ITERATIVE SUBSTRUCTURING METHODS FOR SPECTRAL ELEMENT DISCRETIZATIONS OF ELLIPTIC SYSTEMS. I: COMPRESSIBLE LINEAR ELASTICITY

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Abstract. An iterative substructuring method for the system of linear elasticity in three dimensions is introduced and analyzed. The pure displacement formulation for compressible materials is discretized with the spectral element method. The resulting stiffness matrix is symmetric and positive definite. The method proposed provides a domain decomposition preconditioner constructed from local solvers for the interior of each element, and for each face of the elements and a coarse, global solver related to the wire basket of the elements. As in the scalar case, the condition number of the preconditioned operator is independent of the number of spectral elements and grows as the square of the logarithm of the spectral degree.

Key words. linear elasticity, spectral element methods, preconditioned iterative methods, substructuring, Gauss-Lobatto-Legendre quadrature

AMS(MOS) subject classifications. 65N30, 65N35, 65N55

1. Introduction. Finite element discretizations of problems in structural mechanics require the solution of large and sparse linear systems of equations. In the past, such systems have often been solved by direct methods. These methods are limited by their high arithmetical costs, memory requirements, and poor scalability on parallel computers. In order to overcome these limitations, a great deal of research has in recent years focused on the design and analysis of efficient iterative methods. Solvers which combine a Krylov space accelerator with a robust preconditioner have been shown to outperform direct solvers in large three-dimensional elasticity computations; see Dickinson and Forsyth [11], Farhat and Roux [14] and the references therein. Domain decomposition provides some of the best preconditioners for elliptic problems; see Smith, Bjørstad and Gropp [35] for a general introduction and Le Tallec [20] and Farhat and Roux [14] for a discussion of domain decomposition in structural mechanics. In this paper, we will focus on the system of linear elasticity in three dimensions. For more general problems and methods in nonlinear elasticity, we refer to Ciarlet [10] and Le Tallec [21].

Iterative substructuring methods form an important family of domain decomposition algorithms, with origins in the direct substructuring techniques developed in the structural analysis community over several decades. When an iterative substructuring method is used, the domain of the elliptic problem is decomposed into nonoverlapping subdomains. After the elimination of the interior variables, the discrete problem for the interface variables, known as the Schur complement system, is solved iteratively by

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a preconditioned Krylov space method, such as PCG or GMRES. The preconditioner is constructed from the solutions of local problems and a coarse global problem; see Dryja, Smith, and Widlund [12] and Smith, Bjørstad, and Gropp [35, Ch. 4].

In this paper, we will study iterative substructuring methods for spectral element discretizations of systems of elliptic equations. Spectral elements originated in computational fluid dynamics growing out of earlier work on spectral methods on a single region (see, e.g., Canuto, Hussaini, Quarteroni and Zang [8], Maday, Patera, and Rønquist [22], Bernardi and Maday [3], Funaro [15], Karniadakis and Sherwin [17] and the references therein). The related p and $h-p$ versions of the finite element method were developed for problems in computational structural mechanics (see Szabó and Babuška [36]). We note that the $p-$ and $h-p$ version finite elements differ from spectral elements in the choice of bases and quadrature rules for evaluating the integrals of the Galerkin formulation. For all these methods, improved accuracy is achieved by increasing the spectral degree aas well as the number of elements. We note that iterative solvers for a variety of higher order methods have been developed by Mandel [24], Katz and Hu [18], and Guo [16]; see also the theses of Pavarino [28], Casarin [9], and Bică [4].

In our previous work [29, 31], we considered the scalar case and iterative substructuring methods with a wire basket based coarse space. Each spectral element is then the affine image of a reference cube considered as a subdomain of the domain decomposition method. The wire basket preconditioner for the Schur complement is based on a solver for the interior of each element, a solver for each face (shared by two elements) and a coarse solver for the wire basket (the union of the edges and vertices of the elements). We proved, and verified numerically that the convergence rate of this method is independent of the number of elements and the jumps in the coefficients of the elliptic operator, and that it depends only weakly on the spectral degree. An alternative proof was later given by Casarin [9]. This type of wire basket preconditioner was originally proposed for h-version finite elements by Smith [32, 33, 34]; see also Bramble, Pasciak and Schatz [5] for earlier related work. Other iterative substructuring methods that have been successfully applied to elasticity problems and h-version finite elements are the Neumann-Neumann methods, see, e.g., [20], the balancing domain decomposition method of Mandel [23] and the FETI method of Farhat and Roux [14]. Schwarz methods for nonconforming finite elements in planar elasticity have been considered in Xu and Li [37]. Multigrid methods have also been extended to elasticity problems; see Kočvara and Mandel [19], Parsons and Hall [26, 27], and Brenner [6, 7].

In this paper, we extend the wire basket preconditioner to the system of three-dimensional linear elasticity discretized with spectral elements. We consider compressible materials for which the Poisson ratio ν is bounded away from $1/2$, e.g., $\nu \leq 0.4$. In this case, the pure displacement formulation is valid and it leads to a symmetric positive definite stiffness matrix. For almost incompressible materials, characterized by ν close to $1/2$, the pure displacement formulation breaks down; in addition to problems with *locking* (see Babuška and Suri [1]), the stiffness matrix becomes increasingly ill-conditioned (see Figure 1). A possible solution is to use a mixed formulation. This approach, as well as the related Stokes problem for incompressible fluids, will be the subject of part II of this work [30].

The main result of this paper is a polylogarithmic bound for the condition number of the iteration operator with the wire basket preconditioner. As in the scalar case, this result is obtained by working within the Schwarz framework; see [35, 12, 13].

The paper is organized as follows. In Section 2, we briefly describe the system of linear elasticity. In Section 3, we discretize the problem by the spectral element method and Gauss-Lobatto-Legendre quadrature. In preparation for the definition of the wire basket preconditioner, we introduce, in Section 4, two extension operators from the interface to the interior of each element and in Section 5 an extension operator from the wire basket to the interface and interior of each element. In Section 6, we construct the wire basket preconditioner in matrix form, while in Section 7 we give a variational formulation in the Schwarz framework and prove some technical results and the main theorem. In the concluding Section 8, we report on some numerical experiments in three dimensions.

2. The linear elasticity system. Let $\Omega \subset R^3$ be a polyhedral domain and let Γ_0 be a nonempty subset of its boundary. Let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, subject to a surface force of density $\mathbf{g}$, along $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and a body force $\mathbf{f}$:

$$(1) \qquad 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx + \lambda \int_\Omega \mathrm{div}\mathbf{u} \, \mathrm{div}\mathbf{v} \, dx = < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, and the inner products are defined as

$$\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}), \qquad < \mathbf{F}, \mathbf{v} > = \int_\Omega \sum_{i=1}^{3} f_i v_i \, dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i v_i \, ds.$$

We denote the bilinear form of linear elasticity by

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx + \lambda \int_\Omega \mathrm{div}\mathbf{u} \, \mathrm{div}\mathbf{v} \, dx;$$

see, e.g., Ciarlet [10] for a detailed treatment of nonlinear and linear elasticity. This pure displacement model is a good formulation for compressible materials, for which the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is strictly less than $1/2$, e.g., $\nu \leq 0.4$. In this paper, we confine our analysis to this case. For almost incompressible materials and the related Stokes problem for incompressible fluids, we will use a mixed spectral element formulation; see part II of this work [30].

