Theorem 5.21 Let $U$ be a feasible augmented graph, with the blocks of $B^PQ$-bridges: $B_1, B_2, \ldots, B_l$.

Then $U$ does not have a pair of interlacing bights, if for all $i$ ($1 \leq i \leq l$) $U_{B_i}$ has no pair of interlacing bights.

Proof. Assume to the contrary. Then there exists a feasible U-Fragment with the blocks of overlapping $B^PQ$-bridges: $B_1, B_2, \ldots, B_l$, such that for all $i$ ($1 \leq i \leq l$) $U_{B_i}$ has no pair of interlacing bights, but $U$ has a pair of interlacing bights, $A_1$ and $A_2$. Let $A_1$ be between $a_1$ and $a_2$ and $A_2$, between $b_1$ and $b_2$, where $a_1$, $a_2$, $b_1$ and $b_2$ are four distinct vertices on $J$ such that they are external vertices of attachment on $J$ or one of $s$ and $t$. Further assume that $U$ has the least number of blocks of overlapping $B^PQ$-bridges. Assume that $B_1, B_2, \ldots, B_l$ are so so ordered that if $i < j$ (for $1 \leq i, j \leq l$) then the vertices of attachment of $B_i$ on $P$ and $Q$ are to the left of those of $B_j$ on $P$ and $Q$, respectively.

It then follows that of the four vertices $a_1$, $a_2$, $b_1$ and $b_2$, at least two lie on the subpath $J [t_P(B_1); t_Q(B_1)]$ and at least two lie on the subpath $J [s_P(B_1); s_Q(B_1)]$ and that $l = 2$.

But then it is easy to see that in this case either $U_{B_1}$ or $U_{B_2}$ has a pair of interlacing bights; this contradicts the assumption. □
of $U_B$ obtained by contracting the set of residual paths, $\mathcal{L}$, be as in the step 2 of the Algorithm Analyze-$U$-$U$.

**Theorem 5.18** If $U_B$ contains a $P_4$, $Q_4$, $PQ$- or an ST-Cross-Cut pair then $U_B$ has a $P_4$, $Q_4$, or an ST-Cross-Cut pair.

**Proof.**
Follows immediately from the following two facts:
1. If $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$ are the cross-cuts of $U_B^i$, then $U_B$ has a pair of interlacing cross-cuts, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$.
2. If $L_i[a; b]$ is a contraction of $L_i[a; b] \in \mathcal{L}$, and if $U_B^i$ has an external vertex of attachment at $a$, at $b$ or on $L_i[a; b]$ then $U_B$ has an external vertex of attachment at $a$, at $b$ or on $L_i[a; b]$ respectively. \qed

**Theorem 5.19** If $U_B$ does not have a pair of interlacing bights then $U_B$ does not have a pair of interlacing bights.

**Proof.** Let $U_B^0 (= U_B^1)$, $U_B^1, \ldots, U_B^m (= U_B^i)$ be sequence of minors of the bridge-fragment $U_B$, where $U_B^i$ is obtained from $U_B^{i-1}$ by contracting $L_i$ (for $1 \leq i \leq m$). The theorem follows from the following claim:

**Claim** If $U_B^{i-1}$ has a pair of interlacing bights then $U_B^i$ has a pair of interlacing bights.

**Proof of the claim.** Let $L_i = L[a; b]$. Let $J_0$ be the cycle $J$ of $U_B$, and $J_i$, the cycle in $U_B^i$ obtained by contracting the subpath $L_i$ of $J_{i-1}$ in $U_B^{i-1}$. Assume that $a_1, a_2, b_1$ and $b_2$ are four distinct vertices of $J_{i-1}$ such that there is a pair of interlacing bights, $A_1$ between $a_1$ and $a_2$ and $A_2$ between $b_1$ and $b_2$—here, $a_1, a_2, b_1$ and $b_2$ are either external vertices of attachment or one of $s$ and $t$. Henceforth, assume that $|L[a; b]| \geq 3$, since otherwise $U_B^i = U_B^{i-1}$, and the claim holds trivially.

Consider the case when both $A_1$ and $A_2$ have common sections, $M_1$ and $M_2$, respectively, such that $L \cap M_i \neq \emptyset$ (for $i = 1, 2$).

Then $M_1$ and $M_2$ are common end sections of $A_1$ and $A_2$, respectively. Without loss of generality assume that $M_1 = A_1^i[a_1; a]$ and $M_2 = A_2^i[b_1; b]$. Then $a_1$ must be to the left of $b_1$ on $L[a; b]$ and $|EA(L[a; b])| \geq 2$. Let $[x; y]$ be the contraction of the subpath $L[a; b]$. Then $U_B^i$ has external vertices of attachment at $x$ and $y$.

Note that both $x$ and $y$ are external vertices of attachment of $U_B^i$, and $x$ and $a_2$ separate $y$ and $b_2$ on $J_i$. The vertex-disjoint paths $[x, a] * A_1[a; a_2]$ and $[y, b] * A_2[b; b_2]$ define the appropriate interlacing bights.

Other cases can be handled in a similar fashion. \qed

### §E Interlacing Cross-Cuts and Bights: Augmented Graph

Let $U$ be a feasible augmented graph consisting of the cycle $J = \{P\} \cup \{Q\}$.

**Theorem 5.20** Let $U$ be a feasible augmented graph, with the blocks of $B^{PQ}$-bridges: $B_1, B_2, \ldots, B_l$.

Then $U$ has a $P_4$, $Q_4$, or an ST-cross-cut pair, if for some $i$ ($1 \leq i \leq l$) $U_B$ has a $P_4$, $Q_4$, or an ST-cross-cut pair.

**Proof.**
Obvious. \qed
(4) Since $U$ is feasible, $U$ an external vertex of attachment $d$, on $Q[s_Q; s_Q^*]$ or on $J[t_P^*; t_Q^*]$, where $d$ is distinct from $c$. Assume that $d \in Q[s; s_Q]$. Then, if $C_2$ is defined then we may assume that the bridges of $C_2$ avoid the bridges of $C_1$, since, otherwise, $U$ has $Q$- or $ST$-cross-cuts. Hence $B$ interlaces with a bridge of $C_1$, but in this case, again $U$ has $Q$- or $ST$-cross-cuts.

In the other case, i.e., $d \in J[t_P^*; t_Q^*]$, $B$ interlaces with a bridge of $C_1 \cup C_2$, and $U$ has $P$, $Q$, or $PQ$-cross-cuts. \qed

Lemma 5.16 Let $U$ and $B$ be as in the Theorem 5.14. If $U$ has an external vertex of attachment, $c$, on $P[s_P; t_P]$ and an external vertex of attachment, $d$, on $Q[s_Q; t_Q]$ then $U$ has a $PQ$-, $P$, $Q$- or an ST-cross-cut pair.

Proof. We may assume that $B$ has exactly one vertex of attachment, $s_P = t_P$, on $P[s; t]$ and has exactly one vertex of attachment, $s_Q^* = t_Q^*$ on $Q[s; t]$, since, otherwise, the statement follows from the previous Lemma 5.15. Since $s_P$ is distinct from $t_P$, $s_Q$, distinct from $t_Q$, and since $B$ is not a block of equivalent $BPQ$ 3-bridges, $B$ has a vertex of attachment at $s$ and a vertex of attachment at $t$. Since $B$ is a single proper block of overlapping $BPQ$-bridges, there must be a $BPQ$-bridge, $B \in B$, such that $B$ has vertices of attachment at $s$ and $t$ and $U$ has ST-cross-cuts. \qed

Lemma 5.17 Let $U$ and $B$ be as in the Theorem 5.14. If $U$ has an external vertex of attachment, $c$, on $J[s_P; s_Q]$ and an external vertex of attachment, $d$, on $J[t_P; t_Q]$, and if $B$ has more than one distinct vertices of attachment on $P[s; t]$ and more than one distinct vertices of attachment on $Q[s; t]$ then $U$ has $PQ$-, $P$, $Q$- or an ST-cross-cut pair.

Proof. Since $B$ is not a singleton set, and not a block of equivalent 3-bridges, and since $B$ has more than one distinct vertices of attachment on $P[s; t]$ and more than one distinct vertices of attachment on $Q[s; t]$, it must have two bridges $B_1$ and $B_2 \in B$ such that they provide two interlacing cross-cuts: $N_1[x_1; x_2]$ in $B_1$ and $N_2[y_1; y_2]$ in $B_2$ such that $x_1$ is to the left of $y_1$ on $P[s; t]$ and $y_2$ is to the left of $x_1$ on $Q[s; t]$. Hence $U$ has $PQ$-cross-cuts. \qed

Proof of the Theorem 5.14.

Since $U$ is feasible either $s_P \neq t_P$ or $s_Q \neq t_Q$ (or both). Assume that $s_Q \neq t_Q$.

- Case 1. $U$ has an external vertex of attachment on $Q[s_Q; t_Q]$.

If $s_P = t_P$ then both $s_Q$ and $t_Q$ are distinct from $s$ and $t$; and the theorem follows from Lemma 5.15. Hence assume that $s_P \neq t_P$. If $U$ also has an external vertex of attachment on $P[s_P; t_P]$ then the theorem follows from Lemma 5.16. On the other hand, if $U$ has no external vertex of attachment on $P[s_P; t_P]$ then, since $U$ is feasible, $s_Q^*$ and $t_Q^*$ are distinct, and the theorem follows from Lemma 5.15.

- Case 2. $U$ has no external vertex of attachment on $Q[s_Q; t_Q]$.

Since $U$ is feasible, $s_P^*$ and $t_P^*$ are distinct. We may assume that $U$ has no external vertex of attachment on $P[s_P; t_P]$ (otherwise, it is similar to the previous case.) Then $s_Q^*$ and $t_Q^*$ are distinct. Moreover, since $U$ is feasible, not all the external vertices of attachment lie only on the subpath $J[s_Q; s_P^*]$, or only on the subpath $J[t_P; t_Q]$. But then the theorem follows from Lemma 5.17. \qed

§D Interlacing Cross-Cuts and Bights: Minor of an Augmented Graph

Let $U_B$ be an augmented graph with the single (possibly, degenerate) block of $BPQ$-bridges $B$ of $J = \{P\} \cup \{Q\}$ in $U_B$. Let $\mathcal{L}$, the set of residual paths of $J$ in $U_B$ and $U_B^\prime$, the minor
Note that if Class.1 = 0 then both Class.2 and Class.3 \neq 0.

(1) We may assume that Class.1 = 0. Suppose not, i.e., Class.1 \neq 0. Then if Class.4 \neq 0 then \( U \) has P-cross-cuts. If Class.2 \neq 0 then a bridge of Class.2 overlaps with a bridge of Class.1 and \( U \) has P- or PQ-cross-cuts. If Class.3 \neq 0 then again \( U \) has P- or PQ-cross-cuts.

Hence consider the case when Class.2 = Class.3 = Class.4 = 0, i.e., \( B = \text{Class.1} \). Hence Class.1 is neither a singleton set; nor is it a block of equivalent \( B^{PQ} \) 3-bridges. Hence, either Class.1 has more than two vertices of attachment on \( P \), in which case \( U \) has P-cross-cuts, or Class.1 has more than one vertex of attachment on \( Q \), in which case \( U \) has \( P \)Q-cross-cuts.

(2) We may assume that Class.2 \neq 0 and Class.3 \neq 0 and bridges of Class.2 avoid the bridges of Class.3.

This follows from the facts that Class.1 = 0 and if a bridge of Class.2 interlaces with a bridge of Class.3 then \( U \) has P- or PQ-cross-cuts. Hence the vertices of attachment of Class.2 are to the left of those of Class.3 on \( P \) and \( Q \), respectively. Let \( x_2 \) and \( y_2 \) be the right-most vertices of attachment of Class.2 on \( P \) and \( Q \), respectively; and similarly, \( x_3 \) and \( y_3 \), the left-most vertices of attachment of Class.3 on \( P \) and \( Q \), respectively. Hence \( x_2 \) is to the left of \( x_3 \) on \( P \) and \( y_2 \), to the left of \( y_3 \) on \( Q \).

(3) We may further assume that Class.4 \neq 0.

If Class.4 = 0 then from (2) it follows that \( B \) is not a single block of \( B^{PQ} \)-bridges, which is a contradiction. Let \( v \) be an arbitrary vertex on the path \( Q[y_2, y_3] \). If every bridge of Class.4 has all its vertices of attachment on \( Q[s; t] \) either to the left of \( v \) or to the right of \( v \) on \( Q \), then \( B \) is not a single block of \( B^{PQ} \)-bridges.

Hence Class.4 has a \( B^{PQ} \)-bridge that interlaces with a bridge of Class.2 and has a vertex of attachment on \( Q[y_2, y_3] \). Similarly, Class.4 has a \( B^{PQ} \)-bridge that interlaces with a bridge of Class.3 and has a vertex of attachment on \( Q[s; y_3] \).

Since \( U \) is feasible it has an external vertex of attachment \( d \), distinct from \( c \). If \( d \) lies on \( J[y_2, c[ \) or \( J[c, y_2[ \) then \( U \) has \( P \)-, \( Q \)-, \( PQ \) or \( ST \)-cross-cuts.

Hence, assume that \( d \in Q[y_2, y_3] \). Hence there is a bridge \( B \in \text{Class.4} \) with a vertex of attachment on \( Q[s; d[ \) and a vertex of attachment on \( Q[d; t] \). If in addition, \( B \) has a vertex of attachment on \( Q[s, d[ \) and a vertex of attachment on \( Q[d; t] \) then if we define the Classes (1, 2, 3 and 4) as above, then the Class.1 \neq 0 for \( d \). Hence we can derive a contradiction as in (1). Otherwise it is easy to see that \( U \) has \( Q \)- or \( ST \)-cross-cuts.

- **Case 2.**
  By an argument similar to Case 1, we derive a contradiction, if \( c \in P[s_p, t_p] \) (where \( s_Q \neq t_Q \)).

- **Case 3.**
  Hence assume that \( U \) has no external vertex of attachment on \( P[s_p, t_p] \) or \( Q[s_q, t_q] \), but \( U \) satisfies one of the conditions of the Lemma, say condition(1).

  Without loss of generality assume that \( c \in P[s_p, s^*] \) (where \( s_p \neq s^* \)). Let Class.1, Class.2, Class.3 and Class.4, with respect to \( c \), be defined as before. In this case, Class.1 and Class.2 = 0, but Class.3 \cup Class.4 \neq 0.

  Let \( C_1 \subseteq B \) be the set of bridges of \( B \) with a vertex of attachment at \( s \). Notice that \( C_1 \) is non-empty. Let \( x \) and \( y \) be the left- and right-most vertices of attachment of \( C_1 \) on \( P \) and \( t \).

  (1) If \( B \setminus C_1 \) has a bridge with a vertex of attachment on \( P[s; y[ \) then \( U \) has \( P \)-cross-cuts. Hence every bridge of \( B \setminus C_1 \) has all its vertices of attachment on \( P[s; t[ \) only on \( P[y[ \).

  (2) If \( x \) and \( y \) are distinct then \( C_1 \) is a singleton set. (Otherwise, \( U \) has \( P \)-cross-cuts.) Since \( B \) is a proper block, \( B \setminus C_1 \) has a bridge \( B \) such that \( B \) overlaps with the bridge of \( C_1 \). Since the \( B^{PQ} \)-bridge, \( B \in B \setminus C_1 \), \( B \) has a vertex of attachment on \( P[x; t[ \).

  (3) If \( x = y \) then define \( C_2 \) to be the set of bridges of \( B \) with exactly one vertex of attachment at \( x \) on \( P[s; t[ \). But in this case, since \( B \) has more than one distinct vertices of attachment on \( P \) and \( t \), there is a \( B^{PQ} \)-bridge \( B \in B \setminus (C_1 \cup C_2) \) that overlaps with a bridge of \( C_1 \cup C_2 \). Since the \( B^{PQ} \)-bridge, \( B \in B \setminus (C_1 \cup C_2) \), \( B \) has a vertex of attachment on \( P[x; t[ \).
constituent bights of $J_1$ in $\overline{U}$ to a Jordan curve such that all but its end vertices lie in the face, $F_1$; each of its constituent bights of $J_2$ in $\overline{U}$ to a Jordan curve such that all but its end vertices lie in the face, $F_2$; and each of its common sections to appropriate common sections in the Jordan curve $R(\pi) \text{ or } J(\pi)$. The claim follows from the following observations: every common section on $R(\pi)$ (or $J(\pi)$) is disjoint with every other common section on $R(\pi)$ (or $J(\pi)$, respectively), and do not contain a point of $F_1$, $F_2$ or EXT $J$; every Jordan curve corresponding to a bight of $J_1$ (respectively, $J_2$) can be mapped in $F_1$ (respectively, $F_2$) such that it does not meet any other Jordan curves in $F_1$ (respectively, $F_2$) and meets $R(\pi)$ or $J(\pi)$ at the appropriate ends of its common end sections.

