MULTILEVEL SCHWARZ METHODS FOR ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS IN THREE DIMENSIONS

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Abstract. Multilevel Schwarz methods are developed for a conforming finite element approximation of second order elliptic problems. We focus on problems in three dimensions with possibly large jumps in the coefficients across the interface separating the subregions. We establish a condition number estimate for the iterative operator, which is independent of the coefficients, and grows at most as the square of the number of levels. We also characterize a class of distributions of the coefficients, called quasi-monotone, for which the weighted L^2-projection is stable and for which we can use the standard piecewise linear functions as a coarse space. In this case, we obtain optimal methods, i.e. bounds which are independent of the number of levels and subregions. We also design and analyze multilevel methods with new coarse spaces given by simple explicit formulas. We consider nonuniform meshes and conclude by an analysis of multilevel iterative substructuring methods.

Key words. elliptic problems, Schwarz methods, multigrid methods, interface problems, preconditioned conjugate gradients

AMS(MOS) subject classifications. 65F10, 65N30, 65N55

1. Introduction.

The purpose of this paper is to develop multilevel methods for second order elliptic partial differential equations approximated by conforming finite element methods. A special emphasis is placed on problems in three dimensions with highly discontinuous coefficients. To simplify the presentation only piecewise linear finite elements are considered. Our goal is to design and analyze methods with a rate of convergence which is independent of the jumps of the coefficients, the number of sub-

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structures, and the number of levels.

We consider two classes of the methods, additive and multiplicative. The multiplicative methods are variants of the multigrid V-cycle method. In our design and analysis, we use a general Schwarz method framework developed in Dryja and Widlund [11,12,15], and Dryja, Smith, and Widlund [10] for the additive variant, and Bramble, Pasciak, Xu, and Wang [3] for the multiplicative ones. Among the particular cases, discussed in this paper, are the BPX algorithm, cf. Bramble, Pasciak and Xu [4], and the multilevel Schwarz method with one-dimensional subspaces considered by Zhang [29,30]; see also Dryja and Widlund [13,14]. It is well known that these methods are optimal when the coefficients are regular.

The problems become quite challenging for problems with highly discontinuous coefficients. In Dryja and Widlund [14], the BPX method was modified and applied to a Schur complement system obtained after that the unknowns of the interior nodal points of the substructures had been eliminated. In that case, the condition number of the preconditioned system was shown to be bounded from above by $C\,(1+\ell)^2$, where ℓ is the number of level of the refinement; see further Section 9.

The main question for problems with discontinuous coefficients is the choice of a coarse space. We introduce a coarse triangulation given by the substructures and assume that the coefficients can have large variations only across the interfaces of these substructures. We then design methods with several coarse spaces, sometimes known as *exotic coarse spaces*; cf. Widlund [25]. Some are new and others have previously been discussed; see Dryja, Smith and Widlund [10], Dryja and Widlund [15], and Sarkis [20]. One of our main results is that the condition number of the resulting systems can be estimated from above by $C\,(1+\ell)^2$ with C independent of the jumps of coefficients, of the number of substructures, and also of ℓ. For multiplicative variants such as the V-cycle multigrid, the rate of convergence is bounded from above by $1-C\,(1+\ell)^{-2}, C>0$.

In Section 5, we study in detail the weighted L^2 projection with weights given by the discontinuous coefficients of the elliptic problem. Bramble and Xu [5], and Xu [26] have considered this problem and established that the weighted L^2 projection is not always stable in the presence of interior cross points. In this paper, we introduce a new concept called *quasi-monotone distribution of coefficients* which characterizes cases for which certain optimal estimates for the weighted L^2 projection are possible. For problems with quasi-monotone coefficients, the standard piecewise linear functions can be used as the coarse space and optimal multilevel algorithms are obtained.

In Section 7, we introduce *approximate discrete harmonic extensions* and define new coarses spaces by modifying the previously known exotic coarse spaces; see Sarkis [20] for a case of nonconforming elements. Using these extensions, we can avoid solving a local Dirichlet problem for each substructure when using exotic coarse spaces. We show that the converge rate estimate of our new iterative methods, with approximate discrete harmonic extensions, are comparable to those using exact discrete harmonic extensions. The use of approximate discrete harmonic extensions results in algorithms where the work per iteration is linear in the number of degrees of freedom with the possible exception of the cost of solving the coarse problem.

Elliptic problems with discontinuous coefficients have solutions with singular behavior. Therefore, in Section 8, we consider nonuniform refinements. We begin with a coarse triangulation that is shape regular and possibly nonuniform and then refine it using a local refinement scheme analyzed by Bornemann and Yserentant [1]. We establish a condition number estimate for the iteration operator which is bounded from above by $C\,(1+\ell)^2$. For quasi-monotone coefficients, we obtain an optimal multilevel preconditioner.

2. Differential and Finite Element Model Problems. We consider the following selfadjoint second order problem:

Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v) \quad \forall\ v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \int_\Omega \rho(x)\,\nabla u \cdot \nabla v\,dx \quad \text{and} \quad f(v) = \int_\Omega f v\,dx \quad \text{for} \quad f \in L^2(\Omega).$$

For simplicity, let Ω be a bounded polyhedral region in $\Re^3$ with a diameter of order 1. A triangulation of Ω is introduced by dividing the region into nonoverlapping shape regular simplices $\{\Omega_i\}_{i=1}^N$, with diameters of order H, which are called substructures or subdomains. This partitioning induces a coarse triangulation associated with the parameter H. In Section 8, we consider a case in which the coarse triangulation is shape regular but possibly nonuniform.

We assume that $\rho(x) > 0$ is constant, in each substructure, with possibly large jumps occurring only across substructure boundaries. Therefore, $\rho(x) = \rho_i = \text{const}$ in each substructure Ω_i. The analysis of our methods can easily be extended to the case when $\rho(x)$ varies moderately in each subregion.

We define a sequence of quasi-uniform nested triangulations $\{\mathcal{T}^k\}_{k=0}^{\ell}$ as follows. We start with a coarse triangulation $\mathcal{T}^0 = \{\Omega_i\}_{i=1}^N$ and set $h_0 = H$. A triangulation $\mathcal{T}^k = \{\tau_j^k\}_{j=1}^{N^k}$ on level k is obtained by subdividing each individual element τ_j^{k-1} in the set $\mathcal{T}^{k-1}$ into several elements denoted by τ_j^k. We assume that all the triangulations are shape regular and quasi-uniform. Let $h_j^k = \text{diameter}(\tau_j^k)$, $h_k = \max_j h_j^k$, and $h = h_\ell$, where ℓ is the number of refinement levels. We also assume that there exist constants $\gamma < 1$, $c > 0$, and C, such that if an element τ_j^{n+k} of level $n+k$ is contained in an element τ_j^k of level k, then

$$c\gamma^n \le \frac{\mathrm{diam}(\tau_j^{n+k})}{\mathrm{diam}(\tau_j^k)} \le C\gamma^n.$$

In Section 8, we consider a case in which the refinement is *nonuniform.*

For each level of triangulation, we define a finite element space $V^k(\Omega)$ which is the space of continuous piecewise linear functions associated with the triangulation $\mathcal{T}^k$. Let $V_0^k(\Omega)$ be the subspace of $V^k(\Omega)$ of functions which vanish on $\partial\Omega$, the boundary of Ω. We also use the notation $V_0^h(\Omega) = V_0^\ell(\Omega)$.

The discrete problem associated with (1) is given by:

Find $u \in V_0^h(\Omega)$, such that

$$a(u,v) = f(v) \quad \forall\ v \in \ V_0^h(\Omega). \tag{2}$$

The bilinear form $a(u,v)$ is directly related to a weighted Sobolev space $H_\rho^1(\Omega)$ defined by the seminorm

$$|u|^2_{H^1_\rho(\Omega)} = a(u,u).$$

We also define a weighted L^2 norm by:

$$\|u\|^2_{L^2_\rho(\Omega)} = \int_\Omega \rho(x)\ |u(x)|^2\, dx \quad \text{for } u\ \in L^2(\Omega). \tag{3}$$

Let Σ be a region contained in Ω such that $\partial\Sigma$ does not cut through any element $\tau_j^k \in \mathcal{T}^k$. We denote by $V^k(\Sigma)$ the restriction of $V^k(\Omega)$ to $\bar{\Sigma}$, and by $V_0^k(\Sigma)$ the subspace of $V^k(\Sigma)$ of functions which vanish on $\partial\Sigma$. We also define $H_\rho^1(\Sigma)$ and $L_\rho^2(\Sigma)$ by restricting the domain of integration of the weighted norms to Σ. To avoid unnecessary notations, we drop the parameter ρ when $\rho = 1$, and Σ when the domain of integration is Ω.

In the case of a region Σ of diameter of order h_k, such as an element τ_j^k or the union of few elements, we use a weighted norm,

$$\|u\|^2_{H^1_\rho(\Sigma)} = |u|^2_{H^1_\rho(\Sigma)} + \frac{1}{h_k^2}\|u\|^2_{L^2_\rho(\Sigma)}. \tag{4}$$

We introduce the following notations: $u \preceq v$, $w \succeq x$, and $y \asymp z$ meaning that there are positive constants C and c such that

$$u \le C\,v, \quad w \ge c\,x \quad \text{and} \quad c\,z \le y \le C\,z, \quad \text{respectively.}$$

Here C and c are independent of the variables appearing in the inequalities and the parameters related to meshes, spaces and, especially, the weight ρ. Sometimes, we will use $\le$ to stress that $C = 1$.

3. Multilevel Additive Schwarz Method. Any Schwarz method can be defined by a splitting of the space V_0^h into a sum of subspaces, and by bilinear forms associated with each of these subspaces. We first consider certain multilevel methods based on the MDS-multilevel diagonal scaling introduced by Zhang [30], enriched with a coarse space as in Dryja and Widlund [15], Dryja, Smith, and Widlund [10], or Sarkis [20].

Let $\mathcal{N}^k$ and $\mathcal{N}_0^k$ be the set of nodes associated with the space V^k and V_0^k, respectively. Let ϕ_j^k be a standard nodal basis function of V_0^k, and let $V_j^k = \text{span}\{\phi_j^k\}$. We decompose V_0^h as

$$V_0^h = V_{-1}^X + \sum_{k=0}^{\ell} V_0^k = V_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} V_j^k.$$

We note that this decomposition is not a direct sum and that $\dim(V_j^k) = 1$. Four different types of coarse spaces V_{-1}^X, and associated bilinear forms $b_{-1}^X(u,u) : V_{-1}^X \times V_{-1}^X \rightarrow \Re$, with $X = F, E, NN$, and W, will be considered; see Section 4. We also consider variants of these four coarses spaces using spaces of approximate discrete harmonic extensions given by simple explicit formulas; see Section 7. The case when the coarse space is $V_0^0 = V_0^H$ is considered in Section 5.

We introduce operators $P_j^k : V_0^h \rightarrow V_j^k$, by

$$a(P_j^k u, v) = a(u,v) \quad \forall\ v \in\ V_j^k,$$

and an operator $T_{-1}^X : V_0^h \rightarrow V_{-1}^X$, by

$$b_{-1}^X(T_{-1}^X u, v) = a(u,v) \quad \forall\ v \in\ V_{-1}^X.$$

The analysis can easily be extended to the case when we use approximate solvers for the spaces V_j^k. Thus, we do not need to save, in memory, or recompute, all the values of $a(\phi_j^k, \phi_j^k)$, for $k = 1, \cdots, \ell$, and $\forall j \in \mathcal{N}_0^k$.

Let

$$T^X = T_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k. \tag{5}$$

We now replace (2) by

$$T^X u = g, \quad g = T_{-1}^X u + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k u. \tag{6}$$

By construction, (2) and (6) have the same solution. We notice that $T_{-1}^X u$ (and similarly the $P_j^k u$) can be computed, without the knowledge of u, the solution of (2), since we can find $T_{-1}^X u$ by solving

$$b_{-1}^X(T_{-1}^X u, v) = a(u,v) = f(v) \ \ \forall v \in\ V_{-1}^X. \tag{7}$$

Equation (6) is typically solved by a conjugate gradient method. In order to estimate its rate of convergence, we need to obtain upper and lower bounds for the spectrum of T^X. The bounds are obtained by using the following theorem; cf. Dryja and Widlund [15], Zhang [30,29].

THEOREM 1. *Suppose the following three assumptions hold:*

i) There exists a constant C_0 such that for all $u \in V_0^h$ there exists a decomposition $u = u_{-1} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} u_j^k$, with $u_{-1} \in V_{-1}^X$, $u_j^k \in V_j^k$, such that

$$b_{-1}^X(u_{-1}^X, u_{-1}^X) + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} a(u_j^k, u_j^k) \le C_0^2 a(u,u).$$

ii) There exists a constant ω such that

$$a(u,u) \le \omega\, b_{-1}^X(u,u) \quad \forall u \in V_{-1}^X.$$

iii) There exist constants ϵ_{ij}^{mn}, $m,n = 0, \cdots, \ell$ and $\forall i \in \mathcal{N}_0^m$, $\forall j \in \mathcal{N}_0^n$ such that

$$a(u_i^m, u_j^n) \le \epsilon_{ij}^{mn}\, a(u_i^m, u_i^m)^{1/2} a(u_j^n, u_j^n)^{1/2}$$

$$\forall u_i^m \in V_i^m \quad \forall u_j^n \in V_j^n.$$

Then, T is invertible, $a(Tu, v) = a(u, Tv)$, and

$$(8) \qquad C_0^{-2} a(u,u) \le a(Tu,u) \le (\rho(\mathcal{E}) + 1)\omega a(u,u) \quad \forall u \in V_0^h.$$

Here $\rho(\mathcal{E})$ is the spectral radius of the tensor $\mathcal{E} = \{\epsilon_{ij}^{mn}\}_{i,j,m,n=0}^{\ell}$.