3. Spectral elements and Gauss-Lobatto-Legendre quadrature. Let Ω_{ref} be the reference cube $(-1, 1)^3$, and let $Q_n(\overline{\Omega}_{\mathrm{ref}})$ be the set of polynomials on $\overline{\Omega}_{\mathrm{ref}}$ of degree n in each variable. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an affine image of the reference cube,

$$\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i.$$

Thus, $\Omega_i = \phi_i(\Omega_{\mathrm{ref}})$, where ϕ_i is an affine mapping. The displacement $\mathbf{u}$ is discretized, component by component, by conforming spectral elements, i.e. by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\overline{\Omega}_i} \circ \phi_i \in Q_n(\overline{\Omega}_{\mathrm{ref}}), \ i = 1, \cdots, N, \quad k = 1, 2, 3\}.$$

A very convenient tensor product basis for $\mathbf{V}^n$ can be constructed using Gauss-Lobatto-Legendre (GLL) quadrature points; other bases could be considered, such as those based on integrated Legendre polynomials common in the p-version finite element literature; see Szabó and Babuška [36]. Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points on $[-1,1]^3$, and by σ_i the quadrature weight associated with ξ_i. Let $l_i(\cdot)$ be the Lagrange interpolating polynomial which vanishes at all the GLL nodes except ξ_i, where it equals one. The basis functions on the reference cube are then defined by a tensor product as

$$l_i(x_1)l_j(x_2)l_k(x_3), \qquad 0 \le i,j,k \le n.$$

This is a nodal basis, since every element of $Q_n(\overline{\Omega}_{\text{ref}})$ can be written as

$$u(x_1,x_2,x_3) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k)l_i(x_1)l_j(x_2)l_k(x_3).$$

The reference element can be decomposed into its interior, six faces, twelve edges, and eight vertices. The union of its edges and vertices is called the wire basket of the element and is denoted by W_{ref}. Analogously, each basis function can be characterized as being of interior, face, edge, or vertex type:

- interior: $i,j,k \neq 0$ and $\neq n$;
- face: exactly one index is 0 or n;
- edge: exactly two indices are 0 and/or n;
- vertex: all three indices are 0 and/or n.

Each displacement component can therefore be written as the sum of its vertex, edge, face, and interior components,

$$u = u_V + u_E + u_F + u_I,$$

where each term is expressed in terms of the corresponding basis functions.

We now replace each integral of the continuous model (1) by GLL quadrature. On Ω_{ref},

$$(u,v)_{n,\Omega_{\text{ref}}} = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k)v(\xi_i,\xi_j,\xi_k)\sigma_i\sigma_j\sigma_k,$$

and on all of Ω,

$$(u,v)_{n,\Omega} = \sum_{s=1}^{N}\sum_{i,j,k=0}^{n} (u\circ\phi_s)(\xi_i,\xi_j,\xi_k)(v\circ\phi_s)(\xi_i,\xi_j,\xi_k)|J_s|\sigma_i\sigma_j\sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. This inner product is uniformly equivalent to the standard L_2-inner product on $Q_n(\Omega_{\text{ref}})$; it is shown in Bernardi and Maday [2, 3] that

$$(2) \qquad ||u||^2_{L_2(\Omega_{\text{ref}})} \le (u,u)_{n,\Omega_{\text{ref}}} \le 27||u||^2_{L_2(\Omega_{\text{ref}})} \qquad \forall u \in Q_n(\Omega_{\text{ref}}).$$

These bounds imply an analogous uniform equivalence between the $H^1(\Omega)$-seminorm and the discrete seminorm $(\nabla u, \nabla u)_{n,\Omega}$ based on GLL quadrature. Applying these quadrature rules, we obtain the discrete bilinear form

$$a_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega} + \lambda(\text{div}\mathbf{u}, \text{div}\mathbf{v})_{n,\Omega},$$

and the discrete elasticity problem:
Find $\mathbf{u} \in \mathbf{V}^n$ such that

$$a_n(\mathbf{u}, \mathbf{v}) = < \mathbf{F}, \mathbf{v} >_{n,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V}^n. \tag{3}$$

An analysis of the spectral element discretization for the Laplacian and Stokes problems can be found in Bernardi and Maday [2, 3] and in Maday, Patera, and Rønquist [22]. The same techniques can be applied to provide an analysis and error estimates for the linear elasticity problem.

We denote by K the symmetric and positive definite stiffness matrix associated with the discrete problem (3). It is less sparse than the stiffness matrices obtained by low-order finite elements, but is still well-structured and the corresponding matrix-vector multiplication is relatively inexpensive if advantage is taken of the tensor product structure; see, e.g., Bernardi and Maday [2, 3]. See also Figure 1 for a sparsity plot of the stiffness matrix K on the reference element.

For an interior element, $a_n(\cdot, \cdot)$ has a six-dimensional null space $\mathcal{N}$, spanned by the rigid body motions $\mathbf{r}_j$

$$\mathcal{N} = span\{\mathbf{r}_j, \ j = 1, \cdots, 6\}.$$

On Ω_{ref}, the $\mathbf{r}_j$ are given, componentwise, by three translations

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4}$$

and three rotations

$$\mathbf{r}_4 = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \quad \mathbf{r}_5 = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad \mathbf{r}_6 = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}. \tag{5}$$

We note that it is easy to show that both the divergence and the linearized strain tensor of these six functions vanish.

4. Extensions from the interface. In the construction of our algorithm, we need to extend piecewise polynomials from the boundaries of the elements to their interiors.

Let Γ be the (interior) interface of the decomposition $\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i$:

$$\Gamma = (\cup_{i=1}^{N} \partial\Omega_i) \setminus \partial\Omega.$$

Γ is composed of N_F faces F_k (open sets) of the elements and the wire basket W, defined as the union of the edges and vertices,

$$\Gamma = \cup_{k=1}^{N_F} F_k \cup W.$$

Let $\mathbf{V}_\Gamma^n = \mathbf{V}^n(\Gamma)$ be the space of restrictions of $\mathbf{V}^n$ to the interface. We note that, on each face, any function in $\mathbf{V}_\Gamma^n$ is an affine image of a Q_n space with two variables. We define local subspaces consisting of piecewise polynomials with support in the interior of each element,

$$\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3, \qquad i = 1, \cdots, N. \tag{6}$$

We will consider two ways of extending a piecewise polynomial defined on Γ to the interior of each element Ω_i. These extensions are constructed locally, i.e. element by element.