Arguing in a manner similar to above, we can show that, since $A_1'$ and $A_2'$ are vertex-disjoint simple paths and since constituent bights of neither of them can interlace with one another in $U'$ or $\overline{U}$, it is, additionally, possible to map them such that $A_1'(\pi)$ and $A_2'(\pi)$ do not meet each other. Notice that $A_1'(\pi)$ and $A_2'(\pi)$ are Jordan curves with their endpoints $a_1(\pi)$ and $a_2(\pi)$, and $b_1(\pi)$ and $b_2(\pi)$ of $J(\pi)$, respectively, and with none of their points in EXT $J$. But this is impossible. □

§C Interlacing Cross-Cuts and Bights: Single Proper Block

Notice that if $U$ is an infeasible augmented graph with a single block of $B^{PQ}$-bridges, it follows from the Theorem 3.1 that $U$ does not have a pair of interlacing bights. Hence we may concentrate only on the case when $U$ is a feasible augmented graph with a single (feasible) proper block of $B^{PQ}$-bridge, $B$.

**Theorem 5.14** Let $U$ be a feasible augmented graph with a single (feasible) proper block of $B^{PQ}$-bridges, $B$. Notice that since $U$ is feasible, $B$ is not a block of equivalent 3-bridges. Then $U$ has a $PQ$-, $P$-, $Q$- or an ST-cross-cut pair. □

**Lemma 5.15** Let $U$ and $B$ be as in the Theorem 5.14. If

1. $U$ has an external vertex of attachment, $c$, on $P][s; t_p]\{ and $B$ has more than one distinct vertices of attachment on $P][s; t]$,

2. $U$ has an external vertex of attachment, $c$, on $Q][s; t_q]$ and if $B$ has more than one distinct vertices of attachment on $Q][s; t$]

then $U$ has $PQ$-, $P$-, $Q$- or an ST-cross-cut pair.

**Proof.**
Assume to the contrary, i.e. $U$ satisfies one of the conditions of the lemma, but does not have $P$-, $Q$-, $PQ$- or ST-cross-cuts.

- **Case 1.**
First assume that $c \in P][s; t_p]\{ (where $s_p \neq t_p$). Let us partition the bridges of $B$ in to the following four classes:

1. Class 1 = \{ $B \in B$ | $B$ has at least one vertex of attachment on $P][s; c]$ and at least one vertex of attachment on $P][c; t]$ \}

2. Class 2 = \{ $B \in B$ | $B$ has at least one vertex of attachment on $P][s; c]$ and no vertex of attachment on $P][c; t]$ \}

3. Class 3 = \{ $B \in B$ | $B$ has no vertex of attachment on $P][s; c]$ and at least one vertex of attachment on $P][c; t]$ \}

4. Class 4 = $B \setminus ( $\text{Class 1} \cup $\text{Class 2} \cup $\text{Class 3} ) = \{ $B \in B$ | $B$ has exactly one vertex of attachment on $P][s; t$ [ at $c$. \}
by the Claim 2. $A_1$ must be a bight of $\overline{U}$. Hence if the path associated with $A_1$ contains a vertex of $V(H)$ it must lie on $R[s_Q; t_Q]$.  

(2) Assume that the path associated with $A_1$ contains exactly one vertex $x \in V(H)$, where $x$ lies on $R[s_Q; t_Q]$. Hence $x$ must be $v_{\text{cat}}$. Hence the path associated with $A_2$ does not contain $v_{\text{cat}}$, and the end segment of $A_2$ starting at $b_1$, $A_2[b_1; y_1]$ must terminate in $s_Q$ or $t_Q$. In either case $\overline{U}$ has a pair of interlacing bights.

(3) Hence if the path associated with $A_1$ contains a vertex $x \in V(H)$ then it must be one of $s_Q$ or $t_Q$ (say, $s_Q$). Let the end segment of $A_2$ starting with $b_1$, $A_2[b_1; y_1]$ terminate in $y_1$ where $y_1$ lies on the subpath $Q[s_Q; t_Q]$. Hence $A_2'[b_1; y_1]$ must contain the $v_{\text{cat}}$.

Since $R$ is a path in a proper bridge, $R$ must have an internal vertex, $c$. Let $L$ be a path in $B$ connecting the vertices $c$ and $v_{\text{cat}}$. An appropriate subpath of $L$ meets $R$ only in $c'$ and $A_2'[b_1; y_1]$ only in $y'$ and avoids the path associated with the bight $A_1$. (Follows from reducibility of $B$.) Hence the path $A_2'[b_1; y_1] \ast L[y'; c'] \ast R[c'; t_Q]$ defines a bight of $\overline{U}$ between $b_1$ and $t_Q$ that interlaces with the bight $A_1$.

On the other hand, if the end segment of $A_2'[b_1; y_1]$ terminates in a vertex on the subpath $R[s_Q; t_Q]$ it is easy to show, by an argument as above, that $\overline{U}$ has a pair of interlacing bights. In either case, it results in a contradiction.  

\[ \square \]

**Theorem 5.13** Let $U$ be a feasible augmented graph with a single (feasible) $B^PQ$-bridge, $B$, such that the pair of vertices associated with $B$ is $(s_Q; t_Q)$. Let $R[s_Q; t_Q]$ be a path in $B$, connecting the vertices $s_Q$ and $t_Q$, such that $B$ is irreducible with respect to $R$. Let $B_1, B_2, B_3$ and $B_4$ be as in the definition 4.5. Assume that $|B_3| = 0$; and the path $R$ divides $U$ into $U'$, a $U$-Fragment on $Q$, and $\overline{U}$ a $\overline{U}$-Fragment on $Q$, as in the definition 4.5.

Then $U$ does not have a pair of interlacing bights, if neither $U'$ nor $\overline{U}$ has a pair of interlacing bights.

**Proof**.

In the following we use some simple ideas from ‘point set topology.’ (Cf. [5], [15] and [17].) Let $\pi$ be a plane and $J(\pi)$ be closed Jordan curve in $\pi$: the closed Jordan curve $J(\pi)$ divides the rest of the plane into two connected open residual domains, the interior domain (int $J$) and the exterior domain (ext $J$).

Assume that $a_1, a_2, b_1$ and $b_2$ are four distinct vertices of $J$ such that there is a pair of interlacing bights, $A_1$ between $a_1$ and $a_2$, and $A_2$ between $b_1$ and $b_2$—here, $a_1, a_2, b_1$ and $b_2$ are either external vertices of attachment or one of $s$ and $t$. Let $A'_1$ and $A'_2$ be the paths associated with the bights $A_1$ and $A_2$, respectively. We derive a contradiction.

Let $J = P[s; t] \cup Q[s; t]$, $J_1 = P[s; t] \cup Q[s; s_Q] \ast R[s_Q; t_Q] \ast Q[t_Q; t]$ and $J_2 = Q[s_Q; t_Q] \cup R[s_Q; t_Q]$.

Let $J(\pi)$ be a closed Jordan curve in the plane $\pi$. The vertices on $J$ are mapped onto a point set in $J(\pi)$ such that the mapping preserves their order in $J$. Let $s_Q(\pi)$ and $t_Q(\pi)$ be the points on $J(\pi)$ corresponding to the vertices $s_Q$ and $t_Q$ of $J$, respectively. There exists a Jordan curve in the plane $\pi$ with its end points being $s_Q(\pi)$ and $t_Q(\pi)$ and with all other vertices of it in int $J$; let $R(\pi)$ be such a Jordan curve. The vertices on $R$ are mapped onto a point set in $R(\pi)$ such that the mapping preserves their order in $R$ with respect to $s_Q$ and $t_Q$. As a result the cycles $J_1$ and $J_2$ are mapped to the closed Jordan curves $J_1(\pi)$ and $J_2(\pi)$, uniquely defined by $R(\pi)$. Let the faces $F_1$ and $F_2$ be the residual domains int $J_1$ and int $J_2$, respectively.

Let $A'[a_1; a_2]$ be the path associated with a bight between $a_1$ and $a_2$ of $J$ in $U$. Such a path can always be written as the concatenation of common sections on $R$, common sections on $J$, and bights between vertices of $J_1$ in $\overline{U}$ or bights between vertices of $J_2$ in $U'$. By the assumption, it is impossible for two bights of $J_1$ in $\overline{U}$ or two bights of $J_2$ in $U'$, to interlace. Moreover, since $A'$ is a simple path all its common sections are vertex-disjoint.

We claim that $A'$ can be mapped to a Jordan curve $A'(\pi)$ with its end points being $a_1(\pi)$ and $a_2(\pi)$ such that none of the points on it lie in ext $J$. This can be done by successively mapping each of its
Let $A$ be a bight of $U$ between two vertices $a_1$ and $a_2$, where $a_1$ and $a_2$ are external vertices of attachment of $U$ or one of $s$ and $t$. Let $A'$ be the path associated with the bight $A$. If $A'$ contains one or more vertices of $V(H)$ then the subpath of $A'$, $A'[a_1; a_2] = u_0(= a_1), u_1, \ldots, u_{n-1}, u_n(= a_1)$, where $u_0, \ldots, u_{n-1} \not\in V(H)$ and $u_n \in V(H)$, is called the end segment of $A$ starting at $a_1$. The end segment of $A$ starting at $a_2$ is defined similarly.

![Diagram of three cases for Lemma 5.12](image)

**Figure 23: Three Cases for Lemma 5.12.**

**Claim 1.** If the path associated with a bight $A$ of $U$ contains at least two distinct vertices of $V(H)$ then it contains at least two out of three vertices, $v_{\text{cut}}$, $s_Q$, and $t_Q$.

**Proof of Claim 1.** Let $A$ be a bight of $U$ between two vertices $a_1$ and $a_2$. Let $A'[a_1; x_1]$ and $A'[a_2; x_2]$ be the end segments of $A$ starting with $a_1$ and $a_2$, respectively. Since the path associated with $A$, $A'$ contains two or more distinct vertices of $V(H)$, $x_1$ and $x_2$ are distinct and $A'[a_1; x_1]$ and $A'[a_2; x_2]$ are vertex-disjoint.

Notice that if $x_1$ is distinct from $s_Q$ and $t_Q$ then $A'[a_1; x_1]$ contains $v_{\text{cut}}$. This is due to following: Since $x_1 \in V(H)$ it lies on the path $R|s_Q; t_Q|$. Since none of the other vertices of $A'[a_1; x_1]$ is in $V(H)$, there is a subpath of $A'[a_1; x_1]$ that connects a vertex of $P|s$ with $x_1$, and otherwise avoids $P$, $Q$ and $R$. But this subpath must contain $v_{\text{cut}}$. Similarly, if $x_2$ is distinct from $s_Q$ and $t_Q$ then $A'[a_2; x_2]$ contains $v_{\text{cut}}$.

The claim follows from the observation that $A'[a_1; x_1]$ and $A'[a_2; x_2]$ are vertex-disjoint, and hence, both cannot contain $v_{\text{cut}}$.

**Claim 2.** Let $A$ be a bight of $U$. Then either $A$ is also a bight of $\overline{U}$, or the path associated with it contains at least two distinct vertices of $V(H)$.

**Proof of the Claim 2.** Assume to the contrary, i.e., there is a bight $A$ of $U$ between $a_1$ and $a_2$ such that the path associated with it contains no more than one vertex of $V(H)$, $A$ is not a bight of $\overline{U}$.

This is possible only if $A$ contains only one vertex $x \in V(H)$ such that $x$ lies on $Q\mid s_Q; t_Q\mid$. But, this implies that the end segments $A'[a_1; x]$ and $A'[a_2; x]$ both contain $v_{\text{cut}}$. But since $v_{\text{cut}} \in N(B)$ it is distinct from $x$, and this contradicts the fact that $A'$, the path associated with the bight $A$, is simple.

Assume that $a_1$, $a_2$, $b_1$ and $b_2$ are four distinct vertices on $J$ of $U$ such that there is a pair of interlacing bights $A_1$ between $a_1$ and $a_2$, and $A_2$ between $b_1$ and $b_2$, where $a_1$, $a_2$, $b_1$ and $b_2$ are either external vertices of $U$ or one of $s$ and $t$.

Since $\overline{U}$ does not have a pair of interlacing bights at least one of $A_1$ and $A_2$ (say, $A_2$) is not a bight of $\overline{U}$. Let $a_1$ precede $a_2$ in the clockwise cyclic order of the vertices of $J$. Since $a_1$ and $a_2$ separate $b_1$ and $b_2$ on $J$, one of $b_1$ and $b_2$ (say $b_1$) lies on $J|a_1; a_2|$.

(1) We may assume that $A_1$ contains no more than one vertex of $V(H)$, since, otherwise, by the previous claims each of the paths associated with the bights $A_1$ and $A_2$ must contain at least two out of three vertices, $v_{\text{cut}}$, $s_Q$ and $t_Q$, which is impossible, since these two paths are vertex-disjoint. Moreover,
Figure 22: Cases 2 and 3 of Lemma 5.11.

(2) \( U' \) has a PQ-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1 \) is to the left of \( y_1 \) on \( P \).
Since \( U \) must be PQ- or P-feasible, \( |UA(U')| > 0 \) and since \( B \) is irreducible with respect to \( R \), \( |LA(U')| > 0 \).

If both \( s_Q \) and \( t_Q \) are distinct from \( s \) and \( t \) then, clearly, \( U \) has weak PQ-cross-cuts. Hence assume that \( s_Q = s \). If \( U' \) has a lower external vertex of attachment on \( R[y_2; t_Q] \) and an upper external vertex of attachment on \( P[s; y_1] \) then \( U \) has P-cross-cuts. If not, \( U' \) has a lower external vertex of attachment on \( R[s; x_2] \) and an upper external vertex of attachment on \( P[x_1; t] \). If \( t_Q \) is distinct from \( t \) then \( U \) has weak PQ-cross-cuts. On the other hand, if \( t_Q = t \) then \( U \) has P-cross-cuts.

(3) \( U' \) has a P-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1, x_2 \in P[s; t], y_1 \in P[x_1; x_2[ \) and \( y_2 \in R[s_Q; t_Q] \).

If \( s_Q \) is distinct from \( s \) (or, symmetrically, \( t_Q \) distinct from \( t \) then, since \( N_1 \) avoids \( R[s_Q; t_Q] \), \( U \) has P-cross-cuts. If \( s_Q = s \) and \( t_Q = t \) then \( N_1 \) avoids \( R[s_Q; t_Q] \). Moreover, since \( B \) is irreducible, \( U' \) has a lower external vertex of attachment \( \theta' \in R[s_Q; t_Q] \) and there is a path \( R_4[\theta'; b] \) in \( B_2 \) such that \( R_6 \) meets \( R[s_Q; t_Q] \) only in \( \theta \) and meets \( Q[s_Q; t_Q] \) only in \( b \). Clearly, the cross-cuts, \( N_1 \) and \( N_2[y_1; y_2] \ast R[y_2; \theta'] \ast R_4[\theta'; b] \), form P-cross-cuts in \( U \).

\( B \) Interlacing Bights: Single Bridge

**Theorem 5.12** Let \( U \) be a feasible augmented graph with a single (feasible) \( B^{PQ} \)-bridge, \( B \), such that the pair of vertices associated with \( B \) is \( \langle s_Q; t_Q \rangle \). Let \( R[s_Q; t_Q] \) be a path in \( B \), connecting the vertices \( s_Q \) and \( t_Q \), such that \( B \) is reducible with respect to \( R \). Let \( U' \) be its \( U \)-Fragment, as in the definition 4.5.

Then \( U \) does not have a pair of interlacing bights, if \( U' \) has no pair of interlacing bights.

**Proof.**
First, we prove two useful claims. Since \( B \) is reducible with respect to \( R \), there is a cut vertex, \( v_{cut} \), in the nucleus of the bridge \( B, N(B) \). Let \( H \) be the section graph induced by the edges of the subpaths \( Q[s_Q; t_Q] \) and \( R[s_Q; t_Q] \), i.e.,

\[ H = \langle E(Q[s_Q; t_Q]) \cup E(R[s_Q; t_Q]) \rangle. \]
PROOF.

(1) $U'$ has a $Q$- or an $ST$-cross-cut pair, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$, where $x_1$, $x_2 \in R[s_q; t_q]$ and $y_1 \in P[s]; t[$ and $y_2 \in R[x_1; x_2]$.

By definition of $Q$- and $ST$-cross-cuts, there is a lower external vertex of attachment $c' \in R[x_1; x_2]$ and hence a path $R[c'; c]$ in the $U$-Fragment $U'$ such that $R[c]$ meets $R[s_q; t_q]$ only in $c$ and meets $Q[s_q; t_q]$ only in $c$.

Let us refer to the path $R[s_q; x_1] \ast N_1[x_1; x_2] \ast R[x_2; t_q]$ by $M_1$ and the path $R[x_1; x_2]$ by $M_2$.

By lemma 5.7, it suffices to show that $U$ has a $Y$-$Q$ pair. Since $B$ is irreducible with respect to $R$, and since $|B_s| = 0$, there are two vertex disjoint paths $R_a[a'; a]$ and $R_b[b'; b]$ where $a', b' \in R[s_q; t_q]$ and $a, b \in P[s]; t[$. Let $R_a$ and $R_b$ be the appropriate subpaths of the disjoint paths $R_a$ and $R_b$, respectively, such that $R_a$ and $R_b$ meet $P[s]; t[$ only in $a$ and $b$, respectively; $R_a$ meets only one of $M_1$ and $M_2$ only in $a''$ and $R_b$ meets only one of $M_1$ and $M_2$ only in $b''$, where $a''$ and $b'' \in V(M_1) \cup V(M_2) \\{s_q, t_q\}$.

