\mathbf{u}_\Gamma = \mathbf{u}$ on Γ and $(\mathbf{u}_\Gamma, p_\Gamma - p_0)$ satisfy the saddle point problem (17) defining the discrete Stokes extension. Considering now the remaining test functions $(\mathbf{v}, q) \in \mathbf{V}_{\mathcal{S}}^n \times U_0$, we see that $(\mathbf{u}_\Gamma, p_\Gamma)$ satisfy the saddle point problem

$$(28) \qquad \begin{cases} s_n(\mathbf{u}_\Gamma, \mathbf{v}) + b_n(\mathbf{v}, p_\Gamma) = < \tilde{\mathbf{F}}, \mathbf{v} >_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n \\ b_n(\mathbf{u}_\Gamma, q) = 0 & \forall q \in U_0. \end{cases}$$

This is so for $q \in U_0$ and we can apply the divergence theorem on each element and obtain $b_n(\mathbf{u}_i, q) = 0$. In order for the operator of problem (28) to be equal to the saddle point Schur complement (25), it only remains to prove that $b_n(\mathbf{v}, p_\Gamma) = b_n(\mathbf{v}, p_0) \quad \forall \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n$, i.e. that $b_n(\mathbf{v}, \mathcal{S}_p^n \mathbf{u}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n$. This follows immediately from the definition (17) of the discrete Stokes extension, since $\mathbf{v} = \mathcal{S}^n \mathbf{v}$ and $\mathcal{S}_p^n \mathbf{u} = q \in \sum_{i=1}^{N} U_i^n$. □

We can now prove a uniform bound on the inf-sup constant of the saddle point Schur complement (25) for Stokes systems.

LEMMA 7.2.

$$\sup_{\mathcal{S}^n \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{s_n(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})} \geq \beta_\Gamma^2 \|q_0\|_{L^2}^2 \qquad \forall q_0 \in U^0,$$

where β_Γ is independent of q_0, n, and N.

Proof. Since $\mathbf{V}^2(\Gamma) \subset \mathbf{V}^n(\Gamma)$, we have $\mathcal{S}^n(\mathbf{V}^2) \subset \mathcal{S}^n(\mathbf{V}^n) = \mathbf{V}_{\mathcal{S}}^n$. Therefore,

$$\sup_{\mathcal{S}^n \mathbf{v} \in \mathbf{V}_{\mathcal{S}}^n} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{s_n(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})} \geq \sup_{\mathcal{S}^n \mathbf{v} \in \mathcal{S}^n(\mathbf{V}^2)} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{s_n(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})}$$

$$\geq c \sup_{\mathcal{S}^n \mathbf{v} \in \mathcal{S}^n(\mathbf{V}^2)} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{s(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})}.$$

In the last estimate, we have used the equivalence of $s_n(\cdot, \cdot)$ and $s(\cdot, \cdot) = \mu | \cdot |_{H^1(\Omega)^3}^2$ on $\mathbf{V}^n \times \mathbf{V}^n$. Hence, there remains to prove an inf-sup condition for the mixed spaces $\mathcal{S}^n(\mathbf{V}^2) \times U_0$. According to Brezzi and Fortin [10, pp. 219-221], such an inf-sup bound is equivalent to the existence of a linear operator $\Pi_H^n : H_0^1(\Omega)^3 \longrightarrow \mathcal{S}^n(\mathbf{V}^2)$, with the properties

i) $|\Pi_H^n \mathbf{u}|_{H^1(\Omega)^3} \leq C |\mathbf{u}|_{H^1(\Omega)^3}$, with C independent of n and H,

ii) $\int_\Omega \mathrm{div}(\mathbf{u} - \Pi_H^n \mathbf{u}) q_0 dx = 0, \quad \forall q_0 \in U_0$.

It is well known that $\mathbf{V}^2 \times U_0$ is a stable pair of mixed finite element spaces, since it corresponds to the standard $Q_2 - P_0$ element. This implies the existence of a linear operator $\Pi_H : H_0^1(\Omega)^3 \longrightarrow \mathbf{V}^2$, such that

$|\Pi_H \mathbf{u}|_{H^1(\Omega)^3} \le C|\mathbf{u}|_{H^1(\Omega)^3}$ with C independent of n and H,

$\int_\Omega \mathrm{div}(\mathbf{u} - \Pi_H \mathbf{u}) q_0 dx = 0, \quad \forall q_0 \in U_0.$

If we define $\Pi_H^n = \mathcal{S}^n \circ \Pi_H$, we obtain an operator Π_H^n that satisfies $i)$ and $ii)$. In fact, by Casarin [14, Lemma 5.5.1], we can bound the energy of the discrete Stokes extension of a quadratic polynomial by

$$|\mathcal{S}^n(\mathbf{u})|_{H^1(\Omega)^3} \le C|\mathbf{u}|_{H^1(\Omega)^3} \qquad \forall \mathbf{u} \in \mathbf{V}^2,$$

with C independent of n and N. This bound and the stability of Π_H yield $i)$:

$$|\Pi_H^n \mathbf{u}|_{H^1(\Omega)^3} = |\mathcal{S}^n(\Pi_H \mathbf{u})|_{H^1(\Omega)^3} \le C|\Pi_H \mathbf{u}|_{H^1(\Omega)^3} \le C|\mathbf{u}|_{H^1(\Omega)^3}.$$

Moreover, since any $q_0 \in U_0$ is constant in each element, we obtain $ii)$ by the divergence theorem and the properties of Π_H:

$$\int_\Omega \mathrm{div}(\mathbf{u} - \Pi_H^n \mathbf{u}) q_0 dx = \sum_{i=1}^N q_{0i} \int_{\Omega_i} \mathrm{div}(\mathbf{u} - \mathcal{S}^n(\Pi_H \mathbf{u})) dx = \sum_{i=1}^N q_{0i} \int_{\partial\Omega_i} (\mathbf{u} - \Pi_H \mathbf{u}) \cdot \mathbf{n} ds$$

$$= \sum_{i=1}^N q_{0i} \int_{\Omega_i} \mathrm{div}(\mathbf{u} - \Pi_H \mathbf{u}) dx = \int_\Omega \mathrm{div}(\mathbf{u} - \Pi_H \mathbf{u}) q_0 dx = 0.$$

□

7.2. Incompressible elasticity. The following lemma is the analog of Lemma 7.1 for incompressible elasticity problems. It can be proved in the same way substituting $e_n(\cdot,\cdot)$ for $s_n(\cdot,\cdot)$ and using the definition (19) of the discrete mixed elastic extension.