4. Exotic Coarse Spaces and Condition Numbers.

We now introduce certain geometrical objects in preparation for the description of our exotic coarse spaces V_{-1}^X. Let $\mathcal{F}_{ij}$ represent the open *face* which is shared by two substructures Ω_i and Ω_j. Let $\mathcal{E}_l$ represent an open *edge*, and $\mathcal{V}_m$ a *vertex* of the substructure Ω_i. Let $\mathcal{W}_i$ denote the *wire basket* of the subdomain Ω_i, i.e. the union of the closures of the edges of $\partial\Omega_i$. We denote the *interface* between the subdomains by $\Gamma = \cup \partial\Omega_i \backslash \partial\Omega$, and the *wire basket* by $\mathcal{W} = \cup \mathcal{W}_i \backslash \partial\Omega$. The sets of nodes belonging to $\partial\Omega$, $\bar{\Omega}_i$, $\partial\Omega_i$, $\mathcal{F}_{ij}$, $\mathcal{E}_l$, $\mathcal{W}_i$, and Γ are denoted by $\partial\Omega_h$, $\partial\Omega_{i,h}$, $\bar{\Omega}_{i,h}$, $\mathcal{F}_{ij,h}$, $\mathcal{E}_{l,h}$, $\mathcal{W}_{i,h}$, and Γ_h, respectively.

We now proceed to discuss several alternative coarse spaces and to establish bounds for the condition numbers of the corresponding additive multilevel methods.

4.1. Neumann-Neumann coarse spaces. We first consider the Neumann-Neumann coarse spaces which have been analyzed in Dryja and Widlund [15], Mandel and Brezina [16], and Sarkis [20]. An interesting feature of these coarse spaces is that they are of minimal dimension with only one degree of freedom per substructure, even in the case when the substructures are not simplices. For any $\beta \geq 1/2$, we introduce the weighted counting functions $\mu_{i,\beta}$, for all $i = 1, \cdots, N$, defined by

$$\mu_{i,\beta}(x) = \sum_j \rho_j^\beta, \; x \in \partial\Omega_{i,h} \backslash \partial\Omega_h, \quad \mu_{i,\beta}(x) = 0, \; x \in (\Gamma_h \backslash \partial\Omega_{i,h}) \cup \partial\Omega_h.$$

For each $x \in \partial\Omega_{i,h} \backslash \partial\Omega_h$, the sum is taken over the values of j for which $x \in \partial\Omega_{j,h}$.

The pseudo inverse $\mu_{i,\beta}^+$ of $\mu_{i,\beta}$ is defined by

$$\mu_{i,\beta}^+(x) = (\mu_{i,\beta}(x))^{-1}, \; x \in \partial\Omega_{i,h} \backslash \partial\Omega_h,$$

and

$$\mu_{i,\beta}^+(x) = 0, \; x \in (\Gamma_h \backslash \partial\Omega_{i,h}) \cup \partial\Omega_h.$$

We extend $\mu_{i,\beta}^+$ elsewhere in Ω as a minimal energy, *discrete harmonic* function using the values on $\Gamma_h \cup \partial\Omega_h$ as boundary values. The resulting functions belong to $V_0^h(\Omega)$ and are also denoted by $\mu_{i,\beta}^+$.

We can now define the coarse space $V_{-1}^{NN} \subset V_0^h$ by

$$V_{-1}^{NN} = \text{span}\{\rho_i^\beta \, \mu_{i,\beta}^+\}, \tag{9}$$

i.e. we use one basis functions for each substructure Ω_i. We remark that we can even define a Neumann-Neumann coarse space for $\beta = \infty$ by considering, the limit of the space V_{-1}^{NN} when β approaches ∞, i.e.

$$\rho_i^\infty \, \mu_{i,\infty}^+ := \lim_{\beta \to \infty} \rho_i^\beta \, \mu_{i,\beta}^+.$$

For instance, for $x \in \mathcal{F}_{ij,h}$, we obtain

$$\rho_i^\infty \, \mu_{i,\infty}^+(x) = 1, \quad \text{if} \;\; \rho_i > \rho_j,$$

$$\rho_i^\infty \, \mu_{i,\infty}^+(x) = 0, \quad \text{if} \;\; \rho_i < \rho_j,$$

and

$$\rho_i^\infty \, \mu_{i,\infty}^+(x) = 1/2, \quad \text{if} \;\; \rho_i = \rho_j.$$

We note that V_{-1}^{NN} is also the range of an *interpolator* $I_h^{NN} : V_0^h \rightarrow V_{-1}^{NN}$, given by

$$u_{-1} = I_h^{NN} u(x) = \sum_i u_{-1}^{(i)} = \sum_i \bar{u}_i \, \rho_i^{\beta} \mu_{i,\beta}^{+}. \tag{10}$$

Here, $\bar{u}_i$ is the average of the discrete values of u over $\partial\Omega_{i,h}$.

We note the coarse spaces defined with $\beta = 1/2$, $\beta = 1$, and $\beta \geq 1/2$ have been used by Dryja and Widlund [15], Mandel and Brezina [16], and Sarkis [20], respectively. Recently, Wang and Xie [24] introduced another coarse space which is similar to ours with $\beta = \infty$. However, their basis functions only take the value 0 or 1 on Γ_h.

We introduce the bilinear form $b_{-1}^{NN}(u, v) : V_{-1}^{NN} \times V_{-1}^{NN} \rightarrow \Re$, by

$$b_{-1}^{NN}(u, v) = a(u, v). \tag{11}$$

THEOREM 2. *Let T^{NN} be defined by (5), and let $1/2 \leq \beta \leq \infty$. Then for any $u \in V_0^h(\Omega)$, we have*

$$(1 + \ell)^{-2} a(u, u) \preceq a(T^{NN} u, u) \preceq a(u, u).$$

The bounds are also independent of β.

Proof. We use Theorem 1.

Assumption i). For notational convenience, we introduce, for $k = 0, \cdots, \ell$, the bilinear forms $b_k(u_k, u_k)$: $V_0^k \times V_0^k \rightarrow \Re$, by

$$b_k(u_k, u_k) = \sum_{j \in \mathcal{N}_0^k} u_k^2(x_j) a(\phi_j^k, \phi_j^k) = \sum_{j \in \mathcal{N}_0^k} a(u_j^k, u_j^k), \tag{12}$$

We use a level k decomposition given by $u_k = \sum_j u_j^k$, with $u_j^k = u_k(x_j)\phi_j^k$. Here, x_j is the position of the node $j \in \mathcal{N}_0^k$.

We decompose $u \in V_0^h(\Omega)$ as

$$u = \mathcal{H}u + \mathcal{P}u \;\; \text{in } \Omega,$$

where

$$u = \mathcal{H}^{(i)} u + \mathcal{P}^{(i)} u \;\; \text{in } \Omega_i. \tag{13}$$

Here, $\mathcal{H}^{(i)} u$ is the discrete harmonic part of u, i.e.

$$(\nabla \mathcal{H}^{(i)} u, \nabla v)_{L^2(\Omega_i)} = 0 \;\; \forall v \in V_0^h(\Omega_i),$$

$$\mathcal{H}^{(i)} u = u \text{ on } \partial\Omega_i,$$

and $\mathcal{P}^{(i)}u \in V_0^h(\Omega_i)$ is the H^1-projection associated with the space $V_0^h(\Omega_i)$, i.e.

$$(\nabla(\mathcal{P}^{(i)}u), \nabla\phi)_{L^2(\Omega_i)} = (\nabla u, \nabla\phi)_{L^2(\Omega_i)} \quad \forall\phi \in V_0^h(\Omega_i).$$

We decompose $\mathcal{P}^{(i)}u$ and $\mathcal{H}^{(i)}u$ separately. We start by decomposing $v^{(i)} = \mathcal{P}^{(i)}u$ in Ω_i as

$$v^{(i)} = \mathcal{P}_0^{(i)}v^{(i)} + (\mathcal{P}_1^{(i)} - \mathcal{P}_0^{(i)})\, v^{(i)} + \cdots + (\mathcal{P}_\ell^{(i)} - \mathcal{P}_{\ell-1}^{(i)})\, v^{(i)},$$

where $\mathcal{P}_k^{(i)} : V_0^h(\Omega_i) \to V_0^k(\Omega_i)$, is the H^1-projection defined by

$$(\nabla(\mathcal{P}_k^{(i)}v^{(i)}), \nabla\phi)_{L^2(\Omega_i)} = (\nabla v^{(i)}, \nabla\phi)_{L^2(\Omega_i)} \quad \forall\phi \in V_0^k(\Omega_i).$$

We extend $\mathcal{P}_k^{(i)}v^{(i)}$ by zero to $\Omega\backslash\bar{\Omega}_i$, and also denote this extension by $\mathcal{P}_k^{(i)}\, v^{(i)}$. Thus, $\mathcal{P}_k^{(i)}v^{(i)} \in V_0^k(\Omega)$. Let

$$v_0^{(i)} = \mathcal{P}_0^{(i)}v^{(i)}, \quad v_k^{(i)} = (\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)}, \ k = 1, \cdots, \ell.$$

Hence

$$v^{(i)} = v_0^{(i)} + v_1^{(i)} + \cdots + v_k^{(i)} + \cdots + v_{\ell-1}^{(i)} + v_\ell^{(i)}. \tag{14}$$

We use the decomposition (14) for all $i = 1, \cdots, N$. The global decomposition of v is equal to $v^{(i)}$ in Ω_i, and is defined by

$$v = \sum_{k=0}^{\ell} v_k, \quad v_k = \sum_{i=1}^{N} v_k^{(i)}. \tag{15}$$

We now decompose $\mathcal{H}u$. Let

$$w = \mathcal{H}u - u_{-1},$$

where u_{-1} is defined in (10).

We decompose w as

$$w = \sum_{i=1}^{N} w^{(i)},$$

where

$$w^{(i)} = I_h(u\, \rho_i^\beta \mu_{i,\beta}^+) - \bar{u}_i\, \rho_i^\beta \mu_{i,\beta}^+ = I_h(\rho_i^\beta \mu_{i,\beta}^+\, (u - \bar{u}_i)) \text{ on } \ \Gamma \cup \partial\Omega$$

and extended as a discrete harmonic function in each Ω_j, $j = 1, \cdots, N$. Here, I_h is the standard linear interpolant based on the h-triangulation of Γ. It is easy to show that $w(x) = \sum_i w^{(i)}(x) \ \ \forall x \in \bar{\Omega}$. Note that the support of $w^{(i)}$ is the union of the $\bar{\Omega}_j$ which have a vertex, edge, or face in common with Ω_i.

We decompose $w^{(i)}$ further as

$$w^{(i)} = \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} w^{(i)}_{\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l \subset \partial\Omega_i} w^{(i)}_{\mathcal{E}_l} + \sum_{\mathcal{V}_m \subset \partial\Omega_i} w^{(i)}_{\mathcal{V}_m}, \tag{16}$$

where $\mathcal{F}_{ij}, \mathcal{E}_l$, and $\mathcal{V}_m$ are the faces, edges, and vertices of $\partial\Omega_i$. Here, $w^{(i)}_{\mathcal{F}_{ij}}, w^{(i)}_{\mathcal{E}_l}$, and $w^{(i)}_{\mathcal{V}_m}$ are the discrete harmonic extensions into Ω with possibly nonzero interface values only on $\mathcal{F}_{ij,h}$, $\mathcal{E}_{l,h}$, and $\mathcal{V}_m$, respectively. The support of each function on the right hand side of (16) is the union of a few $\bar{\Omega}_j$ and its interior is denoted by $\Omega_{\mathcal{F}_{ij}}$, $\Omega_{\mathcal{E}_l}$, or $\Omega_{\mathcal{V}_m}$, respectively. We can assume that the regions $\Omega_{\mathcal{F}_{ij}}$, $\Omega_{\mathcal{E}_l}$, and $\Omega_{\mathcal{V}_m}$ are convex. If not, we can extend them to convex regions and use a trick developed in Lemma 3.6. of Zhang [30].

