ITERATIVE SUBSTRUCTURING METHODS FOR SPECTRAL ELEMENT DISCRETIZATIONS OF ELLIPTIC SYSTEMS IN THREE DIMENSIONS.

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Abstract. Spectral element methods are considered for symmetric elliptic systems of second-order partial differential equations, such as the linear elasticity and the Stokes systems in three dimensions. The resulting discrete problems can be positive definite, as in the case of compressible elasticity in pure displacement form, or saddle point problems, as in the case of almost incompressible elasticity in mixed form and Stokes equations. Iterative substructuring algorithms are developed for both cases. They are domain decomposition preconditioners constructed from local solvers for the interior of each element and for each face of the elements and a coarse, global solver related to the wire basket of the elements. In the positive definite case, the condition number of the resulting preconditioned operator is independent of the number of spectral elements and grows at most in proportion to the square of the logarithm of the spectral degree. For saddle point problems, there is an additional factor in the estimate of the condition number, namely, the inverse of the discrete inf-sup constant of the problem.

Key words. linear elasticity, Stokes problem, spectral element methods, mixed methods, preconditioned iterative methods, substructuring, Gauss-Lobatto-Legendre quadrature

AMS(MOS) subject classifications. 65N30, 65N35, 65N55

1. Introduction. The goal of this paper is to formulate and study iterative substructuring methods for symmetric elliptic systems of second-order partial differential equations in three dimensions. Important examples, which are considered in some detail, are the equations of linear elasticity and Stokes. We consider conforming spectral finite element discretizations based on a Galerkin formulation of the problem and Gauss-Lobatto-Legendre quadrature. The resulting discrete systems are either positive definite, as in the case of compressible elasticity in pure displacement form, or of saddle point form, as in the case of almost incompressible elasticity in mixed form and Stokes problems. For these three cases, we introduce iterative substructuring algorithms which extends our earlier work [30, 33] on scalar second-order elliptic equations. We recall that iterative substructuring methods are domain decomposition algorithms, in which we, implicitly, solve a reduced Schur complement system that is obtained by eliminating the variables interior to all the subregions into which the given region has been divided; cf., e.g., Smith, Bjørstad, and Gropp [38] or Dryja, Smith, and Widlund [12]. We consider iterative substructuring methods of wire basket type, where a preconditioner for the Schur complement is built from local solvers for each face (shared by two elements) and a coarse solver related to the wire basket (the union of the edges and

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the vertices of the elements). The main result for positive definite problems is a bound on the condition number of the preconditioned operator, which is independent of the number of spectral elements and is bounded from above by the square of the logarithm of the spectral degree. For saddle point problems, the reduced Schur complement is itself a saddle point problem, involving the interface unknowns and piecewise constant Lagrange multipliers. We then use a Krylov space method with a block-diagonal or block-triangular preconditioner using our wire basket preconditioner for the interface block. The main result for saddle point problems is a bound on the condition numbers of the preconditioned operators, which in this case is the product of a polylogarithmic factor and the inverse of inf-sup constant of the problem. Proofs of our results and additional details will be presented in two articles; see [31, 32].

We note that other iterative substructuring methods have been proposed in recent years. For positive definite systems, see, e.g., Mandel [27, 26], Le Tallec [22], and Farhat and Roux [16] and for saddle point problems, see Bramble and Pasciak [5], Quarteroni [34], Fischer and Rønquist [17], Maday, Meiron, Patera, and Rønquist [24], Rønquist [35], Le Tallec and Patra [23], and Casarin [10]. We also note that alternative iterative methods have been considered for saddle point problems, such as Uzawa's algorithm, multigrid methods, block–diagonal and block-triangular preconditioners; see, e.g., Elman [13, 14], Brenner [6], Klawonn [20], and the references therein.

The rest of the paper is organized as follows. In Section 2, we introduce the three elliptic systems which will serve as model problems throughout the paper: compressible linear elasticity in pure displacement form, incompressible and almost incompressible linear elasticity in mixed form, and the Stokes system. In Section 3, the spectral element discretization of these systems and GLL quadrature are described briefly. In Section 4, we introduce some extension operators from the interface that are needed in the construction of our preconditioners: the discrete harmonic, elastic, Stokes and mixed elastic extensions. An additional extension operator associated with the wire basket is also introduced. In Section 5, we describe our wire basket preconditioner for positive definite systems, both in matrix and variational form, and formulate the main result on the condition number of the preconditioned operator. In Section 6, we turn our attention to saddle point problems, starting with the description of the basic substructuring technique the use of which leads to a saddle point Schur complement. We then study the stability of this Schur complement problem and introduce block preconditioners built on our wire basket preconditioner for the positive definite case. Our main results for both the Stokes and and the incompressible elasticity problems are also formulated. Section 7 concludes the paper with results of some of our numerical experiments for problems in three dimensions.

2. Model elliptic systems. In this section, we will introduce three symmetric elliptic systems: compressible linear elasticity in pure displacement form, incompressible and almost incompressible linear elasticity in mixed form, and the Stokes system. The first is coercive, while the other two provide examples of saddle point problems. We will work with spectral element discretizations of these systems and introduce and study iterative substructuring methods for these concrete cases. However, the same techniques can be applied to other well-posed symmetric elliptic systems as well.

Throughout the paper, we will denote vector quantities by bold face characters.

2.1. Compressible linear elasticity in pure displacement form. Let $\Omega \subset R^3$ be a polyhedral domain, let Γ_0 be a nonempty subset of its boundary, and let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, resulting from a surface force of density $\mathbf{g}$, along $\Gamma_1 = \partial\Omega - \Gamma_0$, and a body force $\mathbf{f}$:

$$(1) \qquad a(\mathbf{u},\mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx + \lambda \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v}\ dx\ = \ <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, and the inner products are defined as

$$\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}), \qquad <\mathbf{F},\mathbf{v}> = \int_\Omega \sum_{i=1}^{3} f_i v_i\ dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i v_i\ ds.$$

This pure displacement model is a good formulation for compressible materials, for which the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is strictly less than 1/2, e.g., $\nu \le 0.4$; see, e.g., Ciarlet [11] for a detailed treatment of nonlinear and linear elasticity.

2.2. Almost incompressible linear elasticity in mixed form. When λ approaches infinity, the pure displacement model describes materials that are almost incompressible. In terms of the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$, such materials are characterized by values of ν close to 1/2. It is well-known that when low order, h-version finite elements are used in the discretization of (1), *locking* can cause a severe deterioration of the convergence rate as $h \to 0$; see, e.g., Babuška and Suri [1]. If the p-version is used instead, locking in $\mathbf{u}$ is eliminated, but it could still occur in quantities of interest such as $\lambda \mathrm{div}\mathbf{u}$. Moreover, the stiffness matrix obtained by discretizing the pure displacement model (1) has a condition number that goes to infinity when $\nu \to 1/2$. Therefore, the convergence rate of any iterative method must also be expected to deteriorate rapidly as the material becomes almost incompressible.

Locking can be eliminated by introducing a space of Lagrange multipliers $U = L^2(\Omega)$ and the new variable $p = -\lambda \mathrm{div}\mathbf{u} \in U$ and by replacing the pure displacement problem with a mixed formulation:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(2) \qquad \begin{cases} 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx & - \int_\Omega \mathrm{div}\mathbf{v}\ p\ dx & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ -\int_\Omega \mathrm{div}\mathbf{u}\ q\ dx & - \frac{1}{\lambda}\int_\Omega pq\ dx & = & 0 & \forall q \in U; \end{cases}$$

see Brezzi and Fortin [7]. Using the notations,

$$e(\mathbf{u},\mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx, \qquad b(\mathbf{v},q) = -\int_\Omega \mathrm{div}\mathbf{v}\ q\ dx, \qquad c(p,q) = \int_\Omega pq\ dx,$$

the problem takes the following form:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(3) \qquad \begin{cases} e(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) & - & \frac{1}{\lambda}c(p,q) & = & 0 & \forall q \in U. \end{cases}$$

When $\lambda \to \infty$ (or, equivalently, $\nu \to 1/2$), we obtain the limiting problem for incompressible linear elasticity; we then simply drop the appropriate term in (3).

2.3. The generalized Stokes system. In case of homogeneous Dirichlet boundary conditions on the whole boundary $\partial\Omega$, problem (2) is equivalent to the following generalized Stokes problem (see Brezzi and Fortin [7]):
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(4) \qquad \left\{ \begin{array}{lclcl} s(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) & - & \frac{1}{\lambda+\mu} c(p,q) & = & 0 \qquad \forall q \in U. \end{array} \right.$$

Here,

$$s(\mathbf{u},\mathbf{v}) = \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \; dx,$$

and U is now defined by

$$U = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\},$$

since it can be shown that the pressure will have a zero mean value as a consequence of $\mathbf{u}$ vanishing on the boundary of Ω. The penalty term in (4) can also originate from stabilization techniques or penalty formulations for Stokes problems. The classical Stokes system, describing the velocity $\mathbf{u}$ and pressure p of a fluid of viscosity μ, can be obtained from (4) by letting $\lambda \to \infty$; again we simply drop one of the terms in formula (4). We refer to Girault and Raviart [18] for an introduction to the Stokes and Navier-Stokes equations and their finite element discretization. See also Yang [40] for an alternative formulation of saddle point problems.