LEMMA 7.3. *The variational form of the saddle point Schur complement (24) is: Find $\mathcal{M}^n \mathbf{u} \in \mathcal{M}^n(\mathbf{V}^n)$ and $p_0 \in U_0$ such that*

$$(29) \begin{cases} e_n(\mathcal{M}^n \mathbf{u}, \mathcal{M}^n \mathbf{v}) + b_n(\mathcal{M}^n \mathbf{v}, p_0) = <\tilde{\mathbf{F}}, \mathbf{v}>_{n,\Omega} & \forall \mathcal{M}^n \mathbf{v} \in \mathcal{M}^n(\mathbf{V}^n) \\ b_n(\mathcal{M}^n \mathbf{u}, q_0) = 0 & \forall q_0 \in U_0. \end{cases}$$

We can now prove a uniform bound on the inf-sup constant of this saddle point Schur complement for incompressible elasticity, using the bound just proved for the Stokes case in Lemma 7.2.

LEMMA 7.4.

$$\sup_{\mathcal{M}^n \mathbf{v} \in \mathbf{V}^n_{\mathcal{M}}} \frac{(\mathrm{div} \mathcal{M}^n \mathbf{v}, q_0)^2}{e_n(\mathcal{M}^n \mathbf{v}, \mathcal{M}^n \mathbf{v})} \ge \beta_\Gamma^2 \|q_0\|_{L^2}^2 \qquad \forall q_0 \in U^0,$$

where β_Γ is independent of $q_0, n,$ and N.

Proof. From the minimization property of the discrete elastic extension and the Cauchy-Schwarz inequality, we have

$$e_n(\mathcal{M}^n \mathbf{u}, \mathcal{M}^n \mathbf{u}) \le e_n(\mathcal{S}^n \mathbf{u}, \mathcal{S}^n \mathbf{u}) \le C s_n(\mathcal{S}^n \mathbf{u}, \mathcal{S}^n \mathbf{u}).$$

As in Lemma 7.2, we can rewrite the nominator using the divergence theorem on each element:

$$\int_{\Omega_i} \mathrm{div}(\mathcal{M}^n \mathbf{u}) q_{0i} dx = q_{0i} \int_{\partial\Omega_i} \mathcal{M}^n \mathbf{u} \cdot \mathbf{n} ds = q_{0i} \int_{\partial\Omega_i} \mathcal{S}^n \mathbf{u} \cdot \mathbf{n} ds = \int_{\Omega_i} \mathrm{div}(\mathcal{S}^n \mathbf{u}) q_{0i} dx.$$

Therefore,

$$\sup_{\mathcal{M}^n \mathbf{v} \in \mathbf{V}^n_{\mathcal{M}}} \frac{(\mathrm{div} \mathcal{M}^n \mathbf{v}, q_0)^2}{e_n(\mathcal{M}^n \mathbf{v}, \mathcal{M}^n \mathbf{v})} \geq \sup_{\mathcal{S}^n \mathbf{v} \in \mathbf{V}^n_{\mathcal{S}}} \frac{(\mathrm{div} \mathcal{S}^n \mathbf{v}, q_0)^2}{e_n(\mathcal{S}^n \mathbf{v}, \mathcal{S}^n \mathbf{v})} \qquad \forall q_0 \in U^0,$$

and we conclude by applying Lemma 7.2. □

8. Block preconditioners for the saddle point Schur complement. Block preconditioners for saddle point problems have been studied by Rusten and Winther [38], Silvester and Wathen [39], Elman and Silvester [19], and Klawonn [24, 22, 23]. Here, we follow Klawonn's approach.

Let S be the coefficient matrix of the reduced saddle point problem (24)

$$S = \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix}. \tag{30}$$

We will consider the following block-diagonal and lower block-triangular preconditioners (an upper block-triangular preconditioner could be considered as well):

$$\widehat{D} = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ 0 & \widehat{C}_0 \end{bmatrix} \qquad \widehat{T} = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ B_0 & -\widehat{C}_0 \end{bmatrix},$$

where $\widehat{S}_\Gamma$ and $\widehat{C}_0$ are good preconditioners for S_Γ and the coarse pressure mass matrix C_0, respectively:

Assumption 1 : $\exists$ constants $a_0, a_1 > 0$ such that

$$a_0^2 \mathbf{v}^t \widehat{S}_\Gamma \mathbf{v} \leq \mathbf{v}^t S_\Gamma \mathbf{v} \leq a_1^2 \mathbf{v}^t \widehat{S}_\Gamma \mathbf{v} \qquad \forall \mathbf{v} \in \mathbf{V}^n(\Gamma);$$

Assumption 2 : $\exists$ constants $m_0, m_1 > 0$ such that

$$m_0^2 q^t \widehat{C}_0 q \leq q^t C_0 q \leq m_1^2 q^t \widehat{C}_0 q \qquad \forall q \in U_0.$$

Even if the coarse pressure mass matrix C_0 is diagonal, we allow for a possible preconditioner $\widehat{C}_0$ because it has been shown by Klawonn [24] that the use of more expensive preconditioners can significantly reduce the iteration counts. We will denote by D and T the operators with exact blocks $\widehat{S}_\Gamma = S_\Gamma$ and $\widehat{C} = C$. With the block-diagonal preconditioner $\widehat{D}$, we can use the preconditioned conjugate residual method (PCR). In the block-triangular case, $\widehat{T}$ is no longer symmetric and we need to use a Krylov space method for nonsymmetric systems, such as GMRES or QMR.

Under Assumptions 1 and 2, we obtain the following convergence bounds, by applying Klawonn's results, see [24, 22, 23].

THEOREM 8.1. *The block-diagonal preconditioner $\widehat{D}$ satisfies the bound*

$$cond(\widehat{D}^{-1} S) \leq \frac{max\{a_1^2, m_1^2\}}{min\{a_0^2, m_0^2\}} \; cond(D^{-1} S)$$

and

$$cond(D^{-1}S) \leq \frac{1/2 + \sqrt{\beta_1^2 + 1/4}}{-1/2 + \sqrt{\beta_\Gamma^2 + 1/4}} \ ,$$

where β_Γ is the inf-sup constant of the reduced saddle point problem (24) and β_1 is the continuity constant of B_0. Here $cond(D^{-1}S)$ is the ratio of the maximum and the minimum absolute value of the eigenvalues of $D^{-1}S$.