![Diagram](image.png)

**Figure 21: Case 1 of Lemma 5.11.**

- **Case 1.**
  
  $a''' \in V(M_1) \\{s_q, t_q\}$ and $b''' \in V(M_2)$. Then the cross-cut $N = R_a[b'; b'] \ast M_2[b'; c'] \ast R_a[c'; c]$ and the $Y$-graph $Y = \{M_1[s_q; a''] \ast \{R_a[a; a'']\} \ast \{M_1[a''; s_q]\}$ form a $Y$-$Q$ pair.

- **Case 2.**
  
  Since $N_1$ is a path in some $B_1$ bridge, say $B_1$, $B_1$ is a proper bridge and $N_1$ has an internal vertex, $d'$. Let $d$ be a vertex of attachment of $B_1$ on $P$. There is a path in $B_1$ whose ends are $d$ and $d'$ and which avoids $P$. A suitable subpath, $R_a[d; d']$, of it meets $N_1$ only in $d''$ an internal vertex of $N_1$, meets $P[s]; t[$ in $d$ and otherwise avoids $P$, $R$ and $N_1$. If $R_a[d]$ is vertex-disjoint with either of $R_a$ and $R_b$, then this case reduces to an instance of the previous case. Suppose not. Then a suitable subpath of $R_a[d; d']$ meets $N_1$ only in $d''$, only one of $R_a$ and $R_b$ only in $z$ and avoids the other. Without loss of generality assume that $z \in R_a[a; a'''']$. Then the pair of paths $R_a[d'''; z] \ast R_a[z; a]$ and $R_b$ satisfies the condition of the previous case.

- **Case 3.**
  
  $a'''$ and $b''' \in V(M_1) \\{s_q, t_q\}$. If $N_2[y_1; y_2]$ is vertex-disjoint with either of $R_a$ and $R_b$, then this case reduces to an instance of the first case. Suppose not. Then a suitable subpath of $N_2$ meets $M_2$ only in $y_1$, meets only one of $R_a$ and $R_b$ only in $z$ and avoids the other. Without loss of generality assume that $z \in R_a[a; a'''']$. Then the pair of paths $N_2[y_1; z] \ast R_a[z; a]$ and $R_b$ satisfies the condition of the first case.
Figure 20: Two Cases for Lemma 5.10.

(1) \( U' \) has a \( PQ \)-cross-cut pair, \( N_1 [x_1; x_2] \) and \( N_2 [y_1; y_2] \).

As a result of the fact that \( N_1 \) and \( N_2 \) form \( PQ \)-cross-cuts and the above claim, we may assume that \( a' \in R[s_Q; y_1] \) and \( b' \in R[x_1; t_Q] \). Then the cross-cut \( N = N_2[y_2; y_1] \ast R[y_1; b'] \ast R[b'; b] \) and the \( Y \)-Graph, \( Y = \{ R[s_Q; x_1] \} \cup \{ R[a; a'] \ast R[a'; x_1] \} \cup \{ N_1[x_1; x_2] \} \) (if \( a' \in R[x_1; y_1] \)), or \( Y = \{ R[s_Q; a'] \} \cup \{ R[a'; x_1] \ast N_1[x_1; x_2] \} \) (if \( a' \in R[s_Q; x_1] \)), form a \( Y-Q \) pair.

(2) \( U' \) has \( P-Q \)-cross-cut pair, \( N_1 [x_1; x_2] \) and \( N_2 [y_1; y_2] \), where \( x_1, x_2 \in R[s_Q; t_Q] \).

As a result of the fact that \( N_1 \) and \( N_2 \) form \( P-Q \)-cross-cuts and the above claim, we may assume that at least one of \( a' \) and \( b' \in R[x_1; x_2] \), say \( a' \).

(2a) Assume that \( b' \in R[y_1; t_Q] \) and \( x_2 \) is distinct from \( t_Q \). Then the cross-cut \( N = N_2[y_2; y_1] \ast R[y_1; a'] \ast R[a'; a] \) and the \( Y \)-graph \( Y = \{ R[s_Q; x_1] \ast N_1[x_1; x_2] \ast R[x_2; b'] \} \cup \{ R_l[b'; b] \} \cup \{ R[b'; t_Q] \} \) (if \( x_2 \in R[t_Q] \)), or \( Y = \{ R[s_Q; x_1] \ast N_1[x_1; x_2] \} \cup \{ R[b'; b'] \ast R[b'; x_2] \} \cup \{ R[t_Q; x_2] \} \) (if \( b' \in R[y_1; x_2] \)), form \( Y-Q \) pair. Similarly, if \( b' \in R[s_Q; y_1] \) and \( x_1 \) is distinct from \( s_Q \) then \( U \) has a \( Y-Q \) pair.

(2b) From (2a), we see that if both \( x_1 \) and \( x_2 \) are both distinct from \( s_Q \) and \( t_Q \) then \( U \) has \( Y-Q \) cross-cuts.

(2c) Henceforth, assume that \( x_1 = s_Q \) and hence \( x_2 \) is distinct from \( t_Q \). Hence \( b' \in R[s_Q; y_1] \) and since \( b' \in R[x_1; x_2] \), also \( a' \in R[s_Q; y_1] \). Moreover, if \( U' \) has an external vertex of attachment on \( R[y_1; t_Q] \) then using the claim above we can show that \( U \) has a \( Y-Q \) pair. Hence all the external vertices of attachment of \( U' \) lie on \( R[s_Q; y_1] \).

Let \( s_R^* \) be the left-most vertex of attachment of \( U' \) on \( R \) distinct from \( s \). Hence, there is a bridge \( b' \in B_1 \) with a vertex of attachment \( s_R^* \) on \( R[s_Q; t_Q] \) and a vertex of attachment, \( y \), on \( Q[s_Q; t_Q] \). Let \( L[s_R^*; y] \) be a cross-cut of \( U' \) between \( s_R^* \) and \( y \). Since \( U' \) is feasible, not all the external vertex of attachment \( U' \) lie on \( R[s_Q; s_R^*] \). If \( L \) avoids both \( N_1 \) and \( N_2 \) then the \( P-Q \)-cross-cuts defined by \( N_1 \) and \( L \), together with an application of the above claim, satisfies (2a).

Hence, assume that \( L \) does not avoid both \( N_1 \) and \( N_2 \). Then there is a vertex \( z \in L[s_R^*; s] \) such that \( L[s_R^*; z] \) meets \( N \) or \( N_2 \) (but not both) in \( z \). If \( L \) meets \( N_2 \) in \( z \) then the \( P-Q \)-cross-cuts \( N_1 \) and \( L[s_R^*; z] \ast N_2[z; y] \), together with an application of the above claim, satisfies (2a). On the other hand, if \( L \) meets \( N_1 \) in \( z \) then the \( P-Q \)-cross-cuts \( L[s_R^*; z] \ast N_1[z; x_2] \) and \( N_2 \) satisfies (2b).

\[ \square \]

**Lemma 5.11** Let \( U, B, R, B_1, B_2 \) and \( B_3 \) be as in the Theorem 5.3. If \( U' \) contains a \( P-, Q-, P-Q- \) or \( ST- \)-cross-cut pair then \( U \) has a \( P- \) or a \( PQ- \)-cross-cut pair.
(3) $a', b' \in R[s_Q; t_Q]$.
Assume that $N[y_1; y_2]$ meets both $R_a$ and $R_b$, since otherwise, we can easily find a Y-Q-pair. Hence a suitable subpath of $N$ meets $Q$ only in $y_2$, meets only one of $R_a$ and $R_b$ in $z$, and avoids the other. Without loss of generality assume that $z \in R_a[a'; a']$. Then $R_a[a; z] \cup N[z; y_2] \cap \{R_b[b; b'] \cup \{R[b'; t_Q]\}$ form a Y-Q-pair.

**Lemma 5.10** Let $U$, $B$, $R$, $B_1$, $B_2$ and $B_3$ be as in the Theorem 5.3. If $U'$ contains a $P\!-, Q\!-, PQ\!-$ or ST-cross-cut pair then $U$ has a $P\!-, Q\!-, PQ\!-$ or an ST-cross-cut pair.

**Proof.**
Since $B$ is irreducible with respect to $R$, $|\Lambda(U')| > 0$. Moreover, since $U'$ must be $P$-feasible, $U'$ cannot have $Q$- or ST-cross-cuts.

Hence it suffices to show that if $U'$ has $PQ$- or $P$-cross-cuts, then $U$ has weak $PQ$-cross-cuts or $Y$-$Q$-cross-cuts. (Cf. Lemma 5.7.) Since $B$ is irreducible with respect to $R$, and since $|B_2| = 0$, there are two vertex disjoint paths $R_a[a'; a]$ and $R_b[b; b]$ where $a', b' \in R[s_Q; t_Q]$ and $a, b \in P[s; t]$ (Cf. Lemma 4.3.) Without loss of generality, we may assume that $a'$ is to the left of $b'$. The following claim will be found useful in the rest of the proof:

**Claim** Let $c'$ be an arbitrary upper external vertex of attachment of $U'$ distinct from $a'$ and $b'$. Then there exists a path $R_c$ in $B_1$, joining $c'$ to an internal vertex of $P$ such that $R_c$ avoids $R_a$ or $R_b$ or both.

**Proof of the Claim.** Let $B' \in B_1$ be a bridge such that $c'$ is a vertex of attachment of $B'$ on $R$ and let $c$ be a vertex of attachment of $B'$ on $P[s; t]$. Then there is a path $R_c[c'; c]$ in $B_1$. If $R_c$ avoids $R_a$ or $R_b$ then it satisfies the claim.

Hence, let $R_c$ meet one or both of the paths $R_a$ and $R_b$. It is possible to find a vertex $z \in R_c[c'; c]$ such that $R_c[z; c]$ meets $R_a[a'; a]$ or $R_b[b; b]$ but not both only in $z$. (This follows from the facts that $R_a$, $R_b$ are vertex disjoint and $a', b'$ and $c'$ are distinct.) Without loss of generality, assume that $z \in R_a[a'; a]$; then the path $R_c[c'; z] \ast R_a[z; a]$ avoids $R_b$.

![Figure 19: Three Cases for Lemma 5.9.](image)
$PQ$-cross-cuts of $\overline{U}$, it has an upper external vertex of attachment on $P[s; y_1]$ and $U$ has $P$-cross-cuts. The case, when $t_Q = t$ and $s_Q$ is distinct from $s$, is similar.

Figure 18: Two Cases for Lemma 5.8.

(2) $\overline{U}$ has a $P$-cross-cut pair, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$, where $x_1, x_2 \in P[s; t], y_1 \in P[x_1; x_2]$ and $y_2 \in R[s_Q; t_Q]$.
If $s_Q$ is distinct from $s$ (or, symmetrically, $t_Q$ distinct from $t$) then, since $N_1$ avoids $R[s_Q; t_Q]$, $U$ has $P$-cross-cuts. If, on the other hand, $s_Q = s$ and $t_Q = t$ then, since $B$ is reducible and since $y_2 \in R[s_Q; t_Q]$, the path $N_2$ must contain the cut vertex $v_{\text{cut}}$ of $N(B)$. Let $b$ be a vertex of attachment of $B$ on $Q[s; t] = Q[s_Q; t_Q]$, say $b$. Let $R_b[b; v_{\text{cut}}]$ be a path in $B$ joining $b$ and $v_{\text{cut}}$; and let $R_b[b; b']$ be a suitable subpath of $R_b$ such that $R_b$ meets $Q[s_Q; t_Q]$ only in $b$ and meets $N_1$ or $N_2$ (but not both) only in $b'$. If $R_b[b; b']$ meets $N_1$ in $b'$ then there is a path from $b$ to $x_1$ and to $x_2$ that does not contain the cut vertex, $v_{\text{cut}}$, where one of $x_1$ and $x_2 \in P[s; t]$—thus contradicting the assumption that $B$ is reducible with respect to $R$. Hence $R_b[b; b']$ meets $N_2$ in $b'$ and the cross-cuts $N_1$ and $N_2[y_1; b'] \ast R_b[b'; b]$ form $P$-cross-cuts.

\begin{lemma}
Let $U$, $B$, $R$, $B_1$, $B_2$ and $B_3$ be as in the Theorem 5.3. If $|B_3| \neq 0$ then $U$ has a $P$, $Q$, $PQ$, or an $ST$-cross-cut pair.
\end{lemma}

\textbf{Proof.}

If $|B_3| \neq 0$ then there is a cross-cut $N[y_1; y_2]$ of $J$ between $y_1 \in P[s; t]$ and $y_2 \in Q[s_Q; t_Q]$ such that $N$ avoids $R$.

We will show that $U$ has a $Y$-$Q$-Pair (Cf. Lemma 5.7.) Since $B$ is irreducible, there are two vertex disjoint paths $R_a$ and $R_b$.

(1) $a' \in Q[s_Q; t_Q]$ and $b' \in R[s_Q; t_Q]$.
Then the cross-cut $R_a$ and the $Y$-Graph $\{R_a[b; b']\} \cup \{R[s_Q; b']\} \cup \{R[t_Q; b']\}$ form a $Y$-$Q$-Pair.

(2) $a', b' \in Q[s_Q; t_Q]$.
Since $R$ belongs to a proper bridge, $B$, $R$ has an internal vertex $d''$. Let $d$ be a vertex of attachment of $B$ on $P[s; t]$. Then there is a path in $B$ that joins $d$ and $d''$, and a suitable subpath, $R_d[d; d']$, of it meets $R[s_Q; t_Q]$ only in $d'$, and meets $P[s; t]$ only in $d$. Assume that $R_d$ meets both $R_a$ and $R_b$; since, otherwise, we can easily find a $Y$-$Q$ pair. Hence, a suitable subpath of $R_d$ meets $R[s_Q; t_Q]$ only in $d'$, meets only one of $R_a$ and $R_b$ only in $z$, and avoids the other. Without loss of generality assume that $z \in R_a[a; a']$, then the cross-cut $R_b$ and the $Y$-graph $\{R_a[a; z] \ast R_d[z; d']\} \cup \{R[s_Q; d']\} \cup \{R[t_Q; d']\}$ form a $Y$-$Q$ pair.
Figure 17: Modifying a Y.Q Pair (Lemma 5.7).

(2) \( U \) has a Y.Q Pair, \( N[x_1; x_2] \) and \( Y = \{ Y_1[z_1; v] \} \cup \{ Y_2[z_2; v] \} \cup \{ Y_3[z_3; v] \} \). Without loss of generality assume that \( x_1 \) is to the left of \( z_1 \). If \( z_2 \) is distinct from \( s \) then the cross-cuts \( N_1 \) and \( Y_1[z_1; v] \ast Y_2[v; z_2] \) form weak P.Q-cross-cuts, in which case the proof proceeds as in (1). However, if \( z_2 = s \) and if there is an external vertex of attachment on \( P[s; z_1] \) then the vertex-disjoint cross-cuts \( N_1 \) and \( Y_1[s; v] \ast Y_2[v; z_1] \) form P.Q-cross-cuts.

Hence assume that \( z_2 = s \) and since \( B \) is \( P \)-feasible, all the external vertices of attachment lie on \( J[z_1; t_Q] \). But since \( U \) is feasible, it has at least two distinct external vertices of attachment and not all its external vertices of attachment lie on \( J[t_1; t_Q] \). This implies that \( U \) has an external vertex of attachment on \( J[z_1; t^*_P] \) and an external vertex of attachment on \( J[z_1; t_Q] \).

Let \( L \) be a path in \( B \) that joins \( t^*_P \) and \( v \) and avoids \( J \). It is possible to find a vertex \( z \in L[t^*_P; v] \) such that \( L[t^*_P; z] \) meets \( N \) or \( Y \) (but not both) only in \( z \), an internal vertex.

If \( z \) is an internal vertex of \( Y \) then the vertex-disjoint cross-cuts \( N \) and \( Y[s; z] \ast L[z; t^*_P] \) form P-Q-cross-cuts. On the other hand, if \( z \) is an internal vertex of \( N \) and if \( z_2 \) is distinct from \( t \), then the cross-cuts \( L[t^*_P; z] \ast N[z; z_2] \) and \( Y_1[z_1; v] \ast Y_2[v; z_3] \) form weak P.Q-cross-cuts; and the proof proceeds as in (1).

Hence, consider the case when \( z \) is an internal vertex of \( N \) and \( z_2 = t \). Observe that in this case \( t_Q = t \) and \( U \) has an external vertex of attachment on \( P[z_1; t] \). Clearly, the cross-cuts \( L[t^*_P; z] \ast N[z; z_2] \) and \( Y_1[z_1; v] \ast Y_2[v; t] \) form P.Q-cross-cuts.

Lemma 5.8 Let \( U \), \( B \), \( R \), \( B_1 \), \( B_2 \) and \( B_3 \) be as in the Theorem 5.2. If \( U \) contains a \( P \)-, \( Q \)-, \( P.Q \)- or \( ST \)-cross-cut pair then \( U \) has a \( P \)-, \( Q \)-, \( P.Q \)- or \( ST \)-cross-cut pair.

Proof.
Notice that since \( B \) is reducible \( U \) has no lower external vertex of attachment on \( R[s_Q; t_Q] \) and hence no \( Q \)- or \( ST \)-cross-cuts.