3. Spectral element methods. Let Ω_{ref} be the reference cube $(-1,1)^3$, let $Q_n(\overline{\Omega}_{\text{ref}})$ be the set of polynomials on $\overline{\Omega}_{\text{ref}}$ of degree n in each variable, and let $P_n(\overline{\Omega}_{\text{ref}})$ be the set of polynomials on $\overline{\Omega}_{\text{ref}}$ of total degree n. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an affine image of the reference cube. Thus, $\Omega_i = \phi_i(\Omega_{\text{ref}})$, where ϕ_i is an affine mapping. The displacement is discretized, component by component, by conforming spectral elements, i.e. by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\overline{\Omega}_i} \circ \phi_i \in Q_n(\overline{\Omega}_{\text{ref}}),\ i = 1, \cdots, N, \quad k = 1,2,3\}.$$

The pressure space can be discretized by piecewise polynomials of degree $n-2$:

$$U^n = \{q \in L_0^2(\Omega) : q|_{\Omega_i} \circ \phi_i \in Q_{n-2}(\Omega_{\text{ref}}),\ i = 1, \cdots, N\}.$$

We note that the elements of U^n are discontinuous across the boundaries of the elements Ω_i. This choice for U^n gives us the $Q_n - Q_{n-2}$ method, proposed by Maday, Patera, and Rønquist [25] for the Stokes system; see further Subsection 3.3 for a discussion of the stability of this method.

Another choice of the discrete pressure space is given by piecewise polynomials of total degree $n-1$:

$$\{q \in U : q|_{\Omega_i} \circ F_i \in P_{n-1}(\Omega_{ref}).\ i = 1, \cdots, N\}.$$

This choice has been analyzed in Stenberg and Suri [39] and is known as the $Q_n - P_{n-1}$ method. For P_{n-1} a standard tensorial basis does not exist but other bases, common in the p-version finite element literature, can be used. We will not work extensively with this space in this paper.

Other interesting choices for U^n have been studied in Canuto [8] and Canuto and Van Kemenade [9] in connection with stabilization techniques for spectral elements using bubble functions.

3.1. GLL quadrature. Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points of the reference cube $[-1, 1]^3$, and by σ_i the quadrature weight associated with ξ_i. Let $l_i(x)$ be the Lagrange interpolating polynomial that vanishes at all the GLL nodes except ξ_i where it equals one. The basis functions on the reference cube are then defined by a tensor product as

$$l_i(x)l_j(y)l_k(z), \qquad 0 \leq i, j, k \leq n.$$

This is a nodal basis, since any element of $Q_n(\Omega_{\text{ref}})$ can be written as

$$u(x, y, z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i, \xi_j, \xi_k)l_i(x)l_j(y)l_k(z).$$

The reference element can be decomposed into its interior, six faces, twelve edges, and eight vertices. The union of its edges and vertices is called the wire basket of the element and is denoted by W_{ref}. Analogously, each basis function can be characterized as being of interior, face, edge, or vertex type:

- interior: $i, j, k \neq 0$ and $\neq n$;
- face: exactly one index is 0 or n;
- edge: exactly two indices are 0 and/or n;
- vertex: all three indices are 0 and/or n.

Each component of the displacement model, and generally any element in V^n, can be written as the sum of its interior, face, edge, and vertex components,

$$u = u_I + u_F + u_E + u_V,$$

where each term is expressed in terms of the corresponding set of basis functions.

For the space U^n, we can similarly use the very convenient basis consisting of tensor-product Lagrangian nodal basis functions associated with just the internal GLL nodes; we note that the degree of the polynomials are now $n-2$. Another basis associated with Gauss-Legendre nodes has been considered in [17] and [24].

We now replace each integral of the continuous models (3) and (4) by using GLL quadrature. On Ω_{ref},

$$(u, v)_{n,\Omega_{\text{ref}}} = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i, \xi_j, \xi_k)v(\xi_i, \xi_j, \xi_k)\sigma_i\sigma_j\sigma_k,$$

and in general on Ω

$$(u, v)_{n,\Omega} = \sum_{s=1}^{N}\sum_{i,j,k=0}^{n} (u \circ \phi_s)(\xi_i, \xi_j, \xi_k)(v \circ \phi_s)(\xi_i, \xi_j, \xi_k)|J_s|\sigma_i\sigma_j\sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. The first inner product is uniformly equivalent to the standard L_2-inner product on $Q_n(\Omega_{\text{ref}})$. Thus, it is shown in Bernardi and Maday [3, 4] that

$$\|u\|^2_{L_2(\Omega_{\text{ref}})} \leq (u, u)_{n,\Omega_{\text{ref}}} \leq 27\|u\|^2_{L_2(\Omega_{\text{ref}})} \qquad \forall u \in Q_n(\Omega_{\text{ref}}). \tag{5}$$

These bounds imply an analogous uniform equivalence between the $L_2(\Omega)-$norm (and the $H^1(\Omega)-$seminorm) and the corresponding discrete norm (and seminorm) based on GLL quadrature.

3.2. The discrete problems. Applying GLL quadrature to the pure displacement model (1), we obtain the discrete bilinear form

$$a_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega} + \lambda(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v})_{n,\Omega},$$

and the *discrete elasticity system in pure displacement form*:
Find $\mathbf{u} \in \mathbf{V}^n$ such that

$$a_n(\mathbf{u},\mathbf{v}) = < \mathbf{F}, \mathbf{v} >_{n,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V}^n. \tag{6}$$

An analysis of the spectral element discretization for the Laplacian and Stokes problems can be found in Bernardi and Maday [3, 4] and in Maday, Patera, and Rønquist [25]. The same techniques can be applied to provide an analysis and error estimates for the linear elasticity problem. The stiffness matrix K associated to the discrete problem (6) is symmetric and positive definite. It is less sparse than the stiffness matrices obtained by low-order finite elements, but still well-structured, and the corresponding matrix-vector multiplication is relatively inexpensive if advantage is taken of its tensor product structure; see, e.g., Bernardi and Maday [3].

For an interior element, $a_n(\cdot,\cdot)$ has a six-dimensional null space $\mathcal{N}$, spanned by the rigid body motions $\mathbf{r}_j$:

$$\mathcal{N} = span\{\mathbf{r}_j,\ j = 1, \cdots, 6\}.$$

On Ω_{ref}, the $\mathbf{r}_j$ are given, component-wise, by three translations

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{7}$$

and three rotations

$$\mathbf{r}_4 = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \quad \mathbf{r}_5 = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad \mathbf{r}_6 = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}. \tag{8}$$

It is easy to show that both the divergence and the linearized strain tensor of these six functions vanish.

Applying GLL quadrature to the mixed models (3) and (4), we obtain the discrete bilinear forms

$$e_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega}, \qquad s_n(\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u} : \nabla\mathbf{v})_{n,\Omega},$$

$$b_n(\mathbf{u},p) = -(\mathrm{div}\mathbf{u}, p)_{n,\Omega}, \qquad c_n(p,q) = (p,q)_{n,\Omega}.$$

We note that, since GLL quadrature in each variable is exact for polynomials of degree up to and including $2n-1$ and we are using affine images of the reference cube, the last two bilinear forms are exact, i.e. $b_n(\mathbf{u},p) = b(\mathbf{u},p)$ and $c_n(p,q) = c(p,q), \quad \forall \mathbf{u} \in \mathbf{V}^n, \ p,q \in U^n$.

We can now obtain the *discrete elasticity system in mixed form*:
Find $(\mathbf{u},p) \in \mathbf{V}^n \times U^n$ such that

$$
\text{(9)} \qquad \begin{cases} e_n(\mathbf{u},\mathbf{v}) & + & b_n(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}>_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}^n \\ b_n(\mathbf{u},q) & - & \frac{1}{\lambda} c_n(p,q) & = & 0 & \forall q \in U^n \end{cases}
$$

In the incompressible case, we remove the $c_n(\cdot,\cdot)$ term, since $1/\lambda = 0$.
The *discrete generalized Stokes problem* is an analogous saddle point problem, with $s_n(\cdot,\cdot)$ in place of $e_n(\cdot,\cdot)$ and the penalty parameter equal to $1/(\lambda+\mu)$.

These are all saddle point problems, and they include a penalty term in the elasticity and generalized Stokes case. Using, for simplicity, the same notation for functions and their coefficient vectors, we can write the saddle point problems in matrix form as

$$
\text{(10)} \qquad K \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -t^2 C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix},
$$

where A, B, and C are the matrices associated with $s_n(\cdot,\cdot)$ or $e_n(\cdot,\cdot)$, and with $b_n(\cdot,\cdot)$, and $c_n(\cdot,\cdot)$, respectively. The penalty parameter is $t^2 = \frac{1}{\lambda}$ for elasticity problems and $t^2 = \frac{1}{\lambda+\mu}$ for generalized Stokes problems. The stiffness matrix K is now symmetric and indefinite.

In the following, we will also use $c > 0$ and $C < +\infty$ to denote generic constants in our inequalities; it will be clear from the context if we are referring to generic constants or to the bilinear form $c(\cdot,\cdot)$ and the associated matrix C.