We now decompose $w^{(i)}_{\mathcal{F}_{ij}}$, $w^{(i)}_{\mathcal{E}_\ell}$, and $w^{(i)}_{\mathcal{V}_m}$ in the same way as the $v^{(i)}$. Let us first consider $w^{(i)}_{\mathcal{F}_{ij}}$. We obtain

$$w^{(i)}_{\mathcal{F}_{ij}} = w^{(i)}_{0,\mathcal{F}_{ij}} + w^{(i)}_{1,\mathcal{F}_{ij}} + \cdots + w^{(i)}_{k,\mathcal{F}_{ij}} + \cdots + w^{(i)}_{\ell-1,\mathcal{F}_{ij}} + w^{(i)}_{\ell,\mathcal{F}_{ij}}. \tag{17}$$

Here

$$w^{(i)}_{0,\mathcal{F}_{ij}} = \mathcal{P}^{(i)}_{0,\mathcal{F}_{ij}} w^{(i)}_{\mathcal{F}_{ij}}, \quad w^{(i)}_{k,\mathcal{F}_{ij}} = (\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} - \mathcal{P}^{(i)}_{k-1,\mathcal{F}_{ij}})\, w^{(i)}_{\mathcal{F}_{ij}}, \; k = 1, \cdots, \ell.$$

$\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} : V^h_0(\Omega_{\mathcal{F}_{ij}}) \to V^k_0(\Omega_{\mathcal{F}_{ij}})$, is the H^1-projection. As before, we extend $\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} w^{(i)}_{\mathcal{F}_{ij}}$ by zero outside $\Omega_{\mathcal{F}_{ij}}$.

We decompose $w^{(i)}_{\mathcal{E}_l}$ and $w^{(i)}_{\mathcal{V}_m}$ in the same way, and obtain

$$w^{(i)}_{\mathcal{E}_l} = w^{(i)}_{0,\mathcal{E}_l} + w^{(i)}_{1,\mathcal{E}_l} + \cdots + w^{(i)}_{k,\mathcal{E}_l} + \cdots + w^{(i)}_{\ell-1,\mathcal{E}_l} + w^{(i)}_{\ell,\mathcal{E}_l}, \tag{18}$$

where

$$w^{(i)}_{0,\mathcal{E}_l} = \mathcal{P}^{(i)}_{0,\mathcal{E}_l} w^{(i)}_{\mathcal{E}_l}, \quad w^{(i)}_{k,\mathcal{E}_l} = (\mathcal{P}^{(i)}_{k,\mathcal{E}_l} - \mathcal{P}^{(i)}_{k-1,\mathcal{E}_l})\, w^{(i)}_{\mathcal{E}_l}, \; k = 1, \cdots, \ell,$$

and

$$w^{(i)}_{\mathcal{V}_m} = w^{(i)}_{0,\mathcal{V}_m} + w^{(i)}_{1,\mathcal{V}_m} + \cdots + w^{(i)}_{k,\mathcal{V}_m} + \cdots + w^{(i)}_{\ell-1,\mathcal{V}_m} + w^{(i)}_{\ell,\mathcal{V}_m}. \tag{19}$$

Here,

$$w^{(i)}_{0,\mathcal{V}_m} = \mathcal{P}^{(i)}_{0,\mathcal{V}_m} w^{(i)}_{\mathcal{V}_m}, \quad w^{(i)}_{k,\mathcal{V}_m} = (\mathcal{P}^{(i)}_{k,\mathcal{V}_m} - \mathcal{P}^{(i)}_{k-1,\mathcal{V}_m})\, w^{(i)}_{\mathcal{V}_m}, \; k = 1, \cdots, \ell.$$

We can now define a global decomposition of the function w as

$$w = w_0 + w_1 + \cdots + w_k + \cdots + w_{\ell-1} + w_\ell, \tag{20}$$

where

$$w_k = \sum_i \Big(\sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{E}_l} + \sum_{\mathcal{V}_m \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{V}_m} \Big).$$

We check straightforwardly that (20) is valid $\forall x \in \Omega$. Using (20) and (15), we define a decomposition for $u \in V_0^h$ as

$$u = u_{-1} + u_0 + \cdots + u_k + \cdots + u_{\ell-1} + u_\ell, \tag{21}$$

where

$$u_k = v_k + w_k, \quad k = 0, \cdots, \ell.$$

We obtain the desired decomposition in Assumption i) by decomposing u_k as in (12). We will now prove that

$$\sum_{k=-1}^{\ell} b_k(u_k, u_k) \preceq (1 + \log(H/h))^2 a(u, u). \tag{22}$$

Therefore, in view of (12), we obtain $C_0^2 = C\,(1+\ell)^2$ in Assumption i) of Theorem 1.

We start with the decomposition of v.

LEMMA 1. *For the decomposition of v given by (15), we have*

$$\sum_{k=0}^{\ell} b_k(v_k, v_k) \preceq a(u, u). \tag{23}$$

Proof. We first show that

$$\sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq \rho_i \, |u|^2_{H^1(\Omega_i)}. \tag{24}$$

Note that for $k \geq 1$, $v_k^{(i)} = (\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)} = (I - \mathcal{P}_{k-1}^{(i)})\, v_k^{(i)}$. Hence,

$$b_k(v_k^{(i)}, v_k^{(i)}) \preceq \frac{1}{h_k^2}\, \rho_i \|v_k^{(i)}\|^2_{L^2(\Omega_i)} \preceq \rho_i \, |v_k^{(i)}|^2_{H^1(\Omega_i)}.$$

The last inequality follows from the well known error estimate for H^1-projections on convex domains; see Ciarlet [8]. For $k = 0$,

$$b_0(v_0^{(i)}, v_0^{(i)}) \preceq \frac{1}{h_0^2} \rho_i \, \|v_0^{(i)}\|^2_{L^2(\Omega_i)} \preceq \rho_i \, |v_0^{(i)}|^2_{H^1(\Omega_i)},$$

using Friedrichs' inequality.

Adding the above inequalities, we obtain

$$\sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq$$

$$\rho_i\{(\nabla \mathcal{P}_0^{(i)} v^{(i)}, \nabla v^{(i)})_{L^2(\Omega_i)} + \sum_{k=1}^{\ell} (\nabla(\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)}, \nabla v^{(i)})_{L^2(\Omega_i)}\}$$

$$= \rho_i\, \|\nabla v^{(i)}\|^2_{L^2(\Omega_i)} = \rho_i\, \|\nabla \mathcal{P}^{(i)} u\|^2_{L^2(\Omega_i)} \le \rho_i\, \|\nabla u\|^2_{L^2(\Omega_i)}.$$

Thus

$$\sum_{i=1}^{N} \sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq a(u,u).$$

□

LEMMA 2. *For the decomposition of* $w^{(i)}_{\mathcal{F}_{ij}}$*, given by (17), we have*

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{F}_{ij}}, w^{(i)}_{k,\mathcal{F}_{ij}}) \preceq (1+\ell)^2 \rho_i\, |u|^2_{H^1(\Omega_i)}. \tag{25}$$

Proof. Note that

$$b_k(w^{(i)}_{k,\mathcal{F}_{ij}}, w^{(i)}_{k,\mathcal{F}_{ij}}) \preceq \frac{1}{h_k^2}(\rho_i + \rho_j)\, \|w^{(i)}_{\mathcal{F}_{ij,k}}\|_{L^2(\Omega_{\mathcal{F}_{ij}})}.$$

Here, $\Omega_{\mathcal{F}_{ij}} = \Omega_i \cup \Omega_j \cup \mathcal{F}_{ij}$. Note that $w^{(i)}_{\mathcal{F}_{ij}} = 0$ in $\Omega \backslash \Omega_{\mathcal{F}_{ij}}$.

By the same argument as in the proof of Lemma 1, we obtain

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{F}_{ij}}, w^{(i)}_{k,\mathcal{F}_{ij}}) \preceq (\rho_i + \rho_j) |w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})}. \tag{26}$$

Note that

$$(\rho_i + \rho_j) |w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})} \preceq (\rho_i + \rho_j) \|w^{(i)}_{\mathcal{F}_{ij}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} =$$

$$(\rho_i + \rho_j) \|(I_h(\rho_i^{\beta} \mu^{+}_{i,\beta}(u - \bar{u}_i)))_{\mathcal{F}_{ij,h}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} =$$

$$\rho_i\, \frac{(1+\frac{\rho_j}{\rho_i})}{(1+(\frac{\rho_j}{\rho_i})^{\beta})^2}\, \|(u - \bar{u}_i)_{\mathcal{F}_{ij,h}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} \preceq \rho_i \|(u - \bar{u}_i)_{\mathcal{F}_{ij,h}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})}.$$

The first inequality above follows from an extension theorem [9]. Here, $(v)_{\mathcal{F}_{ij,h}} = v$ at the nodal points on $\mathcal{F}_{ij,h}$ and $(v)_{\mathcal{F}_{ij,h}} = 0$ on $\partial\mathcal{F}_{ij,h}$. For the last inequality, we have also used the fact that $\beta \geq 1/2$. For $\beta = \infty$, we use a limiting process. Using Lemma 3 of Dryja and Widlund [15], which is an inequality for $H_{00}^{1/2}(\mathcal{F}_{ij})$, and Poincaré's inequality, we obtain

$$\rho_i \|(u - \bar{u}_i)_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} \preceq \rho_i (1 + \log H/h)^2 \|u - \bar{u}_i\|^2_{H^1(\Omega_i)}$$

$$\preceq \rho_i (1 + \log H/h)^2 |u|^2_{H^1(\Omega_i)} \asymp \rho_i (1 + \ell)^2 |u|^2_{H^1(\Omega_i)}.$$

Combining this inequality with (26), we obtain (25). □

The next two lemmas are proved in the same way as Lemma 2; a similar argument will also be given in the proof of Theorem 4.

LEMMA 3. *For the decomposition of* $w^{(i)}_{\mathcal{E}_l}$*, given in (18), we have*

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{E}_l}, w^{(i)}_{k,\mathcal{E}_l}) \preceq (1 + \ell)\, \rho_i\, |u|^2_{H^1(\Omega_i)}. \tag{27}$$

LEMMA 4. *For the decomposition of* $w^{(i)}_{\mathcal{V}_m}$*, given in (19), we have*

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{V}_m}, w^{(i)}_{k,\mathcal{V}_m}) \preceq \rho_i\, |u|^2_{H^1(\Omega_i)}. \tag{28}$$

COROLLARY 1. *For the decomposition of* w*, given in (20), we have*

$$\sum_{k=0}^{\ell} b_k(w_k, w_k) \preceq (1 + \ell)^2 a(u, u). \tag{29}$$

The proof of Corollary 1 follows from Lemmas 2, 3, and 4.

We now estimate $a(u_{-1}, u_{-1})$.

LEMMA 5. *For* $u_{-1} = \sum_i u^{(i)}_{-1},\ u^{(i)}_{-1} = \bar{u}_i\, \rho_i^{\beta} \mu^{+}_{i,\beta}$,

$$a(u_{-1}, u_{-1}) \preceq (1 + \log H/h)^2 a(u, u) \asymp (1 + \ell)^2 a(u, u)$$

The proof of this result, for $\beta = 1/2$, can be found in the proofs of Theorem 6 and 7 of Dryja and Widlund [15]. For different values of β, we use an argument similar to that of the proof of Lemma 2; cf. also Sarkis [20].

Returning to the proof of Theorem 2, we find that (22) follows from Corollary 1, and Lemmas 1 and 5. The bound for C_0 has then been established.

Assumption ii). Trivially, we have $\omega = 1$.

Assumption iii) We need to show that $\rho(\mathcal{E}) \leq$ const. This has been established in Remark 3.3 in Zhang [30]. □

4.2. A face based coarse space. The next exotic coarse space is denoted by $V_{-1}^F \subset V_0^h$, and is based on values on the wire basket $\mathcal{W}_h$ and averages over the faces $\mathcal{F}_{ij}$. This coarse space can conveniently be defined as the range of an interpolation operator $I_h^F : V_0^h \to V_{-1}^F$, defined by

$$I_h^F u(x)_{|\bar{\Omega}_i} = \sum_{p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\varphi_p(x)$ is the discrete harmonic extension, into Ω_i, of the standard nodal basis function associated with a node p. $\bar{u}_{\mathcal{F}_{ij}}$ is the average value of u over $\mathcal{F}_{ij,h}$, and $\theta_{\mathcal{F}_{ij}}(x)$ the discrete harmonic function, defined in Ω_i, which equals 1 on $\mathcal{F}_{ij,h}$ and is zero on $\partial\Omega_{i,h} \backslash \mathcal{F}_{ij,h}$.

We define the bilinear form by

$$b_{-1}^F(u,u) = \sum_i \rho_i \{ \sum_{p \in \mathcal{W}_{i,h}} h\,(u(x_p) - \bar{u}_i)^2$$

$$+ \; H(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}} - \bar{u}_i)^2 \},$$

where $\bar{u}_i$ is the average value of u over $\partial\Omega_{i,h}$.