THEOREM 8.2. *The block-triangular preconditioner T with exact blocks satisfies the inclusion*

$$spectrum(T^{-1}S) \subset [\beta_\Gamma^2, \beta_1^2 + 1] \cup \{1\}.$$

The case of a block-triangular preconditioner with inexact blocks is studied in Klawonn [22, 23], under the previous Assumptions 1 and 2 and the additional scaling assumption $1 < a_0 \leq a_1$. The estimate provided is analogous to the case with exact blocks, but it is more complicated and we therefore refer to [22] for details. In this case, we can define an additional energy norm based on the inexact blocks and a GMRES convergence bound can be proven in this energy norm.

In order to obtain convergence bounds using Theorems 8.1 and 8.2, we need only verify Assumptions 1 and 2 for a choice of preconditioner blocks $\widehat{S}_\Gamma$ and $\widehat{C}_0$. We will do so in the next section, illustrating our results mainly in the block-diagonal case. The construction of an iterative substructuring algorithm is therefore a very modular process in this framework.

8.1. A wire basket preconditioner for Stokes problems. We consider first a Laplacian-based wire basket preconditioner $\widehat{S}_\Gamma$ given, for each component $u^{(i)}$ of $\mathbf{u}$, by the scalar wire basket preconditioner $\widehat{S}_W$ introduced in Pavarino and Widlund [32] and extended to GLL quadrature based approximations in [35],

$$\widehat{S}_\Gamma = \begin{bmatrix} \widehat{S}_W & 0 & 0 \\ 0 & \widehat{S}_W & 0 \\ 0 & 0 & \widehat{S}_W \end{bmatrix}. \tag{31}$$

In those earlier papers, we considered the scalar Laplace equation with piecewise constant coefficients and constructed a preconditioner $\widehat{S}_W$ for the Schur complement $S_\mathcal{H}$ of the discrete harmonic interface variables, obtained by eliminating the interior degrees of freedom. Here, we briefly recall the construction of $\widehat{S}_W$ and refer to [32] for many more details and a full analysis.

The Schur complement system $S_\mathcal{H}$ is obtained by subassembly from its local contributions $S_\mathcal{H}^{(j)}$ on the element Ω_j, $j = 1, \cdots, N$. The interface Γ is decomposed into the N_F faces F_k of the elements and the wire basket W (the union of edges and vertices),

$$\Gamma = \cup_{k=1}^{N_F} F_k \cup W.$$

If the local vector of interface unknowns is reordered accordingly into face and wire basket components (u_F, u_W), the local Schur complement for the element Ω_j can be written as

$$S_\mathcal{H}^{(j)} = \begin{pmatrix} S_{FF}^{(j)} & S_{FW}^{(j)} \\ S_{FW}^{(j)^T} & S_{WW}^{(j)} \end{pmatrix}.$$

We then modify the subspace spanned by the wire basket functions in order to ensure that a function which is constant on the wire basket is also equal to the same constant on all of Γ when expanded in the wire basket basis functions. This is represented by the transformation matrix

$$\begin{pmatrix} I & 0 \\ R^{(j)} & I \end{pmatrix}.$$

Then $S_{\mathcal{H}}^{(j)}$ is transformed into

$$\begin{pmatrix} I & 0 \\ R^{(j)} & I \end{pmatrix} \begin{pmatrix} S_{FF}^{(j)} & S_{FW}^{(j)} \\ S_{FW}^{(j)^T} & S_{WW}^{(j)} \end{pmatrix} \begin{pmatrix} I & R^{(j)^T} \\ 0 & I \end{pmatrix} = \begin{pmatrix} S_{FF}^{(j)} & \text{nonzero} \\ \text{nonzero} & \widetilde{S}_{WW}^{(j)} \end{pmatrix}.$$

The local preconditioner $\widehat{S}_W^{(j)}$ is constructed by:

a) eliminating the coupling between faces and wire basket;

b) eliminating the coupling between faces, i.e. replacing $S_{FF}^{(j)}$ by its block-diagonal part $\widehat{S}_{FF}^{(j)}$;

c) replacing the wire basket block $\widetilde{S}_{WW}^{(j)}$ by a simpler matrix $\widehat{S}_{WW}^{(j)}$. Let $M^{(j)}$ be the mass matrix associated with the local wire basket W_j, defined by $u^T M^{(j)} u = (u, u)_{n,W_j}$ and let z be the vector of wire basket coefficients of the constant function 1. We replace $\widetilde{S}_{WW}^{(j)}$ by a scaled rank-one perturbation of $M^{(j)}$. On the reference element,

$$\widehat{S}_{WW}^{(j)} = (1 + \log n)(M^{(j)} - \frac{(M^{(j)}z)(M^{(j)}z)^T}{z^T M^{(j)} z}). \tag{32}$$

We then return to the original basis:

$$\widehat{S}_W^{(j)} = \begin{pmatrix} I & 0 \\ -R^{(j)} & I \end{pmatrix} \begin{pmatrix} \widehat{S}_{FF}^{(j)} & 0 \\ 0 & \widehat{S}_{WW}^{(j)} \end{pmatrix} \begin{pmatrix} I & -R^{(j)^T} \\ 0 & I \end{pmatrix}. \tag{33}$$

Finally, the wire basket preconditioner is obtained by subassembly:

$$\widehat{S}_W = \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} \begin{pmatrix} \widehat{S}_{FF} & 0 \\ 0 & \widehat{S}_{WW} \end{pmatrix} \begin{pmatrix} I & -R^T \\ 0 & I \end{pmatrix}.$$

We find that,

$$\widehat{S}_W^{-1} = R_0 \widehat{S}_{WW}^{-1} R_0^T + \sum_k R_{F_k} \widehat{S}_{F_k F_k}^{-1} R_{F_k}^T,$$

where $R_0 = (R, I)$ and $R_{F_k}^T$ are restriction matrices returning the degrees of freedom associated with each face F_k. This is an additive preconditioner with independent parts associated with each face and the wire basket.