4.1. The discrete harmonic extension. The discrete harmonic extension is defined as the operator $\mathcal{H}^n : \mathbf{V}_\Gamma^n \longrightarrow \mathbf{V}^n$, that maps an element $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the unique solution $\mathcal{H}^n \mathbf{u} \in \mathbf{V}^n$ of

$$b_n(\mathcal{H}^n \mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_i^n, \qquad \mathcal{H}^n \mathbf{u} = \mathbf{u} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, N,$$

where $b_n(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u} : \nabla \mathbf{v})_{n,\Omega}$ is the discrete bilinear form associated with the vector Laplacian. This is just the component by component discrete harmonic extension used extensively in the study of iterative substructuring methods for scalar elliptic problems. As in the scalar case, the discrete harmonic extension satisfies the minimization property

$$b_n(\mathcal{H}^n \mathbf{u}, \mathcal{H}^n \mathbf{u}) = \min_{\mathbf{v}} b_n(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}^n \text{ with } \mathbf{v} = \mathbf{u} \text{ on } \Gamma.$$

4.2. The discrete elastic extension. We can also extend a piecewise polynomial from Γ to the interior of each element by solving a linear elasticity problem in each element. The discrete elastic extension $\mathcal{E}^n : \mathbf{V}_\Gamma^n \longrightarrow \mathbf{V}^n$, is the operator that maps a piecewise polynomial $\mathbf{u} \in \mathbf{V}_\Gamma^n$ into the unique solution of

$$a_n(\mathcal{E}^n \mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_i^n, \qquad \mathcal{E}^n \mathbf{u} = \mathbf{u} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, N. \tag{7}$$

In our applications to elasticity problems, we will choose the range of this extension operator,

$$\mathbf{V}_\mathcal{E}^n = \mathcal{E}^n(\mathbf{V}_\Gamma^n) \tag{8}$$

as the subspace of interface displacements. The elements in this subspace are completely determined by their values on Γ.

The discrete elastic extension satisfies the minimization property

$$a_n(\mathcal{E}^n \mathbf{u}, \mathcal{E}^n \mathbf{u}) = \min_{\mathbf{v}} a_n(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}^n \text{ with } \mathbf{v} = \mathbf{u} \text{ on } \Gamma.$$

5. Extension from the wire basket. In the construction of our algorithm, we will also need to extend piecewise polynomials from the wire basket to the faces. As this is also a local operation, we can restrict our attention to the reference element. A preliminary extension operator $\tilde{I}^W$ from the wire basket is constructed for any function $\mathbf{u} \in \mathbf{V}^n$ by expanding its restriction to the wire basket, using the vertex and edge basis functions described in Section 3,

$$\tilde{I}^W \mathbf{u} = \mathbf{u}_V + \mathbf{u}_E.$$

Given that we are using a nodal basis, $\tilde{I}^W \mathbf{u}$ will simply vanish at all the face GLL points outside the wire basket. Therefore this extension operator does not preserve the rigid body motions $\mathbf{r}_j, j = 1, \cdots, 6$. In order to construct a scalable algorithm, we must define an extension operator that preserves the rigid body motions on the interface; see Mandel [23] and or Smith, Bjørstad, and Gropp, [35, p. 132] for a discussion of this *null space property.*

We start by considering the difference between each of the $\mathbf{r}_j$ and the function obtained by using the preliminary extension. They can all be expressed in terms of four scalar functions, defined on each face by

$$\mathcal{F}^0 = 1 - \tilde{I}^W 1, \quad \mathcal{F}^1 = x_1 - \tilde{I}^W x_1, \quad \mathcal{F}^2 = x_2 - \tilde{I}^W x_2, \quad \mathcal{F}^3 = x_3 - \tilde{I}^W x_3.$$

We remark that in our previous study of the scalar case, see [29, 31], only $\mathcal{F}^0$ was needed, because the null space of the discrete bilinear form on an interior element consists only of constants. Each of our four functions, just defined, vanishes on the wire basket and each can be split into six face terms,

$$\mathcal{F}^0 = \sum_{k=1}^{6} \mathcal{F}_k^0, \qquad \mathcal{F}^1 = \sum_{k=1}^{6} \mathcal{F}_k^1, \qquad \mathcal{F}^2 = \sum_{k=1}^{6} \mathcal{F}_k^2, \qquad \mathcal{F}^3 = \sum_{k=1}^{6} \mathcal{F}_k^3.$$

Here, the $\mathcal{F}_k^j, j = 0, 1, 2, 3$, vanish on all faces except F_k. For each scalar component $u^{(i)}$ of $\mathbf{u}$, we define a new extension $I^W u$ from the wire basket to the interface as follows: On a face F_k, for which the two relevant variables are x_1 and x_2, the restriction of $I^W u^{(i)}$ to F_k has the form

$$I^W u^{(i)} = \tilde{I}^W u^{(i)} + a_k \mathcal{F}_k^0 + b_k^1 \mathcal{F}_k^1 + b_k^2 \mathcal{F}_k^2. \tag{9}$$

The weights a_k, b_k^1, b_k^2, and b_k^3 are given by the following moments (the factors $\frac{1}{8}$ and $\frac{3}{16}$ come from the fact that we work on the reference element):

$$a_k = \frac{(u^{(i)}, 1)_{n,\partial F_k}}{(1,1)_{n,\partial F_k}} = \frac{1}{8}(u^{(i)}, 1)_{n,\partial F_k},$$

$$b_k^j = \frac{(u^{(i)}, x_j)_{n,\partial F_k}}{(x_j, x_j)_{n,\partial F_k}} = \frac{3}{16}(u^{(i)}, x_j)_{n,\partial F_k}, \qquad j = 1, 2, 3.$$

We note that on each face only three correction terms are used; see (9). For a vector valued displacement $\mathbf{u}$, the extension operator is then defined as the discrete elastic extension of the scalar face functions given by (9), i.e.

$$I^W \mathbf{u} = \mathcal{E}^n (I^W u^{(1)}, I^W u^{(2)}, I^W u^{(3)}).$$

A simple computation shows that, on each face, the new extension operator reproduces all P_1 polynomials and therefore also all the rigid body motions. If, e.g., $u = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3$, we have on the face $F_k = \{x_3 = 1\}$,

$$a_k = \frac{1}{8}(c_0 + c_1 x_1 + c_2 x_2 + c_3, 1)_{n,\partial F_k} = c_0 + c_3,$$

$$b_k^1 = \frac{3}{16}(c_0 + c_1 x_1 + c_2 x_2 + c_3, x_1)_{n,\partial F_k} = c_1,$$

$$b_k^2 = \frac{3}{16}(c_0 + c_1 x_1 + c_2 x_2 + c_3, x_2)_{n,\partial F_k} = c_2,$$

as required. Moreover, any rigid body motion $\mathbf{r}$ is also reproduced inside each element, i.e. $\mathcal{E}^n \mathbf{r} = \mathbf{r}$. This follows from the minimization property of the elastic extension and the fact that $a_n(\mathbf{r}, \mathbf{r}) = 0$. Therefore, $I^W \mathbf{r} = \mathbf{r}, \ \ \forall \mathbf{r} \in \mathcal{N}$. We note that the extension operator I^W defines a change of basis in $\mathbf{V}_\Gamma^n$; the face basis functions are unchanged, but the wire basket basis functions are transformed according to (9).

6. A wire basket preconditioner for linear elasticity. In this section, we describe our wire basket preconditioner for linear elasticity problems in matrix form. A variational form of the method and an analysis using the Schwarz framework will be given in the next section. We model the wire basket preconditioner on our previous work on the scalar case; see [29, 31].