(1) \( U \) has a P.Q-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1 \) is to the left of \( y_1 \) on \( P \).

If both \( s_Q \) and \( t_Q \) are distinct from \( s \) and \( t \) then \( U \) has weak P.Q-cross-cuts.

Notice that it is not possible that \( s_Q = s \) and \( t_Q = t \), since, otherwise, \( N_1 \) and \( N_2 \) are two vertex-disjoint paths in \( B \) such that they meet \( P[s; t] \) only in \( x_1 \) and \( y_1 \), respectively, and meet \( R[s_Q; t_Q] \) only in \( x_2 \) and \( y_2 \) and \( B \) is not reducible with respect to \( R \).

Hence assume that \( s_Q = s \) and \( t_Q \) is distinct from \( t \). Hence \( U \) has a lower external vertex of attachment at \( t_Q \) and has no lower external vertex of attachment on \( R[s; t_Q] \). Since \( N_1 \) and \( N_2 \) form
Definition 5.7 Y-Q Pair.
Let $U$ be a U- or a U-Fragment with a cycle $J = \{P\} \cup \{Q\}$. Let $Y = \{Y_1[z_1; v]\} \cup \{Y_2[z_2; v]\} \cup \{Y_3[z_3; v]\}$ be a Y-Graph of $J$ and let $N[x_1; x_2]$ be a cross-cut of $J$, vertex-disjoint with $Y$. Such a pair of subgraphs is said to be a Y-Q pair of $U$, if it satisfies the following two conditions:

1. $z_1 \in P s; t[ and $z_2, z_3 \in Q s; t$.
2. $x_1 \in P s; t[, x_2 \in Q s; t$. [1]

Lemma 5.7 Let $U$ be a feasible augmented graph with a single $P$-feasible $B^{PQ}$-bridge, $B$, such that the pair of vertices associated with $B$ is $(s_Q; t_Q)$.

If $U$ has a weak $PQ$-cross-cut pair or a Y-Q pair, then $U$ is guaranteed to have a PQ- or a $P$-cross-cut pair.

![Diagram](image)

**Figure 16:** Modifying a Weak PQ-Cross-Cut Pair (Lemma 5.7).

**Proof.**
(1) $U$ has a weak PQ-cross-cut pair, $N_1[x_1, x_2]$ and $N_2[y_1, y_2]$, where $x_1$ is to the left of $y_1$ on $P$.
Assume that $N_1$ and $N_2$ do not form PQ-cross-cut, that is all the external vertices of attachment lie on $J[y_2; x_1]$ (the case when the external vertices of attachment lie on $J[y_1; x_2]$ is symmetric.) But since $B$ is $P$-feasible, not all the external vertices of attachment of $U$ lie on $J[s_Q; s_P]$. Hence $U$ has an external vertex of attachment on $P[s_P^{*}; x_1]$. Since the cross-cuts $N_1$ and $N_2$ belong to $B$, $B$ is a proper bridge and $N_2$ has an internal vertex, $z'$. Let $L$ be a path in $B$ that joins $s_P^{*}$ and $z'$ and avoids $J$. It is possible to find a vertex $z \in L[s_P^{*}; z']$ such that $L[s_P^{*}; z]$ meets $N_1$ or $N_2$ (but not both) only in $z$, an internal vertex of the appropriate cross-cut.

If $z$ is an internal vertex of $N_1$ then the vertex-disjoint cross-cuts $L[s_P^{*}; z] * N_1[z; x_2]$ and $N_2$ form PQ-cross-cuts of $U$. If $z$ is an internal vertex of $N_2$ then vertex-disjoint cross-cuts $L[s_P^{*}; z] * N_2[z; y_1]$ and $N_1$ form $P$-cross-cuts of $U$. [1]
of attachment on $R[s;x_2]$ and an upper external vertex of attachment on $P[x_1;t]$. If $t_Q$ is distinct from $t$ then $U$ has $PQ$-cross-cuts. On the other hand, if $t_Q = t$ then $U$ has $P$-cross-cuts.

![Diagram](image)

Figure 14: Case(2) of Lemma 5.6.

(2) $\overline{U}$ has a $P$-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where $x_1, x_2 \in P[s;t]$, $y_1 \in P[x_1;x_2]$ and $y_2 \in R[s_Q;t_Q]$. If $s_Q$ is distinct from $s$ (or, symmetrically, $t_Q$ distinct from $t$) then, since $N_1$ avoids $R[s_Q;t_Q]$, $U$ has $P$-cross-cuts. If $s_Q = s$ and $t_Q = t$ then $N_1$ avoids $R[s_Q;t_Q]$. Moreover, since $B$ is irreducible, $\overline{U}$ has a lower external vertex of attachment $b' \in R[s_Q;t_Q]$ (by definition) and there is a path $R_b[b';b]$ in $B_2$ such that $R_b$ meets $R[s_Q;t_Q]$ only in $b'$ and meets $Q[s_Q;t_Q]$ only in $b$. Clearly, the cross-cuts, $N_1$ and $N_2[y_1;y_2] \ast R[b';b] \ast R_b[b';b]$, form $P$-cross-cuts in $U$. □

§A.2 Case(2): $B$ is $P$-feasible.

Before presenting the proofs for this case, we present the following definitions and a technical Lemma.

![Diagram](image)

Figure 15: Weak $PQ$-Cross-Cut Pair and $Y$-$Q$ Pair.

**Definition 5.6 Weak $PQ$-Cross-Cut Pair.**

Let $U$ be a $U$- or a $\overline{U}$-Fragment with a cycle $J = \{P\} \cup \{Q\}$. A pair of interlacing vertex-disjoint cross-cuts $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$ is said to be a weak $PQ$-cross-cut pair of $U$, if $x_1, y_1 \in P[s;t]$ and $x_2, y_2 \in Q[s;t]$. □
(2) \( U' \) has an ST-cross-cut pair, \( N_1[s_Q; t_Q] \) and \( N_2[y_1; y_2] \). Let \( a' \) be an upper external vertex of attachment and \( \theta' \) be a lower external vertex of attachment of \( U' \); and let \( R_s \) be as above. Then the paths \( L_1 = N_1[s_Q; t_Q] \) and \( L_2 = R_s[a'; a'] \ast R[a'; y_1] \ast N_2[y_1; y_2] \) form either \( Q \)-cross-cuts or ST-cross-cuts of \( U \).

(3) \( U' \) has a \( P \)-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1, x_2 \in R[s_Q; t_Q] \). Let \( a' \in R[x_1; x_2] \) be an upper external vertex of attachment and \( \theta' \) be any lower external vertex of attachment and let \( R_s \) be as above. The paths \( L_1 = R[s_Q; x_1] \ast N_1[x_1; x_2] \ast R[x_2; t_Q] \) and \( L_2 = R_s[a'; a'] \ast R[a'; y_1] \ast N_2[y_1; y_2] \) form either \( Q \)-cross-cuts or ST-cross-cuts of \( U \).

(4) \( U' \) has a \( Q \)-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1, x_2 \in Q[s_Q; t_Q] \). Let \( a' \) be any upper external vertex of attachment and \( \theta' \in Q[x_1; x_2] \) be a lower external vertex of attachment and let \( R_s \) be as above. The paths \( L_1 = N_1[x_1; x_2] \) and \( L_2 = R_s[a'; a'] \ast R[a'; y_1] \ast N_2[y_1; y_2] \) form \( Q \)-cross-cuts of \( U \).  

**Lemma 5.6** Let \( U, B, R, B_1, B_2 \) and \( B_3 \) be as in the Theorem 5.3. If \( U' \) contains a \( P \)-, \( Q \)-, \( PQ \)- or an ST-cross-cut pair then \( U \) has a \( P \)-, \( Q \)-, \( PQ \)- or an ST-cross-cut pair.

**Proof.**
Since \( B \) is \( PQ \)-feasible, it is easy to see that if \( U' \) has \( Q \)- or \( ST \)-cross-cuts, then \( U \) has \( Q \)- or \( ST \)-cross-cuts. Hence, we need to consider the remaining two cases only.

![Figure 13: Case(1) of Lemma 5.6.](image)

(1) \( U' \) has a \( PQ \)-cross-cut pair, \( N_1[x_1; x_2] \) and \( N_2[y_1; y_2] \), where \( x_1 \) is to the left of \( y_1 \) on \( P \). Since \( B \) of \( U \) has two distinct vertices of attachment \( x_1 \) and \( y_1 \) on \( P \), \( s_P \) and \( t_P \) must be distinct and \( U \) is \( PQ \)-feasible. Hence, \( |U(A(U'))| > 0 \). Moreover, since \( B \) is irreducible with respect to \( R \), \( |LA(U')| > 0 \) (by definition). If both \( s_Q \) and \( t_Q \) are distinct from \( s \) and \( t \) then, clearly, \( U \) has \( PQ \)-cross-cuts.

Hence assume that \( s_Q = s \). If \( U' \) has a lower external vertex of attachment on \( R[y_2; t_Q] \) and an upper external vertex of attachment on \( P \) then \( U \) has \( P \)-cross-cuts. If not, \( U' \) has a lower external vertex...
Figure 11: Case: B is PQ-feasible and \(|B_3| \neq 0\).

**Lemma 5.5** Let \(U, B, R, B_1, B_2\) and \(B_3\) be as in the Theorem 5.3. If \(U'\) contains a \(P-, Q-, PQ-\) or \(ST\)-cross-cut pair then \(U\) has a \(Q\)- or an \(ST\)-cross-cut pair.

**Proof.**
Since \(B\) is irreducible with respect to \(R\), \(|U \setminus (U')| > 0\). Moreover, since \(B\) is \(PQ\)-feasible, and since \(U'\) is feasible, it must be a \(PQ\)-feasible \(U\)-Fragment of \(U\) on \(Q\). Note that if \(s_Q = s\) and \(t_Q = t\), then \(s_P = s_Q\) and \(t_P = t_Q\) are distinct and by definition, \(U\) is \(PQ\)-feasible and has an upper external vertex of attachment on \(P\).

Figure 12: Four Cases for Lemma 5.5.

1) \(U'\) has a \(PQ\)-cross-cut pair, \(N_1[x_1; x_2]\) and \(N_2[y_1; y_2]\).
Without loss of generality, assume that there exist external vertices of attachment \(a' \in R]\{s_Q; y_1]\) and \(b' \in Q]\{y_2; t_Q\}. Let \(R_a[a; a']\) be a path in \(B_1\) such that \(R_a\) meets \(P]\{s; t\} only in \(a\) and meets \(R]\{s_Q; t_Q]\) only in \(a'\). The cross-cuts \(L_1 = N_1[y_2; y_1] * R[y_1; t_Q]\) and \(L_2 = R_a[a; a'] * R[a'; x_1] * N_1[x_1; x_2]\) form \(Q\)-cross-cuts of \(U\).
Appendix: Interlacing Cross-Cuts and Bights

In this appendix we present a proof for Theorem 4.2. The proof is by a complete induction on the graphs, where the well-ordering is the one induced by the lexicographic ordering of the signature of the graphs. This appendix is organized in a fashion closely resembling the structure of the algorithms Analyze-Block and Analyze-U-\overline{U}.

§ A Interlacing Cross-Cuts: Single Bridge

Notice that if \( U \) is an infeasible augmented graph, then it does not have a pair of interlacing bights. Hence we may assume that \( U \) is feasible.

**Theorem 5.2** Let \( U \) be a feasible augmented graph with a single (feasible) \( B^{PQ} \)-bridge, \( B \), such that the pair of vertices associated with \( B \) is \( \langle s_Q; t_Q \rangle \). Let \( R[s_Q; t_Q] \) be a path in \( B \), connecting the vertices \( s_Q \) and \( t_Q \), such that \( B \) is reducible with respect to \( R \). Let \( U' \) be its \( \overline{U} \)-Fragment, as in the definition 4.5.

Then \( U \) has a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair, if \( \overline{U'} \) is a \( \overline{U} \)-Fragment containing a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair. \( \square \)

**Theorem 5.3** Let \( U \) be a feasible augmented graph with a single (feasible) \( B^{PQ} \)-bridge, \( B \), such that the pair of vertices associated with \( B \) is \( \langle s_Q; t_Q \rangle \). Let \( R[s_Q; t_Q] \) be a path in \( B \), connecting the vertices \( s_Q \) and \( t_Q \), such that \( B \) is irreducible with respect to \( R \). Let \( B_1, B_2, B_3 \) and \( B_4 \) be as in the definition 4.5. Notice that if \( |B_3| = 0 \) then the path \( R \) divides \( U \) into \( U' \), a \( U \)-Fragment on \( Q \), and \( \overline{U'} \) a \( \overline{U} \)-Fragment on \( Q \), as in the definition 4.5.

Then \( U \) has a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair, if one of the three following conditions is satisfied:

1. \( |B_3| \neq 0 \).
2. \( |B_3| = 0 \), and \( U' \) is a feasible \( U \)-Fragment containing a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair.
3. \( |B_3| = 0 \), and \( \overline{U'} \) is a \( \overline{U} \)-Fragment containing a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair. \( \square \)

We have three cases to consider: (1) \( B \) is \( PQ \)-feasible (if \( |LA(U)| \neq 0 \) and \( |UA(U)| \neq 0 \), (2) \( B \) is \( P \)-feasible (if \( |LA(U)| = 0 \), and (3) \( B \) is \( Q \)-feasible (if \( |UA(U)| = 0 \), the last two cases being symmetric.

§ A.1 Case(1): \( B \) is \( PQ \)-feasible.

First notice that if \( B \) is \( PQ \)-feasible, then \( B \) is irreducible with respect to path \( R \); and the Theorem 5.2 is vacuously true. Hence for this case we only need to prove the Theorem 5.3.

**Lemma 5.4** Let \( U, B, R, B_1, B_2 \) and \( B_3 \) be as in the Theorem 5.3. If \( |B_3| \neq 0 \) then \( U \) has a \( P-, Q-, PQ- \) or a \( ST \)-cross-cut pair.

**Proof.**

If \( |B_3| \neq 0 \) then there is a cross-cut \( N[y_1; y_2] \) of \( J \) between \( y_1 \in P[s; t] \) and \( y_2 \in Q[s_Q; t_Q] \) such that \( N \) avoids \( R \). Since \( B \) is a \( PQ \)-feasible \( B^{PQ} \)-bridge, the cross-cuts \( R \) and \( N \) form either \( Q \)-cross-cuts or \( ST \)-cross-cuts of \( U \). \( \square \)


References


In the first example, since \( B_3 = \emptyset \), we determine its \( U \)- and \( \overline{U} \)-fragments and analyze them further to notice that the \( B^{PQ} \)-bridge in the first example does not have \( P \)-, \( Q \)-, \( PQ \)- or \( ST \)- cross-cuts and the edges of \( P[s; t] \) (and similarly, edges of \( Q[s; t] \)) are all unidirectional. Only the edges of the \( B^{PQ} \)-bridge are bidirectional.

In the second example, since \( B_3 \neq \emptyset \), we see that all the edges of \( P][s; t] \) are admissible. By a similar analysis, we see that all the edges of \( Q][s; t] \) are admissible. As all the edges of the \( B^{PQ} \)-bridge are bidirectional, we see that only the edges directly incident on \( s \) and \( t \) are unidirectional.

Acknowledgement

I wish to acknowledge with gratitude the considerable help and advice I have received from Profs. E.M. Clarke, M. Foster, E. Frank, A. Frieze, M. Furst, R. Kannan, L. Rudolph, D. Sleator and R. Statman. Special thanks go to Prof. R.E. Tarjan without whose help this work would not have been possible. Thanks are also due to Prof. J.T. Schwartz for his encouragement to publish these results.
RECUR:  
if \(|\mathcal{R}_2| \neq 0\) then  
\[ \mathcal{A}_2 := E(P[s_P; t_P]) ; \]
else  
Let \( U' \) and \( \overline{U'} \) be its \( U- \) and \( U- \) Fragment, respectively, as in the Definition 4.5. Let \( \mathcal{A}(U' : P) \) be the set of admissible edges of \( U' \) obtained by calling FIND-AE-U-\( \overline{U}((U' : P)) \);  
\[ \mathcal{A}_2 := \mathcal{A}(U' : P) ; \]
if \( \text{ANALYZE-U-}((U')) \) returns ‘YES’ then  
\[ \mathcal{A}_2 := E(P[s_P; t_P]) ; \]
else  
\[ \mathcal{A}_2 := \emptyset ; \]
end {if} ;  
\[ \mathcal{A}_2 := \mathcal{A}_2 \cup \mathcal{A}_2 ; \]
end {if} ;  
\[ \mathcal{A}(U_B : P) := FE(U_B : P) \cap E(\mathcal{A}_2) ; \]
return \( \mathcal{A}(U_B : P) \);  
end {case} ;

end {FIND-AE-BLOCK.}  

\[ \square \]

Remark 5.4 As before, we see that the algorithm reduces the augmented graph \( U \) to \( U' \) and \( \overline{U'} \) in \( O(|E|) \) time; and \( U' \prec U \) and \( \overline{U'} \prec U \).  

Theorem 5.1 Let \( U \) be an augmented graph with the path \( P \) associated with it. Then the Algorithm FIND-AE-U-\( \overline{U} \) computes \( \mathcal{A}(U : P) \), the set of Admissible Edges of \( (U : P) \) in time \( O(|E| \cdot |V|) \).