THEOREM 3. *Let T^F be defined by (5). Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^F u, u) \preceq a(u,u).$$

Proof. Assumption i). The decomposition here is the same as before, except that we now choose $w^{(i)} = I_h(\rho_i^{1/2}\mu_{i,1/2}^+(u - I_h^F u))$ on $\Gamma \cup \partial\Omega$, and extending these boundary values as a discrete harmonic function elsewhere. We note that this decomposition is simpler since $w^{(i)}$ vanishes on the wire basket. Therefore, $w_{\mathcal{E}_l}^{(i)} = 0$ and $w_{\mathcal{V}_m}^{(i)} = 0$. A counterpart of Lemma 2 holds since

$$(\rho_i + \rho_j)|w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 \preceq (\rho_i + \rho_j)\|(I_h(\rho_i^{1/2}\mu_{i,1/2}^+(u - I_F^h u)))_{\mathcal{F}_{ij,h}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2$$

$$\preceq \rho_i\,\|(u - \bar{u}_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 \preceq \rho_i\,(1+\log H/h)^2 \|u - \bar{u}_{\mathcal{F}_{ij}}\|_{H^1(\Omega_i)}^2$$

$$\preceq \rho_i\,(1+\log H/h)^2 |u|_{H^1(\Omega_i)}^2 \asymp \rho_i\,(1+\ell)^2 |u|_{H^1(\Omega_i)}^2.$$

Here we have used the same arguments as in the proof of Lemma 2. Finally, Lemma 5 is replaced by

LEMMA 6. *For $u \in V_0^h(\Omega)$*

$$b_{-1}^F(I_h^F u, I_h^F u) \preceq (1+\log H/h) a(u,u) \asymp (1+\ell) a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.7 of Dryja, Smith, and Widlund [10].

Assumption ii). We have $\omega \preceq 1$; see Dryja, Smith, and Widlund [10].

Assumption iii). As in subsection 4.1. □

REMARK 1. *Another possible decomposition for w is given by*

$$w = \sum_{\mathcal{F}_{ij} \subset \Gamma} w_{\mathcal{F}_{ij}}.$$

Here, $w_{\mathcal{F}_{ij}}$ is the discrete harmonic function on Ω with possibly nonzero interface values only on $\mathcal{F}_{ij,h}$. We note that the support of $w_{\mathcal{F}_{ij}}$ is $\bar{\Omega}_{\mathcal{F}_{ij}}$. We can decompose $w_{\mathcal{F}_{ij}}$ as in (17), and obtain

$$b_k(w_{k,\mathcal{F}_{ij}}, w_{k,\mathcal{F}_{ij}}) \preceq (\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} =$$

$$(\rho_i + \rho_j) \, \|(u - \bar{u}_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} \preceq (1 + \log H/h)^2 \|u - \bar{u}_{\mathcal{F}_{ij}}\|^2_{H^1_\rho(\Omega_{\mathcal{F}_{ij}})}$$

$$\preceq (1 + \log H/h)^2 |u|^2_{H^1_\rho(\Omega)} \asymp (1 + \ell)^2 |u|^2_{H^1_\rho(\Omega)}.$$

4.3. An edge based coarse space. We can decrease the dimension of V^F_{-1} and define another coarse space. Rather than using the values at all the nodes on the edges as degrees of freedom, only one degree of freedom per edge, an average value, is used. The resulting space, denoted by $V^E_{-1} \subset V^h_0$, is the range of the interpolation operator $I^E_h : V^h_0 \rightarrow V^E_{-1}$, defined by

$$I^E_h u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{V}_m \in \partial\Omega_i} u(\mathcal{V}_m) \varphi_m(x) +$$

$$\sum_{\mathcal{E}_l \subset \mathcal{W}_i} \bar{u}_{\mathcal{E}_l} \theta_{\mathcal{E}_l}(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\bar{u}_{\mathcal{E}_l}$ is the average value of u over $\mathcal{E}_{l,h}$, and $\theta_{\mathcal{E}_l}$ the discrete harmonic function which equals 1 on $\mathcal{E}_{l,h}$ and is zero on $\partial\Omega_{i,h} \setminus \mathcal{E}_{l,h}$. $\varphi_m(x)$ is the discrete harmonic extension into Ω_i of the boundary values of standard nodal basis function associated with the vertex $\mathcal{V}_m$.

We define a bilinear form by

$$b^E_{-1}(u, u) = \sum_i \rho_i \{ h \sum_{\mathcal{V}_m \in \partial\Omega_i} (u(\mathcal{V}_m) - \bar{u}_i)^2 +$$

$$H \sum_{\mathcal{E}_m \subset \partial\Omega_i} (\bar{u}_{\mathcal{E}_m} - \bar{u}_i)^2 + H(1 + \ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}} - \bar{u}_i)^2 \}.$$

THEOREM 4. *Let T^E be defined by (5). Then, for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^E u, u) \preceq a(u,u).$$

Proof. Assumption i). Here, we choose $w^{(i)} = I_h(\rho_i^{1/2}\mu_{i,1/2}^{+}(u - I_h^E u))$ on $\Gamma \cup \partial\Omega$ and extend these boundary values as a discrete harmonic function elsewhere. Note also that, we have $w_{\mathcal{V}_m}^{(i)} = 0$. The proof of a counterpart of Lemma 2 is similar to that given in the proof of Theorem 3.

The proof of a variant of Lemma 3 for this case proceeds as follows. Let $w_{\mathcal{E}_l}^{(i)}$ be the edge component of $w^{(i)}$, given similarly as in (16), and let $w_{k,\mathcal{E}_l}^{(i)}$ be the decomposition of $w_{\mathcal{E}_l}^{(i)}$ given as in (18). Using similar arguments as in Lemma 2, we have

$$\sum_{k=0}^{\ell} b_k(w_{k,\mathcal{E}_l}^{(i)}, w_{k,\mathcal{E}_l}^{(i)}) \preceq \sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2,$$

where the sum $\sum_m$ is taken over all substructures, which share the open edge $\mathcal{E}_l$. We now use the fact that $\mu_{i,1/2}^{+} = (\sum_m \rho_m^{1/2})^{-1}$ on $\mathcal{E}_{l,h}$, and an inverse inequality, and obtain

$$\sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2 \preceq \rho_i \sum_{p \in \mathcal{E}_{l,h}} h\,(\bar{u}_{\mathcal{E}_l} - u(x_p))^2.$$

We next use a Sobolev type inequality (see Lemma 4.3 of [10]) to obtain

$$\rho_i \sum_{p \in \mathcal{E}_{l,h}} h\,(\bar{u}_{\mathcal{E}_l} - u(x_p))^2 \preceq \rho_i\,(1 + \log H/h)\|u\|_{H^1(\Omega_i)}^2 \asymp \rho_i\,(1+\ell)\|u\|_{H^1(\Omega_i)}^2.$$

We note that $w_{\mathcal{E}_l}^{(i)}$ does not change if we add a constant to u. Therefore, we can use Poincaré's inequality to obtain

$$\sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2 \preceq \rho_i\,(1+\ell)|u|_{H^1(\Omega_i)}^2. \tag{30}$$

Finally, we use

LEMMA 7. *For $u \in V_0^h$*

$$b_{-1}^E(I_h^E u, I_h^E u) \preceq (1+\ell)\,a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.10 of Dryja, Smith, and Widlund [10].

Assumption ii). We have $\omega \preceq 1$; see Dryja, Smith, Widlund [10].

Assumption iii). As in subsection 4.1. □

REMARK 2. *We can also simplify the proof by decomposing w as*

$$w = \sum_{\mathcal{F}_{ij} \subset \Gamma} w_{\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l \subset \Gamma} w_{\mathcal{E}_l}.$$

Here, $w_{\mathcal{F}_{ij}}$ is chosen as in Remark 1 and $w_{\mathcal{E}_l}$ is the piecewise discrete harmonic function with possibly nonzero interface values only on $\mathcal{E}_{l,h}$. We note that the support of $w_{\mathcal{E}_l}$ is in $\bar{\Omega}_{\mathcal{E}_l}$. We decompose $w_{\mathcal{E}_l}$ as in (18) and obtain

$$\sum_m \rho_m |w_{\mathcal{E}_l}|^2_{H^1(\Omega_{\mathcal{E}_l})} \preceq \sum_m \rho_m \sum_{p\in\mathcal{E}_{l,h}} h\left(\bar{u}_{\mathcal{E}_l} - u(x_p)\right)^2$$

$$\preceq (1+\ell)\,|u|^2_{H^1_\rho(\Omega_{\mathcal{E}_l})}.$$

4.4. A wire basket based coarse space. Finally, we consider a coarse space $V_{-1}^W \subset V_0^h$, due to Smith [21]. It is based only on the values on the wire basket $\mathcal{W}_h$. The interpolation operator $I_h^W : V_0^h \to V_{-1}^W$, and is defined by

$$I_W^h u(x)_{|\bar{\Omega}_i} = \sum_{p\in\mathcal{W}_{i,h}} u(x_p)\varphi_p(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \bar{u}_{\partial\mathcal{F}_{ij}}\theta_{\mathcal{F}_{ij}}(x).$$

Here, $\bar{u}_{\partial\mathcal{F}_{ij}}$ is the discrete average value of u on $\partial\mathcal{F}_{ij,h}$. Let $\bar{u}_{\mathcal{W}_i}$ be the discrete average value of u on $\mathcal{W}_{i,h}$. We define the bilinear form by

$$b_{-1}^W(u,u) = (1+\ell)\sum_i \rho_i \sum_{p\in\mathcal{W}_{i,h}} h(u(p) - \bar{u}_{\mathcal{W}_i})^2.$$

THEOREM 5. *Let T^W be defined by (5). Then, for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2}a(u,u) \preceq a(T^W u,u) \preceq a(u,u).$$

Proof. Assumption i). Let $w^{(i)} = I_h(\rho_i^{1/2}\mu_{i,1/2}^+(u - I_h^W u))$. Therefore, we have $w^{(i)}_{\mathcal{E}_l} = 0$ and $w^{(i)}_{\mathcal{V}_m} = 0$. The proof of the counterpart of Lemma 2 is as follows:

$$(\rho_i+\rho_j)|w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})} \preceq \rho_i\, \|(I_h((u-\bar{u}_{\partial\mathcal{F}_{ij}})_{\mathcal{F}_{ij}}\,\theta_{\mathcal{F}_{ij}}))_{\mathcal{F}_{ij}}\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})}$$

$$\preceq \rho_i\{\|I_h(u\,\theta_{\mathcal{F}_{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})} + (\bar{u}_{\partial\mathcal{F}_{ij}})^2\|I_h(\theta_{\mathcal{F}_{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}_{ij})}\}$$

$$\preceq \rho_i\,\{(1+\log H/h)^2\|u\|^2_{H^1(\Omega_i)} + \frac{1}{H}\,(1+\log H/h)\|u\|^2_{L^2(\partial\mathcal{F}_{ij})}\}$$

$$\preceq \rho_i(1+\log H/h)^2\|u\|^2_{H^1(\Omega_i)} \asymp \rho_i(1+\ell)^2\|u\|^2_{H^1(\Omega_i)}$$

Here we have used Lemma 3, and Lemmas 4.3, 4.4, and 4.5 of Dryja, Smith, and Widlund [10].

To get the seminorm bound, we use the same arguments as for (30). Finally, we also use

LEMMA 8. *For* $u \in V_0^h(\Omega)$

$$b_{-1}^W(I_h^W u, I_h^W u) \preceq (1 + \log H/h)^2 a(u,u) \asymp (1+\ell)^2 a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.4 of Dryja, Smith, and Widlund [10].

Assumption ii). We have $\omega \preceq 1$; see Dryja, Smith, Widlund [10].

Assumption iii). As in subsection 4.1. □

Remark 1 also applies in this case.

5. Special Coefficients and an Optimal Algorithm. In this section, we show that if the coefficients ρ_i satisfy certain assumptions, the $L^2_\rho-$projection is stable and we can use the space of piecewise linear functions $V^H(\Omega)$ as a coarse space and obtain an *optimal* multilevel preconditioner. It should be pointed out that the $L^2_\rho-$projection is not stable in general; see the counterexample given in Xu [26].

5.1. Quasi-monotone coefficients. Let $\mathcal{V}_m, m = 1, \cdots, L$, be the set of substructure vertices. We also include the vertices on $\partial\Omega$. Let $\Omega_{m_i},\ i = 1, \cdots, s(m)$, denote the substructures that have the vertex $\mathcal{V}_m$ in common, and let ρ_{m_i} denote their coefficients. Let Ω'_m be the interior of the closure of the union of the substructures Ω_{m_i}, i.e. the interior of $\cup_{i=1}^{s(m)} \bar{\Omega}_{m_i}$. By using the fact that all substructures are simplices, we see that each Ω_{m_i} has a whole face in common with $\partial\Omega'_m$. Thus, the vertex $\mathcal{V}_m$ is the only internal cross point in $\bar{\Omega}'_m$, i.e. the only point that belongs to more than two $\bar{\Omega}_{m_i}$. Two-dimensional illustrations of $\bar{\Omega}'_m = \cup_i \bar{\Omega}_{m_i}$ are given by Fig. 1 and Fig. 2.