The main result of [32] and [35] is the proof of a polylogarithmic bound for the condition number of the scalar wire basket preconditioner (see in [32, Theorem 3.1] and its GLL extensions, [35, Theorems 1 and 2]):

$$c(1 + \log n)^{-2} u_\Gamma^{(i)^T} \widehat{S}_W u_\Gamma^{(i)} \le u_\Gamma^{(i)^T} S_{\mathcal{H}} u_\Gamma^{(i)} \le C u_\Gamma^{(i)^T} \widehat{S}_W u_\Gamma^{(i)} \qquad \forall u_\Gamma^{(i)} \in V^n(\Gamma). \tag{34}$$

Applying this bound to each component, we obtain the analogous bound

$$(35)\quad c(1+\log n)^{-2}\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \begin{bmatrix} S_{\mathcal{H}} & 0 & 0 \\ 0 & S_{\mathcal{H}} & 0 \\ 0 & 0 & S_{\mathcal{H}} \end{bmatrix} \mathbf{u}_\Gamma \le C \mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \quad \forall \mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma).$$

This result allows us to prove a convergence bound for the reduced saddle point problem (24) with block-diagonal preconditioner.

THEOREM 8.3. *Let the blocks of the block-diagonal preconditioner $\widehat{D}_W$ be the wire basket preconditioner $\widehat{S}_\Gamma$ defined in (31) and the coarse mass matrix C_0. Then the Stokes saddle point Schur complement S preconditioned by $\widehat{D}_W$ satisfies*

$$cond(\widehat{D}_W^{-1} S) \le C \frac{(1+\log n)^2}{\beta_n},$$

where C is independent of n and N.

Proof. We estimate the constants a_0, a_1, m_0, m_1 in Assumptions 1 and 2, the inf-sup constant β_Γ and apply Theorem 8.1.

Assumption 1: In the Stokes case (Lemma 7.1),

$$\mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma = s_n(\mathcal{S}^n \mathbf{u}, \mathcal{S}^n \mathbf{u}).$$

By Lemma 5.1, we can compare the energy of the discrete Stokes and (componentwise) discrete harmonic extension

$$(36)\quad c_0 \beta_n \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \begin{bmatrix} S_{\mathcal{H}} & 0 & 0 \\ 0 & S_{\mathcal{H}} & 0 \\ 0 & 0 & S_{\mathcal{H}} \end{bmatrix} \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma).$$

Combining (35) and (36), we obtain

$$c(1+\log n)^{-2}\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma \le \frac{C}{c_0 \beta_n} \mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma),$$

i.e. $a_0^2 = c(1+\log n)^{-2}$ and $a_1^2 = \frac{C}{c_0\beta_n}$.

Assumption 2: Since we use the exact coarse mass matrix as our pressure block, this assumption holds with $m_0 = m_1 = 1$.

Estimate of β_Γ: By Lemma 7.2, β_Γ is uniformly bounded away from zero, independently of n and N. □

An analogous bound for the block-triangular preconditioner follows from the estimates of the constants in the Assumptions 1 and 2 given in the previous proof.

8.2. Neumann-Neumann and other preconditioners for Stokes problems. In the Stokes case, we can use any other scalar substructuring preconditioner for each component of $\mathbf{u}$ in (31), instead of using the wire basket preconditioner. For example, we could use a Neumann-Neumann preconditioner; see Dryja and Widlund [18] for a detailed analysis of this family of preconditioners for $h-$version finite elements and Pavarino [29] for an extension to spectral elements. We then obtain a Laplacian-based Neumann-Neumann preconditioner $\widehat{S}_\Gamma$ with a scalar Neumann-Neumann preconditioner $\widehat{S}_{NN}$ in each diagonal block

$$(37)\qquad \widehat{S}_\Gamma = \begin{bmatrix} \widehat{S}_{NN} & 0 & 0 \\ 0 & \widehat{S}_{NN} & 0 \\ 0 & 0 & \widehat{S}_{NN} \end{bmatrix}.$$

We recall that

$$\widehat{S}_{NN}^{-1} = R_H^T K_H^{-1} R_H + \sum_{j=1}^{N} R_{\partial\Omega_j}^T D_j^{-1} \widehat{S}_j^{\dagger} D_j^{-1} R_{\partial\Omega_j}$$

is an additive preconditioner with an independent coarse solver K_H^{-1} and local solvers $\widehat{S}_j^{\dagger}$, respectively associated with the coarse triangulation determined by the elements and with the boundary $\partial\Omega_j$ of each element. Here $R_{\partial\Omega_j}$ are restriction matrices returning the degrees of freedom associated with the boundary of Ω_j, D_j are diagonal matrices and $\dagger$ denotes an appropriate pseudo-inverse for the singular Schur complements associated with the interior elements; see [18, 29] for more details.

The polylogarithmic bound proven in the scalar case, carries over to the case now under consideration:

$$c(1+\log n)^{-2}\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T \begin{bmatrix} S_{\mathcal{H}} & 0 & 0 \\ 0 & S_{\mathcal{H}} & 0 \\ 0 & 0 & S_{\mathcal{H}} \end{bmatrix} \mathbf{u}_\Gamma \leq C\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma).$$

It is then possible to prove a result analogous to Theorem 8.3.

THEOREM 8.4. *Let the blocks of the block-diagonal preconditioner $\widehat{D}_{NN}$ be the Neumann-Neumann preconditioner $\widehat{S}_\Gamma$ defined in (37) and the coarse mass matrix C_0. Then the Stokes saddle point Schur complement S preconditioned by $\widehat{D}_{NN}$ satisfies*

$$cond(\widehat{D}_{NN}^{-1}S) \leq C\frac{(1+\log n)^2}{\beta_n},$$

where C is independent of n and N.

Other scalar iterative substructuring preconditioners could also be applied in this fashion to the Stokes system; see Dryja, Smith, and Widlund [17] for many alternatives.

8.3. A wire basket preconditioner for incompressible elasticity problems. The block-diagonal preconditioners of the form (31) introduced in the previous sections do not take any coupling between the three components of $\mathbf{u}$ into account. This works for Stokes problems, but for elasticity problems such an approach would lead to non-scalable algorithms. In fact, the saddle point Schur complement for linear elasticity for an interior element Ω_i has a six dimensional nullspace, spanned by the rigid body motions (three translations and three rotations). In order to obtain a scalable algorithm, the local contribution from Ω_i to the wire basket preconditioner must have the same six dimensional nullspace. This condition is of course violated by the componentwise preconditioner of the previous section, that has only a three dimensional nullspace of componentwise translations. In this section, we introduce a scalable wire basket preconditioner for mixed elasticity problems, using the techniques and the analysis of [33]. The basic changes consist in:

a) using the bilinear form

$$e_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega}$$

instead of

$$2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega} + \lambda(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v})_{n,\Omega}$$

used in [33] for compressible elasticity;

b) using the mixed elastic extension $\mathcal{M}^n$ instead of the elastic extension $\mathcal{E}^n$. This implies that the extension from the wire basket is now defined by