3.3. The inf-sup condition for spectral elements. The convergence of mixed methods depends not only on the approximation properties of the discrete spaces $\mathbf{V}^n$ and U^n, but also on a stability condition known as the inf-sup (or LBB) condition; see, e.g., Brezzi and Fortin [7]. While many important h-version finite elements for Stokes problems satisfy the inf-sup condition with a constant independent of h, several important spectral elements proposed for Stokes problems, such as the $Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ methods, satisfy only the following inf-sup condition:

$$
\text{(11)} \qquad \sup_{\mathbf{v} \in \mathbf{V}^n} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}} \geq C n^{-(\frac{d-1}{2})} \|q\|_{L^2} \quad \forall q \in U^n,
$$

where $d = 2,3$ and the constant C is independent of n and q. This result has been proven for the $Q_n - Q_{n-2}$ method by Maday, Patera, and Rønquist [25] and by Stenberg and Suri [39] for more general discrete mixed spaces. For the $Q_n - Q_{n-2}$ method, an example is also given in [25] showing that the estimate is sharp, i.e. the inf-sup constant indeed approaches zero as $n^{-(d-1)/2}$ $(d = 2,3)$. However, numerical experiments, reported in Maday, Meiron, Patera, and Rønquist, see [24] and [25], have also shown that for practical values of n, e.g., $n \leq 16$, the inf-sup constant β_n of the $Q_n - Q_{n-2}$ method decays much slower than would be expected from the theoretical bound. Our own numerical experiments, reported in [28, 29], indicate that the situation is even better for the $Q_n - P_{n-1}$ method; see further Section 7, in particular Table 2. For numerical studies of the inf-sup constant of various h-version finite elements, see Bathe and Chapelle [2].

We can rewrite the inf-sup condition in matrix form as

$$q^t B A^{-1} B^t q \geq \beta_n^2 q^t C q \quad \forall q \in U^n, \tag{12}$$

where β_n is the inf-sup constant of the method; see Brezzi and Fortin [7]. Therefore β_n^2 scales as $\lambda_{min}(C^{-1}BA^{-1}B^t)$. Similarly, if $\tilde{\beta}$ is the continuity constant of the bilinear form $b(\cdot,\cdot)$, we have

$$\mathbf{v}^t B^t q \leq \tilde{\beta}(q^t C q)^{1/2}(\mathbf{v}^t A \mathbf{v})^{1/2} \quad \forall \mathbf{v} \in \mathbf{V}^n, \forall q \in U^n. \tag{13}$$

From (12) and (13), it follows that

$$\beta_n^2 \leq \frac{q^t B A^{-1} B^t q}{q^t C q} \leq \tilde{\beta}^2 \quad \forall q \in U^n.$$

We remark that the dependence on n of the inf-sup constant implies only a loss (of order $n^{-(d-1)/2}$) in the order of convergence for the pressure p, but not for the velocity $\mathbf{u}$; see the classical error estimates as given in Bernardi and Maday [3, Theorems 2.5 and 7.7] and Stenberg and Suri [39, Theorem 5.2 and Remark 5.3]. Therefore, for problems with regular solutions, for which spectral methods are most appropriate, we still have spectral convergence for both components of the discrete solution.

4. Extensions from the interface. In the construction and analysis of our algorithms, we will need to consider a number of subspaces of the space $\mathbf{V}^n$. Many of them involve extensions into the interior of the elements of the interface values of elements of the spectral finite element space $\mathbf{V}^n$. The interface Γ of the decomposition $\{\Omega_i\}$ of Ω is defined by

$$\Gamma = (\cup_{i=1}^N \partial\Omega_i) \setminus \partial\Omega.$$

The space of restrictions to the interface is defined by

$$\mathbf{V}_\Gamma^n = \{\mathbf{v}|_\Gamma, \quad \mathbf{v} \in \mathbf{V}^n\}.$$

Γ is composed of N_F faces F_k (open sets) of the elements and the wire basket W, defined as the union of the edges and vertices of the elements, i.e.

$$\Gamma = \cup_{k=1}^{N_F} F_k \cup W.$$

We first define local subspaces consisting of elements of V^n with support in the interior of individual elements,

$$\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3, \qquad i = 1, \cdots, N. \tag{14}$$

We will often also use related local subspaces of pressures, with support and zero mean value in individual elements, defined by

$$U_i^n = U^n \cap L_0^2(\Omega_i), \qquad i = 1, \cdots, N. \tag{15}$$

We will now examine several useful ways of extending elements of $\mathbf{V}_\Gamma^n$. These extensions are all constructed locally, i.e. element by element.

4.1. The discrete harmonic extension. The discrete harmonic extension $\mathcal{H}^n : \mathbf{V}_\Gamma^n \longrightarrow \mathbf{V}^n$, is defined as the operator that maps any element $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the unique solution $\mathcal{H}^n\mathbf{u} \in \mathbf{V}^n$ of

$$s_n(\mathcal{H}^n\mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_i^n, \qquad \mathcal{H}^n\mathbf{u} = \mathbf{u} \ \text{ on } \ \partial\Omega_i, \quad i = 1, \cdots, N.$$

This is just an application, for each of the three components separately, of the well-known scalar discrete harmonic extension. As in the scalar case, the discrete harmonic extension satisfies the minimization property

$$s_n(\mathcal{H}^n\mathbf{u}, \mathcal{H}^n\mathbf{u}) = \min_{\mathbf{v}\in\mathbf{V}^n,\ \mathbf{v}|_\Gamma = \mathbf{u}} s_n(\mathbf{v}, \mathbf{v}).$$

4.2. The discrete elastic extension. We can also extend any element of $\mathbf{V}_\Gamma^n$ to the interior of each element by solving a linear elasticity problem in each element. The discrete elastic extension $\mathcal{E}^n : \mathbf{V}_\Gamma^n \longrightarrow \mathbf{V}^n$, is the operator that maps any $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the unique solution of

$$a_n(\mathcal{E}^n\mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_i^n, \qquad \mathcal{E}^n\mathbf{u} = \mathbf{u} \ \text{ on } \ \partial\Omega_i, \quad i = 1, \cdots, N. \tag{16}$$

In our applications to elasticity problems, we will choose the range of this extension operator,

$$\mathbf{V}_\mathcal{E}^n = \mathcal{E}^n(\mathbf{V}_\Gamma^n), \tag{17}$$

as the subspace of interface displacements. The elements in this subspace are completely determined by their values on Γ.

The discrete elastic extension satisfies the minimization property

$$a_n(\mathcal{E}^n\mathbf{u}, \mathcal{E}^n\mathbf{u}) = \min_{\mathbf{v}\in\mathbf{V}^n,\ \mathbf{v}|_\Gamma = \mathbf{u}} a_n(\mathbf{v}, \mathbf{v}).$$

4.3. The discrete Stokes extension. We can also extend any element of $\mathbf{V}_\Gamma^n$ to the interior of each element by solving a Stokes problem in each element. The discrete Stokes extension $(\mathcal{S}^n, \mathcal{S}_p^n) : \mathbf{V}_\Gamma^n \longrightarrow \mathbf{V}^n \times U^n$, is the operator that maps any $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the solution of the following Stokes problem on each element:
Find $\mathcal{S}^n\mathbf{u} \in \mathbf{V}^n$ and $\mathcal{S}_p^n\mathbf{u} \in (\sum_{i=1}^N U_i^n)$ such that on each Ω_i

$$\begin{cases} s_n(\mathcal{S}^n\mathbf{u}, \mathbf{v}) + b_n(\mathbf{v}, \mathcal{S}_p^n\mathbf{u}) = 0 & \forall \mathbf{v} \in \mathbf{V}_i^n \\ b_n(\mathcal{S}^n\mathbf{u}, q) = 0 & \forall q \in U_i^n \\ \mathcal{S}^n\mathbf{u} = \mathbf{u} \quad \text{on} \quad \partial\Omega_i \end{cases} \tag{18}$$

In our applications to Stokes problems, we will choose the range of this extension operator,

$$\mathbf{V}_\mathcal{S}^n = \mathcal{S}^n(\mathbf{V}_\Gamma^n), \tag{19}$$

as the subspace of interface velocities. As with the discrete harmonic extension, the velocities in this subspace are completely determined by their values on Γ.

The discrete Stokes extension satisfies the minimization property

$$s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) = \min_{\mathbf{v}|_\Gamma=\mathbf{u}} s_n(\mathbf{v},\mathbf{v}) \qquad \forall \mathbf{v} \in \{\mathbf{v} \in \mathbf{V}^n \ : \ b_n(\mathbf{v}, q) = 0 \ \ \forall q \in \sum_{i=1}^N U_i^n\}.$$

The following comparison of the energy of the discrete Stokes and harmonic extensions can be found in [18], [5], [23], and [10].

LEMMA 4.1.

$$c\beta_n s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) \le s_n(\mathcal{H}^n\mathbf{u}, \mathcal{H}^n\mathbf{u}) \le s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{u}) \qquad \forall \mathbf{u} \in \mathbf{V}^n_\Gamma.$$

4.4. The discrete mixed elastic extension. We can also extend any element of $\mathbf{V}^n_\Gamma$ to the interior of each element by solving an incompressible linear mixed form elasticity problem in each element. The discrete elastic extension $(\mathcal{M}^n, \mathcal{M}^n_p) : \mathbf{V}^n_\Gamma \longrightarrow \mathbf{V}^n \times U^n$, is the operator that maps any $\mathbf{u} \in \mathbf{V}^n_\Gamma$ into the solution of the following incompressible elasticity problem:
Find $\mathcal{M}^n\mathbf{u} \in \mathbf{V}^n$ and $\mathcal{M}^n_p\mathbf{u} = p \in (\sum_{i=1}^N U_i^n)$ such that on each Ω_i

$$(20) \qquad \begin{cases} e_n(\mathcal{M}^n\mathbf{u}, \mathbf{v}) \ + \ b_n(\mathbf{v}, p) \ = \ 0 & \forall \mathbf{v} \in \mathbf{V}_i^n \\ b_n(\mathcal{M}^n\mathbf{u}, q) \ = \ 0 & \forall q \in U_i^n \\ \mathcal{M}^n\mathbf{u} = \mathbf{u} \quad \text{on} \quad \partial\Omega_i & \end{cases}$$

In our applications to elasticity problems, we will choose the range of this extension operator,

$$(21) \qquad \mathbf{V}^n_{\mathcal{M}} = \mathcal{M}^n(\mathbf{V}^n_\Gamma),$$

as our subspace of interface displacements. As with the other extensions, the displacements in this subspace are completely determined by their values on Γ.