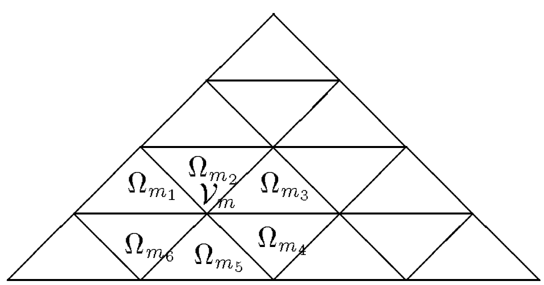

Fig. 1

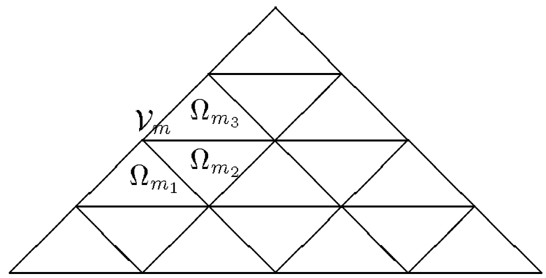

Fig. 2

DEFINITION 1. *For each Ω'_m, order its substructures such that $\rho_{m_1} = \max_{i:1,\cdots,s(m)} \rho_{m_i}$. We say that a distribution of ρ_{m_i} is quasi-monotone in Ω'_m if for every $i = 1, \cdots, s(m)$, there exists a sequence $i_j,\ j = 1, \cdots, R$, with*

$$\rho_{m_i} = \rho_{m_{i_R}} \leq \cdots \leq \rho_{m_{i_{j+1}}} \leq \rho_{m_{i_j}} \leq \cdots \leq \rho_{m_{i_1}} = \rho_{m_1}, \tag{31}$$

where the substructures $\Omega_{m_{i_j}}$ and $\Omega_{m_{i_{j+1}}}$ have a face in common. If the vertex $\mathcal{V}_m \in \partial\Omega$, then we additionally assume that $\partial\Omega_{m_1} \cap \partial\Omega$ contains a face for which $\mathcal{V}_m$ is a vertex.

A distribution ρ_i on Ω is quasi-monotone with respect to the coarse triangulation $\mathcal{T}^0$ if it is quasi-monotone for each Ω'_m.

We also define quasi-monotonicity with respect to a triangulation $\mathcal{T}^k$, as before, by replacing the Ω_i and the substructure vertices $\mathcal{V}_m$ by elements τ_j^k and nodes in $\mathcal{N}^k$, respectively.

In two dimensions, quasi-monotonicity with respect to $\mathcal{T}^0$ implies, for the same distribution of the ρ_i, quasi-monotonicity with respect to $\mathcal{T}^k$. We can check this as follows. The nodes of $\mathcal{N}^k$ divide into three sets: i) those which coincide with vertices of the substructures (nodes of the coarse triangulation), ii) those which belong to edges of the substructures, and iii) those which belong to the interior of the substructures. By examing the three cases, it is now easy to see that a distribution of the coefficients is quasi-monotone with respect to $\mathcal{T}^k$ if it is quasi-monotone with respect to $\mathcal{T}^0$.

In three dimensions there are cases in which a distribution of ρ_i is quasi-monotone with respect to $\mathcal{T}^0$ but not quasi-monotone with respect to $\mathcal{T}^k$. In this case, the nodes of $\mathcal{N}^k$ are divided in four sets: those at vertices, edges, faces, and interiors of the substructures. There are no problems for those of the vertices, faces and interior sets but there can be complications with the edge set of nodes. Quasi-monotonicity of the coefficients for nodes belonging to the edge set does not follow from the quasi-monotonicity with respect to the coarse triangulation. To see that, let $\mathcal{E}_l$ be an edge of a substructure Ω_i, and let $\mathcal{V}_m$ be a vertex of Ω_i and an end point of $\mathcal{E}_l$. There are more substructures sharing $\mathcal{V}_m$ than substructures sharing the whole edge $\mathcal{E}_l$. From this, it easy to distribute coefficients in such a way that they are quasi-monotone with respect to the coarse triangulation but not quasi-monotone with respect to a finer triangulation.

We note that in Theorem 6 only the quasi-monotonicity of the coefficients ρ_i with respect to $\mathcal{T}^0$ is needed. We have introduced quasi-monotonicity with respect to $\mathcal{T}^k$ only to obtain a complete theory for the stability of the L^2_ρ-projection on finer levels; see Lemma 9 and Corollary 2

5.2. A new interpolator. We define an interpolation operator $I_k^M : V^h(\Omega) \rightarrow V^k(\Omega)$, as follows

Definition 2. *Given $u \in V^h(\Omega)$, define $u_k = I_k^M u \in V^k(\Omega)$ by the values of u_k at two sets of nodes of $\mathcal{N}^k$:*

- *i) For a nodal point $P \in \mathcal{N}_0^k$, let $u_k(P)$ be the average of u over an element $\tau_{j_P}^k \in \mathcal{T}^k$.*
- *ii) For a nodal point $P \in \mathcal{N}_{\partial\Omega}^k$, let $u_k(P)$ be the average of u over $\bar{\tau}_{j_P}^k \cap \partial\Omega$.*

Here, $\tau_{j_P}^k$ is the element, or one of the elements, with the vertex P with the largest coefficient ρ_i. $\mathcal{N}_{\partial\Omega}^k$ is the set of nodes of $\mathcal{N}^k$ which belong to $\partial\Omega$, and $\mathcal{N}_0^k = \mathcal{N}^k \backslash \mathcal{N}_{\partial\Omega}^k$.

It is easy to see that, for any constant c, $I_k^M(u-c) = I_k^M(u) - c \quad \forall c$, and also that u_k vanishes on $\partial\Omega$ whenever u vanishes on $\partial\Omega$.

We note that $\bar{\tau}_{j_P}^k \cap \partial\Omega$ is a face of $\tau_{j_P}^k$ for a quasi-monotone distribution of coefficients ρ_i with respect to the triangulation $\mathcal{T}^0$. $\bar{\tau}_{j_P}^k \cap \partial\Omega$ might be an edge or a vertex for coefficients that are not quasi-monotone with respect to $\mathcal{T}^0$.

Lemma 9. *For a quasi-monotone distribution of coefficients ρ_i with respect to the triangulation $\mathcal{T}^k$, we have $\forall u \in V^h(\Omega)$*

$$\|(I - I_k^M)u\|_{L^2_\rho(\tau_j^k)} \preceq h_k \, |u|_{H^1_\rho(\bar{\tau}_j^{k,M})} \tag{32}$$

and

$$|I_k^M u|_{H^1_\rho(\tau_j^k)} \preceq |u|_{H^1_\rho(\bar{\tau}_j^{k,M})} \quad \forall \tau_j^k \in \mathcal{T}^k. \tag{33}$$

Here, $\bar{\tau}_j^{k,M} \subset \bar{\tau}_j^{k,ext}$ is a connected Lipschitz region given explicitly in the proof of this lemma. $\bar{\tau}_j^{k,ext}$ is the union of the $\bar{\tau}_i^k$ which have a vertex, edge, or face in common with τ_j^k.

Furthermore,

$$I_k^M u \in V_0^k(\Omega) \quad \textit{if} \quad u \in V_0^h(\Omega).$$

Proof. We have

$$\|u - I_k^M u\|^2_{L^2_\rho(\tau_j^k)} = \rho(\tau_j^k)\, \|u - I_k^M u\|^2_{L^2(\tau_j^k)}$$

$$\preceq \rho(\tau_j^k)\, (\|I_k^M u - c\|^2_{L^2(\tau_j^k)} + \|u - c\|^2_{L^2(\tau_j^k)}).$$

Using the definition and properties of I_k^M, we obtain

$$\|I_k^M u - c\|^2_{L^2(\tau_j^k)} = \|I_k^M (u - c)\|^2_{L^2(\tau_j^k)} \asymp \sum_{P \in \tau_j^k} h_k^3\, |(I_k^M(u-c))(P)|^2.$$

Here, the P are the vertices of the element τ_j^k.

For a case in which $P \in \mathcal{N}_0^k$, $I_k^M u(P)$, the average value of u over an element $\tau_{j_P}^k$, can be bounded from above in terms of the L^2 norm of u in $\tau_{j_P}^k$, i.e.

$$h_k^3\, |(I_k^M(u-c))(P)|^2 \preceq \|u - c\|^2_{L^2(\tau_{j_P}^k)}.$$

Here, $\tau_{j_P}^k$ is the element given in Definition 2.

For a case in which $P \in \mathcal{N}_{\partial\Omega}^k$, $I_k^M u(P)$, the average value of u over a triangle $\bar{\tau}_{j_P}^k \cap \partial\Omega$, can be bounded from above in terms of the energy norm (4) of u in $\tau_{j_P}^k$. Indeed,

$$h_k^3\, |(I_k^M(u-c))(P)|^2 \preceq h_k\, \|u - c\|^2_{L^2(\bar{\tau}_{j_P}^k \cap \partial\Omega)} \preceq h_k^2\, \|u - c\|^2_{H^1(\tau_{j_P}^k)}.$$

From the definition of quasi-monotonicity with respect to the triangulation $\mathcal{T}^k$, there exists for each P a sequence of elements $\tau_{j_i}^k$, $i = 1, \cdots, n$, with

$$\rho(\tau_j^k) = \rho(\tau_{j_n}^k) \le \cdots \le \rho(\tau_{j_2}^k) \le \rho(\tau_{j_1}^k) = \rho(\tau_{j_P}^k). \tag{34}$$

Let $\bar{\tau}_{j_P}^{k,M} = \cup_{i=1}^n \bar{\tau}_{j_n}^k$ and $\bar{\tau}_j^{k,M} = \cup_{P \in \tau_j^k} \bar{\tau}_{j_P}^{k,M}$. Then,

$$\|u - I_k^M u\|^2_{L^2_\rho(\tau_j^k)} \preceq \rho(\tau_j^k)\, h_k^2\, \|u - c\|^2_{H^1(\bar{\tau}_j^{k,M})}.$$

Note that $\bar{\tau}_j^{k,M}$ is a connected Lipschtz region with a diameter of order h_k. Thus, we can use Poincaré's inequality to obtain

$$\inf_c \rho(\tau_j^k)\, h_k^2\, \|u - c\|^2_{H^1(\bar{\tau}_j^{k,M})} \preceq \rho(\tau_j^k)\, h_k^2\, |u|^2_{H^1(\bar{\tau}_j^{k,M})},$$

and use (34) to obtain

$$\rho(\tau_j^k)\, h_k^2\, |u|^2_{H^1(\bar{\tau}_j^{k,M})} \le h_k^2\, |u|^2_{H^1_\rho(\bar{\tau}_j^{k,M})}.$$

To obtain (33), we use

$$|I_k^M u|^2_{H^1_\rho(\tau_j^k)} \preceq \sum_{i=1}^{4} \rho(\tau_j^k)\, h_k\, |(I_k^M(u-c))(P_i)|^2.$$

Here, the P_i are the vertices of the element τ_j^k. For the rest of the proof, we use the same arguments as before. □

COROLLARY 2. *For a quasi-monotone distribution of coefficients ρ_i with respect to $\mathcal{T}^k$, we have*

$$\|(I - Q_\rho^k)u\|_{L^2_\rho(\Omega)} \preceq h_k\, |u|_{H^1_\rho(\Omega)} \quad \forall u \in V_0^h(\Omega), \tag{35}$$

and

$$|Q_\rho^k u|_{H^1_\rho(\Omega)} \preceq |u|_{H^1_\rho(\Omega)} \quad \forall u \in V_0^h(\Omega). \tag{36}$$

Here, Q_ρ^k is the weighted L^2-projection from $V_0^h(\Omega)$ to $V_0^k(\Omega)$.

Proof. We obtain (35) from (32), since Q_ρ^k gives the best approximation with respect to $L^2_\rho(\Omega)$. Finally, we have (36) since L^2_ρ stability implies H^1_ρ stability; see Theorem 3.4 in Bramble and Xu [5]. □

REMARK 3. *The Lemma 9 and Corollary 2 can easily be extended to functions which do not vanish on the whole boundary $\partial\Omega$. Using Lemma 9, we can also establish optimal multilevel algorithms for problems with Neumann or mixed boundary conditions, and quasi-monotone coefficients with respect to $\mathcal{T}^0$.*

5.3. An optimal algorithm. We prove that the MDS algorithm, using the space of piecewise linear functions, V_0^0, as a coarse space, is optimal if the coefficient is quasi-monotone with respect to the coarse mesh $\mathcal{T}^0$. It is important to note that to prove our next theorem, we do not need to have quasi-monotonicity with respect to the fine meshes $\mathcal{T}^k$.

THEOREM 6. *Let T^0 be defined by (5) with $V_{-1} = V_0^0 = V_0^H$, $b_{-1}(\cdot,\cdot) = a(\cdot,\cdot)$. For a quasi-monotone distribution of the coefficients ρ_i with respect to $\mathcal{T}^0$, we have*

$$a(T^0 u, u) \asymp a(u,u) \quad \forall u \in V_0^h(\Omega).$$

Proof. We start by considering *Assumption i)*; *Assumptions ii)* and *iii)* have been checked in the proofs of the previous theorems.

Let the $\{\theta_m\}$ be a partition of unity over Ω with $\theta_m \in C_0^\infty(\Omega'_m)$. Because of the size of the overlap of the subregions Ω'_m, these functions can be chosen such that $|\nabla\theta_m|$ is bounded by C/H. We decompose $w = u - I_0^M u$ as

$$w = \sum_{m=1}^{L} w_m, \quad \text{where} \quad w_m = I_h(\theta_m w). \tag{37}$$

Here, I_h is the standard linear interpolant with respect to the triangulation $\mathcal{T}^\ell$.