The stiffness matrix K of the discrete linear elasticity problem (3) is built by subassembly from the individual contributions from each element Ω_i,

$$\mathbf{u}^T K \mathbf{u} = \sum_{i=1}^{N} \mathbf{u}^{(i)^T} K^{(i)} \mathbf{u}^{(i)}.$$

In each element, we order first the interior variables and then the interface variables, obtaining local stiffness matrices of the form

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{I\Gamma}^{(i)} \\ K_{I\Gamma}^{(i)^T} & K_{\Gamma\Gamma}^{(i)} \end{bmatrix}.$$

The interior unknowns are eliminated by solving local linear elasticity problems, obtaining local Schur complement matrices

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{I\Gamma}^{(i)^T} K_{II}^{(i)^{-1}} K_{I\Gamma}^{(i)}.$$

The global Schur complement is also built by subassembly from the local contributions

$$\mathbf{u}_\Gamma^T S \mathbf{u}_\Gamma = \sum_{i=1}^{N} \mathbf{u}_\Gamma^{(i)^T} S^{(i)} \mathbf{u}_\Gamma^{(i)}. \tag{10}$$

We solve the interface problem with the coefficient matrix S using a preconditioned Krylov space method, such as CG. We can then avoid forming S explicitly, since only the matrix-vector product $S\mathbf{v}$ is needed and this product can be evaluated by solving N local linear elasticity problems.

We now introduce a wire basket preconditioner $\widehat{S}$ for S, based on the solution of local problems for each face and a coarse, global problem associated with the wire basket. If the interface unknowns are ordered by placing the face variables first, and then the wire basket variables, the local Schur complements can be written as

$$S^{(i)} = \begin{bmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ S_{FW}^{(i)^T} & S_{WW}^{(i)} \end{bmatrix}.$$

We then perform a change of basis in the space spanned by the wire basket functions in order to satisfy the null space property, i.e. in order to ensure that the null space of the local contribution $\widehat{S}^{(i)}$ to the preconditioner is the space of rigid body motions $\mathcal{N}$. This can be done by using the extension operator I^W defined by (9), since I^W reproduces rigid body motions. In matrix form, this change of basis is represented locally by the transformation matrix

$$\begin{bmatrix} I_{FF}^{(i)} & 0 \\ R^{(i)} & I_{WW}^{(i)} \end{bmatrix},$$

where the $I^{(i)}$ are identity matrices of the appropriate oreder. Then $S^{(i)}$ is transformed into

$$\begin{bmatrix} I_{FF}^{(i)} & 0 \\ R^{(i)} & I_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ {S_{FW}^{(i)}}^T & S_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} I_{FF}^{(i)} & {R^{(i)}}^T \\ 0 & I_{WW}^{(i)} \end{bmatrix} = \begin{bmatrix} S_{FF}^{(i)} & \text{nonzero} \\ \text{nonzero} & \widetilde{S}_{WW}^{(i)} \end{bmatrix}.$$

The local preconditioner $\widehat{S}^{(i)}$ is constructed by

a) eliminating the coupling between faces and the wire basket;

b) eliminating the coupling between all pairs of faces, i.e. by replacing $S_{FF}^{(i)}$ by its block-diagonal part $\widehat{S}_{FF}^{(i)}$;

c) replacing the wire basket block $\widetilde{S}_{WW}^{(i)}$ by a simpler matrix $\widehat{S}_{WW}^{(i)}$: Let $M^{(i)}$ be the mass matrix of the local wire basket $W^{(i)}$, defined by $\mathbf{u}^T M^{(i)} \mathbf{u} = (\mathbf{u}, \mathbf{u})_{n,W^{(i)}}$. We replace $\widetilde{S}_{WW}^{(i)}$ by a scaled rank-six perturbation of $M^{(i)}$. On the reference element,

$$\widehat{S}_{WW}^{(i)} = (1 + \log n)(M^{(i)} - \sum_{j=1}^{6} \frac{(M^{(i)}\mathbf{r}_j)(M^{(i)}\mathbf{r}_j)^T}{\mathbf{r}_j^T M^{(i)} \mathbf{r}_j}). \tag{11}$$

This corresponds to using a simpler approximate solver for the wire basket variables; see the next section for further details. We then return to the original basis:

$$\widehat{S}^{(i)} = \begin{bmatrix} I_{FF}^{(i)} & 0 \\ -R^{(i)} & I_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} \widehat{S}_{FF}^{(i)} & 0 \\ 0 & \widehat{S}_{WW}^{(i)} \end{bmatrix} \begin{bmatrix} I_{FF}^{(i)} & -{R^{(i)}}^T \\ 0 & I_{WW}^{(i)} \end{bmatrix}. \tag{12}$$

The action of $R^{(j)}$ and $R^{(i)}$ on a face shared by two elements Ω_j and Ω_i is the same, because the extension of any function defined on the wire basket to a face, using the operator I^W, is determined only by the values on the boundary of that face. Therefore the preconditioner can be obtained by subassembly

$$\widehat{S} = \begin{bmatrix} I_{FF} & 0 \\ -R & I_{WW} \end{bmatrix} \begin{bmatrix} \widehat{S}_{FF} & 0 \\ 0 & \widehat{S}_{WW} \end{bmatrix} \begin{bmatrix} I_{FF} & -R^T \\ 0 & I_{WW} \end{bmatrix},$$

and

$$\widehat{S}^{-1} S = R_0 \widehat{S}_{WW}^{-1} R_0^T S + \sum_k R_{F_k} \widehat{S}_{F_k F_k}^{-1} R_{F_k}^T S,$$

where $R_0 = (R, I_{WW})$; see Dryja, Smith, and Widlund [12]. We have thus obtained an additive preconditioner, with independent parts associated with each face and the wire basket. Multiplicative and hybrid variants can also be defined and analyzed in a completely routine way once that the analysis of the additive method has been completed; see, e.g., Smith, Bjørstad, and Gropp, [35].

7. Variational formulation and analysis of the method. Working inside the standard Schwarz framework, see, e.g., Smith, Bjørstad, and Gropp, [35], we define an iterative substructuring method by first decomposing the space $\mathbf{V}^n$ into subspaces associated with the interiors and a space associated with the interface, which, in turn, will be further decomposed:

$$\mathbf{V}^n = \sum_{i=1}^{N} \mathbf{V}_i^n + \mathbf{V}_{\mathcal{E}}^n.$$

Here $\mathbf{V}_i^n = \mathbf{V}^n \cap H_0^1(\Omega_i)^3$ are the interior spaces and $\mathbf{V}_\mathcal{E}^n = \mathcal{E}^n(\mathbf{V}_\Gamma^n)$ the interface space defined in (8). It is easy to see that

$$a_n(\mathcal{E}^n \mathbf{u}_\Gamma, \mathcal{E}^n \mathbf{u}_\Gamma) = \mathbf{u}_\Gamma^T S \mathbf{u}_\Gamma,$$

where S is the Schur complement defined in (10). Our wire basket method is defined by the following decomposition of the interface space:

$$\mathbf{V}_\mathcal{E}^n = \mathbf{V}_0 + \sum_k \mathbf{V}_{F_k}^n,$$

where

$$\mathbf{V}_0 = range(I^W)$$

is the wire basket space consisting of discrete elastic extensions of piecewise polynomials with given values on the wire basket. Their extension to the faces is determined using the interpolation operator I^W given in (9), and where

$$\mathbf{V}_{F_k}^n = \{\mathbf{v} \in \mathbf{V}^n : \mathbf{v} = \mathcal{E}^n \mathbf{w},\ \mathbf{w} \in \mathbf{V}_\Gamma^n \ \text{ with } \ \mathbf{w} = 0 \ \text{ on } \ \Gamma \setminus F_k\}$$

are face spaces consisting of piecewise discrete elastic extensions of polynomials associated with individual faces.