$$I^W \mathbf{u} = \mathcal{M}^n(I^W u^{(1)}, I^W u^{(2)}, I^W u^{(3)}),$$

where the single scalar components are given in [33, (9)], and the subspace of interface displacements is now $\mathbf{V}^n_{\mathcal{M}} = \mathcal{M}^n(\mathbf{V}^n)$. We note that the nullspaces of $e_n(\cdot,\cdot)$ and the bilinear form of compressible elasticity, on an interior element, are the same set $\mathcal{N}$ spanned by the rigid body motions. Moreover, I^W still reproduces the rigid body motions. Therefore, the same construction as in [33, Section 6] can be used to obtain a wire basket preconditioner

$$\widehat{S}_\Gamma^{-1} = R_0 \widehat{S}_{WW}^{-1} R_0^T + \sum_k R_{F_k} \widehat{S}_{F_k F_k}^{-1} R_{F_k}^T. \tag{38}$$

Here we use a different scaling of the wire basket inexact solver $\widehat{S}_{WW}^{-1}$; on an interior element Ω_j, which we, for simplicity assume to be the reference element, we set

$$\widehat{S}_{WW}^{(j)} = \frac{(1+\log n)}{\beta_n}\Big(M^{(j)} - \sum_{i=1}^{6} \frac{(M^{(j)}\mathbf{r}_i)(M^{(j)}\mathbf{r}_i)^T}{\mathbf{r}_i^T M^{(j)} \mathbf{r}_i}\Big).$$

We can then prove a bound analogous to the main result of [33].

THEOREM 8.5. *The wire basket preconditioner* $\widehat{S}_\Gamma^{-1}$ *satisfies the bounds*

$$c\beta_n(1+\log n)^{-2}\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma \le C \mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma).$$

Proof. We recall that in the mixed elasticity case

$$\mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma = e_n(\mathcal{M}^n \mathbf{u}, \mathcal{M}^n \mathbf{u});$$

see Lemma 7.3. We also recall that using the standard Schwarz theory, it is enough to prove the upper and lower bounds of the theorem locally on an interior element, which we for simplicity assume to be the reference element; see Smith, Bjørstad, and Gropp [40]. We decompose $\mathbf{u}_\Gamma \in \mathbf{V}^n(\Gamma)$ as

$$\mathbf{u}_\Gamma = \mathbf{u}_0 + \sum_{k=1}^{6} \mathbf{u}_{F_k},$$

where $\mathbf{u}_0 = I^W \mathbf{u}$ and $\mathbf{u}_{F_k} = \mathbf{u} - I^W \mathbf{u}$ on F_k and vanishes on the other faces and on the wire basket. We also define a simplified bilinear form defined on the wire basket space by the approximate solver $\widehat{S}_{WW}$ given by

$$\tilde{e}_0(\mathbf{u},\mathbf{u}) = \frac{(1+\log n)}{\beta_n} \sum_{i=1}^{N} \inf_{c_{ij}} \|\mathbf{u} - \sum_{j=1}^{6} c_{ij}\mathbf{r}_j\|^2_{n,W_i}.$$

Then the lower bound of the theorem can be formulated variationally as

$$\tilde{e}_{0,\text{ref}}(\mathbf{u}_0,\mathbf{u}_0) + \sum_{k=1}^{6} e_{n,\text{ref}}(\mathcal{M}^n \mathbf{u}_{F_k}, \mathcal{M}^n \mathbf{u}_{F_k}) \le C \frac{(1+\log n)^2}{\beta_n} e_{n,\text{ref}}(\mathbf{u},\mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{V}^n_{\mathcal{M}},$$

and the upper bound as

$$e_{n,\mathrm{ref}}(\mathbf{u},\mathbf{u}) \le C(\tilde{e}_{0,\mathrm{ref}}(\mathbf{u}_0,\mathbf{u}_0) + \sum_{k=1}^{6} e_{n,\mathrm{ref}}(\mathcal{M}^n \mathbf{u}_{F_k}, \mathcal{M}^n \mathbf{u}_{F_k})) \quad \forall \mathbf{u} \in \mathbf{V}^n_{\mathcal{M}}.$$

a) Lower bound: We note that from the minimization property of the mixed elastic extension and the energy comparison given in Lemma 5.1, we obtain

$$e_{n,\mathrm{ref}}(\mathcal{M}^n \mathbf{u}_{F_k}, \mathcal{M}^n \mathbf{u}_{F_k}) \le e_{n,\mathrm{ref}}(\mathcal{S}^n \mathbf{u}_{F_k}, \mathcal{S}^n \mathbf{u}_{F_k}) \le$$

$$C a_{n,\mathrm{ref}}(\mathcal{S}^n \mathbf{u}_{F_k}, \mathcal{S}^n \mathbf{u}_{F_k}) \le \frac{C}{\beta_n} a_{n,\mathrm{ref}}(\mathcal{H}^n \mathbf{u}_{F_k}, \mathcal{H}^n \mathbf{u}_{F_k}).$$

We can then repeat, step by step, the proof of the lower bound in [33, Theorem 7.1, Section 7.2] .

b) Upper bound: Similarly, we have

$$e_{n,\mathrm{ref}}(I^W \mathbf{u}, I^W \mathbf{u}) \le e_{n,\mathrm{ref}}(\mathcal{S}^n(I^W \mathbf{u}), \mathcal{S}^n(I^W \mathbf{u})) \le$$

$$C a_{n,\mathrm{ref}}(\mathcal{S}^n(I^W \mathbf{u}), \mathcal{S}^n(I^W \mathbf{u})) \le \frac{C}{\beta_n} a_{n,\mathrm{ref}}(\mathcal{H}^n(I^W \mathbf{u}), \mathcal{H}^n(I^W \mathbf{u})).$$

Therefore, we can also repeat the proof of the upper bound in [33, Theorem 7.1, Section 7.2]. We note that the simplified wire basket bilinear form $\tilde{e}_{0,\mathrm{ref}}$ is now scaled by $\frac{(1+\log n)}{\beta_n}$. □

Using Theorem 8.5 to bound the constants of Assumption 1, we can then prove the following result.

THEOREM 8.6. *Let the blocks of the block-diagonal preconditioner $\widehat{D}_W$ be the wire basket preconditioner $\widehat{S}_\Gamma$ defined in (38) and the coarse mass matrix C_0. Then the incompressible mixed elasticity saddle point Schur complement S preconditioned by $\widehat{D}_W$ satisfies*

$$cond(\widehat{D}_W^{-1} S) \le C \frac{(1+\log n)^2}{\beta_n},$$

where C is independent of n and N.