We note that $w_m = 0$, on and outside of $\partial\Omega'_m$, $m = 1, \cdots, L$. Using standard arguments, cf. Dryja and Widlund [11], we can show that

$$|w_m|^2_{H^1_\rho(\Omega'_m)} \preceq |w|^2_{H^1_\rho(\Omega'_m)} + \frac{1}{H^2}\|w\|^2_{L^2_\rho(\Omega'_m)},$$

and by using Lemma 9, we obtain

$$|w_m|^2_{H^1_\rho(\Omega'_m)} \preceq |u|^2_{H^1_\rho(\bar{\Omega}'^{,ext}_m)}.$$

Here, $\bar{\Omega}'^{,ext}_m$ is the closure of the union of Ω'_m and the Ω_i which have a vertex, edge, or face in common with $\partial\Omega'_m$. By assumption, we have quasi-monotone coefficients with respect to $\mathcal{T}^0$.

We now remove the substructure Ω_{m_1} from Ω'_m obtaining $\Omega^c_{m_1} = \Omega'_m \backslash \bar{\Omega}_{m_1}$.

We decompose w_m as

$$w_m = \mathcal{H}^{(m)} w_m + (w_m - \mathcal{H}^{(m)} w_m).$$

Here, $\mathcal{H}^{(m)} w_m$ is the piecewise discrete harmonic function on Ω_{m_1} and $\Omega^c_{m_1}$ that equals w_m on $\partial\Omega_{m_1} \cup \partial\Omega'_m$. We stress that we use the weight $\rho = 1$ in obtaining this piecewise discrete harmonic function.

We decompose $\mathcal{H}^{(m)} w_m$ as in (19), and obtain

$$\mathcal{H}^{(m)} w_m = (\mathcal{H}^{(m)} w_m)_\ell + \cdots + (\mathcal{H}^{(m)} w_m)_1 + (\mathcal{H}^{(m)} w_m)_0. \tag{38}$$

Therefore, by using the same arguments as in the beginning of the proof of lemma earlier sections,

$$\sum_{k=0}^{\ell} b_k((\mathcal{H}^{(m)} w_m)_k, (\mathcal{H}^{(m)} w_m)_k) \preceq \rho_{m_1} |\mathcal{H}^{(m)} w_m|^2_{H^1(\Omega'_m)}$$

$$\preceq \rho_{m_1} \|\mathcal{H}^{(m)} w_m\|^2_{H^{1/2}_{00}(\partial\Omega_{m_1} \cap \partial\Omega^c_{m_1})} \preceq \rho_{m_1} |\mathcal{H}^{(m)} w_m|^2_{H^1(\Omega_{m_1})} \tag{39}$$

$$\leq \rho_{m_1} |w_m|^2_{H^1(\Omega_{m_1})} \leq |w_m|^2_{H^1_\rho(\Omega'_m)}.$$

Let $\tilde{w}_m = w_m - \mathcal{H}^{(m)} w_m$. Using the triangular inequality, we obtain

$$|\tilde{w}_m|_{H^1_\rho(\Omega'_m)} \preceq |w_m|_{H^1_\rho(\Omega'_m)}.$$

Note that $\tilde{w}_m$ vanishes on $\partial\Omega_{m_1} \cup \partial\Omega'_m$. Therefore, we can decompose $\tilde{w}_m$ in Ω_{m_1} and $\Omega^c_{m_1}$, independently. For the decomposition in Ω_{m_1}, we have no difficulties, since we have constant coefficients. For the decomposition in $\Omega^c_{m_1}$, we can try to remove the substructure with largest coefficient ρ_{m_i} in $\Omega^c_{m_1}$, and repeat the analysis just described. It is easy to show that we can remove all substructures, recursively, if we have a quasi-monotone distribution with respect to $\mathcal{T}^0$. We note that the argument in (39) fails if we do not have quasi-monotone coefficients. □

REMARK 4. *In the proof of Theorem 6, we just need to carry out the analysis locally for each Ω'_m. In a case of quasi-monotone coefficients with respect to $\mathcal{T}^0$ and with the coarse spaces V^F_{-1} or V^E_{-1}, we can derive a bound on the condition number of the multilevel additive Schwarz algorithm that is linear with respect to the number of levels ℓ. The analysis also works if we use different coarse spaces in different parts of the domain Ω. We can also use the coarse space V^0_0 and an exotic space V^X_{-1} simultaneously. The resulting multilevel algorithm is optimal if the coefficient is quasi-monotone, and is almost optimal with a condition number bounded in terms of $(1+\ell)^2$ otherwise. The same arguments can also be used to prove that we also have an optimal multilevel algorithm with Neumann or mixed boundary condition and quasi-monotone coefficients.*

6. Multiplicative Versions. In this section, we discuss some multiplicative versions of the multilevel additive Schwarz methods; they correspond to certain multigrid methods. Following Zhang [30], we consider two algorithms defined by their error propagation operators

$$E_G = (\prod_{k=0}^{\ell} \prod_{j \in \mathcal{N}_0^k} (I - P_j^k))(I - \eta T_{-1}), \tag{40}$$

and

$$E_J = \prod_{k=-1}^{\ell} (I - \tilde{T}^k) = (\prod_{k=0}^{\ell} (I - \eta \sum_{j \in \mathcal{N}_0^k} P_j^k))(I - \eta T_{-1}), \tag{41}$$

where η is a damping factor chosen such that $\|\tilde{T}^k\|_{H^1_\rho} \le w < 2$.

The products in the above expressions can be arranged in any order; different orders result in different schemes; see Zhang [30]. When the product is arranged in an appropriate order, the operators E_G and E_J correspond to the error propagation operators of V-cycle multigrid methods using Gauss-Seidel and damped Jacobi method as smoothers for the refined spaces, respectively.

By applying techniques developed in Zhang [30], and Dryja and Widlund [15], we can show that the norm of the error propagation operators $\|E_G\|_{H^1_\rho}$ and $\|E_F\|_{H^1_\rho}$ can be estimated from above by $1 - C\,(1+\ell)^{-2}$. In a case in which we have quasi-monotone coefficients and use the standard coarse space V^H, we can establish that the V-cycle multigrid methods, given by (40) and (41), are *optimal*.

7. Approximate discrete harmonic extensions. A disadvantage of using the coarse spaces V_{-1}^{X}, with $X = F, E, NN$, and W, is that we have to solve a local Dirichlet problem exactly for each substructure to obtain the discrete harmonic extensions. However, we can define new exotic coarse spaces, called $V_{-1}^{\tilde{X}}$, with $\tilde{X} = \tilde{F}, \tilde{E}, \widetilde{NN}$, and $\tilde{W}$ by introducing *approximate discrete harmonic extensions.* They are given by simple explicit formulas [10,20] and have the same $H_{\rho}^{1}-$stability property estimates as the discrete harmonic extensions. Here we use strongly the fact that our exotic spaces $V_{-1}^{\tilde{X}}$ have constant values at the nodal points of the faces of the substructures. We prove that the MDS, with these new coarse spaces, have condition number estimate proportional to $(1+\ell)^2$.

Let C_k, $k = 1, \cdots, 4$, be the barycenters of the faces $\mathcal{F}_{ik}$ of $\partial\Omega_i$, and let V_k be the vertex of Ω_i that is opposite to C_k. Let C be the centroid of Ω_i, i.e. the intersection of the line segments connecting the V_k to the C_k. Let E_{kl}, $l = 1, 2, 3$, be the open edges of $\partial\mathcal{F}_{ik}$; see Fig. 3.

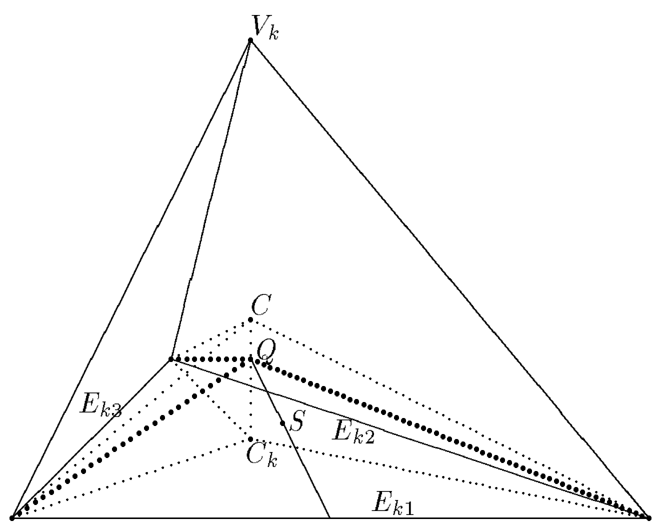

Fig. 3

To approximate the discrete harmonic function $\theta_{\mathcal{F}_{ij}}$ in Ω_i, we use the a finite element function $\vartheta_{\mathcal{F}_{ij}}$ introduced in the proof of Lemma 4.4 of Dryja, Smith and Widlund [10] (see also Extension 3 in Sarkis [20]), given by (see Fig. 3):

Definition 3.

i) Let

$$u_{ij}(C) := \frac{1}{4}.$$

ii) For a point Q that belongs to a line segment connecting C to $C_k, k = 1, \cdots, 4$, define $u_{ij}(Q)$ by linear interpolation between the values $u_{ij}(C) = 1/4$ and $u_{ij}(C_k) = \delta_{jk}$, i.e. by

$$u_{ij}(Q) := \lambda(Q)\frac{1}{4} + (1 - \lambda(Q))\delta_{jk}.$$

Here $\lambda(Q)$ =distance(Q, C_k)/distance(C, C_k) and $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$, otherwise.

iii) For a point S that belongs to a triangle defined by the previous Q as a vertex, and the edge E_{kl} as a side opposite to Q, $l = 1, \cdots, 3$, let

$$u_{ij}(S) := u_{ij}(Q).$$

iv) Let $\vartheta_{\mathcal{F}_{ij}} = I_h^0 u_{ij}$, where I_h^0 is the interpolation operator into the space $V^h(\Omega_i)$ that preserves the values of a function u_{ij} at the nodal points of $\bar{\Omega}_{i,h} \backslash \mathcal{W}_{i,h}$ and set them to zero on $\mathcal{W}_{i,h}$.

v) In Ω_j, which has a common face $\mathcal{F}_{ij}$ with Ω_i, $\vartheta_{\mathcal{F}_{ij}}$ is defined as in Ω_i. Finally, $\vartheta_{\mathcal{F}_{ij}}$ is extended by zero outside $\Omega_{\mathcal{F}_{ij}}$.

Note that $\vartheta_{\mathcal{F}_{ij}} \in V_0^h$, and

$$\sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \vartheta_{\mathcal{F}_{ij}} = 1 \quad \text{on} \quad \bar{\Omega}_{i,h} \backslash \mathcal{W}_{i,h}.$$

We remark that other extensions are also possible; see, e.g., Extension 2 of [20].

Using ideas in the proof of Lemma 4.4 in [10], we obtain

LEMMA 10.

$$|\theta_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \leq |\vartheta_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \preceq H\,(1 + \log H/h).$$

For the wire basket contributions, we replace the piecewise discrete harmonic function $\sum_{p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p$, by $\sum_{p \in \mathcal{W}_{i,h}} u(x_p)\phi_p^\ell$, where ϕ_p^ℓ is the standard nodal basis function associated with a node p. Using the definition of ϕ_p^ℓ and a Sobolev type inequality (see Lemma 4.3 of [10]), we obtain

LEMMA 11.

$$|\sum_{p \in \mathcal{W}_{i,h}} (u(x_p) - \bar{u}_i)\,\phi_p^\ell|^2_{H^1(\Omega_i)} \preceq \|u - \bar{u}_i\|^2_{L^2(\mathcal{W}_i)}$$

$$\preceq (1 + \ell)\,|u|^2_{H^1(\Omega_i)}.$$

New exotic coarse spaces $V_{-1}^{\tilde{X}}$, with $\tilde{X} = \tilde{F}, \tilde{E}, \widetilde{NN}$, and $\tilde{W}$ are introduced by combining the approximate discrete harmonic functions $\vartheta_{\mathcal{F}_{ij}}$ and $\sum_{p\in\mathcal{W}_{i,h}} u(x_p)\phi_p^\ell$. We define $V_{-1}^{\tilde{X}}$ as the range of the following interpolators $I_h^{\tilde{X}} : V_0^h \rightarrow V_{-1}^{\tilde{X}}$:

- **Modified Neumann-Neumann coarse spaces**

$$I_h^{\widetilde{NN}} u = \tilde{u}_{-1} = \sum_i \tilde{u}_{-1}^i = \sum_i \bar{u}_i \, \rho_i^\beta \tilde{\mu}_{i,\beta}^+.$$

Here, $\tilde{\mu}_{i,\mu}^+ = \mu_{i,\beta}^+$ on $\Gamma_h \cup \partial\Omega$ and is extended elsewhere in Ω as an approximate discrete harmonic function given by:

$$\tilde{\mu}_{i,\beta}^+(x) = \sum_{p\in\mathcal{W}_{i,h}} \mu_{i,\beta}^+(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \mu_{i,\beta}^+(\mathcal{F}_{ij})\,\vartheta_{\mathcal{F}_{ij}}(x) \;\; \forall x.$$

- **A modified face based coarse space**

$$I_h^{\tilde{F}} u(x)_{|\bar{\Omega}_i} = \sum_{p\in\mathcal{W}_{i,h}} u(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \vartheta_{\mathcal{F}_{ij}}(x).$$

- **A modified edge based coarse space**

$$I_h^{\tilde{E}} u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{V}_m\in\partial\Omega_i} u(\mathcal{V}_m)\phi_{\mathcal{V}_m}^\ell(x) +$$

$$\sum_{\mathcal{E}_l\subset\mathcal{W}_i} \bar{u}_{\mathcal{E}_l} \sum_{p\in\mathcal{E}_{l,h}} \phi_p^\ell(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \vartheta_{\mathcal{F}_{ij}}(x).$$

- **A modified wire basket based coarse space**

$$I_h^{\tilde{W}} u(x)_{|\Omega_i} = \sum_{p\in\mathcal{W}_{i,h}} u(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \bar{u}_{\partial\mathcal{F}_{ij}} \vartheta_{\mathcal{F}_{ij}}(x).$$

It is important to note that our approximate discrete harmonic extensions recover constant functions, because

$$\sum_{p\in\mathcal{W}_{i,h}} \phi_p^\ell(x) + \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} \vartheta_{\mathcal{F}_{ij}}(x) = 1 \;\; \forall x \in \bar{\Omega}_i.$$

We define the bilinear forms exactly as before, i.e.

$$b_{-1}^{\tilde{X}} = b_{-1}^X,$$

and operators $T_{-1}^{\tilde{X}} : V^h \rightarrow V_{-1}^{\tilde{X}}$, by

$$b_{-1}^{\tilde{X}}(T_{-1}^{\tilde{X}} u, v) = a(u,v) \;\;\; \forall \; v \in \; V_{-1}^{\tilde{X}}.$$

Let

$$T^{\tilde{X}} = T_{-1}^{\tilde{X}} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k.$$

THEOREM 7. *For any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^{\tilde{X}} u, u) \preceq a(u,u).$$

Proof. Let us first consider a case with $V_{-1}^{\tilde{F}}$ as the coarse space.