Proof.  
The proof of the time-complexity of the algorithm is similar to that of \( \text{ANALYZE-U-} \overline{U} \).

The rest of the proof is by a simple case by case analysis, which shows that our algorithm returns exactly those edges \( e \in \mathcal{A}(U : P) \) for which \( \text{ANALYZE-U-} \overline{U}(u(U : e)) \) returns ‘YES’. We omit a detailed proof. (See [11].)  

Example 5.5 Consider the example graphs that were shown in the companion paper[12] (also, see figure 9.) Since the graphs posses mirror symmetric, we might simply consider only one of the paths, say \( P \). First note that both the graphs are Type IV.

Proceeding with the algorithms described in this section, we first find a path \( R[s_Q; t_Q] \) connecting the two extreme vertices of attachment of the bridge on \( Q \)s; \( t \). Note that both the graphs are irreducible with respect to \( R \).
Section 5  FINDING THE SET OF ADMISSIBLE EDGES  23

Algorithm FIND-AE-BLOCK((U_B : P)):

**step1. Feasibility-Test:** If FE(U_B : P) = ∅ then return AE(U_B : P) := ∅. Otherwise go to the next step.

**step2.**

case  
| B | 1:  
| --- | --- | 
| | AE(U_B : P) := FE(U_B : P);  
| | return AE(U_B : P);  
| |  
| B | 1:  
| --- | --- | 
| | Let B = {B};  
| | if the pair of vertices associated with B is (s_Q, t_Q) then  
| | Divide: Find a path R in B connecting the vertices s_Q and t_Q. Modify R such that the bridges with vertices of attachment solely on R avoid other bridges. Let B_1, B_2 and B_3 be as in the Definition 4.5.  
| | Recur:  
| | if B is reducible with respect to R then  
| | Let U be the U-Fragment as in the Definition 4.5. Let AE(U : P) be the set of admissible edges of U obtained by calling FIND-AE-U(U : P);  
| | AS_1 := AE(U : P);  
| | else (* B is irreducible with respect to R. *)  
| | if |B_1| ≠ 0 then  
| | AS_1 := E(P[s; t]);  
| | else  
| | Let U' and U be its U- and U-Fragment, respectively, as in the Definition 4.5.  
| | if ANALYZE-U(U') returns 'YES' then  
| | AS_1 := E(P[s; t]);  
| | else  
| | AS_1 := ∅;  
| | end {if};  
| | Let AE(U : P) be the set of admissible edges of U obtained by calling FIND-AE-U-U(U : P);  
| | AS_1 := AE(U : P);  
| | AS_1 := AS_1 \cup AS_1;  
| | end {if};  
| | end {if};  
| | AE(U_B : P) := FE(U_B : P) \cap AS_1;  
| | else if the pair of vertices associated with B is (s_P, t_P) then  
| | Divide: Find a path R in B connecting the vertices s_P and t_P. Modify R such that the bridges with vertices of attachment solely on R avoid other bridges. Let B_1, B_2 and B_3 be as in the Definition 4.5.
Now we are ready to describe the algorithm that computes all the admissible edges.

Algorithm \textsc{Find-AE-U-}\overline{\text{U}}((U : P)):

\textbf{step1.} If \text{FE}(U : P) = \emptyset then return \text{AE}(U : P) := \emptyset.

\textbf{step2.} Let $B_1, B_2, \ldots, B_l$ be the set of blocks of $B^{PQ}$-bridges of $U$. Let $B$ stand for an arbitrary block of $B^{PQ}$-bridges $B_i$ (for $i = 1, 2, \ldots, l$). Let $U_B = \{P\} \cup \{Q\} \cup \{B\}$ be the subgraph of $U$ whose external vertices of attachment are same as those of $U$ on $Q$; $t$. Let $J = \{P\} \cup \{Q\}$. The graph minor $U_B'$ of $U_B$ is obtained as follows:

Let $v_1, \ldots, v_{p-1}$ and $w_1, \ldots, w_{q-1}$ be the vertices of attachment of the set of $B^{PQ}$-bridges of $B$ on $P$; $t$; and $Q$; $s$; $t$. ordered in their left-to-right order, respectively. Let $L = \{P[v_0(= s); v_1], \ldots, P[v_{p-1}; v_{p}(= t)], Q[w_0(= s); w_1], \ldots, Q[w_{q-1}; w_q(= t)]\}$ be the set of residual paths of $J$.

Let $L[a;b] \in L$ be a residual path. If $|L[a;b]| \geq 3$ then contract the subpath $L[a;b] = L[x;y]$ to a single edge, $[x,y]$. Let $\text{EA}(L[a;b])$ be the set of external vertices of attachment of $U$ on the subpath $L[a;b]$.

1. If $|\text{EA}(L[a;b])| \geq 2$ then $U_B'$ has external vertices of attachment at $x$ and $y$.
2. If $|\text{EA}(L[a;b])| = 1$ then $U_B'$ has external vertices of attachment at $x$ or $y$, the choice being arbitrary.
3. If $|\text{EA}(L[a;b])| = 0$ then $U_B'$ has no external vertices of attachment at $x$ or $y$.

Each of the edges of the cycle, $J'$, of $U_B'$ is a pseudo-edge.

\textbf{step3.} For each block $B_i$, find \text{AE}(U_B' : P')$, the set of admissible edges of $U_B'$, by calling \textsc{Find-AE-Block}(U_B' : P'). Let

\[ \text{AE}_1 := \{e \in E(P) : \text{ for some } i, \exists e' \in \text{AE}(U_B' : P'_i) \text{ such that } e' \text{ is a contraction of a subpath of } P \text{ containing } e.$\]

\[ \text{AE}(U : P) := \text{FE}(U : P) \cap \text{AE}_1; \]

\text{return } \text{AE}(U : P).

end \{\textsc{Find-U-}\overline{\text{U}}.\} \hfill \Box

\textbf{Remark 5.3} Note that since all of the non-recursive steps can be done in time $O(|E| + |pE|)$ time, we see that the algorithm reduces the augmented graph $U$ to $U_{B_1}' \cup U_{B_2}' \cup \cdots \cup U_{B_l}'$ in $O(|E|)$ time; and $U_{B_i}' \prec U$, for $1 \leq i \leq l$. \hfill \Box
5 Finding the Set of Admissible Edges

However, it is easy to see that a straight-forward application of the algorithm Analyze-U-Û will result in an $O(|E| \cdot |V|^2)$ algorithm. Instead, if we simply compute the set of edges, $AE$, of $P[s; t]$ and $Q[s; t]$ in $G$ such that $e \in AE$ if and only if Analyze-U-Û($u(G : e)$) returns 'yes' then these are exactly the bidirectional edges of $P$ and $Q$ in $G$. The set $AE$ can be computed efficiently in time $O(|E| \cdot |V|)$. The algorithm is developed in the rest of this section.

Remark 5.1 It is easy to see that $FE(U : P)$ can be computed in time $O(|E|)$. The set $FE(U : P)$ can be completely described as follows:

If $U$ has less than four distinct vertices of attachment then $FE(U : P) = \emptyset$, otherwise $FE(U : P) = FE_1 \cap FE_2$, where $FE_1$ and $FE_2$ are as follows:

1. If $s_Q \neq t_Q$ and $U$ has no external vertex of attachment on $Q|s_Q; t_Q|$ then

   $$FE_1 = E\left(FS_1^1 \cup FS_1^2 \cup FS_1^3\right),$$

   otherwise $FE_1 = E(P[s; t])$. $FS_1^1$, $FS_1^2$ and $FS_1^3$ are as follows:

   (a) If $s_P^* \neq t_P^*$ and $t_P^*$ are distinct then $FS_1^1 = P[s_P^*; t_P^*]$ else $FS_1^1 = \emptyset$.

   (b) If $s_P^*$ and $t_P^*$ are distinct and $U$ has an external vertex of attachment on $Q[t_Q; t]$ then $FS_1^2 = P[s; t_P^*]$ else $FS_1^2 = \emptyset$.

   (c) If $s_P^*$ and $t_P^*$ are distinct and $U$ has an external vertex of attachment on $Q[s; s_Q]$ then $FS_1^3 = P[s_P^*; t]$ else $FS_1^3 = \emptyset$.

2. If $s_P \neq t_P$ then

   $$FE_2 = E\left(FS_2^1 \cup FS_2^2 \cup FS_2^3\right),$$

   otherwise $FE_2 = E(P[s; t])$. $FS_2^1$, $FS_2^2$ and $FS_2^3$ are as follows:

   (a) $FS_2^1 = P[s_P; t_P]$.

   (b) If $s_Q^* \neq t_Q^*$ and $t_Q^*$ are distinct and $U$ has an external vertex of attachment on $Q|s_Q^*; t|$ then $FS_2^2 = P[s; s_Q]$ else $FS_2^2 = \emptyset$.

   (c) If $s_Q^*$ and $t_Q^*$ are distinct and $U$ has an external vertex of attachment on $Q[s; t_Q]^*$ then $FS_2^3 = P[t_P; t]$ else $FS_2^3 = \emptyset$. □

Definition 5.2 The Vertices Associated with a $B^{PQ}$-Bridge of $(U : P)$.

Let $U$ be an augmented graph, with a path $P$ associated with it and with a single $B^{PQ}$-bridge, $B$. If $FE(U : P) \neq \emptyset$, we associate a pair of left- and right-most vertices (either $(s_P, t_P)$ or $(s_Q, t_Q)$) with the bridge $B$ as follows:

1. $s_P = t_P$ and $s_Q$ and $t_Q$ are distinct: The pair of vertices associated with $B$ is $(s_Q, t_Q)$.

2. $s_Q = t_Q$ and $s_P$ and $t_P$ are distinct: The pair of vertices associated with $B$ is $(s_P, t_P)$.

3. $s_P$ and $t_P$ as well as $s_Q$ and $t_Q$ are distinct: The pair of vertices associated with $B$ is $(s_Q, t_Q)$. □
Let $u, v \in V(J)$. Then the distance between $u$ and $v$ is defined to be

$$\text{distance}(u, v) = \min \left( |J[u; v]|, |J[v; u]| \right).$$

Let $e = [u, v] \in E(J)$ and $w \in V(J)$. Then, similarly, the distance of $e$ from $w$ is defined as

$$\text{distance}(e, w) = \min \left( \text{distance}(u, w), \text{distance}(v, w) \right).$$

We partition the pseudo-edges, $pE(H_j)$, of $H_j$ into the following two classes: $pE^{(1)}(H_j)$ and $pE^{(2)}(H_j) = pE(H_j) \setminus pE^{(1)}(H_j)$, where

$$pE^{(1)}(H_j) = \{ e \in pE(H_j) : e \in E(J) \text{ and there exists a vertex } w, \text{ a vertex of attachment of } \ B^{PQ} \text{-bridge or one of } s \text{ and } t, \text{ such that}$$

$$\text{distance}(e, w) \leq 2 \}.$$ 

(a) We define a function $g_1$ that maps a pseudo-edge, $e' = [u, v] \in pE^{(1)}(H_j)$ to a graph edge, $e \in gE(H_j)$. If $H_j$ has a $B^{PQ}$-bridge, $B$ with a vertex of attachment at $w'$ such that $\text{distance}(e', w') \leq 2$ then $g_1(e') = e$, where $e$ is a graph edge of $B$ incident at $w'$; otherwise, $g_1(e') = \text{undefined}$. Since $g_1$ maps at most eight distinct pseudo-edges of $pE^{(1)}(H_j)$ to one graph edge of $gE(H_j)$, and since $g_1$ is not defined for at most eight edges, we have $\frac{1}{8} \cdot |pE^{(1)}(H_j)| \leq 8 \cdot |gE(H_j)| + 8$.

(b) We define a function $g_2$ that maps a pseudo-edge, $e' = [u, v] \in pE^{(2)}(H_j)$ to a graph edge, $e \in gE(H)$. In this case, $H_j$ must be a $U$- or a $\overline{U}$-fragment of the augmented graph $H$, and there is a path $R$ in $H$ such that an edge $e \in E(R)$ corresponds to the edge $e'$. Moreover, every such edge is a graph-edge. Define $g_2(e') = e$. Since $g_2$ maps at most two distinct pseudo-edges of $\bigcup_{j=1}^{m} pE^{(2)}(H_j)$ to one graph edge of $gE(H)$, we have $\sum_{j=1}^{m} |pE^{(2)}(H_j)| \leq 2 \cdot |gE(H)|$.

Hence

$$\sum_{j=1}^{m} |pE(H_j)| \leq \sum_{j=1}^{m} \left[ 8 \cdot gE(H_j) + 8 \right] + 2 \cdot gE(H) \leq 26 \cdot gE(H),$$

since each graph edge of $H$ occurs in one graph $H_j$ and since,

$$m \leq \max(2, |gE(H)|) \leq 2 \cdot |gE(H)|.$$ 

Now, by our previous observations, we know that the algorithms spend the following amount of time in stage $i$:

$$O \left( \sum_{j=1}^{n} \left[ |gE(U_j)| + |pE(U_j)| \right] + \sum_{j=1}^{n} \sum_{k=1}^{m_j} \left[ |gE(U_{j,k})| + |pE(U_{j,k})| \right] \right) = O \left( |E| \right),$$

where the graphs of stage $i$ be $\{U_1, \ldots, U_n\}$ and the graphs of stage $i + 1$ be $\{U_{1,1}, \ldots, U_{1,m_1}, \ldots, U_{n,1}, \ldots, U_{n,m_n}\}$. As noted earlier, as there are at most $O(|V|)$ stages, the algorithm has the desired $O(|E| \cdot |V|)$ time complexity. $\square$
Section 4 Analyzing Augmented Graphs

The correctness of the algorithm will follow from the following theorem. The proof of the theorem is rather technical and is presented in several parts in the appendix. The proof is by a complete induction on the following well-ordering on augmented graphs.

**Definition 4.9 Signature of an Augmented Graph.**

Let $U$ be an augmented graph, consisting of the cycle $J = \{P\} \cup \{Q\}$ and a set of $B_{PQ}$-bridges, $B$. To $U$, we assign a pair of positive integers $(i_1, i_2)$, called its signature, where

$$i_1 = \sum_{B \in B} |V(N(B))| \leq |V(U)|,$$

and $i_2 = 1$ or $0$, depending, respectively, on whether $U$ is an arbitrary augmented graph, or an augmented graph with exactly one block of $B_{PQ}$-bridges.

We say augmented graphs $U_1 \preceq U_2$, if

$$\text{signature}(U_1) \leq \text{signature}(U_2).$$

This defines a well-ordering among the augmented graphs. Let $U_0 \succ U_1 \succ \cdots \succ U_n$ be a decreasing chain of graphs ordered by the above ordering. Hence $n \leq 2 \cdot |V(U_0)|$.

**Theorem 4.2** Let $U$ be an augmented graph.

1. If the algorithm $\text{ANALYZE-U-}\overline{U}(U)$ returns ‘YES’, then $U$ is feasible and has a $P$, $Q$, $PQ$- or an $ST$-cross-cut pair.

2. If the algorithm $\text{ANALYZE-U-}\overline{U}(U)$ returns ‘NO’ then $U$ does not have a pair of interlacing bights.

**Corollary 4.3** An edge $e$ on $P[s; t]$ is bidirectional if and only if $\text{ANALYZE-U-}\overline{U}(u(G : e))$ returns ‘YES’.

§4.3 Complexity Analysis.

**Theorem 4.4** Let $U$ be a $U$-Fragment or a $\overline{U}$-Fragment. The algorithm $\text{ANALYZE-U-}\overline{U}(U)$ terminates in $O(|E| \cdot |V|)$ time.

**Proof.**

Let the pseudo-edges of a graph $U$ be denoted by $pE(U)$ and the graph-edges of $U$, by $gE(U)$. We define a set of graphs of a stage of the algorithm as follows: graphs of stage $1 = \{U\}$. Let graphs of stage $i$ be $\{U_1, \ldots, U_n\}$. Suppose, we apply the appropriate algorithm (one of $\text{ANALYZE-U-}\overline{U}$ and $\text{ANALYZE-Block}$) to $U_j$ to reduce it to a set of graphs $\{U_{j,1}, \ldots, U_{j,m_j}\}$ such that $U_{j,k} \prec U_j$. Then, the graphs of stage $i + 1$ be $\{U_{i+1,1}, \ldots, U_{i+1,m_{i+1}}, \ldots, U_{n,1}, \ldots, U_{n,m_n}\}$.

It is easy to see that the total number of stages is $\leq 2 \cdot |V(U)|$. If we now show that the total amount of time spent on the graphs of stage $i$ is $O(|E(U)|)$ (for all $i$), then we have exhibited an $O(|E| \cdot |V|)$ time complexity for the complete algorithm.

We start with the following observation: Let $H$ be a graph in stage $i - 1$, and $\{H_1, \ldots, H_m\}$ be the graphs of stage $i$, obtained from $H$. It can be readily checked that all the pseudo-edges of $H_j$ lie on the cycle $J$. 


Algorithm \texttt{Analyze-Block}(U_B):

\textbf{step1. Feasibility-Test:} If \(U_B\) is infeasible, return \textquoteleft \textit{no}\textquoteright; otherwise go to the next step.