9. Numerical results. In this last section, we report the results of numerical experiments for both Stokes and mixed elasticity problems in three dimensions. All computations were carried out in Matlab 5.0 on Sun workstations.

We first computed the discrete inf-sup constant β_n of the whole Stokes problem on the reference cube with zero Dirichlet boundary conditions. β_n is computed as the square root of the minimum nonzero eigenvalue of $C^{-1}B^T A^{-1} B$, where A, B, and C are the blocks in (7). The results reported in Table 1 and plotted in Figure 1 indicate that the inf-sup parameter of the $Q_n - P_{n-1}$ method is much better than that of the $Q_n - Q_{n-2}$ method, in agreement with the theoretical results of [5] and the experiments in [30, 31].

We then compute the discrete inf-sup constant β_Γ of the saddle point Schur complement (30) for both the mixed elasticity and Stokes system on the reference cube

FIG. 1. *Inf-sup constant β_n for the discrete Stokes problem ($Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ spectral elements)*

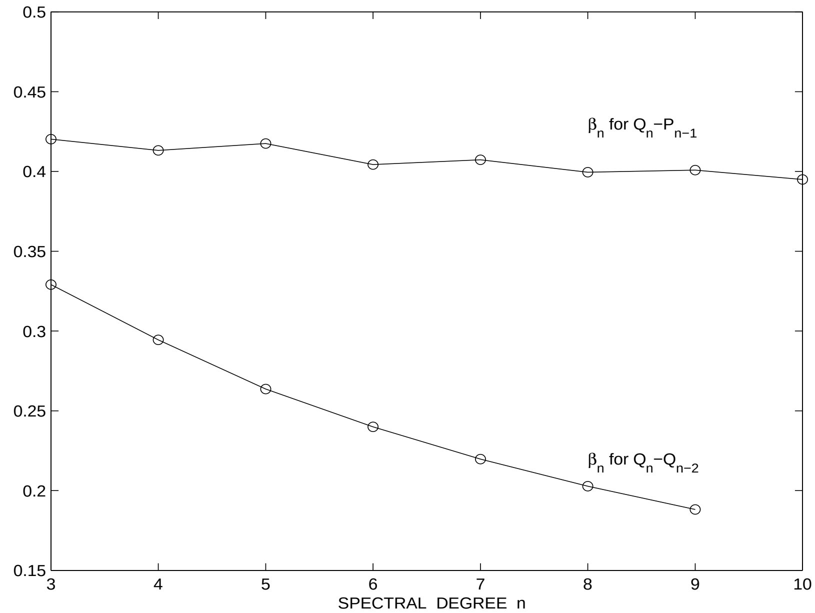


FIG. 2. *Inf-sup constant β_Γ for the Stokes and incompressible mixed elasticity saddle point Schur complement ($Q_n - Q_{n-2}$ spectral elements)*

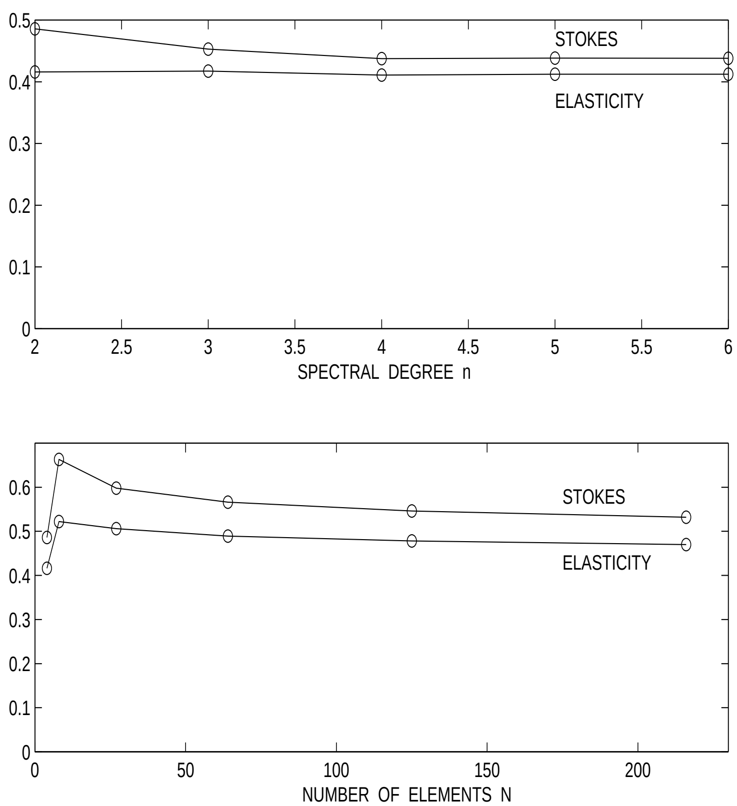


TABLE 1

Inf-sup constant $\beta_n = \lambda_{min}^{1/2}(C^{-1}B^TA^{-1}B)$ *and other spectral information for the Stokes system*

	$Q_n - Q_{n-2}$				$Q_n - P_{n-1}$			
n	β_n	λ_{min}	λ_{max}	$\frac{\lambda_{max}}{\lambda_{min}}$	β_n	λ_{min}	λ_{max}	$\frac{\lambda_{max}}{\lambda_{min}}$
3	0.3291	0.1083	0.2284	2.1084	0.4095	0.1677	0.3611	2.1527
4	0.2944	0.0867	0.6334	7.3040	0.4132	0.1707	0.4570	2.6771
5	0.2636	0.0695	0.6447	9.2670	0.4175	0.1743	0.5973	3.4258
6	0.2400	0.0576	0.6500	11.2829	0.4044	0.1635	0.6097	3.7291
7	0.2198	0.0483	0.6500	13.4537	0.4073	0.1659	0.6499	3.9161
8	0.2027	0.0411	0.6500	15.8016	0.3995	0.1596	0.6499	4.0713
9	0.1881	0.0354	0.6500	18.3445	0.4009	0.1607	0.6500	4.0446
10	-	-	-	-	0.3950	0.1560	0.6500	4.1653

with zero Dirichlet boundary conditions. Here we only considered $Q_n - Q_{n-2}$ spectral elements. β_Γ is computed as the square root of the minimum nonzero eigenvalue of $C_0^{-1}B_0^TS_\Gamma^{-1}B_0$, where S_Γ and B_0 are the blocks in (30) and C_0 is the coarse pressure mass matrix. The upper plot in Figure 2 shows β_Γ as a function of the spectral degree n while keeping fixed a small number of elements, $N = 2 \times 2 \times 1$. The lower plot in Figure 2 shows β_Γ as a function of the number of spectral elements N for a small fixed spectral degree $n = 2$. Both figures indicate that β_Γ is bounded by a constant independent of N and n, in agreement with Lemma 7.2 and 7.4.