We now define a projection-like operator for V_0 and a projection for each of the other subspaces

$$T_0 : \mathbf{V}_\mathcal{E}^n \to \mathbf{V}_0 \qquad \text{by} \qquad \tilde{a}_0(T_0\mathbf{u}, \mathbf{v}) = a_n(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0,$$

$$T_{F_k} : \mathbf{V}_\mathcal{E}^n \to \mathbf{V}_{F_k}^n \qquad \text{by} \qquad a_n(T_{F_k}\mathbf{u}, \mathbf{v}) = a_n(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_{F_k}^n.$$

On the wire basket space, we use the special bilinear form

$$\tilde{a}_0(\mathbf{u}, \mathbf{u}) = (1 + \log n) \sum_{i=1}^{N} \inf_{c_{ij}} ||\mathbf{u} - \sum_{j=1}^{6} c_{ij} \mathbf{r}_j ||_{n, W^{(i)}}^2,$$

which leads to a simplified solver for this space, constructed from the matrix $\widehat{S}_{WW}^{(i)}$ defined in (11). This can be seen by a computation analogous to that of the scalar case. In fact, the minimizing c_{ij} are given on the reference element by

$$c_{ij} = \frac{(\mathbf{u}, \mathbf{r}_j)_{n, W_{\text{ref}}}}{(\mathbf{r}_j, \mathbf{r}_j)_{n, W_{\text{ref}}}}. \tag{13}$$

When deriving this formula, we use the fact that the $\mathbf{r}_j$ are L^2−orthogonal on W_{ref}. Therefore,

$$\inf_{c_{ij}} ||\mathbf{u} - \sum_{j=1}^{6} c_{ij} \mathbf{r}_j ||_{n, W^{(i)}}^2 = (\mathbf{u}, \mathbf{u})_{n, W^{(i)}} - \sum_{j=1}^{6} \frac{(\mathbf{u}, \mathbf{r}_j)_{n, W^{(i)}}^2}{(\mathbf{r}_j, \mathbf{r}_j)_{n, W^{(i)}}} \tag{14}$$

$$= \mathbf{u}^T (M^{(i)} - \sum_{j=1}^{6} \frac{(M^{(i)}\mathbf{r}_j)(M^{(i)}\mathbf{r}_j)^T}{\mathbf{r}_j^T M^{(i)} \mathbf{r}_j}) \mathbf{u} = \mathbf{u}^T \widehat{S}_{WW}^{(i)} \mathbf{u}.$$

We are now ready to define the additive Schwarz operator

$$T = T_0 + \sum_{F_k} T_{F_k},$$

and to formulate the main result of this paper.

THEOREM 7.1. *The condition number of the iteration operator T is bounded by*

$$cond(T) \le C(1 + \log n)^2,$$

where C is a constant independent of n and N.

By explicitly computing the matrix form of the operators T_0 and T_{F_k}, we see that the matrix form of the operator T is given by $\widehat{S}^{-1}S$. Therefore, Theorem 7.1 provides a polylogarithmic bound on $cond(\widehat{S}^{-1}S)$.

7.1. Technical tools. Before proving the main theorem, we need the following technical results. In the following estimates, C will denote a generic constant independent of n and N.

LEMMA 7.2.

$$|\tilde{I}^W \mathbf{u}|^2_{H^1(\Omega_{\text{ref}})^3} \le C \|\mathbf{u}\|^2_{L^2(W_{\text{ref}})^3}.$$

The proof consists of applying the scalar result, [31, Lemma 4.4], to each displacement component:

$$|\tilde{I}^W \mathbf{u}|^2_{H^1(\Omega_{\text{ref}})^3} = \sum_{i=1}^{3} |\tilde{I}^W u^{(i)}|^2_{H^1(\Omega_{\text{ref}})} \le C \sum_{i=1}^{3} \|u^{(i)}\|^2_{L^2(W)} = C\|\mathbf{u}\|^2_{L^2(W)^3}.$$

LEMMA 7.3. *For each face F_k of Ω_{ref}*

$$\|\mathbf{u} - \tilde{I}^W \mathbf{u}\|^2_{H^{1/2}_{00}(F_k)^3} \le C(1 + \log n)^2 \|\mathbf{u}\|^2_{H^1(\Omega_{\text{ref}})^3}.$$

Again the proof follows from the scalar result, [31, Lemma 4.6].

LEMMA 7.4.

$$\|\mathcal{F}^i\|^2_{H^{1/2}_{00}(F_k)} \le C(1 + \log n), \qquad i = 0, 1, 2, 3.$$

The bound for $\mathcal{F}^0$ is identical to [31, Lemma 4.8]. The other bounds follows from a slight modification of the proof of [29, Lemma 5.7]. We just have to notice that since $u = x_i$ on $\partial\Omega_{\text{ref}}$, we are again able to avoid a second $(1 + \log n)$ factor, since the maximum of the absolute value of this function is 1.

LEMMA 7.5. *a) The squares of the coefficients a_k, b_k^j, $j = 1, 2, 3$, of the extension operator I^W are all bounded by*

$$C(1 + \log n)\|\mathbf{u}\|^2_{H^1(\Omega_{\text{ref}})^3}.$$

b) The same bound holds for the analogous quantities

$$\tilde{a} = \frac{(u^{(i)}, 1)_{n, W_{\text{ref}}}}{(1, 1)_{n, W_{\text{ref}}}}, \qquad \tilde{b}^j = \frac{(u^{(i)}, x_j)_{n, W_{\text{ref}}}}{(x_j, x_j)_{n, W_{\text{ref}}}}, \qquad j = 1, 2, 3,$$

computed over the whole wire basket W_{ref} instead of just the boundary of one face.