The discrete elastic extension satisfies the minimization property

$$e_n(\mathcal{M}^n\mathbf{u}, \mathcal{M}^n\mathbf{u}) = \min_{\mathbf{v}|_\Gamma=\mathbf{u}} e_n(\mathbf{v},\mathbf{v}) \qquad \forall \mathbf{v} \in \{\mathbf{v} \in \mathbf{V}^n \ : \ b_n(\mathbf{v}, q) = 0 \ \ \forall q \in \sum_{i=1}^N U_i^n\}.$$

4.5. Extension from the wire basket. In the construction of our algorithm, we will also need to extend the restriction of elements of $\mathbf{V}^n$ to the wire basket to the faces. As this is also a local operation, we can restrict our attention to the reference element. A preliminary extension operator $\tilde{I}^W$ from the wire basket is constructed for any function $\mathbf{u} \in \mathbf{V}^n$ by expanding its restriction to the wire basket, using the vertex and edge basis functions described in Subsection 3.1,

$$\tilde{I}^W\mathbf{u} = \mathbf{u}_V + \mathbf{u}_E.$$

Given that we are using a nodal basis, $\tilde{I}^W\mathbf{u}$ will simply vanish at all the face GLL points outside the wire basket. Therefore this extension operator does not preserve the rigid body motions $\mathbf{r}_j, j = 1, \cdots, 6$. In order to construct a scalable algorithm, we must define an extension operator that satisfies this condition on the interface; see Mandel

[26] and or Smith, Bjørstad, and Gropp, [38, p. 132] for a discussion of this *null space property.*

We start by considering the difference between each of the $\mathbf{r}_j$ and the function obtained by using the preliminary extension. They can all be expressed in terms of four scalar functions, defined on each face in terms of

$$\mathcal{F}^0 = 1 - \tilde{I}^W 1, \quad \mathcal{F}^1 = x_1 - \tilde{I}^W x_1, \quad \mathcal{F}^2 = x_2 - \tilde{I}^W x_2, \quad \mathcal{F}^3 = x_3 - \tilde{I}^W x_3.$$

We remark that in our previous study of the scalar case, see [30, 33], only $\mathcal{F}^0$ was needed, because the null space of the discrete bilinear form on an interior element consists only of constants. Each of our four functions, just defined, vanishes on the wire basket and each can be split into six face terms,

$$\mathcal{F}^0 = \sum_{k=1}^{6} \mathcal{F}^0_k, \qquad \mathcal{F}^1 = \sum_{k=1}^{6} \mathcal{F}^1_k, \qquad \mathcal{F}^2 = \sum_{k=1}^{6} \mathcal{F}^2_k, \qquad \mathcal{F}^3 = \sum_{k=1}^{6} \mathcal{F}^3_k.$$

Here, the $\mathcal{F}^j_k, j = 0, 1, 2, 3$, vanish on all faces except F_k. For each scalar component $u^{(i)}$ of $\mathbf{u}$, we define a new extension $I^W u^{(i)}$ from the wire basket to the interface as follows: On a face F_k, for which the two relevant variables are x_1 and x_2, the restriction of $I^W u^{(i)}$ to F_k has the form

$$I^W u^{(i)} = \tilde{I}^W u^{(i)} + a_k \mathcal{F}^0_k + b^1_k \mathcal{F}^1_k + b^2_k \mathcal{F}^2_k. \tag{22}$$

The weights a_k, b^1_k, b^2_k, and b^3_k are given by the following moments (the factors $\frac{1}{8}$ and $\frac{3}{16}$ come from the fact that we work on the reference element):

$$a_k = \frac{(u^{(i)}, 1)_{n,\partial F_k}}{(1, 1)_{n,\partial F_k}} = \frac{1}{8}(u^{(i)}, 1)_{n,\partial F_k},$$

$$b^j_k = \frac{(u^{(i)}, x_j)_{n,\partial F_k}}{(x_j, x_j)_{n,\partial F_k}} = \frac{3}{16}(u^{(i)}, x_j)_{n,\partial F_k}, \qquad j = 1, 2, 3.$$

We note that on each face only three correction terms are used; see (22). For a vector valued displacement $\mathbf{u}$, the extension operator is then defined as the discrete elastic extension of the scalar face functions given by (22), i.e.

$$I^W \mathbf{u} = \mathcal{E}^n(I^W u^{(1)}, I^W u^{(2)}, I^W u^{(3)}).$$

A simple computation shows that, on each face, the new extension operator reproduces all P_1 polynomials and therefore also all the rigid body motions. If, e.g., $u = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3$, we have on the face $F_k = \{x_3 = 1\}$,

$$a_k = \frac{1}{8}(c_0 + c_1 x_1 + c_2 x_2 + c_3, 1)_{n,\partial F_k} = c_0 + c_3,$$

$$b^1_k = \frac{3}{16}(c_0 + c_1 x_1 + c_2 x_2 + c_3, x_1)_{n,\partial F_k} = c_1,$$

$$b^2_k = \frac{3}{16}(c_0 + c_1 x_1 + c_2 x_2 + c_3, x_2)_{n,\partial F_k} = c_2,$$

as required. Moreover, any rigid body motion $\mathbf{r}$ is also reproduced inside each element, i.e. $\mathcal{E}^n \mathbf{r} = \mathbf{r}$. This follows from the minimization property of the elastic extension and the fact that $a_n(\mathbf{r}, \mathbf{r}) = 0$. Therefore, $I^W \mathbf{r} = \mathbf{r} \;\; \forall \mathbf{r} \in \mathcal{N}$. We note that the extension operator I^W defines a change of basis in $\mathbf{V}^n_\Gamma$; the face basis functions are unchanged, but the wire basket basis functions are transformed according to (22).

5. A wire basket preconditioner for the pure displacement model. In this section, we describe our wire basket preconditioner for linear elasticity problems in pure displacement form. We first write it in matrix form and we then outline the main ideas involved in its analysis, based on the standard Schwarz framework. We model the wire basket preconditioner on our previous work on the scalar case; see [30, 33].

5.1. Matrix form of the preconditioner. The stiffness matrix K of the discrete linear elasticity problem (6) is built by subassembly from the individual contributions from each element Ω_i,

$$\mathbf{u}^T K \mathbf{u} = \sum_{i=1}^{N} \mathbf{u}^{(i)^T} K^{(i)} \mathbf{u}^{(i)}.$$

In each element, we order the interior variables first and then the interface variables obtaining local stiffness matrices of the form

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{I\Gamma}^{(i)} \\ K_{I\Gamma}^{(i)^T} & K_{\Gamma\Gamma}^{(i)} \end{bmatrix}.$$

The interior unknowns are eliminated by solving local linear elasticity problems, obtaining local Schur complement matrices

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{I\Gamma}^{(i)^T} K_{II}^{(i)^{-1}} K_{I\Gamma}^{(i)}.$$

The global Schur complement can also be built by subassembly from the local contributions

$$\mathbf{u}_\Gamma^T S \mathbf{u}_\Gamma = \sum_{i=1}^{N} \mathbf{u}_\Gamma^{(i)^T} S^{(i)} \mathbf{u}_\Gamma^{(i)}. \tag{23}$$

We solve the interface problem, with the coefficient matrix S, using a preconditioned Krylov space method, such as CG. We can then avoid forming S explicitly, since only the matrix-vector product $S\mathbf{v}$ is needed and this product can be evaluated by solving N local linear elasticity problems.

We now introduce a wire basket preconditioner $\widehat{S}$ for S, based on the solution of local problems for each face and a coarse, global problem associated with the wire basket. If the interface unknowns are ordered by placing the face variables first, and then the wire basket variables, the local Schur complements can be written as

$$S^{(i)} = \begin{bmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ S_{FW}^{(i)^T} & S_{WW}^{(i)} \end{bmatrix}.$$

We then perform a change of basis in the space spanned by the wire basket functions in order to satisfy the null space property, i.e. in order to ensure that the null space of the local contribution $\widehat{S}^{(i)}$ to the preconditioner is the space of rigid body motions $\mathcal{N}$. This can be done by using the extension operator I^W defined by (22), since I^W reproduces the rigid body motions. In matrix form, this change of basis is represented locally by the transformation matrix

$$\begin{bmatrix} I_{FF}^{(i)} & 0 \\ R^{(i)} & I_{WW}^{(i)} \end{bmatrix},$$

where the $I^{(i)}$ are identity matrices of appropriate order. Then $S^{(i)}$ is transformed into

$$\begin{bmatrix} I_{FF}^{(i)} & 0 \\ R^{(i)} & I_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ S_{FW}^{(i)\,T} & S_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} I_{FF}^{(i)} & R^{(i)T} \\ 0 & I_{WW}^{(i)} \end{bmatrix} = \begin{bmatrix} S_{FF}^{(i)} & \text{nonzero} \\ \text{nonzero} & \widetilde{S}_{WW}^{(i)} \end{bmatrix}.$$