\textbf{step2.}

\texttt{case}

\[|B| > 1:\]
\texttt{return \textquoteleft \textit{yes}\textquoteright;}

\[|B| = 1:\]
Let \(B = \{B\}\);

\texttt{Divide:} Let \((s', t')\) be the pair of vertices associated with \(B\). Find a path \(R\) in \(B\) connecting the vertices \(s'\) and \(t'\) such that the bridges with vertices of attachment solely on \(R\) avoid other bridges. Let \(B_1\), \(B_2\) and \(B_3\) be as in the Definition 4.5.

\texttt{Recur:}
if \(B\) is reducible with respect to \(R\) then
Let \(U'\) be the \(U\)-Fragment as in Definition 4.5.
if \texttt{Analyze-U-\(U'(U)\)} returns \textquoteleft \textit{yes}\textquoteright; then
\texttt{return \textquoteleft \textit{yes}\textquoteright;}
\texttt{end \{}\texttt{if}\};
else (* \(B\) is irreducible with respect to \(R\), *)
if \(|B_3| \neq 0\) then
\texttt{return \textquoteleft \textit{yes}\textquoteright;}
else (* \(|B_3| = 0\)*)
Let \(U'\) and \(\overline{U'}\) be its \(U\) and \(\overline{U}\)-Fragment, respectively, as in Definition 4.5.
if \texttt{Analyze-U-\(U'(U)\)} returns \textquoteleft \textit{yes}\textquoteright; cor
\texttt{Analyze-U-\(\overline{U}'(U')\)} returns \textquoteleft \textit{yes}\textquoteright; then
\texttt{return \textquoteleft \textit{yes}\textquoteright;}
\texttt{end \{}\texttt{if}\};
\texttt{end \{}\texttt{if}\};
\texttt{return \textquoteleft \textit{no}\textquoteright;}
\texttt{end \{}\texttt{case}\)

\texttt{end}\{\texttt{Analyze-Block.}\} \hfill \Box

\textbf{Remark 4.8} It is easily seen that the algorithm reduces the graph \(U\) to graphs \(U'\) and \(\overline{U}'\) in \(O(|E|)\) time. Furthermore, since the \(B^PQ\)-bridge, \(B\), of \(U\) is proper, and since the path \(R\) must contain at least one vertex of \(N(B)\),

\[\sum_{B \in B_i} |V(N(B'))| < |V(N(B))|, \quad i = 1, 2. \hfill \Box\]
Algorithm Analyze-U-U(U):

step1. If $U$ is infeasible; return ‘NO’.

step2. Let $B_1, B_2, \ldots, B_l$ be the set of blocks of $B^PQ$-bridges of $U$. Let $B$ stand for an arbitrary block of $B^PQ$-bridges $B_i$ (for $i = 1, 2, \ldots, l$.) Let $U_B = \{P\} \cup \{Q\} \cup \{B\}$ be the subgraph of $U$ whose external vertices of attachment are same as those of $U$ on $P$; $t$ and on $Q$; $t$. Let $J = \{P\} \cup \{Q\}$. The graph minor $U_B'$ of $U_B$ is obtained as follows:

Let $v_1, \ldots, v_{p-1}$ and $w_1, \ldots, w_{q-1}$ be the vertices of attachment of the set of $B^PQ$-bridges of $B$ on $P$; $t$ and $Q$; $t$, ordered in their left-to-right order, respectively. Let $L = \{P[v_0(= s); v_1], \ldots, P[v_{p-1}; v_p(= t)]; Q[w_0(= s); w_1], \ldots, Q[w_{q-1}; w_q(= t)]\}$ be the set of residual paths of $J$.

Let $L[a;b] \in L$ be a residual path. If $|L[a;b]| \geq 3$ then contract the subpath $L[a;b][= L[x;y]$ to a single edge, $[x, y]$. Let $EA(L)a;b]$ be the set of external vertices of attachment of $U$ on the subpath $L[a;b]$.

1. If $|EA(L)a;b]| \geq 2$ then $U_B'$ has external vertices of attachment at $x$ and $y$.
2. If $|EA(L)a;b]| = 1$ then $U_B'$ has external vertices of attachment at $x$ or $y$, the choice being arbitrary.
3. If $|EA(L)a;b]| = 0$ then $U_B'$ has no external vertices of attachment at $x$ or $y$.

Each of the edges of the cycle, $J'$, of $U_B'$ is a pseudo-edge.

step3. For each block $B$, analyze $U_B'$ by calling Analyze-Block($U_B'$). If the answer is ‘YES’ for any of the blocks of $B$, return ‘YES’; otherwise, return ‘NO’.

end{Analyze-U-U.}  

Remark 4.7 Since the step1 and step2 take $O(|E|)$ time, the algorithm reduces the augmented graph $U$ to $U_{B_1}', U_{B_2}', \ldots, U_{B_l}'$, in $O(|E|)$ time. Also, note that, for each $1 \leq i \leq l$, since

$$\sum_{B \in B_i} |V(N(B))| \leq \sum_{B \in B} |V(N(B))|,$$

and in this sense, the non-recursive part of the algorithm does not increase the “complexity” of the original problem. This simplification is formalized through the notion of the “signature” of a graph subsequently.  


of $U$ and $\overline{U}'$, corresponding to the edges of $R[s_Q; t_Q]$ of $U$ are considered to be pseudo-edges.

The following proposition can be easily verified:

**Proposition 4.1** Let $U$ and $\overline{U}$ be, respectively, the $U$- and $\overline{U}$-Fragments of an augmented graph $U'$. Then both $U$ and $\overline{U}$ are themselves augmented graphs.

§4.2 **Analyzing a U-Fragment** Let $U$ be an augmented graph. In this section, we present an algorithm to analyze an augmented graph and determine if it has a $P$-, $Q$-, $PQ$- or an $ST$-crosscut pair. It consists of two mutually recursive algorithms `ANALYZE-U-\overline{U}`, and `ANALYZE-BLOCK`.
attachment of $B_2$ on $R[s_Q; t_Q]$ and (iii) the vertex $s_Q$ (if distinct from $s$) and the vertex $t_Q$ (if distinct from $t$). The vertices $s_Q$ and $t_Q$ of $U'$ are marked, $l'_Q$ and $r'_Q$, respectively. $l'_P = l_P$ and $r'_P = r_P$.

- $B$ is reducible with respect to $R$. (See Figure 8.)

Let $U'$ be the maximal nonseparable subgraph of $U$ containing the vertices $s$ and $t$, but without the path $Q[s_Q; t_Q]$ and any of the $B_2$ or $B_4$ bridges. Notice that $U'$ contains the cycle $J = \{P[s; l]\} \cup \{Q[s; s_Q] \ast R[s_Q; t_Q] \ast Q[t_Q; l]\}$ and all its bridges have attachment on both $P$ and $R$. The subgraph $U'$ is called a $U$-Fragment of $U$ on $Q$. The set of external vertices of attachment, $EA$ consist of (i) the external vertices of attachment of $U$ on $J[s_Q; t_Q]$, and (ii) the vertex $s_Q$ (if distinct from $s$) and the vertex $t_Q$ (if distinct from $t$). The vertices $s_Q$ and $t_Q$ of $U'$ are marked, $l'_Q$ and $r'_Q$, respectively. $l'_P = l_P$ and $r'_P = r_P$. \hfill \Box

**Remark 4.6**

1. Given a $U$- or a $\overline{U}$-Fragment, $U$, with a single $B^{PQ}$-bridge, $B$, in linear time, we can either divide $U$ into a $U$-Fragment, $U'$ and a $\overline{U}$-Fragment, $\overline{U}'$ (if $B$ is irreducible with respect to $R$) or reduce $U$ to a $U$-Fragment, $\overline{U}'$ (if $B$ is reducible with respect to $R$).

2. Let $U$ and $U'$ be two distinct $U$- or $\overline{U}$-Fragments. In order to distinguish between the paths and vertices of $U'$ from those of $U$, we use the primed versions for $U'$: for instance, $P', Q', s'_P, t'_P, s'_U, t'_U$, etc. refer to those of $U'$ where as $P, Q, s_P, t_P, s_U, t_U$, etc. refer to those of $U$. In what follows, either $U$ is a $U$- or a $\overline{U}$-Fragment of $U'$, or vice versa.

3. If $U$ and $\overline{U}$ are the $U$- and $\overline{U}$-Fragments of an augmented graph, $U$, then the edges
second kind. By our assumption the total number of such disjoint ties and hence the separation number \( \lambda \) must not exceed one. But, since \( B \) is a bridge, \( \lambda(U^-; E_1, E_2) \) must be at least one; let \( \nu \in V(F_1 \cap F_2) \), where \( \langle F_1, F_2 \rangle \) is the cutting pair corresponding to this \( \lambda \).

Furthermore, \( \nu \in N(B) \). This is a consequence of the following reasoning: Since \( U \) is feasible, and has no external vertex of attachment on \( Q \), \( s_P \) and \( t_P \) must be distinct, i.e., \( B \) has at least two vertices of attachment on \( P[s_1]t_1 \). If \( \nu \in V(P[s_1]t_1) \) or \( V(Q)s_Qt_Q \) then there is a vertex of attachment \( x \in V(P[s_1]t_1) \), where \( x \) is distinct from \( \nu \). Hence there is a path from \( x \) to a vertex of \( R[s_Q]t_Q \) in \( B \); but such a path avoids \( \nu \), resulting in a contradiction.

As a result, it is easy to see that in order to determine if a bridge \( B \) is irreducible with respect to a path \( R \), we only have to find if the separation number \( \lambda(U^-; E_1, E_2) > 1 \). This can be done in \( O(|E|) \) time. \( \square \)

**Notation 4.4** It will be customary to associate following named vertices with an augmented graph. An augmented graph \( U \) has two distinguished vertices \( l_P \) and \( r_P \) on \( P[s_1]t_1 \), and two distinguished vertices \( l_Q \) and \( r_Q \) on \( Q[s_1]t_1 \), such that all the vertices of attachment of all of its \( P_Q \)-bridge lie on \( P[l_P]r_P \) and \( Q[l_Q]r_Q \). Also, the left-most and right-most upper external vertices of attachment on \( P \) are denoted by \( s_U \) and \( t_U \) and the left-most and right-most lower external vertices of attachment on \( Q \) by \( s_L \) and \( t_L \), respectively. \( \square \)

**Definition 4.5** U-Fragment and Complementary U-Fragment.

Let \( U \) be a feasible augmented graph with a single (feasible) \( P_Q \)-bridge, \( B \), such that the pair of vertices associated with \( B \) is \( \langle s_Q, t_Q \rangle \). Let \( R[s_Q]t_Q \) be a path in \( B \), connecting \( s_Q \) and \( t_Q \), and let \( J' = \{ P \} \cup \{ Q \} \cup \{ R \} \) be a subgraph of \( U \). As a result \( B \) will be decomposed into the following sets of bridges of \( J' \) in \( U \):

1. The set of bridges, \( B_1 \), with vertices of attachment on \( P \) and \( R \).
2. The set of bridges, \( B_2 \), with vertices of attachment on \( Q \) and \( R \).
3. The set of bridges, \( B_3 \), with vertices of attachment on \( P \), \( Q \) and \( R \).
4. The set of bridges, \( B_4 \), with vertices of attachment solely on \( R \).

Assume that the path \( R \) is such that the bridges of \( B_4 \) avoid those of \( B_1, B_2 \) and \( B_3 \).

Note that a path \( R' \) in \( B \) connecting \( s_Q \) and \( t_Q \) can be found in \( O(|E|) \) time and then, using an algorithm similar to the ambitus-finding algorithm (see Tarjan and Misra[13]), \( R' \) can be modified to \( R \) in \( O(|E|) \) time such that the bridges of \( B_4 \) avoid those of \( B_1, B_2 \) and \( B_3 \).

- \( B \) is irreducible with respect to \( R \). (See Figure 7.)

If \( |B_3| \neq 0 \) then the U- and \( \overline{U} \)-Fragment of \( B \) are undefined; otherwise, they are as follows:

The subgraph \( U' = \{ R[s_Q]t_Q \} \cup \{ Q[s_Q]t_Q \} \cup B_2 \) of \( U \), is called a U-Fragment of \( U \) on \( Q \). The set of external vertices of attachment, \( EA \), of \( U' \) consists of (i) the external vertices of attachment of \( U \) on \( Q \) \( s_Q \) \( t_Q \), and (ii) the vertices of attachment of \( B_1 \) on \( R[s_Q]t_Q \). \( l'_P = l'_Q = s \) and \( r'_P = r'_Q = t \).

The subgraph \( \overline{U} = Q[s_Q]R[s_Q]t_Q \) \cup \{ P \} \cup B_1 \) of \( U \), is called a complementary U-Fragment (or simply \( \overline{U} \)-Fragment) of \( U \) on \( Q \). The set of external vertices of attachment, \( EA \), of \( \overline{U} \) consists of (i) the external vertices of attachment of \( U \) on \( J[s_Q]t_Q \), (ii) the vertices of
vertices \( s_Q \) and \( t_Q \). \( B \) is said to be reducible with respect to \( R \), if (i) \( U \) has no external vertex of attachment on \( Q \)\( s_Q ; t_Q \) and (ii) has a vertex, \( v_{\text{cut}} \in N(B) \), (called the cut vertex) in the nucleus of \( B \), \( N(B) \), satisfying the following condition:

If \( N \) is an arbitrary path in \( B \) that connects a vertex of \( P \)\( s \); \( t \) with a vertex of \( Q \)\( s_Q ; t_Q \) or \( R \)\( s_Q ; t_Q \) and meets \( P \), \( Q \) and \( R \) only in its end vertices then \( N \) contains \( v_{\text{cut}} \).

Otherwise, \( B \) is said to be irreducible with respect to \( R \). \( \square \)

When \( U \) has a single reducible \( B^{PQ} \)-bridge, the analysis of \( U \) is carried on by analyzing only a subgraph \( \overline{B} \) of \( U \) (\( \overline{U} \)-fragment). Otherwise, the divide-and-conquer algorithm may need to explore at most two subgraphs to determine if it has the required structure. In the former case, we say the graph \( U \) reduces to the subgraph \( \overline{U} \).

**Remark 4.3** Let \( U \) be a feasible augmented graph with a single (feasible) \( B^{PQ} \)-bridge, \( B \), as in the preceding definition. Let \( R[s_Q ; t_Q] \) be a path in \( B \) connecting \( s_Q \) and \( t_Q \) the vertices associated with \( B \). Whether \( B \) is reducible with respect to \( R \) can easily be determined in \( O(|E|) \) time as a consequence of the following observation.

It can be seen that \( B \) is irreducible with respect to \( R \) iff the following holds:

There are at least two vertex-disjoint paths \( R_a[a; a'] \) and \( R_b[b; b'] \) in \( B \) such that (i) \( R_a \) and \( R_b \) meet \( P[s]; t[t] \) only in \( a \) and \( b \), respectively; (ii) \( R_a \) meets only one of \( Q \)\( s_Q ; t_Q \) and \( R \)\( s_Q ; t_Q \) only in \( a' \); and (iii) \( R_b \) meets only one of \( Q \)\( s_Q ; t_Q \) and \( R \)\( s_Q ; t_Q \) only in \( b' \).

Suppose that there are no two vertex-disjoint paths \( R_a \) and \( R_b \) as in the above statement. We shall show that \( B \) has a cut vertex \( v_{\text{cut}} \in N(B) \), and \( B \) is reducible with respect to \( R \).

Let \( U^- \) be the subgraph of \( U \) obtained by deleting the residual subpaths \( J[s_Q ; a] \) and \( J[t; t_Q] \). Let \( E_1 = E(P); t[t] \) and \( E_2 = E(R)\)\( s_Q ; t_Q \)\( \cup \cup E(Q)\)\( s_Q ; t_Q \) be disjoint subsets of \( E(U^-) \). Let \( \lambda = \lambda(U^-; E_1, E_2) \) be the separation number of \( E_1 \) and \( E_2 \) in \( U^- \) (cf. pp. 43, Tutte(1984)[20]). By a version of Menger’s Theorem, due to Nash-William and Tutte (Theorem II.36[20], also see [9] and [14]), there exists a set of \( \lambda \) disjoint ties between between \( E_1 \) and \( E_2 \). Since the vertices of \( P[s_P; t_P] \) are disjoint with those of \( R \)\( s_Q ; t_Q \) and \( Q \)\( s_Q ; t_Q \), these must be ties of the

---

\( ^2 \)We consider a graph \( G \) in which two disjoint subsets \( E_1 \) and \( E_2 \) of \( E(G) \) are specified. Following Tutte, we define a cutting pair of \( E_1 \) and \( E_2 \) in \( G \) as an ordered pair \( \langle F_1, F_2 \rangle \) of edge-disjoint subgraphs of \( G \) such that

\[
E_1 \subseteq E(F_1), \quad E_2 \subseteq E(F_2), \quad \text{and} \quad E_1 \cup E_2 = G.
\]

We define the order of a cutting pair \( \langle F_1, F_2 \rangle \) of \( E_1 \) and \( E_2 \) as \( |V(F_1 \cap F_2)| \); the number of common vertices of \( F_1 \) and \( F_2 \). We now define \( \lambda(G; E_1, E_2) \) as the least number of vertices required to separate \( E_1 \) and \( E_2 \). Thus

\[
\lambda(G; E_1, E_2) = \min\{\ell_1, \ell_2\} |V(F_1 \cap F_2)|,
\]

where the minimum is taken over all cutting pairs \( \langle F_1, F_2 \rangle \) of \( E_1 \) and \( E_2 \) in \( G \).