Proof. a) A bound for a_k is obtained as in the scalar case (see [29, (29)]); we again work on the reference element:

$$a_k^2 = \frac{1}{8}(u^{(i)},1)^2_{n,\partial F_k} \le C\|u^{(i)}\|^2_{n,\partial F_k} \le C(1+\log n)\|u^{(i)}\|^2_{H^1(\Omega_{\mathrm{ref}})},$$

where we have used [29, Lemma 5.3] and the equivalence of the continuous and quadrature based L^2-norm for functions of $\mathbf{V}^n$. A bound for the b_k^j is obtained similarly:

$$(b_k^j)^2 = (\frac{3}{16})^2(u^{(i)},x_j)^2_{n,\partial F_k} \le (\frac{3}{16})^2\|x_j\|^2_{n,\partial F_k}\|u^{(i)}\|^2_{n,\partial F_k} \le C\|u^{(i)}\|^2_{n,\partial F_k},$$

and we conclude as before by using [29, Lemma 5.3].
b) The same bound follows as in part a) by applying [29, Lemma 5.3] to each edge of W_{ref}. □

LEMMA 7.6. *Consider $\mathbf{u} \in \mathbf{V}^n$ and its contribution from the faces, $\mathbf{u}_F = \mathbf{u} - I^W\mathbf{u}$. For each face F_k, we have*

$$\|\mathbf{u} - I^W\mathbf{u}\|^2_{H^{1/2}_{00}(F_k)^3} \le C(1+\log n)^2|\mathbf{u}|^2_{H^1(\Omega_{\mathrm{ref}})^3}.$$

Proof. Consider a face F_k where the two relevant variables are x_1 and x_2. We apply Lemmas 7.3, 7.4, and 7.5 to each displacement component, obtaining

$$\|u^{(i)}_{F_k}\|^2_{H^{1/2}_{00}(F_k)} = \|u^{(i)} - I^W u^{(i)}\|^2_{H^{1/2}_{00}(F_k)} = \|u^{(i)} - \tilde{I}^W u^{(i)} - a_k\mathcal{F}_k^0 - b_k^1\mathcal{F}_k^1 - b_k^2\mathcal{F}_k^2\|^2_{H^{1/2}_{00}(F_k)}$$

$$\le C(\|u^{(i)} - \tilde{I}^W u^{(i)}\|^2_{H^{1/2}_{00}(F_k)} + a_k^2\|\mathcal{F}_k^0\|^2_{H^{1/2}_{00}(F_k)} + (b_k^1)^2\|\mathcal{F}_k^1\|^2_{H^{1/2}_{00}(F_k)} + (b_k^2)^2\|\mathcal{F}_k^2\|^2_{H^{1/2}_{00}(F_k)})$$

$$\le C(1+\log n)^2\|u^{(i)}\|^2_{H^1(\Omega_{\mathrm{ref}})}.$$

Since I^W reproduces constant functions, the left hand side does not change if $u^{(i)}$ is shifted by a constant and we can conclude our proof by using Poincaré's inequality. □

We will also need the following form of Korn's inequality; see Ciarlet [10].

LEMMA 7.7.

$$\|\mathbf{v}\|^2_{H^1(\Omega_{\mathrm{ref}})^3} \le C(\|\epsilon(\mathbf{v})\|^2_{L^2(\Omega_{\mathrm{ref}})^3} + \|\mathbf{v}\|^2_{L^2(\Omega_{\mathrm{ref}})^3}) \qquad \forall \mathbf{v} \in H^1(\Omega_{\mathrm{ref}})^3.$$

The reverse inequality is of course also true.

7.2. Proof of Theorem 7.1. We now prove a bound on the condition number $cond(T)$ by proving upper and lower bounds for a related generalized Rayleigh quotient. Since the subspace decomposition defining T satisfies the null space property (see, e.g., Smith, Bjørstad, and Gropp, [35, p. 132]), it is enough to prove upper and lower bounds locally on an interior element and then apply the standard Schwarz theory. For simplicity, we assume that the element is the reference element Ω_{ref}. We denote by $a_{n,\mathrm{ref}}(\cdot,\cdot)$ the restriction of $a_n(\cdot,\cdot)$ to Ω_{ref} and by F_k the faces of Ω_{ref}.

Consider $\mathbf{u} \in \mathbf{V}^n_{\mathcal{E}}$ and its decomposition into wire basket and face components. Locally (on the reference element), we have

$$\mathbf{u} = \mathbf{u}_0 + \sum_{k=1}^{6}\mathbf{u}_{F_k},$$

defined by $\mathbf{u}_0 = I^W \mathbf{u}$ and $\mathbf{u}_{F_k} = \mathcal{E}^n(\tilde{\mathbf{u}}_{F_k})$, where $\tilde{\mathbf{u}}_{F_k} = \left\{ \begin{array}{ll} \mathbf{u} - I^W \mathbf{u} & \text{on } F_k \\ 0 & \text{on } \partial\Omega_{\text{ref}} \setminus F_k \end{array} \right.$.
The lower bound can be written as

$$\tilde{a}_{0,\text{ref}}(\mathbf{u}_0, \mathbf{u}_0) + \sum_{k=1}^{6} a_{n,\text{ref}}(\mathbf{u}_{F_k}, \mathbf{u}_{F_k}) \leq C_1(1 + \log n)^2 a_{n,\text{ref}}(\mathbf{u}, \mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{V}^n_{\mathcal{E}},$$

and the upper bound as

$$a_{n,\text{ref}}(\mathbf{u}, \mathbf{u}) \leq C_2(\tilde{a}_{0,\text{ref}}(\mathbf{u}_0, \mathbf{u}_0) + \sum_{k=1}^{6} a_{n,\text{ref}}(\mathbf{u}_{F_k}, \mathbf{u}_{F_k}) \quad \forall \mathbf{u} \in \mathbf{V}^n_{\mathcal{E}}.$$

We note that if $\mathbf{u}$ is a rigid body motion, then there is nothing to prove because both sides of both inequalities vanish.

a) Lower bound. We first consider the wire basket term. From the definition of $\tilde{a}_{0,\text{ref}}$, equations (13) and (14), we have

$$\tilde{a}_{0,\text{ref}}(\mathbf{u}_0, \mathbf{u}_0) = (1 + \log n) \inf_{c_j} ||\mathbf{u} - \sum_{j=1}^{6} c_j \mathbf{r}_j||^2_{L^2(W_{\text{ref}})^3}$$

$$= (1 + \log n) \sum_{i=1}^{3} ||u^{(i)} - \frac{(u^{(i)}, 1)_{n,W_{\text{ref}}}}{(1,1)_{n,W_{\text{ref}}}} - \sum_{j=1}^{3} \frac{(u^{(i)}, x_j)_{n,W_{\text{ref}}}}{(x_j, x_j)_{n,W_{\text{ref}}}} x_j||^2_{L^2(W_{\text{ref}})},$$

where only two terms in the last sum differ from zero; consider the componentwise structure of the rotations given in (5). Using [29, Lemma 5.3] and Lemma 7.5, we can bound the i–th component of the displacement by

$$C(||u^{(i)} - \frac{(u^{(i)}, 1)_{n,W_{\text{ref}}}}{(1,1)_{n,W_{\text{ref}}}}||^2_{L^2(W_{\text{ref}})} + \sum_{j=1}^{3} (\frac{(u^{(i)}, x_j)_{n,W_{\text{ref}}}}{(x_j, x_j)_{n,W_{\text{ref}}}})^2 ||x_j||^2_{L^2(W_{\text{ref}})})$$

$$\leq C(1 + \log n) ||u^{(i)}||^2_{H^1(\Omega_{\text{ref}})}.$$

Therefore, by adding the three components and by using Korn's inequality, (Lemma 7.7), we obtain

$$\tilde{a}_{0,\text{ref}}(\mathbf{u}_0, \mathbf{u}_0) \leq C(1 + \log n)^2 ||\mathbf{u}||^2_{H^1(\Omega_{\text{ref}})^3}$$

$$\leq C(1 + \log n)^2 (a_{n,\text{ref}}(\mathbf{u}, \mathbf{u}) + ||\mathbf{u}||^2_{L^2(\Omega_{\text{ref}})^3}).$$