The local preconditioner $\widehat{S}^{(i)}$ is constructed by

a) eliminating the coupling between faces and the wire basket;

b) eliminating the coupling between all pairs of faces, i.e. by replacing $S_{FF}^{(i)}$ by its block-diagonal part $\widehat{S}_{FF}^{(i)}$;

c) replacing the wire basket block $\widetilde{S}_{WW}^{(i)}$ by a simpler matrix $\widehat{S}_{WW}^{(i)}$: Let $M^{(i)}$ be the mass matrix of the local wire basket $W^{(i)}$, defined by $\mathbf{u}^T M^{(i)} \mathbf{u} = (\mathbf{u}, \mathbf{u})_{n,W^{(i)}}$. We replace $\widetilde{S}_{WW}^{(i)}$ by a scaled rank-six perturbation of $M^{(i)}$. On the reference element,

$$\widehat{S}_{WW}^{(i)} = (1 + \log n)(M^{(i)} - \sum_{j=1}^{6} \frac{(M^{(i)}\mathbf{r}_j)(M^{(i)}\mathbf{r}_j)^T}{\mathbf{r}_j^T M^{(i)} \mathbf{r}_j}). \tag{24}$$

This corresponds to using a simpler, approximate solver for the wire basket variables; see the next subsection for further details. We finally return to the original basis:

$$\widehat{S}^{(i)} = \begin{bmatrix} I_{FF}^{(i)} & 0 \\ -R^{(i)} & I_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} \widehat{S}_{FF}^{(i)} & 0 \\ 0 & \widehat{S}_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} I_{FF}^{(i)} & -R^{(i)T} \\ 0 & I_{WW}^{(i)} \end{bmatrix}. \tag{25}$$

The action of $R^{(j)}$ and $R^{(i)}$ on a face shared by two elements Ω_j and Ω_i is the same, because the extension of any function defined on the wire basket to a face, using the operator I^W, is determined solely by the values on the boundary of that face. Therefore the preconditioner can be obtained by subassembly

$$\widehat{S} = \begin{bmatrix} I_{FF} & 0 \\ -R & I_{WW} \end{bmatrix} \begin{bmatrix} \widehat{S}_{FF} & 0 \\ 0 & \widehat{S}_{WW} \end{bmatrix} \begin{bmatrix} I_{FF} & -R^T \\ 0 & I_{WW} \end{bmatrix},$$

and

$$\widehat{S}^{-1} S = R_0 \widehat{S}_{WW}^{-1} R_0^T S + \sum_k R_{F_k} \widehat{S}_{F_k F_k}^{-1} R_{F_k}^T S,$$

with $R_0 = (R, I_{WW})$; see Dryja, Smith, and Widlund [12]. We have thus obtained an additive preconditioner, with independent parts associated with each face and the wire basket. Multiplicative and hybrid variants can also be defined and analyzed in a completely routine way once that the analysis of the additive method has been completed; see, e.g., Smith, Bjørstad, and Gropp [38].

5.2. Variational formulation and the main result. Working inside the standard Schwarz framework, see, e.g., Smith, Bjørstad, and Gropp [38], we define an iterative substructuring method by first decomposing the space $\mathbf{V}^n$ into subspaces associated with the interiors and a space associated with the interface, which, in turn, is further decomposed:

$$\mathbf{V}^n = \sum_{i=1}^{N} \mathbf{V}_i^n + \mathbf{V}_{\mathcal{E}}^n.$$

Here $\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3$ are the interior spaces and $\mathbf{V}_{\mathcal{E}}^n = \mathcal{E}^n(\mathbf{V}_\Gamma^n)$ the interface space defined in (17). It is easy to see that

$$a_n(\mathcal{E}^n \mathbf{u}_\Gamma, \mathcal{E}^n \mathbf{u}_\Gamma) = \mathbf{u}_\Gamma^T S \mathbf{u}_\Gamma,$$

where S is the Schur complement defined in (23). Our wire basket method is defined by the following decomposition of the interface space:

$$\mathbf{V}_{\mathcal{E}}^n = \mathbf{V}_0 + \sum_k \mathbf{V}_{F_k}^n,$$

where

$$\mathbf{V}_0 = range(I^W)$$

is the wire basket space consisting of discrete elastic extensions of elements of the restriction of $\mathbf{V}^n$ to the wire basket. The extension to the faces is determined using the interpolation operator I^W given in (22). The others, the face spaces, are defined by

$$\mathbf{V}_{F_k}^n = \{\mathbf{v} \in \mathbf{V}^n : \mathbf{v} = \mathcal{E}^n \mathbf{w},\ \ \mathbf{w} \in \mathbf{V}_\Gamma^n \ \text{ with } \ \mathbf{w} = 0 \ \text{ on } \ \Gamma \setminus F_k\}$$

and consist of elements of $\mathbf{V}^n$ which are elastic extensions of polynomials associated with individual faces.

We now define a projection-like operator for V_0 and a projection for each of the face subspaces:

$$T_0 : \mathbf{V}_{\mathcal{E}}^n \to \mathbf{V}_0 \qquad \text{by} \qquad \tilde{a}_0(T_0\mathbf{u}, \mathbf{v}) = a_n(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0,$$

$$T_{F_k} : \mathbf{V}_{\mathcal{E}}^n \to \mathbf{V}_{F_k}^n \qquad \text{by} \qquad a_n(T_{F_k}\mathbf{u}, \mathbf{v}) = a_n(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_{F_k}^n.$$

On the wire basket space $\mathbf{V}_0$, we use the special bilinear form

$$\tilde{a}_0(\mathbf{u}, \mathbf{u}) = (1 + \log n) \sum_{i=1}^N \inf_{c_{ij}} \|\mathbf{u} - \sum_{j=1}^6 c_{ij} \mathbf{r}_j\|_{n, W^{(i)}}^2,$$

which leads to a simplified solver for this space, constructed from the matrix $\widehat{S}_{WW}^{(i)}$ defined in (24). This can be seen by a computation analogous to that in the scalar case. In fact, the minimizing c_{ij} are given by

$$c_{ij} = \frac{(\mathbf{u}, \mathbf{r}_j)_{n, W_{\text{ref}}}}{(\mathbf{r}_j, \mathbf{r}_j)_{n, W_{\text{ref}}}}, \tag{26}$$

on the reference element. When deriving this formula, we use the fact that the $\mathbf{r}_j$ are L^2-orthogonal on W_{ref}. Therefore,

$$\inf_{c_{ij}} \|\mathbf{u} - \sum_{j=1}^6 c_{ij}\mathbf{r}_j\|_{n, W^{(i)}}^2 = (\mathbf{u}, \mathbf{u})_{n, W^{(i)}} - \sum_{j=1}^6 \frac{(\mathbf{u}, \mathbf{r}_j)_{n, W^{(i)}}^2}{(\mathbf{r}_j, \mathbf{r}_j)_{n, W^{(i)}}} \tag{27}$$

$$= \mathbf{u}^T (M^{(i)} - \sum_{j=1}^6 \frac{(M^{(i)}\mathbf{r}_j)(M^{(i)}\mathbf{r}_j)^T}{\mathbf{r}_j^T M^{(i)} \mathbf{r}_j}) \mathbf{u} = \mathbf{u}^T \widehat{S}_{WW}^{(i)} \mathbf{u}.$$

We are now ready to define the additive Schwarz operator by

$$T = T_0 + \sum_{F_k} T_{F_k},$$

and to formulate the main result for the displacement model; a proof of this result is given in [31].

THEOREM 5.1. *The condition number of the iteration operator T is bounded by*

$$cond(T) \leq C(1 + \log n)^2,$$

where C is a constant independent of n and N.

By explicitly computing the matrix form of the operators T_0 and T_{F_k}, we see that the matrix form of the operator T is given by $\widehat{S}^{-1}S$. Therefore, Theorem 5.1 provides a polylogarithmic bound on $cond(\widehat{S}^{-1}S)$.

6. Iterative substructuring methods for saddle point problems. We now turn our attention to the mixed formulation of the elasticity and Stokes problems, i.e. to the discrete saddle point problems (9). We start by describing how to eliminate the interior unknowns in our saddle point problems. The remaining interface unknowns and constant pressures in each spectral element satisfy a reduced saddle point problem, analogous to the Schur complement in the positive definite case. This process is the starting point of several substructuring methods for Stokes problems; see Bramble and Pasciak [5] for the case of the h-version finite elements, Le Tallec and Patra [23] for $h-p$-version finite elements, and Casarin [10] for spectral elements. The following description applies to both generalized Stokes and almost incompressible elasticity problems, but for simplicity we consider only the incompressible Stokes case and adopt the Stokes terminology (velocity and pressure).