There are two kinds of tie between \( E_1 \) and \( E_2 \). A tie of the first kind is a vertex graph contained in both \( \langle E_1 \rangle \) and \( \langle E_2 \rangle \). A tie of the second kind is a path in \( G \) with one end in \( \langle E_1 \rangle \) but not \( \langle E_2 \rangle \), with its other end in \( \langle E_2 \rangle \) but not \( \langle E_1 \rangle \), and with no edge or internal vertex in either \( \langle E_1 \rangle \) or \( \langle E_2 \rangle \).
end{Label-Type-IV.} \hfill \square

In order to prove an $O(|E| \cdot |V|)$ time-complexity of the algorithm, we need to show that each substep of \textbf{step3} takes $O(|E| \cdot |V|)$ time. In the next two sections, we devise a divide-and-conquer algorithm for this purpose.

4 Analyzing Augmented Graphs

Now we are ready to study an algorithm to determine if an augmented graph is feasible and if so, if it has one of the requisite disjoint cross-cut pairs. The techniques developed to analyze an augmented graph can then be easily modified to devise an algorithm to determine the set of admissible edges.

We start by introducing some further new concepts: Of particular interest to us, will be the following special kinds of augmented subgraphs: (i) \textit{U-Fragment} and (ii) its accompanying \textit{T-Fragment}. These augmented subgraphs allow us to develop an $O(|E| \cdot |V|)$ time algorithm to detect if a feasible augmented graph has a $P$-cross-cut pair, $Q$-cross-cut pair, $PQ$-cross-cut pair or $ST$-cross-cut pair.

§4.1 U-Fragment and Complementary U-Fragment. Consider an augmented graph $U$ with a set of $B^PQ$-bridges. If $U$ is feasible and contains a single block of $B^PQ$-bridges, $B$, then the block $B$ is said to be a \textit{feasible block of $B^PQ$-bridges}. Similarly, if $U$ is feasible and contains a single $B^PQ$-bridge, $B$. Then the bridge $B$ is said to be a \textit{feasible $B^PQ$-bridge}.

Also, there will be occasions where we distinguish among the feasible augmented graphs by referring to them as: $PQ$-feasible (if $|LA(U)| \neq 0$ and $|UA(U)| \neq 0$), $P$-feasible (if $|LA(U)| = 0$), and $Q$-feasible (if $|UA(U)| = 0$).

\textbf{Definition 4.1 The Pair of Vertices Associated with a $B^PQ$-Bridge.}

Let $U$ be a feasible augmented graph, with a single (feasible) $B^PQ$-bridges $B$. We associate a pair of left- and right-most vertices (either $<s_p,t_p>$ or $<s_Q,t_Q>$) with the bridge $B$ as follows:

1. $s_p = t_p$ and $s_Q$ and $t_Q$ are distinct: The pair of vertices associated with $B$ is $<s_Q,t_Q>$.
2. $s_Q = t_Q$ and $s_p$ and $t_p$ are distinct: The pair of vertices associated with $B$ is $<s_p,t_p>$.
3. $s_p$ and $t_p$ as well as $s_Q$ and $t_Q$ are distinct:
   (a) $U$ is $PQ$-feasible: The pair of vertices associated with $B$ is $<s_p,t_p>$ or $<s_Q,t_Q>$, the choice being arbitrary.
   (b) $U$ is $P$-feasible: the pair of vertices associated with $B$ is $<s_Q,t_Q>$.
   (c) $U$ is $Q$-feasible: the pair of vertices associated with $B$ is $<s_p,t_p>$. \hfill \square

\textbf{Definition 4.2 Reducible and Irreducible $B^PQ$-Bridge (with respect to $R$).}

Let $U$ be a feasible augmented graph with a single (feasible) $B^PQ$-bridge, $B$, such that the pair of vertices associated with $B$ is $<s_Q,t_Q>$. Let $R[s_Q ; t_Q]$ be a path in $B$ connecting the
Let $U$ be an augmented graph. We associate one of the paths $P$ and $Q$ with $U$ such that if $P$ (respectively $Q$) is associated with $U$ then $U$ has no external vertex of attachment on $P$ (respectively, $Q$). We represent an augmented graph with the associated path, $P$ or $Q$, by $(U : P)$ and $(U : Q)$, respectively.

**Definition 3.5 Set of Feasible Edges.**

Let $U$ be an augmented graph with the path $P$ associated with it. A set of feasible edges of $(U : P)$, $FE(U : P)$, is a set of distinct edges of $P[s; t]$, $\{e_1, e_2, \ldots, e_k\}$, such that activation of an edge $e = [u, v] \in P[s; t]$ results in a feasible augmented graph $u(U : e)$ if and only if

$$e \in FE(U : P).$$

The set of feasible edges of $(U : Q)$ is defined in an identical manner.

Clearly, if $e \in P[s; t] \setminus FE(U : P)$, it is unidirectional.

**Definition 3.6 Set of Admissible Edges.**

Let $U$ be an augmented graph with the path $P$ associated with it. A set of admissible edges of $(U : P)$, $AE(U : P)$, is a set of distinct edges of $P[s; t]$, $\{e_1, e_2, \ldots, e_k\}$, such that

1. Activation of an edge $e = [u, v] \in P[s; t]$ results in a (feasible) augmented graph $u(U : e)$ with a $P$, $Q$, $PQ$- or an ST-cross-cut pair, if $e \in AE(U : P)$.
2. Activation of an edge $e = [u, v] \in P[s; t]$ results in an augmented graph $u(U : e)$ with no pair interlacing bights, if $e \notin AE(U : P)$.

The set of admissible edges of $(U : Q)$ is defined in an identical manner.

Finally, using the notion of admissible edges, the Label-Type-IV algorithm may be expressed as follows:

§3.4 Labeling the edges of a Type.IV Graph  Let $G$ be Type.IV graph with the paths $P$ and $Q$.

---

**Algorithm Label-Type-IV(G):**

**step1.** Label every edge $[s, u]$ incident on $s$, $(s, u)$ and every edge $[u, t]$ incident on $t$, $(u, t)$.

**step2.** Label every edge $e$ of the $B^{PQ}$-bridge not incident on $s$ or $t$, bidirectional.

**step3.** Let $AE(G : P) \subseteq E(P[s; t])$ and $AE(G : Q) \subseteq E(Q[s; t])$ be the sets of admissible edges of the augmented graph $G$, when the paths associated with $G$ are $P$ and $Q$, respectively.

1. Label every edge $e = [u, v] \in AE(G : P)$ bidirectional and every edge $e = [u, v] \in E(P[s; t]) \setminus AE(G : P)$, $(u, v)$, if $u$ is to the left of $v$ on $P$.
2. Label every edge $e = [u, v] \in AE(G : Q)$ bidirectional and every edge $e = [u, v] \in E(Q[s; t]) \setminus AE(G : Q)$, $(u, v)$, if $u$ is to the left of $v$ on $Q$. 
t, without loss of generality, assume that \( x_2 \) is distinct from \( t \). Since \( U \) has an external vertex of attachment \( e \in P[x_1; x_2] \), \( e \) must be one of \( u \) and \( v \); and hence, \( e \in P[x_1; x_2] \).

If \( e \in P[y_1; x_2] \), the the simple paths \( P[s; t] \) and \( P[s; x_2] \) traverse \( e \) in either direction, and \( e \) is bidirectional. (Figure 6(a).)

Next, if \( e \in P[x_1; y_1] \), then we treat this case differently, depending on whether \( x_1 \) is distinct from \( s \) or not. If \( x_1 \neq s \), then the simple paths \( P[s; y_2] \) and \( Q[s; y_2] \) traverse \( e \) in either direction, and \( e \) is bidirectional.(Figure 6(b).)

If, on the other hand, \( x_1 = s \), we shall see that the cross-cuts can be so modified that this case reduces to one of the earlier cases. Clearly, \( e \in P[s_p; y_1] \) and \( B \) has a vertex of attachment at \( s_p \) and a vertex of attachment at \( y \in Q \). Let \( L[s_p; y] \) be a cross-cut of \( J \) between \( s_p \) and \( y \). If \( L \) avoids both \( N_1 \) and \( N_2 \) then the \( P \)-cross-cut pair defined by \( N_1 \) and \( L \) satisfies the condition shown in Figure 6(a).

Hence assume that \( L \) does not avoid both \( N_1 \) and \( N_2 \). Then there is a vertex \( z \in L[s_p; y] \) such that \( L[s_p; z] \) meets \( N_1 \) or \( N_2 \) (but not both) in \( z \). If \( L \) meets \( N_2 \) in \( z \), then the \( P \)-cross-cut pair \( N_1 \) and \( L[s_p; z] \) satisfy the first condition. On the other hand, if \( L \) meets \( N_1 \) in \( z \) then the \( P \)-cross-cut pair \( L[s_p; z] \) satisfies the condition shown in Figure 6(b).

(2) We show that there is no simple path \( A'[s; t] \) in \( G \) such that \( A' \) traverses \( e \) in the order \( v \) and \( u \). Assume to the contrary, then the subpaths \( A'[s; v] \) and \( A'[u; t] \) are vertex disjoint paths in \( U \). Notice that \( u \) and \( v \) are distinct from \( s \) and \( t \), and hence, are external vertices of attachment of \( U \) introduced as a result of the activation of the edge \( e \). Moreover, since \( A'[s; v] \) does not contain \( u \) or \( t \), it is a not a subpath of \( J \) and hence a path associated with a bight, \( A_1 \) of \( U \) between \( s \) and \( v \). By a similar argument, \( A'[u; t] \) is a path associated with a bight, \( A_2 \) of \( U \) between \( u \) and \( t \). But since \( u \) is to the left of \( v \) on \( P \), \( s \) and \( v \) separate \( u \) and \( t \) on \( J \); and \( A_1 \) and \( A_2 \) form a pair of interlacing bights. This however is a contradiction. \( \square \)

§3.3 Feasible and Admissible Edges. Now, based on the characterization of "bidirectionality" developed in the preceding section, we are ready to introduce the concept of feasible and admissible edges.
Section 3  Characterization of Bidirectionality

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure.png}
\caption{Case a: PQ-Cross-Cut Pair.}
\end{figure}

is, \(a_1, a_2, b_1 \) and \(b_2 \) are four distinct vertices on \(J \) such that there is a pair of interlacing bights \(A_1 \) between \(a_1 \) and \(a_2 \), and \(A_2 \) between \(b_1 \) and \(b_2 \), where \(a_1, a_2, b_1 \) and \(b_2 \) are either external vertices or one of \(s \) and \(t \). Hence \(U \) has at least two distinct external vertices of attachment and the bridges of of \(B \) have at least four distinct vertices of attachment.

Since \(U \) is infeasible, we may assume that \(U \) has no external vertex of attachment on \(Q(s; t) \), say. First, if \(s_P^* = t_P^* \), then each of the paths associated with \(A_1 \) and \(A_2 \) must contain at least two of the three vertices \(s_P^* (= t_P^*) \), \(s_Q \) and \(t_Q \), contradicting the vertex-disjointness of \(A_1 \) and \(A_2 \). Hence, we assume that \(s_P^* \neq t_P^* \) and that all the external vertices of attachment lie only on \(J[s_Q; s_P^*] \). Then \(a_1 \) and \(a_2 \) lie on \(J[s_Q; s_P^*] \), and the path associated with \(A_1 \) must contain the vertices \(s_Q \) and \(s_P^* \), and every path associated with any other bight \(A_2 \) must meet \(A_1 \) — again a contradiction. \(\square\)

\section*{3.2 Bidirectionality of an Edge}

\textbf{Theorem 3.2} Let \(e = [u, v] \) be an arbitrary edge on the path \(P[s; t] \) or \(Q[s; t] \) of a Type IV graph, \(G \); and let \(U = u(G : e) \) be the augmented graph obtained from \(G \) by the activation of the edge \(e \).

1. If \(U \) is feasible and has a \(P-, Q-, PQ-\) or \(ST\)-cross-cut pair then \(e \) is a bidirectional edge.

2. If \(U \) has no pair of interlacing bights then \(e \) is a unidirectional edge.

\textbf{Proof.}

Without loss of generality assume that \(e \in P[s; t] \) and \(u \) is to the left of \(v \) on \(P \).

(1) Since \(U \) is feasible, and since \(U \) has no external vertex of attachment on \(Q \), \(e \in P[s_P^*; t_P^*] \). Clearly \(U \) can have only \(P\)- or \(PQ\)-cross-cut pairs.

\textbf{(Case A)} \(U \) has \(PQ\)-cross-cut pair, \(N_1[x_1; x_2] \) and \(N_2[y_1; y_2] \), where \(x_1 \) is to the left of \(y_1 \) on \(P[s; t] \). Since not all the external vertices of attachment lie only on the subpath \(J[y_1; x_1] \) or only on the subpath \(J[y_1; x_2] \), we have \(e \in P[x_1; y_1] \). But since the simple paths \(P[s; t] \) and \(Q[s; y_2] \) traverse \(e \) in either direction, \(e \) is bidirectional. (Figure 5.)

\textbf{(Case B)} \(U \) has a \(P\)-cross-cut pair, \(N_1[x_1; x_2] \) and \(N_2[y_1; y_2] \) where \(x_1 \) and \(x_2 \in P[s; t] \), \(y_1 \in P[x_1; x_2] \) and \(y_2 \in Q[s; t] \). Since at least one of the vertices \(x_1 \) and \(x_2 \) is distinct from \(s \) and
3. There is a lower external vertex of attachment \( c \in Q \cup \{x_1, x_2\} \).

**Definition 3.3 Bights and Interlacing Bights.**

Let the augmented graph \( U \) and the cycle \( J = \{P\} \cup \{Q\} \) be as defined earlier. Let \( a \) and \( b \) be two distinct vertices on \( J \) such that they are either external vertices of attachment or one of \( a \) and \( b \). Let \( A \) be a path in \( G \) connecting the vertices \( a \) and \( b \), where \( A \) is not a subpath of \( J \).

This path can be dissected uniquely into an alternating sequence of (possibly empty) common sections (subpaths of \( J \)) and cross-cuts of \( J \). A suitable subpath of \( A \), \( A[a'; b'] \), meets \( J \) in its end vertices \( a' \) and \( b' \) such that the common end sections \( A[a; a'] \) and \( A[b'; b] \) are subpaths of \( J \). We say \( A[a'; b'] \) is a bight of \( J \) between the vertices \( a \) and \( b \); and \( A[a; b] = J[a; a'] * A[a'; b'] * J[b'; b] \) the path associated with the bight.

Let \( a_1, a_2, b_1 \) and \( b_2 \) be four distinct vertices on \( J \) such that they are external vertices of attachment or one of \( a \) and \( b \). Let \( A_1[a_1'; a_2'] \) and \( A_2[b_1'; b_2'] \) be the bights of \( J \) between \( a_1 \) and \( a_2 \), and \( b_1 \) and \( b_2 \), respectively. We say the bights \( A_1 \) and \( A_2 \) interlace, if the associated paths are vertex-disjoint and \( a_1 \) and \( a_2 \) separate \( b_1 \) and \( b_2 \) in the cycle \( J \). (Figure 4.)

**Definition 3.4 Feasible Augmented Graph.**

Let \( U \) be an augmented graph with the set of \( B^{PQ} \)-bridges, \( B \), such that \( U \) has at least two distinct external vertices of attachment and the bridges of \( B \) have at least four distinct vertices of attachment. \( U \) is said to be feasible if it satisfies the following two conditions:

1. If \( U \) has no external vertex of attachment on \( Q \cup P \) then (i) \( s_Q \neq t_Q \) and \( s_Q \) are distinct and (ii) all its external vertices lie only on \( J[s_Q; s_P] \) or only on \( J[t_P; t_Q] \).

2. If \( U \) has no external vertex of attachment on \( P \cup Q \) then (i) \( s_P \neq t_P \) and \( s_P \) are distinct and (ii) all its external vertices lie only on \( J[s_Q; s_P] \) or only on \( J[t_P; t_Q] \).

Otherwise, \( U \) is said to be infeasible.

**Lemma 3.1** Let \( U \) be an augmented graph. If \( U \) is infeasible then \( U \) does not have a pair of interlacing bights.

**Proof.**

Let \( U \) be an augmented graph with the set of \( B^{PQ} \)-bridges, \( B \). Assume to the contrary—that
Figure 3: Interlacing Cross-Cuts.

1. \( x_1, y_1 \in P[s; t] \) and \( x_2, y_2 \in Q[s; t] \), where \( x_1 \) is to the left of \( y_1 \) on \( P \).
2. Not all the external vertices of attachment lie only on the subpath \( J[y_2; x_1] \) or only on the subpath \( J[y_1; x_2] \).

- **ST-Cross-Cut Pair**, if it satisfies the following two conditions:
  1. \( x_1 = s, x_2 = t, y_1 \in P[s; t] \) and \( y_2 \in Q[s; t] \).
  2. There are an upper external vertex of attachment on \( P[s; t] \) and a lower external vertex of attachment on \( Q[s; t] \).