We next report on the local condition numbers and extreme nonzero eigenvalues of $\widehat{S}_\Gamma^{-1}S_\Gamma$ for one interior element. Here S_Γ is the velocity block in the saddle point Schur complement (30) and $\widehat{S}_\Gamma^{-1}$ is the wire basket preconditioner described in Section 8.1 for Stokes problems and in Section 8.3 for mixed elasticity problems. We report only the results obtained with the original wire basket block of the preconditioner.

We consider first $Q_n - Q_{n-2}$ spectral elements. The results are plotted in Figure 3. In both the Stokes and elasticity cases, the incompressible limit is clearly the hardest, yielding condition numbers approximately three times as large as those of the corresponding compressible case. For a given value of ν, the condition number seems to grow linearly with n, which is consistent with our theoretical results in Theorems 8.3 and 8.6, since the theoretical bound for the inf-sup constant for $Q_n - Q_{n-2}$ approaches zero as $1/n$. This is reflected in the decay of the minimum eigenvalue λ_{min}, while the maximum eigenvalue λ_{max} seems to be bounded by a constant independent of n. Even if the asymptotic behavior is the same, the condition numbers for the elasticity problem are always larger than those for the Stokes problem.

We then consider $Q_n - P_{n-1}$ spectral elements. Figure 4 presents the results for the generalized Stokes problem and the mixed elasticity problem, respectively. Figure 5 compares the $Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ results for local condition numbers in the incompressible case. The condition numbers for $Q_n - P_{n-1}$ spectral elements are smaller than the corresponding ones for $Q_n - Q_{n-2}$ spectral elements. Again the incompressible limit is the hardest, yielding condition numbers approximately three times as large as those of the corresponding compressible case. From our theoretical results, the growth of the condition numbers, for a fixed value of ν, should now be only polylogarithmic in n, since the inf-sup constant for $Q_n - P_{n-1}$ spectral elements is uniformly bounded away from zero. More results for higher values of n are needed in order to confirm this theoretical result numerically.

Fig. 3. *Local condition number* κ*, inverse of the minimum nonzero eigenvalue, and maximum eigenvalue of* $\widehat{S}_\Gamma^{-1} S_\Gamma$ *for an interior element (original wire basket block); Stokes problem (left), mixed elasticity (right);* $Q_n - Q_{n-2}$ *method*

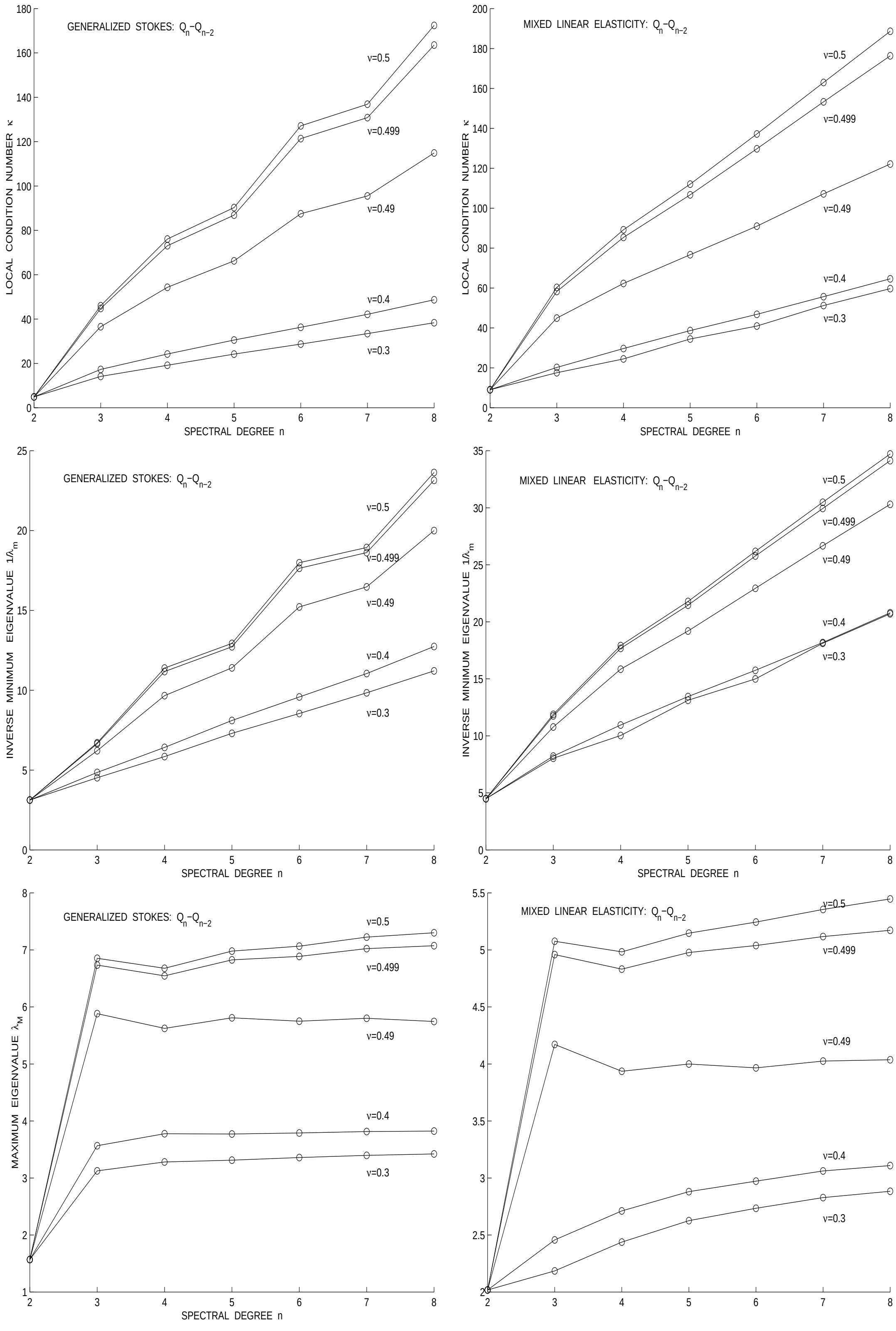


FIG. 4. *Local condition number κ, inverse of the minimum nonzero eigenvalue, and maximum eigenvalue of $\widehat{S}_\Gamma^{-1} S_\Gamma$ for an interior element (original wire basket block); Stokes problem (left), mixed elasticity (right); $Q_n - P_{n-1}$ method*