The velocity space $\mathbf{V}^n$ is decomposed as

$$\mathbf{V}^n = \mathbf{V}_1^n + \mathbf{V}_2^n + \cdots + \mathbf{V}_N^n + \mathbf{V}_{\mathcal{S}}^n,$$

where the local spaces $\mathbf{V}_i^n$ have been defined in (14) and the interface space $\mathbf{V}_{\mathcal{S}}^n$ has been defined in (19). In the elasticity case the interface space is $\mathbf{V}_{\mathcal{M}}^n$ and has been defined in (21)). The pressure space U^n is decomposed as

$$U^n = U_1^n + U_2^n + \cdots + U_N^n + U_0,$$

where the local spaces U_i^n have been defined in (15) and

$$U_0 = \{q \in U^n : q|_{\Omega_i} = \text{ constant, } i = 1, \cdots, N\}$$

consists of piecewise constant pressures in each element. The vector of unknowns is now reordered placing first the interior unknowns, element by element, and then the interface velocities and the piecewise constant pressures in each element:

$$(\mathbf{u}, p)^T = (\mathbf{u}_1 \; p_1, \mathbf{u}_2 \; p_2, \; \cdots, \; \mathbf{u}_N \; p_N, \; \mathbf{u}_\Gamma \; p_0)^T.$$

After this reordering, our saddle point problem (10) has the following matrix structure:

$$
(28) \qquad \begin{bmatrix} A_{11} & B_{11}^T & \cdots & 0 & 0 & A_{1\Gamma} & 0 \\ B_{11} & 0 & \cdots & 0 & 0 & B_{1\Gamma} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{NN} & B_{NN}^T & A_{N\Gamma} & 0 \\ 0 & 0 & \cdots & B_{NN} & 0 & B_{N\Gamma} & 0 \\ A_{\Gamma 1} & B_{1\Gamma}^T & \cdots & A_{\Gamma N} & B_{N\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & \cdots & 0 & 0 & B_0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ p_1 \\ \vdots \\ \mathbf{u}_N \\ p_N \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ 0 \\ \vdots \\ \mathbf{b}_N \\ 0 \\ \mathbf{b}_\Gamma \\ 0 \end{bmatrix}
$$

The leading block of this matrix is the direct sum of N local saddle point problems for the interior velocities and pressures $(\mathbf{u}_i, p_i)$. In addition there is a diagonal block representing a reduced saddle point problem for the interface velocities and piecewise constant pressures $(\mathbf{u}_\Gamma, p_0)$. These subsystems are given by

$$
(29) \qquad \left\{ \begin{array}{lclcl} A_{ii}\mathbf{u}_i & + & B_{ii}^T p_i & = & \mathbf{b}_i - A_{i\Gamma}\mathbf{u}_\Gamma \\ B_{ii}\mathbf{u}_i & & & = & -B_{i\Gamma}\mathbf{u}_\Gamma \end{array} \right. \qquad i = 1, 2, \cdots, N,
$$

and

$$
(30) \quad \left\{ \begin{array}{lcl} A_{\Gamma\Gamma}\mathbf{u}_\Gamma + A_{\Gamma 1}\mathbf{u}_1 + \cdots + A_{\Gamma N}\mathbf{u}_N + B_{1\Gamma}^T p_1 + \cdots + B_{N\Gamma}^T p_N + B_0^T p_0 & = & \mathbf{b}_\Gamma \\ B_0\mathbf{u}_\Gamma & = & 0 \end{array} \right.
$$

The local saddle point problems (29) are uniquely solvable because the local pressures are constrained to have zero mean value. The reduced saddle point problem (30) can be written more clearly by introducing the linear operators R_i^b, R_i^Γ and P_i^b, P_i^Γ representing the solutions of the $i-$th local saddle point problem:

$$
\mathbf{u}_i = R_i^b \mathbf{b}_i + R_i^\Gamma \mathbf{u}_\Gamma, \qquad p_i = P_i^b \mathbf{b}_i + P_i^\Gamma \mathbf{u}_\Gamma, \qquad i = 1, 2, \cdots, N.
$$

Then (30) can be rewritten as

$$
(31) \qquad \left\{ \begin{array}{lclcl} S_\Gamma \mathbf{u}_\Gamma & + & B_0^T p_0 & = & \tilde{\mathbf{b}}_\Gamma \\ B_0 \mathbf{u}_\Gamma & & & = & 0, \end{array} \right.
$$

where

$$
S_\Gamma = A_{\Gamma\Gamma} + \sum_{i=1}^N A_{\Gamma i} R_i^\Gamma + \sum_{i=1}^N B_{i\Gamma}^T P_i^\Gamma, \qquad \tilde{\mathbf{b}}_\Gamma = \mathbf{b}_\Gamma - \sum_{i=1}^N A_{\Gamma i} R_i^b \mathbf{b}_i - \sum_{i=1}^N B_{i\Gamma}^T P_i^b \mathbf{b}_i.
$$

As always, the matrices R_i^b, R_i^Γ and P_i^b, P_i^Γ need not be assembled explicitly; their action on given vectors is computed by solving the corresponding local saddle point problem. Analogously, S_Γ need not be assembled, since its action on a given vector can be computed by solving the N local saddle point problems (29) with $\mathbf{b}_i = 0$. The right-hand side $\tilde{\mathbf{b}}_\Gamma$ is formed from an additional set of solutions of the N local saddle point problems (29) with $\mathbf{u}_\Gamma = 0$.

We solve the saddle point Schur complement system (31) by some preconditioned Krylov space method such as PCR if we use a symmetric positive definite preconditioner or GMRES if we use a more general preconditioner.

6.1. Stability of the saddle point Schur complement. We now study the inf-sup constant β_Γ of the saddle point Schur complement (31).

The Stokes problem. A proof that that problem (31) is uniformly stable, i.e. that it satisfies an inf-sup condition with a constant β_Γ bounded away from zero independently of n and N, is given in [32]. We remark that Bramble and Pasciak [5] have established the same type of result for (31) for h-version finite elements. However, their proof bounds β_Γ in terms of the inf-sup constant of the original system (in our case β_n), which would lead to a nonuniform bound in the spectral element case, since β_n approaches zero when n increases. In our proof, we first give a variational formulation of the saddle point Schur complement (31).

LEMMA 6.1. *The variational form of the saddle point Schur complement (31) is: Find $\mathcal{S}^n\mathbf{u} \in \mathcal{S}^n(\mathbf{V}_\Gamma^n)$ and $p_0 \in U_0$ such that*

$$(32) \quad \begin{cases} s_n(\mathcal{S}^n\mathbf{u}, \mathcal{S}^n\mathbf{v}) + b_n(\mathcal{S}^n\mathbf{v}, p_0) = \, < \tilde{\mathbf{F}}, \mathbf{v} >_{n,\Omega} & \forall \mathcal{S}^n\mathbf{v} \in \mathcal{S}^n(\mathbf{V}_\Gamma^n) \\ b_n(\mathcal{S}^n\mathbf{u}, q_0) = 0 & \forall q_0 \in U_0 \end{cases}$$

The following result is also proven in [32].

LEMMA 6.2.

$$\sup_{\mathcal{S}^n\mathbf{v} \in \mathbf{V}_S^n} \frac{(\mathrm{div}\mathcal{S}^n\mathbf{v}, q_0)^2}{s_n(\mathcal{S}^n\mathbf{v}, \mathcal{S}^n\mathbf{v})} \geq \beta_\Gamma^2 ||q_0||_{L^2}^2 \qquad \forall q_0 \in U^0,$$

where β_Γ is independent of q_0, n, and N.

Incompressible elasticity. The following lemma is the analog of Lemma 6.1 for incompressible elasticity problems. It can be proved in the same way substituting $e_n(\cdot,\cdot)$ for $s_n(\cdot,\cdot)$ and using the definition (20) of the discrete mixed elastic extension.

LEMMA 6.3. *The variational form of the saddle point Schur complement (31) is: Find $\mathcal{M}^n\mathbf{u} \in \mathcal{M}^n(\mathbf{V}^n)$ and $p_0 \in U_0$ such that*

$$(33) \quad \begin{cases} e_n(\mathcal{M}^n\mathbf{u}, \mathcal{M}^n\mathbf{v}) + b_n(\mathcal{M}^n\mathbf{v}, p_0) = \, < \tilde{\mathbf{F}}, \mathbf{v} >_{n,\Omega} & \forall \mathcal{M}^n\mathbf{v} \in \mathcal{M}^n(\mathbf{V}^n) \\ b_n(\mathcal{M}^n\mathbf{u}, q_0) = 0 & \forall q_0 \in U_0 \end{cases}$$

We can also prove a uniform bound on the inf-sup constant of this saddle point Schur complement for incompressible elasticity, using the bound for the Stokes case given in Lemma 6.2; see [32] for a proof.

LEMMA 6.4.

$$\sup_{\mathcal{M}^n\mathbf{v} \in \mathbf{V}_\mathcal{M}^n} \frac{(\mathrm{div}\mathcal{M}^n\mathbf{v}, q_0)^2}{e_n(\mathcal{M}^n\mathbf{v}, \mathcal{M}^n\mathbf{v})} \geq \beta_\Gamma^2 ||q_0||_{L^2}^2 \qquad \forall q_0 \in U^0,$$

where β_Γ is independent of q_0, n, and N.

6.2. Block preconditioners for the saddle point Schur complement. Block preconditioners for saddle point problems have been studied by Rusten and Winther [36], Silvester and Wathen [37], Elman and Silvester [15], and Klawonn [21, 19, 20]. Here, we follow Klawonn's approach.

Let S be the coefficient matrix of the reduced saddle point problem (31)

$$(34) \qquad S = \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix}.$$

We will consider the following block-diagonal and lower block-triangular preconditioners (an upper block-triangular preconditioner could be considered as well):

$$\widehat{D} = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ 0 & \widehat{C}_0 \end{bmatrix} \qquad \widehat{T} = \begin{bmatrix} \widehat{S}_\Gamma & 0 \\ B_0 & -\widehat{C}_0 \end{bmatrix},$$

where $\widehat{S}_\Gamma$ and $\widehat{C}_0$ are good preconditioners for S_Γ and the coarse pressure mass matrix C_0, respectively:

Assumption 1 : $\exists$ constants $a_0, a_1 > 0$ such that

$$a_0^2 \mathbf{v}^t \widehat{S}_\Gamma \mathbf{v} \leq \mathbf{v}^t S_\Gamma \mathbf{v} \leq a_1^2 \mathbf{v}^t \widehat{S}_\Gamma \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}_\Gamma^n;$$

Assumption 2 : $\exists$ constants $m_0, m_1 > 0$ such that

$$m_0^2 q^t \widehat{C}_0 q \leq q^t C_0 q \leq m_1^2 q^t \widehat{C}_0 q \quad \forall q \in U_0.$$

We will denote by D and T the operators with exact blocks $\widehat{S}_\Gamma = S_\Gamma$ and $\widehat{C} = C$. With the block-diagonal preconditioner $\widehat{D}$, we can use the preconditioned conjugate residual method (PCR). In the block-triangular case, $\widehat{T}$ is no longer symmetric and we need to use a Krylov space method for nonsymmetric systems, such as GMRES or QMR.