Assumption ii) Using the triangle inequality, the explicit formulas for the approximate discrete harmonic functions, and Lemma 10, we obtain

$$|u|^2_{H^1_\rho(\Omega_i)} \preceq \sum_i \rho_i \{ \sum_{p \in \mathcal{W}_{i,h}} h\,(u(x_p))^2$$

$$+ \;\; H(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}})^2 \} \;\; \forall u \in V_{-1}^{\tilde{F}}.$$

We now use that the approximate discrete harmonic extension recovers constant functions to obtain

$$a(u,u) \preceq b_{-1}^{\tilde{F}}(u,u). \tag{42}$$

Assumption i) Note that the bilinear form $b_{-1}^{\tilde{F}}(u,u)$ depends only on the values of u on Γ_h, and let $\tilde{u}_{-1} = I_h^{\tilde{F}} u$ and $u_{-1} = I_h^F u$. We can therefore use Lemma 6 to obtain

$$b_{-1}^{\tilde{F}}(\tilde{u}_{-1}, \tilde{u}_{-1}) = b_{-1}^F(u_{-1}, u_{-1}) \preceq (1+\ell)\, a(u,u) \;\; \forall u \in V_0^h. \tag{43}$$

We now modify the decomposition in the proof of Theorem 3

$$u = u_{-1} + \sum_k \sum_j u_j^k$$

and construct a decomposition for the current theorem by

$$u = \tilde{u}_{-1} + (u_{-1} - \tilde{u}_{-1}) + \sum_k \sum_j u_j^k =$$

$$\tilde{u}_{-1} + \sum_k \sum_j \tilde{u}_j^k + \sum_k \sum_j u_j^k = \tilde{u}_{-1} + \sum_k \sum_j v_j^k.$$

Here we use that $(u_{-1} - \tilde{u}_{-1})$ vanishes on $\Gamma_h \cup \partial\Omega_h$, and then decompose $(u_{-1} - \tilde{u}_{-1})$ as in Lemma 1 to obtain

$$\sum_k \sum_j a(\tilde{u}_j^k, \tilde{u}_j^k) \preceq a(u_{-1} - \tilde{u}_{-1}, u_{-1} - \tilde{u}_{-1}) \preceq a(\tilde{u}_{-1}, \tilde{u}_{-1}).$$

We now use (42) and (43) and obtain

$$a(\tilde{u}_{-1}, \tilde{u}_{-1}) \preceq b_{-1}^{\tilde{F}}(\tilde{u}_{-1}, \tilde{u}_{-1}) \preceq (1+\ell) a(u,u).$$

Finally, we use the proof of Theorem 3 to obtain

$$\sum_k \sum_j a(u_j^k, u_j^k) \preceq (1+\ell)^2 \, a(u,u).$$

The proof of this theorem for $V_{-1}^{\tilde{E}}$ or $V_{-1}^{\tilde{W}}$ as the coarse space is quite similar.

Let us finally consider the case with $V_{-1}^{\widetilde{NN}}$ as the coarse space. For *Assumption ii)*, we trivially have $\omega = 1$. *Assumption i)* is handled exactly as before. The only nontrivial part is to show that

$$a(I_h^{\widetilde{NN}} u, I_h^{\widetilde{NN}} u) \preceq (1+\ell)^2 a(u,u) \quad \forall u \in V_0^h(\Omega). \tag{44}$$

The idea of the proof of (44) is the same as in Dryja and Widlund [15]. We reduce the estimates to bounds related to the vertices, edges, and faces and use Lemmas 10 and 11, and Lemma 4 in [10]. □

8. Nonuniform refinements. We now consider finite element approximation with locally nested refinement. Such refinements can be used to improve the accuracy of the solutions of problems with singular behavior which arise in elliptic problems with discontinuous coefficients, nonconvex domains, or singular data. We note, in particular, that solutions of elliptic problems with highly discontinuous coefficients are very likely to become increasingly singular when we approach the wire basket.

Nested local refinements have previously been analyzed by Bornemann and Yserentant [1], Bramble and Pasciak [2], Cheng [6,7], Oswald [18,19], and Yserentant [27,28]. By nested local refinement we mean that an element, which is not refined at level j, cannot be a candidate for further refinement. Under certain assumptions on the local refinement, optimal multilevel preconditioners have been obtained for problems with nearly constant coefficients in two and three dimensions. For problems in two dimensions with highly discontinuous coefficients, the standard piecewise linear function can be used as a coarse space to design multilevel preconditioners. A bound on the condition number can be derived, which is independent of the coefficients, and which grows at most as the square of the number of levels; see, e.g., Yserentant [28]. Here, we extend the analysis to the case where the coefficients are quasi-monotone with respect to the coarse triangulation or are highly discontinuous in two or in three dimensions.

Let us begin by a shape regular but possibly nonuniform coarse triangulation $\mathcal{T}^0 = \mathcal{T}^{0^*}$, which defines substructures Ω_i with diameters H_i. It follows from shape regularity that neighboring substructures are of comparable size.

We introduce the following refinement procedure: For $k = 1, \cdots, \ell$, subdivide all the tetrahedra $\tau_j^{k-1} \in \mathcal{T}^{k-1}$ into eight tetrahedra (see, e.g., Ong [17]); these are elements of level k and belong, by definition, to $\mathcal{T}^k$. A shape regular refinement is obtained by connecting properly the midpoints of the edges τ_j^{k-1}. We note that this refinement, restricted to each $\bar{\Omega}_i$, is quasi-uniform.

Let V_0^k be the space of piecewise linear functions associated with $\mathcal{T}^k$, which vanish on $\partial\Omega$, and let $\mathcal{N}_0^k$ be the set of nodal points associated with the space V_0^k. Let ϕ_j^k, $j \in \mathcal{N}_0^k$, be a standard nodal basis function of V_0^k, and let $V_j^k = \text{span}\{\phi_j^k\}$.

We define a locally nested refinement in terms of a sequence of open subregions $\mathcal{O}_k \subset \Omega$ such that

$$\mathcal{O}_\ell \subset \mathcal{O}_{\ell-1} \subset \cdots \subset \mathcal{O}_k \subset \cdots \subset \mathcal{O}_1 \subset \mathcal{O}_0 = \Omega,$$

and assume that the $\partial\mathcal{O}_k$, the boundary of $\mathcal{O}_k$, align with element boundaries of $\mathcal{T}^{k-1}$, for $k \geq 1$.

We define a nested, nonconforming triangulations $\mathcal{T}^{k^*}$, $k = 0, \cdots, \ell$, as follows:

$$\mathcal{T}^{k^*} = \mathcal{T}^k \quad \text{on} \quad \bar{\mathcal{O}}_k,$$

and

$$\mathcal{T}^{k^*} = \mathcal{T}^j \quad \text{on} \quad \bar{\mathcal{O}}_j \backslash \mathcal{O}_{j+1} \quad \forall j < k.$$

Assumption 1. *The levels of two elements of a triangulation $\mathcal{T}^{k^*}$, which have at least one common point, differ by at most one.*

We note that Assumption 1 guarantees that all elements in $\mathcal{T}^{\ell^*}(\Omega_i)$ with a common vertex are of comparable size. This type of refinement is exactly the same as that analyzed by Bornemann and Yserentant [1].

Let $V_0^{k^*}$ be the space of piecewise linear functions associated with the triangulation $\mathcal{T}^{k^*}$, which are continuous on Ω and vanish on $\partial\Omega$. By construction, $V_0^{k^*} \subset V_0^{k+1^*} \subset V_0^{k+1}$. The vertices of the elements of $\mathcal{T}^{k^*}$ are called nodes. From the requirement of continuity, it follows that we can distinguish between the set of free nodes $\mathcal{N}_0^{k^*}$ and the remaining set of slave nodes. A function $u \in V_0^{\ell^*}$ is determined uniquely by its values at the free nodes $\mathcal{N}_0^{k^*}$; the values of u at the slave nodes are determined, by interpolation, from the values at $\mathcal{N}_0^{k*}$. Therefore, we have the following representation:

$$u = \sum_{j \in \mathcal{N}_0^{\ell*}} u(x_j)\, \phi_j^{\ell*} \quad \forall u \in V_0^{\ell*} \tag{45}$$

Here, $\phi_j^{\ell*} \in V_0^{\ell*}$ is a nodal basis function, with respect to $\mathcal{T}^{\ell*}$, which equals 1 at one free node and vanishes at all other free nodes of $\mathcal{N}_0^{\ell*}$.

The discrete problem is given by:

Find $u \in V_0^{\ell*}$, such that

$$a(u,v) = f(v) \quad \forall\ v \in\ V_0^{\ell*}. \tag{46}$$

In order to obtain a preconditioner, we consider the following splitting:

$$V_0^{\ell*} = V_{-1}{}^* + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k*}} V_j^k. \tag{47}$$

Here, $\mathcal{V}^{k*}$ is the set of nodes of $\mathcal{N}_0^k$ which belong to the interior of $\mathcal{O}_k$.

We consider in detail only the exotic coarse space $V_{-1}{}^* = V_{-1}^{\tilde{F}}{}^*$, i.e. the counterpart of the modified face based coarse space introduced in Section 7. We note that the same ideas can be extended straightforwardly to define and analyze algorithms using the other exotic coarse spaces introduced in sections 4 and 7.

The coarse space $V_{-1}^{\tilde{F}}{}^*$ can be defined as the range of an interpolation operator $I_\ell^{\tilde{F}^*} : V_0^{\ell*} \to V_{-1}^{\tilde{F}}{}^*$, defined by

$$I_\ell^{\tilde{F}^*} u(x)_{|\bar{\Omega}_i} = \sum_{p \in (\mathcal{W}_i \cap \mathcal{N}_0^{\ell*})} u(x_p)\phi_j^{\ell*}(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}}^* \vartheta_{\mathcal{F}_{ij}}^*(x),$$

where

$$\bar{u}_{\mathcal{F}_{ij}}^* = \frac{\sum_{p \in (\mathcal{F}_{ij} \cap \mathcal{N}_0^{\ell*})} u(x_p) \int_{\text{supp }(\phi_p^{\ell*}(\mathcal{F}_{ij}))} \phi_p^{\ell*}\, dS}{\sum_{p \in (\mathcal{F}_{ij} \cap \mathcal{N}_0^{\ell*})} \int_{\text{supp }(\phi_p^{\ell*}(\mathcal{F}_{ij}))} \phi_p^{\ell*}\, dS}.$$

$\vartheta_{\mathcal{F}_{ij}}^*$ is defined in a way similar to $\vartheta_{\mathcal{F}_{ij}}$ except that in Step iv), of Definition 3, we interpolate at the free nodes $\mathcal{N}_0^{\ell*}$ which belong to $\bar{\Omega}_i \backslash \mathcal{W}_i$ and set $\vartheta_{\mathcal{F}_{ij}}^*$ to zero on $\mathcal{W}_i$.