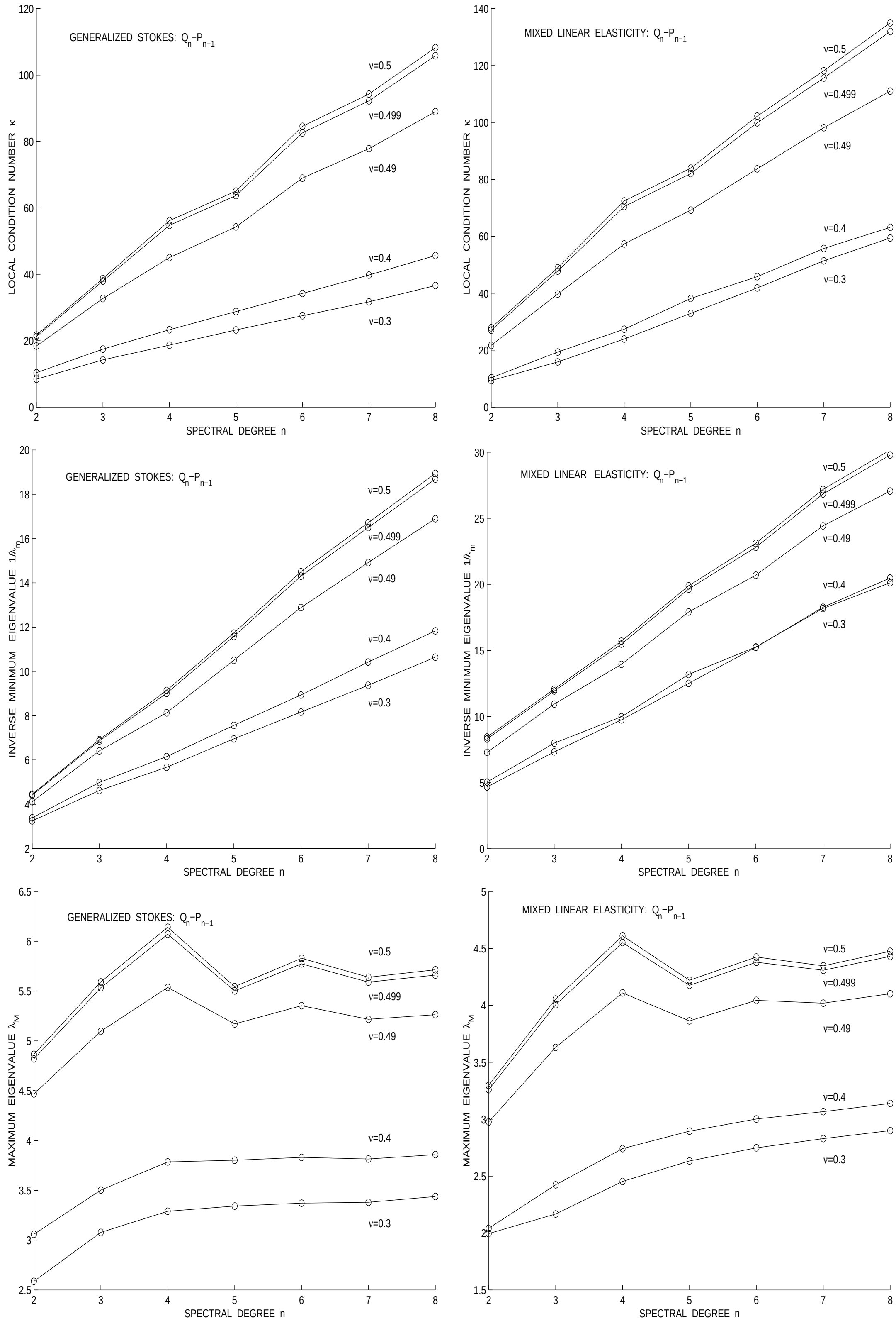
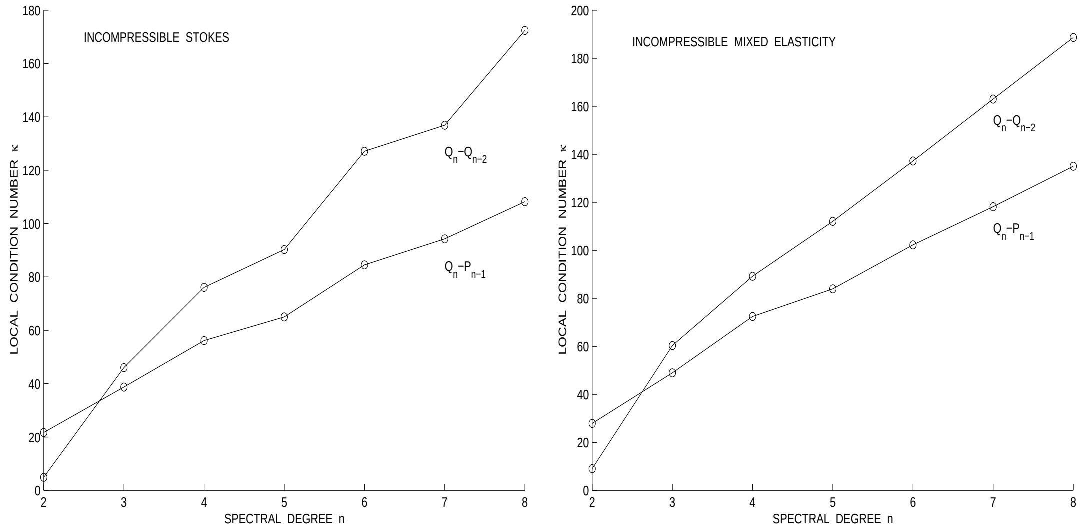


FIG. 5. *Comparison of local condition numbers κ of $\widehat{S}_\Gamma^{-1} S_\Gamma$ for an interior element (original wire basket block) for the incompressible case; Stokes problem (left), mixed elasticity (right)*



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