Under Assumptions 1 and 2, we obtain the following convergence bounds; cf. Klawonn [21, 19, 20].

THEOREM 6.5. *(Block-diagonal preconditioner)*

$$cond(\widehat{D}^{-1}S) \leq \frac{max\{a_1^2, m_1^2\}}{min\{a_0^2, m_0^2\}}\; cond(D^{-1}S)$$

and

$$cond(D^{-1}S) \leq \frac{1/2 + \sqrt{\beta_1^2 + 1/4}}{-1/2 + \sqrt{\beta_\Gamma^2 + 1/4}},$$

where β_Γ is the inf-sup constant of the reduced saddle point problem (31) and β_1 is the continuity constant of B_0. Here $cond(D^{-1}S)$ is the ratio of the maximum and the minimum absolute value of the eigenvalues of $D^{-1}S$.

THEOREM 6.6. *(Block-triangular preconditioner, exact blocks)*

$$spectrum(T^{-1}S) \subset [\beta_\Gamma^2, \beta_1^2 + 1] \cup \{1\}.$$

The case of a block-triangular preconditioner with inexact blocks is studied in Klawonn [19, 20], under the previous Assumptions 1 and 2, assuming additionally that $1 < a_0 \leq a_1$. The estimate provided is analogous to the case with exact blocks, but is more complicated; we refer to [19] for details. In this case, we can define an additional energy norm based on the inexact blocks and a GMRES convergence bound, in this norm, has been established.

In order to obtain convergence bounds from Theorems 6.5 and 6.6, we need only to verify Assumptions 1 and 2 for a choice of the preconditioner blocks $\widehat{S}_\Gamma$ and $\widehat{C}_0$. We will outline how this can be done in the next few subsections, illustrating our results mainly in the block-diagonal case. We note that the construction of these iterative substructuring algorithms is a very modular process.

6.3. A wire basket preconditioner for Stokes problems. We first consider a Laplacian-based wire basket preconditioner $\widehat{S}_\Gamma$ given, for each component $u^{(i)}$ of $\mathbf{u}$, by the scalar wire basket preconditioner $\widehat{S}_W$ introduced in Pavarino and Widlund [30] and extended to GLL quadrature based approximations in [33],

$$\widehat{S}_\Gamma = \begin{bmatrix} \widehat{S}_W & 0 & 0 \\ 0 & \widehat{S}_W & 0 \\ 0 & 0 & \widehat{S}_W \end{bmatrix}. \tag{35}$$

In our earlier work, we considered the scalar Laplace equation with piecewise constant coefficients and constructed a preconditioner $\widehat{S}_W$ for the Schur complement $S_{\mathcal{H}}$ of the discrete harmonic interface variables, obtained by eliminating the interior degrees of freedom:

$$\widehat{S}_W^{-1} = R_0 \widetilde{S}_{WW}^{-1} R_0^T + \sum_k R_{F_k} S_{F_k F_k}^{-1} R_{F_k}^T.$$

Here $R_0 = (R, I)$ is a matrix representing a change of basis in the wire basket space, $R_{F_k}^T$ is the restriction matrix returning the degrees of freedom associated with the face F_k, and $\widetilde{S}_{WW}$ is an approximation of the original wire basket block. This is an additive preconditioner with independent parts associated with each face and the wire basket.

The condition number of this scalar wire basket preconditioner satisfies a polylogarithmic bound

$$c(1+\log n)^{-2} u_\Gamma^{(i)^T} \widehat{S}_W u_\Gamma^{(i)} \le u_\Gamma^{(i)^T} S_{\mathcal{H}} u_\Gamma^{(i)} \le C u_\Gamma^{(i)^T} \widehat{S}_W u_\Gamma^{(i)} \qquad \forall u_\Gamma^{(i)} \in V_\Gamma^n; \tag{36}$$

see [30, Theorem 3.1] and [33, Theorems 1 and 2]. We can obtain an analogous bound by applying this bound to each component:

$$c(1+\log n)^{-2} \mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \begin{bmatrix} S_{\mathcal{H}} & 0 & 0 \\ 0 & S_{\mathcal{H}} & 0 \\ 0 & 0 & S_{\mathcal{H}} \end{bmatrix} \mathbf{u}_\Gamma \le C \mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \quad \forall \mathbf{u}_\Gamma \in \mathbf{V}_\Gamma^n. \tag{37}$$

This result allows us to prove a convergence bound for the reduced saddle point problem (31) with the block-diagonal preconditioner.

THEOREM 6.7. *Let the blocks of the block-diagonal preconditioner $\widehat{D}_W$ be the wire basket preconditioner $\widehat{S}_\Gamma$ defined in (35) and the coarse mass matrix C_0. Then the Stokes saddle point Schur complement S preconditioned by $\widehat{D}_W$ satisfies*

$$cond(\widehat{D}_W^{-1} S) \le C \frac{(1+\log n)^2}{\beta_n},$$

where C is independent of n and N.

An analogous bound for the block-triangular preconditioner follows from the estimates of the constants in the Assumptions 1 and 2 required in the proof of Theorem 6.7.

6.4. A wire basket preconditioner for incompressible elasticity problems. The block-diagonal preconditioners (35) introduced in the previous subsections do not take any coupling among the three components of $\mathbf{u}$ into account. This works for Stokes problems, but for elasticity problems such an approach would lead to non-scalable algorithms. In fact, the saddle point Schur complement for linear elasticity on one interior element Ω_i has a six dimensional null space, spanned by the rigid body motions (three translations and three rotations). In order to obtain a scalable algorithm, the local contribution from Ω_i to the wire basket preconditioner must have the same six dimensional null space. This condition is of course violated by the component-wise preconditioner of the previous section, that has only a three dimensional null space of component-wise translations. In this section, we introduce a scalable wire basket preconditioner for mixed elasticity problems, using the techniques and the analysis of [31]. The basic changes consist in:

a) using the bilinear form

$$e_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega}$$

instead of the bilinear form

$$2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega} + \lambda(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v})_{n,\Omega}$$

of compressible elasticity;

b) using the mixed elastic extension $\mathcal{M}^n$ instead of the elastic extension $\mathcal{E}^n$.

This means that the extension from the wire basket is now defined by

$$I^W\mathbf{u} = \mathcal{M}^n(I^W u^{(1)}, I^W u^{(2)}, I^W u^{(3)}),$$

where the single scalar components are given by (22), and the subspace of interface displacements is now $\mathbf{V}^n_{\mathcal{M}} = \mathcal{M}^n(\mathbf{V}^n_\Gamma)$. We note that the null space of both $e_n(\cdot,\cdot)$ and the bilinear form $a_n(\cdot,\cdot)$ of compressible elasticity, on an interior element, is the same space $\mathcal{N}$ of rigid body motions and we recall that I^W reproduces this space. Therefore, the same construction as in [31, Section 6] can be used to obtain a wire basket preconditioner

$$\widehat{S}_\Gamma^{-1} = R_0 \widehat{S}_{WW}^{-1} R_0^T + \sum_k R_{F_k} \widehat{S}_{F_k F_k}^{-1} R_{F_k}^T. \tag{38}$$

We now use a different scaling of the wire basket inexact solver $\widehat{S}_{WW}^{-1}$; on an interior element Ω_i, which we, for simplicity, suppose to be the reference element, we define

$$\widehat{S}_{WW}^{(i)} = \frac{(1+\log n)}{\beta_n}\left(M^{(i)} - \sum_{j=1}^{6} \frac{(M^{(i)}\mathbf{r}_j)(M^{(i)}\mathbf{r}_j)^T}{\mathbf{r}_j^T M^{(i)}\mathbf{r}_j}\right).$$

The following bound, analogous to the main result of [31], can be established.

THEOREM 6.8. *The wire basket preconditioner* $\widehat{S}_\Gamma^{-1}$ *satisfies the bounds*

$$c\beta_n(1+\log n)^{-2}\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma \le C\mathbf{u}_\Gamma^T \widehat{S}_\Gamma \mathbf{u}_\Gamma \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}_\Gamma^n.$$

A proof of this result can be found in [32].

Using Theorem 6.8 to bound the constants of Assumption 1, we can then prove the following result.

THEOREM 6.9. *Let the blocks of the block-diagonal preconditioner* $\widehat{D}_W$ *be the wire basket preconditioner* $\widehat{S}_\Gamma$ *defined in (38) and the coarse mass matrix* C_0. *Then the incompressible mixed elasticity saddle point Schur complement* S, *preconditioned by* $\widehat{D}_W$, *satisfies*

$$cond(\widehat{D}_W^{-1}S) \le C\frac{(1+\log n)^2}{\beta_n},$$

where C *is independent of* n *and* N.