Since the left-hand side is invariant if $\mathbf{u}$ is shifted by a rigid body motion $\mathbf{r} \in \mathcal{N}$, we can remove the L^2-term by a quotient space argument:

$$\tilde{a}_{0,\text{ref}}(\mathbf{u}_0, \mathbf{u}_0) = \tilde{a}_{0,\text{ref}}((\mathbf{u} - \mathbf{r})_0, (\mathbf{u} - \mathbf{r})_0)$$

$$\leq C(1 + \log n)^2 (a_{n,\text{ref}}(\mathbf{u}, \mathbf{u}) + ||\mathbf{u} - \mathbf{r}||^2_{L^2(\Omega_{\text{ref}})^3}).$$

Therefore

$$\tilde{a}_{0,\mathrm{ref}}(\mathbf{u}_0,\mathbf{u}_0) \le C(1+\log n)^2 (a_{n,\mathrm{ref}}(\mathbf{u},\mathbf{u}) + \inf_{\mathbf{r}\in\mathcal{N}} \|\mathbf{u}-\mathbf{r}\|^2_{L^2(\Omega_{\mathrm{ref}})^3}),$$

and we conclude by using a Poincaré-type inequality; see Nečas [25].

Consider now each face term individually. Lemma 7.6 and Korn's inequality (Lemma 7.7) imply that

$$a_{n,\mathrm{ref}}(\mathbf{u}_{F_k},\mathbf{u}_{F_k}) \le a_{n,\mathrm{ref}}(\mathcal{H}^n\mathbf{u}_{F_k},\mathcal{H}^n\mathbf{u}_{F_k}) \le C|\mathcal{H}^n\mathbf{u}_{F_k}|^2_{H^1(\Omega_{\mathrm{ref}})^3}$$

$$= C\|\mathbf{u}_{F_k}\|^2_{H^{1/2}_{00}(F_k)^3} \le C(1+\log n)^2 |\mathbf{u}|^2_{H^1(\Omega_{\mathrm{ref}})^3}$$

$$\le C(1+\log n)^2 (a_{n,\mathrm{ref}}(\mathbf{u},\mathbf{u}) + \|\mathbf{u}\|^2_{L^2(\Omega_{\mathrm{ref}})^3}).$$

As before, the L^2-term can be removed with a quotient space argument, since the left-hand side is invariant if $\mathbf{u}$ is shifted by a rigid body motion $\mathbf{r}\in\mathcal{N}$.

b) Upper bound. We have

$$a_{n,\mathrm{ref}}(\mathbf{u},\mathbf{u}) \le 7(a_{n,\mathrm{ref}}(I^W\mathbf{u}, I^W\mathbf{u}) + \sum_{k=1}^{6} a_{n,\mathrm{ref}}(\mathbf{u}_{F_k},\mathbf{u}_{F_k})).$$

Since

$$a_{n,\mathrm{ref}}(I^W\mathbf{u}, I^W\mathbf{u})) \le C|\mathcal{H}^n(I^W\mathbf{u})|^2_{H^1(\Omega_{\mathrm{ref}})^3} = C\sum_{i=1}^{3} |\mathcal{H}^n(I^W u^{(i)})|^2_{H^1(\Omega_{\mathrm{ref}})},$$

there remains to bound each displacement component of the wire basket extension operator:

$$|\mathcal{H}^n(I^W u^{(i)})|^2_{H^1(\Omega_{\mathrm{ref}})} = |\mathcal{H}^n(\tilde{I}^W u^{(i)} + \sum_{k=1}^{6}(a_k\mathcal{F}_k^0 + b_k^1\mathcal{F}_k^1 + b_k^2\mathcal{F}_k^2 + b_k^3\mathcal{F}_k^3))|^2_{H^1(\Omega_{\mathrm{ref}})}.$$

Here only two of the weights b_k^j in each term of the sum differ from zero. By Lemma 7.4, this last quantity is bounded by

$$7(|\tilde{I}^W u^{(i)}|^2_{H^1(\Omega_{\mathrm{ref}})} + \sum_{k=1}^{6}(a_k^2|\mathcal{H}^n(\mathcal{F}_k^0)|^2_{H^1(\Omega_{\mathrm{ref}})} + \sum_{j=1}^{3}(b_k^j)^2|\mathcal{H}^n(\mathcal{F}_k^j)|^2_{H^1(\Omega_{\mathrm{ref}})})$$

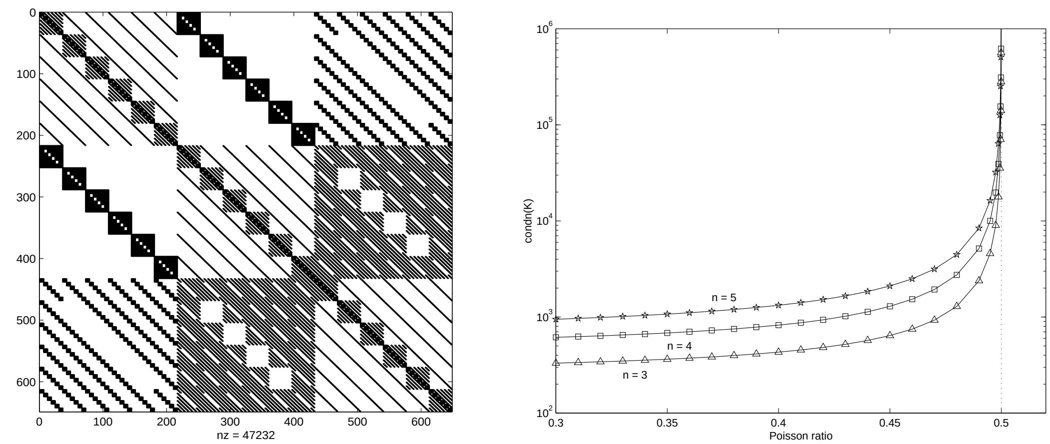
$$\le C(\|u^{(i)}\|^2_{L^2(W_{\mathrm{ref}})} + (1+\log n)\sum_{k=1}^{6}(a_k^2 + (b_k^1)^2 + (b_k^2)^2 + (b_k^3)^2)$$

$$\le C(1+\log n)\|u^{(i)}\|^2_{L^2(W_{\mathrm{ref}})}.$$

The last inequality holds because the coefficients, a_k^2, $(b_k^j)^2$, are all bounded by

$$((u^{(i)})^2, 1)_{n,\partial F_k} \le ((u^{(i)})^2, 1)_{n,W_{\mathrm{ref}}} \le C\|u^{(i)}\|^2_{L^2(W_{\mathrm{ref}})};$$

FIG. 1. *Left: sparsity pattern of the stiffness matrix K, for $n = 5$, with the lexicographic order for each component of the displacement.*
Right: semi-logarithmic plot of the condition number $cond(K)$ of K for $n = 3, 4, 5$ as a function of the Poisson ratio ν; when $\nu \to 0.5, cond(K) \to \infty$.