We consider the following bilinear form:

$$b_{-1}^{\tilde{F}}{}^*(u,u) = \sum_i \rho_i \{ \|u - \bar{u}_i^*\|_{L^2(\mathcal{W}_i)}^2 + \quad H_i(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}}^* - \bar{u}_i^*)^2 \},$$

where,

$$\bar{u}_i^* = \frac{\sum_{p \in (\partial\Omega_i \cap \mathcal{N}_0^{\ell*})} u(x_p) \int_{\text{supp }(\phi_p^{\ell*}(\partial\Omega_i))} \phi_p^{\ell*}\, dS}{\sum_{p \in (\partial\Omega_i \cap \mathcal{N}_0^{\ell*})} \int_{\text{supp }(\phi_p^{\ell*}(\partial\Omega_i))} \phi_p^{\ell*}\, dS},$$

and introduce an operator $T_{-1}^{\tilde{F}^*} : V_0^{\ell *} \to V_{-1}^{\tilde{F}^*}$, by

$$b_{-1}^{\tilde{F}^*}(T_{-1}^{\tilde{F}^*} u, v) = a(u, v) \quad \forall\ v \in\ V_{-1}^{\tilde{F}^*}.$$

Let

$$T^{\tilde{F}^*} = T_{-1}^{\tilde{F}^*} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k*}} P_j^k. \tag{48}$$

THEOREM 8. *For any $u \in V_0^{\ell *}$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^{\tilde{F}^*} u, u) \preceq a(u,u).$$

Proof. Assumptions i) and ii). We first introduce pseudo inverses $\mu_{i,1/2}^{+\ *}$ and apply Sobolev type inequalities and extension theorems to obtain the required results on the coarse space approximation (see Lemma 16) and to reduce our problem to local problems with constant coefficients. For the local problems with constant coefficients, we use the decomposition given in Bornemann and Yserentant [1]. Note that our refinement is a particular case of the local refinement NLR2 considered by Oswald [19]; we can then apply Theorem 6 of Oswald [19] to obtain a good decomposition.

The Sobolev type inequalities and extension theorems required for our particular nonuniform refinement are given below in several lemmas.

Assumption iii). Note that, on each $\bar{\Omega}_i$, the strengthened Cauchy-Schwarz tensor $\mathcal{E}^*$ associated with the splitting (47) can be obtained by symmetrically deleting columns and rows from the tensor $\mathcal{E}$ associated with the case of quasi-uniform refinement. Therefore, we obtain $\rho(\mathcal{E}^*) \leq C$ by a standard Rayleigh quotient argument. □

We now slightly modify some lemmas that are well known for quasi-uniform refinement to show that they hold in our nonuniform refinement case.

LEMMA 12. *For $u \in V^{\ell *}(\Omega_i)$,*

$$| \sum_{p \in (\mathcal{W}_i \cap \mathcal{N}^{\ell *})} (u(x_p) - \bar{u}_i^*)\, \phi_p^{\ell *} |_{H^1(\Omega_i)}^2 \preceq \|u - \bar{u}_i^*\|_{L^2(\mathcal{W}_i)}^2$$

$$\preceq (1+\ell)\, |u|_{H^1(\Omega_i)}^2.$$

Proof. For the proof of the first inequality, we use Assumption 1 and the inverse inequality. For the second inequality, we use that $V^{\ell *}(\Omega_i) \subset V^{\ell}(\Omega_i)$ and then apply a standard Sobolev type inequality (see Lemma 4.3 of [10]). To obtain the seminorm, we use the fact that for any constant c, $\bar{u}_i^* = c$, if $u = c$ on $\mathcal{W}_i \cap \mathcal{N}^{\ell *}$. □

LEMMA 13.

$$|\theta^*_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \leq |\vartheta^*_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \preceq H_i\,(1+\ell),$$

where $\theta^*_{\mathcal{F}_{ij}} \in V^{\ell^*}(\Omega_i)$ *is the discrete harmonic function, in the sense of* $V^{\ell^*}(\Omega_i)$*, which equals 1 on* $(\mathcal{F}_{ij} \cap \mathcal{N}^{\ell^*})$ *and is* 0 *on* $(\partial\Omega_i \backslash \mathcal{F}_{ij})$*.*

Proof. The first inequality is trivial since the discrete harmonic function has minimal energy. For the second inequality, we use the same ideas as in Lemma 4.4 of Dryja, Smith and Widlund [10]. Assumption 1 is crucial in this proof. □

We denote by $V^{\ell^*}(\partial\Omega_i)$ the restriction of $V^{\ell^*}(\Omega_i)$ to $\partial\Omega_i$. Let $\mathcal{H}^{(i)^*}$: $V^{\ell^*}(\partial\Omega_i) \rightarrow V^{\ell^*}(\Omega_i)$, be the discrete harmonic extension operator in the sense of $V^{\ell^*}(\Omega_i)$.

LEMMA 14. *Let* $u \in V^{\ell^*}(\partial\Omega_i)$*. Then*

$$|\mathcal{H}^{(i)^*}u|_{H^1(\Omega_i)} \preceq |u|_{H^{1/2}(\partial\Omega_i)}. \tag{49}$$

Proof. Let $\tilde{u} \in H^1(\Omega_i)$ be the harmonic extension of u defined by

$$(\nabla\tilde{u}, \nabla v)_{L^2(\Omega_i)} = 0 \quad \forall v \in H^1_0(\Omega_i),$$

$$\tilde{u} = u \text{ on } \partial\Omega_i.$$

Therefore, by the definition of the $H^{1/2}$−seminorm,

$$|\tilde{u}|_{H^1(\Omega_i)} = |u|_{H^{1/2}(\partial\Omega_i)}. \tag{50}$$

We now slightly modify the interpolator I^M_k introduced in Definition 2 and define another interpolation operator $I^{M^*}_\ell : H^1(\Omega_i) \rightarrow V^{\ell^*}(\Omega_i)$, as follows

DEFINITION 4. *Given* $\tilde{u} \in H^1(\Omega_i)$*, such that* $\tilde{u}|_{\partial\Omega_i} \in V^{\ell^*}(\partial\Omega_i)$*, define* $u^* = I^M_k\tilde{u} \in V^{\ell^*}(\Omega_i)$ *by the values of* u^* *at two sets of the free nodal points* $\mathcal{N}^{\ell^*}(\bar{\Omega}_i)$*:*

i) For a free nodal point $P \in \mathcal{N}^{\ell^*}(\bar{\Omega}_i)\backslash\mathcal{N}^{\ell^*}(\partial\Omega_i)$*, let* $u^*(P)$ *be the average of* $\tilde{u}$ *over an element* $\tau^\ell_{j_P} \in \mathcal{T}^{\ell^*}(\Omega_i)$*.*

ii) For a free nodal point $P \in \mathcal{N}^{\ell^*}(\partial\Omega_i)$*, let* $u^*(P) = \tilde{u}(P)$*.*

Here, $\tau^k_{j_P}$ *is any element* $\mathcal{T}^{\ell^*}(\Omega_i)$ *with vertex* P*.*

Using the same arguments as in Lemma 9, we obtain

$$|u^*|_{H^1(\Omega_i)} \preceq |\tilde{u}|_{H^1(\Omega_i)}. \tag{51}$$

Finally, we use (50), (51), and the fact that $\mathcal{H}^{(i)^*}u$ has minimal energy, to obtain (49). □

Let I^{ℓ^*} be the interpolation operator into the space V^{ℓ^*} that preserves the values of a function at the free nodal points $\mathcal{N}^{\ell^*}$. Using the same ideas as in Lemma 4.4 of Dryja, Smith and Widlund [10], we obtain

LEMMA 15. *Let $u \in V^{\ell^*}(\Omega_i)$. Then*

$$|I^{\ell^*}(\vartheta^*_{\mathcal{F}_{ij}}\, u)|^2_{H^1(\Omega_i)} \preceq (1+\ell)^2 \|u\|^2_{H^1(\Omega_i)}.$$

Using Lemmas 12, we obtain the necessary results on the coarse space approximation:

LEMMA 16. *Let $u \in V_0^{\ell^*}$. Then*

$$b_{-1}^{\tilde{F}^*}(I_\ell^{\tilde{F}^*} u, I_\ell^{\tilde{F}^*} u) \preceq (1+\ell)\, a(u,u).$$

We now consider the case of quasi-monotone coefficients with respect to the coarse triangulation $\mathcal{T}^{0^*}$. Let

$$T^{0^*} = P^{0^*} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k^*}} P_j^k.$$

Here, $P^{0^*} : V_0^{\ell^*} \rightarrow V_0^{0^*}$, is the $H^1_\rho(\Omega)$-projection.

THEOREM 9. *For a quasi-monotone distribution of the coefficients ρ_i with respect to $\mathcal{T}^{0^*}$, we have*

$$a(\mathcal{T}^{0^*} u, u) \asymp a(u,u) \quad \forall u \in V_0^{\ell^*}.$$

Proof. The proof is very similar to that of Theorem 6 using now Lemmas 12-16. We note that Lemma 9, for $k = 0$, holds for a locally quasi-uniform triangulation $\mathcal{T}^{0^*}$. We now replace the decomposition (38) by a decomposition analyzed by Bornemann and Yserentant [1], or Oswald [19].

We note that Assumption 1 is needed to prove the first inequality in (39). □

9. Multilevel methods on the interface. In this section, we extend our results to multilevel iterative substructuring algorithms for problems with discontinuous coefficients. We recall that iterative substructuring methods provide preconditioners for the reduced system of equations that remains after that all the interior variables of the substructures have been eliminated. We focus only on variants of the algorithms developed in Section 4. Other algorithms, based on other coarse spaces of Sections 4 and with nonuniform refinements, can be designed and analyzed in the same way.

Let $V_0^h(\Gamma)$ be the restriction of $V_0^h(\Omega)$ to Γ. The iterative substructuring method associated with (2) is of the form: Find $u \in V_0^h(\Gamma)$ such that

$$s(u,v) = \tilde{f}(v) \quad \forall v \in V_0^h(\Gamma), \tag{52}$$

where

$$s(u,v) = a(\mathcal{H}u, \mathcal{H}v) = \sum_i \rho_i \int_{\Omega_i} \nabla \mathcal{H}^{(i)} u \cdot \nabla \mathcal{H}^{(i)} v \, dx,$$

and

$$\tilde{f}(v) = \sum_i \int_{\Omega_i} f \, \mathcal{H}^{(i)} v.$$

Let $V_0^k(\Gamma)$, $k = 0, \cdots, \ell$, be the restriction of $V_0^k(\Omega)$ to Γ and let $\mathcal{N}_0^k(\Gamma)$ be the set of nodal points associated with the space $V_0^k(\Gamma)$. Let $V_j^k(\Gamma), j \in \mathcal{N}_0^k(\Gamma)$, be the restriction of $V_j^k(\Omega)$ to Γ.

We introduce the bilinear forms $b_k^j(u,v)$: $V_j^k(\Gamma) \times V_j^k(\Gamma) \rightarrow \Re$, for $k = 0, \cdots, \ell$, and $j \in \mathcal{N}_0^k(\Gamma)$ by

$$b_k^j(u,v) = u(x_j) v(x_j) \, a(\phi_j^k, \phi_j^k). \tag{53}$$

Here, ϕ_j^k is the nodal basis functions that span $V_j^k(\Omega)$. We can easily extend the analysis to the case in which we use a good approximation of $a(\phi_j^k, \phi_j^k)$.

Let $V_{-1}^X(\Gamma)$, with $X = F, E, NN$, and W, be the restriction of $V_{-1}^X(\Omega)$, to Γ, and let the associated bilinear form be given by $b_{-1}^X(\cdot,\cdot)$. Note that b_{-1}^X is well defined for $u \in V_0^h(\Gamma)$, since the computation of $b_{-1}^X(u,u)$ depends only on the values of u on Γ_h.

We introduce the operators $T_j^k : V_0^h(\Gamma) \rightarrow V_j^k(\Gamma)$, $k = 0, \cdots, \ell$, and $j \in \mathcal{N}_0^k(\Gamma)$, by

$$b_k^j(T_j^k \, u, v) = s(u,v) \;\; \forall \; v \in V_j^k(\Gamma),$$

and the operator $T_{-1}^{X,\Gamma} : V_0^h(\Gamma) \rightarrow V_{-1}^X(\Gamma)$, by

$$b_{-1}^X(T_{-1}^{X,\Gamma} u, v) = s(u,v) \;\; \forall \; v \in V_{-1}^X(\Gamma).$$

Let

$$T^{X,\Gamma} = T_{-1}^{X,\Gamma} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k(\Gamma)} T_j^k$$

THEOREM 10. *For* $u \in V_0^h(\Gamma)$

$$(1+\ell)^2 s(u,u) \preceq s(T^{X,\Gamma} u, u) \preceq s(u,u).$$

Proof. The proof for $X = W$, with a condition number estimate of $C(1+\ell)^3$ is given in Dryja and Widlund [14] (Theorem 6.2). To obtain an improved, quadratic estimate, we can use a result of Zhang (see Remark 3.3 in [30]). Using similar arguments as in [14] and in previous sections, we can prove our current theorem for the other exotic coarse spaces as well.

Another technique for estimating condition numbers for preconditioned Schur complement systems was introduced by Smith and Widlund [22]. They showed that the condition number of the preconditioned Schur complement is bounded from above by the condition number of the full linear system preconditioned by a related preconditioner. Using the same technique, Tong, Chan, and Kuo [23] gave an upper bound for the condition number for a Schur complement system preconditioned by a BPX preconditioner. They only considered elliptic problems with nearly constant coefficients. Here, we can also use the same technique to prove our theorem. □

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