TABLE 1

Compressible elasticity in pure displacement form ($\nu = 0.3$): local condition numbers for the wire basket method with original and approximate wire basket block $S^{(i)}_{WW}$

	original wire basket block			approximate wire basket block		
n	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m
2	12.2708	2.3493	0.1915	16.6074	3.6828	0.2218
3	17.4251	2.3915	0.1372	30.3822	3.5840	0.1180
4	24.9668	2.5550	0.1023	40.2729	3.3967	0.0843
5	34.0775	2.6995	0.0792	51.7355	3.4773	0.0672
6	42.5610	2.8032	0.0659	63.2516	3.5712	0.0565
7	52.6813	2.8805	0.0547	76.3325	3.6988	0.0485
8	61.3649	2.9369	0.0479	89.9607	3.8827	0.0432
9	70.3584	2.9810	0.0424	105.0605	4.0638	0.0387
10	78.2626	3.0163	0.0385	119.4171	4.2311	0.0354

7. Numerical results. In this section, we report on results of some numerical experiments concerning local condition numbers and inf-sup constants for our model problems in three dimensions; the computations have been carried out in Matlab 5.1 on Sun workstations.

We consider first the system of compressible elasticity in pure displacement form. We recall that S is the Schur complement of the stiffness matrix K for the discrete compressible elasticity problem (6) and that $\widehat{S}$ denotes the wire basket preconditioner for S. The local contributions from an element Ω_i are denoted by $S^{(i)}$ and $\widehat{S}^{(i)}$, respectively. As we have pointed out before, our wire basket algorithm satisfies the null space property and therefore the local condition number $cond(\widehat{S}^{(i)^{-1}} S^{(i)})$ for an interior element is an upper bound for the condition number $cond(\widehat{S}^{-1}S)$. We note that for an interior element, $S^{(i)}$ and $\widehat{S}^{(i)}$ have the common six-dimensional null space $\mathcal{N}$ spanned by the rigid body motions. The local condition numbers are computed as the ratio of the extreme eigenvalues λ_M and λ_m of $\widehat{S}^{(i)^{-1}} S^{(i)}$ in the space orthogonal to $\mathcal{N}$. Table 1 reports on the local condition numbers for $\nu = 0.3$ when the wire basket preconditioner contains the original wire basket block $S^{(i)}_{WW}$ (left panel) and the approximate rank-six wire basket block $\widehat{S}^{(i)}_{WW}$, defined in (24) (right panel). As in the scalar case, the simplified wire basket block is less expensive but yields higher condition numbers than the original block. In both cases, it is difficult to discern a difference between a linear and a polylogarithmic growth of the condition numbers.

We now consider the system of almost incompressible elasticity in mixed form and the Stokes system. We first computed the discrete inf-sup constant β_Γ of the saddle point Schur complement (34) for both the mixed elasticity and Stokes systems. β_Γ is computed as the square root of the minimum nonzero eigenvalue of $C_0^{-1} B_0^T S_\Gamma^{-1} B_0$ on the reference cube, where S_Γ and B_0 are the blocks of the saddle point Schur complement (34) and C_0 is the coarse pressure mass matrix. The results are plotted in Figure 1, first varying the spectral degree n while keeping fixed a small number of elements, $N = 2 \times 2 \times 1$ (upper plot) and then varying N while keeping $n = 2$ fixed (lower plot). β_Γ appears to be bounded by a constant independent of N and n in both cases. We note that it is well known that the β_n of the original saddle point problem (10) is inversely proportional to n; see section 3.3. In Table 2, we report on the values of β_n for $n = 3, \cdots, 10$, for both the $Q_n - Q_{n-2}$ and the $Q_n - P_{n-1}$ method. Here, β_n is computed

FIG. 1. *Inf-sup constant β_Γ for the saddle point Schur complement*

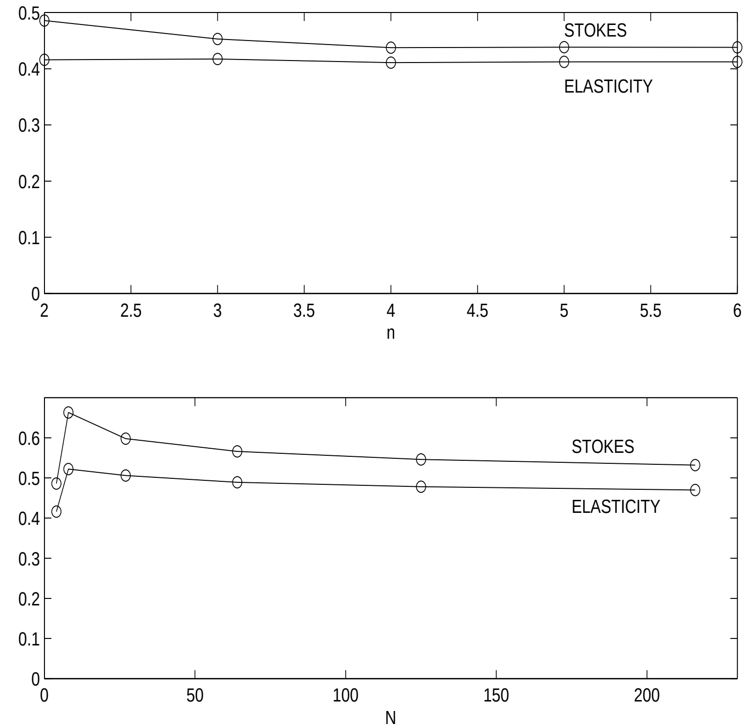

TABLE 2
Inf-sup constant $\beta_n = \lambda_{min}^{1/2}(C^{-1}B^TA^{-1}B)$

n	$Q_n - Q_{n-2}$				$Q_n - P_{n-1}$			
	β_n	λ_{min}	λ_{max}	$\frac{\lambda_{max}}{\lambda_{min}}$	β_n	λ_{min}	λ_{max}	$\frac{\lambda_{max}}{\lambda_{min}}$
3	0.3291	0.1083	0.2284	2.1084	0.4095	0.1677	0.3611	2.1527
4	0.2944	0.0867	0.6334	7.3040	0.4132	0.1707	0.4570	2.6771
5	0.2636	0.0695	0.6447	9.2670	0.4175	0.1743	0.5973	3.4258
6	0.2400	0.0576	0.6500	11.2829	0.4044	0.1635	0.6097	3.7291
7	0.2198	0.0483	0.6500	13.4537	0.4073	0.1659	0.6499	3.9161
8	0.2027	0.0411	0.6500	15.8016	0.3995	0.1596	0.6499	4.0713
9	0.1881	0.0354	0.6500	18.3445	0.4009	0.1607	0.6500	4.0446
10	-	-	-	-	0.3950	0.1560	0.6500	4.1653

as the square root of the minimum nonzero eigenvalue of $C^{-1}B^TA^{-1}B$ on the reference cube, where A, B, and C are the blocks of the original saddle point problem (10). The inf-sup parameter of the $Q_n - P_{n-1}$ method is much better than that of the $Q_n - Q_{n-2}$ method. We refer to [28] and [29] for a comparison of block preconditioners for the two methods.

We next report on the local condition numbers of $\widehat{S}_\Gamma^{-1}S_\Gamma$ for one interior element. Here S_Γ is the velocity block in the saddle point Schur complement (34) and $\widehat{S}_\Gamma$ is the wire basket preconditioner described in Section 6.4 for the mixed elasticity case and in Section 6.3 for the Stokes case. In both cases, we report only on results obtained with the original wire basket block of the preconditioner. Table 3 presents the results for the

TABLE 3

Linear elasticity in mixed form: local condition number of the local saddle point Schur complement with wire basket preconditioner (with original wire basket block) $\widehat{S}_{\Gamma}^{-1}S_{\Gamma}$ *on one interior element*

n	ν							
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
2	9.06	9.06	9.06	9.06	9.06	9.06	9.06	9.06
3	17.54	20.19	44.92	58.26	60.12	60.31	60.33	60.33
4	24.45	29.69	62.30	85.35	88.77	89.13	89.17	89.17
5	34.44	38.68	76.69	106.72	111.49	111.99	112.05	112.05
6	40.97	46.84	90.97	129.73	136.38	137.09	137.16	137.17
7	51.23	55.65	107.19	153.29	161.97	162.90	162.99	162.99
8	59.70	64.60	122.13	176.32	187.45	188.66	188.66	188.66

TABLE 4

Generalized Stokes problem: local condition number of the local saddle point Schur complement with wire basket preconditioner (with original wire basket block) $\widehat{S}_{\Gamma}^{-1}S_{\Gamma}$ *on one interior element*

n	ν							
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
2	4.89	4.89	4.89	4.89	4.89	4.89	4.89	4.89
3	14.13	17.31	36.55	44.79	45.88	45.99	46.00	46.00
4	19.18	24.24	54.33	73.08	75.76	76.04	76.07	76.07
5	24.18	30.56	66.25	86.85	89.92	90.24	90.27	90.28
6	28.71	36.29	87.52	121.36	126.52	127.07	127.12	127.13
7	33.44	42.15	95.50	130.82	136.25	136.82	136.88	136.89
8	38.36	48.71	114.89	163.55	171.49	172.34	172.42	172.43

mixed elasticity problem, while Table 4 gives results for the generalized Stokes problem. In both cases, the incompressible limit is clearly the hardest yielding condition numbers three or four times as large as those of the corresponding compressible case. For a given value of ν, the condition number seems to grow linearly with n, which is consistent with our theoretical result. It is interesting to note that the results for $\nu = 0.3$ for the linear elasticity system in mixed form are better than the corresponding results for the linear elasticity system in pure displacement form. However, the results for the mixed case only concern the velocity block of the preconditioner and we actually need to solve a saddle point problem involving both velocities and constant pressures in each element to be able to make a more complete comparison.

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