see the proof of Lemma 7.5. Therefore,

$$|I^W \mathbf{u}|^2_{H^1(\Omega_{\mathrm{ref}})^3} \leq C(1 + \log n)||\mathbf{u}||^2_{L^2(W_{\mathrm{ref}})^3}$$

and

$$a_{n,\mathrm{ref}}(\mathbf{u}, \mathbf{u}) \leq C((1 + \log n)||\mathbf{u}||^2_{L^2(W_{\mathrm{ref}})^3} + \sum_{k=1}^{6} a_{n,\mathrm{ref}}(\mathbf{u}_{F_k}, \mathbf{u}_{F_k})).$$

If we shift $\mathbf{u}$ by the rigid body motion $\mathbf{r} = \sum_{j=1}^{6} c_j \mathbf{r}_j$, with the coefficients c_j given by (13), we can conclude that

$$a_{n,\mathrm{ref}}(\mathbf{u}, \mathbf{u}) \leq C((1 + \log n)||\mathbf{u} - \mathbf{r}||^2_{L^2(W_{\mathrm{ref}})^3} + \sum_{k=1}^{6} a_{n,\mathrm{ref}}(\mathbf{u}_{F_k}, \mathbf{u}_{F_k}))$$

$$= C(\tilde{a}_{0,\Omega_{\mathrm{ref}}}(\mathbf{u}_0, \mathbf{u}_0) + \sum_{k=1}^{6} a_{n,\mathrm{ref}}(\mathbf{u}_{F_k}, \mathbf{u}_{F_k})).$$

□

8. Numerical results. In this section, we report on the results of a numerical study of the local condition number using the wire basket preconditioner. All computations were carried out in Matlab 5.0. We recall that S is the Schur complement of the stiffness matrix K for the discrete linear elasticity problem (3) and that $\widehat{S}$ denotes the wire basket preconditioner for S. The local contribution from an element Ω_i are denoted by $S^{(i)}$ and $\widehat{S}^{(i)}$, respectively. Figure 1, left panel, shows the sparsity pattern of K when $n = 5$ and the lexicographic order is used for each component of the displacement. The right panel is a semi-logarithmic plot of the condition number $cond(K)$, the ratio of the largest to the smallest nonzero eigenvalues of

TABLE 1

Local condition numbers for the wire basket method with original wire basket block $S_{WW}^{(i)}$

n	$\nu = 0.3$			$\nu = 0.4$		
	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m
2	12.2708	2.3493	0.1915	18.1491	2.7501	0.1515
3	17.4251	2.3915	0.1372	22.3718	2.8175	0.1259
4	24.9668	2.5550	0.1023	30.9845	2.9122	0.0940
5	34.0775	2.6995	0.0792	40.1506	3.0032	0.0748
6	42.5610	2.8032	0.0659	49.6922	3.0765	0.0619
7	52.6813	2.8805	0.0547	59.2671	3.1298	0.0528
8	61.3649	2.9369	0.0479	68.0866	3.1723	0.0466
9	70.3584	2.9810	0.0424	77.6509	3.2043	0.0413
10	78.2626	3.0163	0.0385	85.7246	3.2318	0.0377

TABLE 2

Local condition numbers for the wire basket method with approximate wire basket block $\widehat{S}_{WW}^{(i)}$

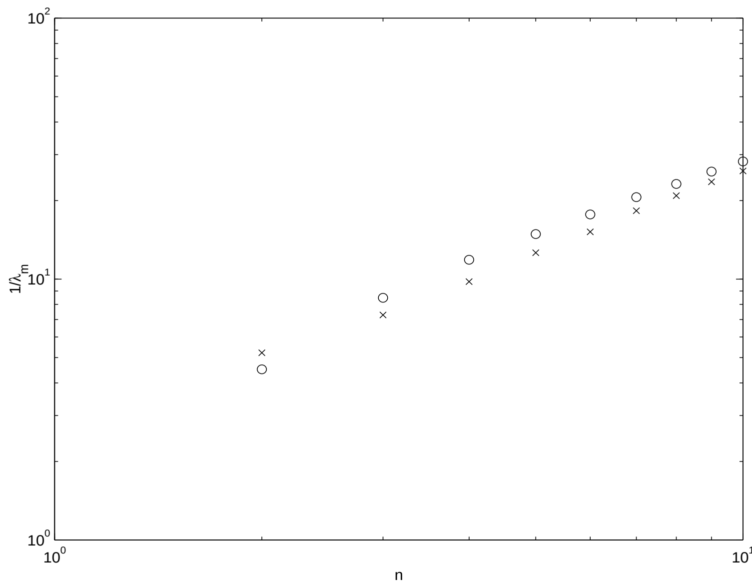
n	$\nu = 0.3$			$\nu = 0.4$		
	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m	$cond(\widehat{S}^{(i)^{-1}} S^{(i)})$	λ_M	λ_m
2	16.6074	3.6828	0.2218	26.3776	5.0205	0.1903
3	30.3822	3.5840	0.1180	51.1020	5.5494	0.1086
4	40.2729	3.3967	0.0843	65.3653	5.1190	0.0783
5	51.7355	3.4773	0.0672	80.7182	5.1366	0.0636
6	63.2516	3.5712	0.0565	93.0737	5.1010	0.0548
7	76.3325	3.6988	0.0485	110.8017	5.1580	0.0466
8	89.9607	3.8827	0.0432	124.2012	5.1945	0.0418
9	105.0605	4.0638	0.0387	139.9723	5.2592	0.0376
10	119.4171	4.2311	0.0354	153.3947	5.3092	0.0346

the stiffness matrix on the reference element for $n = 3, 4, 5$, which clearly shows the limitation of the pure displacement formulation for almost incompressible materials: when $\nu \to 0.5, cond(K) \to \infty$ and a mixed formulation should be used instead.

As we have pointed out in Sections 6 and 7.2, our wire basket algorithm satisfies the null space property and therefore the local condition number $cond(\widehat{S}^{(i)^{-1}} S^{(i)})$ for an interior element is an upper bound for the condition number of the whole preconditioned operator $cond(\widehat{S}^{-1} S)$. For an interior element, both $S^{(i)}$ and $\widehat{S}^{(i)}$ have the six-dimensional null space $\mathcal{N}$ spanned by the rigid body motions. The local condition numbers are computed as the ratio of the extreme eigenvalues λ_M and λ_m of $\widehat{S}^{(i)^{-1}} S^{(i)}$ in the space orthogonal to $\mathcal{N}$. Table 1 provides the local condition numbers and extreme eigenvalues for $\nu = 0.3$ and 0.4 when the wire basket preconditioner contains the original wire basket block $S_{WW}^{(i)}$. Table 2 provides the same quantities when the wire basket preconditioner contains the simplified wire basket block $\widehat{S}_{WW}^{(i)}$, defined in (11).

As in the scalar case, the simplified wire basket block is less expensive but yields higher condition numbers than the original block. In both cases, it is difficult to discern a difference between a linear and a polylogarithmic growth of the condition numbers. However, the log-log plot of Figure 2 seems to indicate that the growth of λ_m^{-1} is less than linear.

FIG. 2. *Log-log plot of $1/\lambda_m$ as a function of n for $\nu = 0.3$; '$\times$' denote values for the method with the original wire basket block, while 'o' denote values for the method with simplified wire basket block*



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