- **P-Cross-Cut Pair**, if it satisfies the following three conditions:
  1. \( x_1, x_2 \in P[s; t] \) and at least one of them is distinct from \( s \) and \( t \).
  2. \( y_1 \in P[s; t] \) and \( y_2 \in Q[s; t] \).
  3. There is an upper external vertex of attachment \( c \in P[x_1; x_2] \).

- **Q-Cross-Cut Pair**, if it satisfies the following three conditions:
  1. \( x_1, x_2 \in Q[s; t] \) and at least one of them is distinct from \( s \) and \( t \).
  2. \( y_1 \in Q[s; t] \) and \( y_2 \in P[s; t] \).
§3.1

**Definition 3.1** Let $U$ be a graph, consisting of the cycle $J = \{P\} \cup \{Q\}$ and a set (possibly, empty) of $B_{PQ}$-bridges, $B$. Some (possibly none) of its vertices on $P$; if may be labeled as **upper external vertices of attachment**, $UA$, and some (possibly none) of its vertices on $Q$; if, as **lower external vertices of attachment**, $LA$. The set $\{UA\} \cup \{LA\}$ is its set of **external vertices of attachment**, $EA$. We refer to $U$ as an **Augmented Graph**. (See Figure 2: $UA = \{e_1, e_2, e_3\}$ and $LA = \{e_4\}$.) □

Note that every Type.IV graph with no external vertex of attachment is trivially an augmented graph.

Usually, a non-trivial augmented graph may be created by activation of an edge as follows:

Let $e = [u, v]$ be an arbitrary edge on the path $P[s; t]$. Let $u$ be to the left of $v$ on $P[s; t]$. We **activate** the edge $e$ by introducing:

1. An external vertex of attachment at $u$, if $u$ is distinct from $s$.
2. An external vertex of attachment at $v$, if $v$ is distinct from $t$.

We say that the resulting graph, an augmented graph, is obtained from $G$ by the activation of the edge $e$, and represent it by $u(G; e)$.

In general, the external vertices will be created in the process of recursive analysis of the graph and represent conduits through which certain vertex disjoint paths may enter and exit the augmented graph.

**Definition 3.2** **Interlacing Cross-Cuts**.

Let $U$ be an augmented graph with a cycle $J = \{P\} \cup \{Q\}$ and let $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$ be a pair of interlacing vertex-disjoint cross-cuts such that not all the vertices $x_1, x_2, y_1$ and $y_2$ lie only on $P[s; t]$ or lie only on $Q[s; t]$. Such a pair of cross-cuts (Figure 3) is said to be a:

- **PQ-Cross-Cut Pair**, if it satisfies the following two conditions:
common steps and retain enough informations from one labeling to another future labeling, we
could improve the efficiency of the over-all algorithm.

Here, we will see how to exploit the intuitions noted above in order to devise an algorithm
can label the edges of $P$ and $Q$ of a Type.IV graph correctly in time $O(|E| \cdot |V|)$. Such
an algorithm would be adequate for our stated goal of finding an $O(|E| \cdot |V|)$ time algorithm for
the general "bidirectional edges problem."

Towards this goal, we proceed by forming a characterization of the bidirectionality of an edge
of the path $P[s; t]$ in terms two properties: feasibility property and admissibility property, the
later being a stronger condition. These properties lead to the definitions of two subsets of the
edges $E(P[s; t])$: the feasible edges, $FE(G : P)$ and the admissible edges, $AE(G : P)$.

$$E(P[s; t]) \supset FE(G : P) \supset AE(G : P).$$

The feasible edges are relatively easy to compute; but the admissible edges cause more compli-
cations as they require one to detect if the graph has certain pairs of interlacing vertex-disjoint
cross-cuts (with respect to the cycle $J = \{P\} \cup \{Q\}$). However, since the bidirectional edges on
$P$ and $Q$ of a Type.IV graph are simply

$$AE(G : P) \cup AE(G : Q),$$

it suffices to show how to compute $AE(G : P)$ in time $O(|E| \cdot |V|)$.

§2.3 Some Notations Let $G$, $s$, $t$, $P[s; t]$, $Q[s; t]$ and $J = \{P\} \cup \{Q\}$ be as before. If $B$
is a bridge of the cycle $J$ (and similarly, for $B$, a block of bridges) with at least one vertex of
attachment on $P[s; t]$, then the left- and the right-most vertices of attachment of $B$ on $P[s; t]$ are
referred to by $sp(B)$ and $tp(B)$ (and, in case of a block of bridges, $B$, $sp(B)$ and $tp(B)$),
and the left-most and the right-most vertices of attachment of $B$ on $P[s; t]$ are referred to by
$s_{\bar{p}}(B)$ and $t_{\bar{p}}(B)$ (and, in case of a block of bridges, $B$, $s_{\bar{p}}(B)$ and $t_{\bar{p}}(B)$).

If, on the other hand, $B$ is a bridge of the cycle $J$ with at least one vertex of attachment on
$Q[s; t]$, then $s_Q(B)$, $t_Q(B)$, $s_Q(B)$, $t_Q(B)$, etc. are defined in an identical manner.

If the bridge or the block of bridges under consideration is clear from the context then we
simply write $sp$, $s_Q$, $s_{\bar{p}}$, $s_Q$, $t_p$, $t_Q$, $t_{\bar{p}}$ and $t_{\bar{p}}$.

3 Characterization of Bidirectionality

Let $G$ be a Type.IV graph consisting of the cycle of $J = \{P\} \cup \{Q\}$, and a single $B^{PQ}$-bridge,
$B$. In this section, we provide a new characterization for an edge $e \in E(P)$ to be bidirectional.

Before presenting the main theorem, we introduce the following notions: (i) an augmented
graph, (ii) and the conditions under which such a graph is feasible. The main admissibility
criterion is based on the condition whether a feasible augmented graph has certain pairs of
interlacing vertex-disjoint cross-cuts. Such pairs of cross-cuts are classified into four categories:
P-cross-cut pair, $Q$-cross-cut pair, $PQ$-cross-cut pair and $ST$-cross-cut pair.

Finally, we introduce the concepts of (i) a Set of Feasible Edges and (ii) a Set of Admissible
Edges of an Augmented Graph with an associated path.
• $B^Q$-BRIDGES: The set of bridges with no vertex of attachment on $P]$s;[ and at least one vertex of attachment on $Q]$s;[.

If a bridge $B$ of $J = P \cup Q$ in $G$ is not a $B^{PQ}$-, $B^P$- or $B^Q$-bridge then it has only $s$ or $t$ as vertices of attachment.

Example 2.4 In the figure 1, we show $B^{PQ}$-, $B^P$- and $B^Q$- bridges of the paths $P$ and $Q$. Bridges $B_1$, $B_2$ and $B_3$ are $B^{PQ}$-bridges; $B_4$ is a $B^P$-bridge and $B_5$, a $B^Q$-bridge.

Definition 2.5 AMBITUS.

Let $J$, $P$ and $Q$ be as in the previous definition. Then $J$ is called an ambitus if every $B^P$- or $B^Q$-bridge avoids every $B^{PQ}$-bridge.

See Mishra and Tarjan[13] for a linear time algorithm to compute an ambitus.

Let $B = B_1, \ldots, B_k$ be the bridges of $J$ in $G$. A non-empty subset of bridges $B \subseteq B$ is called a block of bridges if it satisfies the following two conditions: (1) If $B_i \in B$ and $B_i$ and $B_j$ overlap, then $B_j \notin B$. (2) No non-empty proper subset of $B$ satisfies the above condition.

We say $B$ is proper, if it contains more than one bridge of $J$ in $G$, otherwise, it is degenerate.

§2.2 Henceforth, we assume that the graph $G$ is a Type.IV graph satisfying the following conditions:

Definition 2.6 Type.IV Graphs.

A Type.IV graph $G$ is a nonseparable graph $G$ consisting of a cycle $J$ containing the vertices $s$ and $t$ and exactly one $B^{PQ}$-bridge, such that if the vertices $s$ and $t$ together with their incident edges are deleted from $G$ then the resulting derived subgraph is also nonseparable.

The following observation follows from the algorithmic results of [12]:

Theorem 2.1 Suppose we have an algorithm that correctly labels the edges of $P]$s;[ and $Q]$s;[ of a Type.IV graph $G = (E,V)$ in time $O(T(|E|,|V|)) \geq O(|E| \cdot |V|)$, where $T(\cdot, \cdot)$ is a monotonically-nondecreasing convex function in both its arguments, i.e.,

$$T(x, \cdot) + T(y, \cdot) \leq T(x + y, \cdot) \quad\text{and}\quad T(\cdot, x) + T(\cdot, y) \leq T(\cdot, x + y),$$

where $x \geq 0$ and $y \geq 0$.

Then there is a set of mutually recursive algorithms that correctly labels the edges of an undirected connected strict graph $G = (E,V)$ in time $O(T(|E|,|V|))$.

We also observed that using well-known algorithms for “two vertex disjoint paths problem,” it is easy to label the edges of $P$ and $Q$ of a Type.IV graph correctly in time $O(|E| \cdot |V|^2)$; this insight leads to an over-all $O(|E| \cdot |V|^2)$ time algorithm for an arbitrary undirected graph.

Essentially, such an algorithm “examines” each edge of the path $P$ (and $Q$) individually, in order to find appropriate paths in the graph $G$ that would yield a correct labeling. However, many of the steps performed by the algorithm to label an edge of $P$ are needlessly replicated when a “near-by” edge of $P$ is examined subsequently. Clearly, if we can economize on the
edges; $B$ is a bridge of $J$ in $G$. The component $C$ of $G^-$ is the nucleus of $B$ (denoted, $N(B)$). Such a bridge is called proper; if a bridge $B$ does not have a nucleus (i.e., $B$ is an edge), it is degenerate.

If $J$ is a cycle of the graph $G$, then a path $N$ in $G$ avoiding $J$ but having its two ends $x$ and $y$ in $J$ is called a cross-cut of $J$ between $x$ and $y$. If $B$ is a bridge of the cycle $J$ in $G$, then the vertices of attachment of $B$ dissect $J$ into subpaths called the residual paths of $B$ in $J$.

**Definition 2.2 Relations between Bridges.**

Let $B_1$ and $B_2$ be two distinct bridges of a cycle $J$ of $G$.

- We say $B_1$ avoids $B_2$ if and only if one of the following two conditions is satisfied:
  
  1. $|W(G, B_1)| \leq 1$ or $|W(G, B_2)| \leq 1$.
  2. All the vertices of attachment of $B_1$ are contained in a single residual path $L$ of $B_2$.

- If $B_1$ and $B_2$ do not avoid one another we say that they overlap.
- If there exist two vertices of attachment $x_1$ and $x_2$ of $B_1$ and two vertices of attachment $y_1$ and $y_2$ of $B_2$, all four distinct, such that $x_1$ and $x_2$ separate $y_1$ and $y_2$ in the cycle $J$, then we say that they interlace.

**Definition 2.3 Bridges with respect to the Paths.**

Let $G$ be an undirected graph with two distinguished vertices $s$ and $t$ with two internally vertex disjoint paths $P[s; t]$ and $Q[s; t]$, which meet each other only in their end vertices, $s$ and $t$; $J = \{P\} \cup \{Q\}$ is a cycle in $G$. We consider three different classes of bridges with respect to $J$:

- $B^{PQ}$-bridges: The set of bridges with at least one vertex of attachment on $P[s; t]$ and at least one vertex of attachment on $Q[s; t]$.
- $B^P$-bridges: The set of bridges with at least one vertex of attachment on $P[s; t]$ and no vertex of attachment on $Q[s; t]$. 

graph, and shows how this characterization can be used to devise an efficient algorithm. In the last two sections (4 and 5), the algorithms are developed, followed by arguments for their correctness and an analysis of their time-complexity. A key technical theorem is proven in the appendix.

2 Overview

We begin by recalling some of the key notations and ideas developed in the companion paper[12] and next, present an overview of an efficient algorithm for the “bidirectional edges problem.” In this paper, standard graph theoretic terminology is used without explicit definitions here; readers unfamiliar with the terminology may consult [11,12] or [13].

Consider an undirected graph $G = (V, E)$ consisting of a finite set $V$ of vertices and a set $E$ of pairs of vertices, called edges. A path in $G$ from $u$ to $v$ ($u, v \in V$) in $G$ is a sequence of vertices in $V$ ($u = u_0, u_1, \ldots, u_k = v$) such that $\langle u_i, u_{i+1} \rangle \in E$ for $0 \leq i < k$. (Sometimes denoted by $u \rightarrow v$.) The vertices $u$ and $v$ are called the ends of the path $P$. All other vertices of the path (i.e., $u_i$’s for $0 < i < k$) are the internal vertices of the path.

If $P$ is a path from $u$ to $v$, $u = u_0, u_1, \ldots, u_k = v$, and $0 \leq i \leq j \leq k$ then the subpath from $u_i$ to $u_j$, including both $u_i$ and $u_j$ is represented by $P[u_i; u_j]$; the subpath excluding $u_i$ but including $u_j$, by $P[u_i; u_j]$; the subpath including $u_i$ but excluding $u_j$, by $P[u_i; u_j]$; and the subpath excluding both $u_i$ and $u_j$, by $P[u_i; u_j]$. If $P_1 = u_0, u_1, \ldots, u_i$ and $P_2 = u_i, u_{i+1}, \ldots, u_k$ are two paths then the concatenation of $P_1$ and $P_2$ is $P_1 \ast P_2 = u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_k$.

The path is simple if $u_0, \ldots, u_k$ are distinct (except possibly $u_0 = u_k$) and the path is a cycle if $u_0 = u_k$. By convention there is a path of no edges from every vertex to itself (null path), but a cycle must contain at least two edges. Two simple paths $P_1$ and $P_2$ are said to be vertex disjoint, if the vertices of $P_1$ and $P_2$ are mutually distinct; internally vertex disjoint, if the internal vertices of $P_1$ and $P_2$ are mutually distinct.

§2.1 We recall the following definitions.

**Definition 2.1 Bridges. [Tutte]**

Let $J$ be a fixed subgraph of $G$. For a subgraph $G_1$ of $G$, a vertex of attachment of $G_1$ in $G$ is a vertex of $G_1$ that is incident in $G$ with some edge not belonging to $G_1$; subgraph $G_1$ is said to be $J$-detached in $G$, if all its vertices of attachment are in $J$. We define a bridge of $J$ in $G$ as any subgraph $B$ that satisfies the following three conditions:

- $B$ is not a subgraph of $J$.
- $B$ is $J$-detached in $G$.
- No proper subgraph of $B$ satisfies both (1) and (2).

The set of vertices of attachment of a bridge $B$ of a subgraph $J$ in $G$ is denoted by $W(G, B) = \{v_0, v_1, \ldots, v_{k-1}\}$. \(\square\)

Let $G^-$ be the graph derived from $G$ by deleting the vertices of $J$ and all their incident edges. Let $C$ be any component of $G^-$. Let $B$ be the subgraph of $G$ obtained from $C$ by adjoining to it each edge of $G$ having one end in $C$ and one in $J$, and adjoining also the ends in $J$ of all such
1 Introduction

Let \( G = (V, E) \) be a finite undirected graph with two distinguished vertices, the source, \( s \), and the sink, \( t \). We call an edge \( e = [u, v] \) of \( G \) ‘bidirectional’, if there are two simple paths connecting \( s \) and \( t \) and traversing \( e \) in either order—\( u, v \) and \( v, u \). Similarly, we call an edge \( e = [u, v] \) of \( G \) ‘unidirectional’, if every simple path connecting \( s \) and \( t \) and containing \( e \), traverses \( e \) only in one order, say \( u, v \); additionally, \( e \) is labeled \( \langle u, v \rangle \). The ‘bidirectional edges problem’ is to find all the ‘bidirectional’ and ‘unidirectional’ edges of \( G \), together with the labelings of the ‘unidirectional’ edges.

The notions of ‘unidirectional’ and ‘bidirectional’ edges can be formalized in terms of the labeling function, \( \ell \), that maps each undirected edge \( [u, v] \) to a subset of \( \{\langle u, v \rangle, \langle v, u \rangle\} \).

**Definition 1.1** The edge-labeling function, \( \ell \), is defined as follows:

\[
\ell([u, v]) \supseteq \begin{cases} 
\langle u, v \rangle, & \text{iff there is a simple path} \\
\langle v, u \rangle, & \text{iff there is a simple path} \\
( s = ) \ w_0, \ldots, w_i, w_{i+1}, \ldots, w_n \ ( = t) \\
\text{such that } w_i = u \text{ and } w_{i+1} = v; \\
( s = ) \ w_0, \ldots, w_i, w_{i+1}, \ldots, w_n \ ( = t) \\
\text{such that } w_i = v \text{ and } w_{i+1} = u. 
\end{cases}
\]

Clearly, an edge \( e = [u, v] \) is bidirectional, if \( \ell([u, v]) = \{\langle u, v \rangle, \langle v, u \rangle\} \); and unidirectional, if \( \ell([u, v]) = \{\langle u, v \rangle\} \) or \( \{\langle v, u \rangle\} \). □

The relation between bidirectional edges problem and the classical two vertex disjoint paths problem is elucidated in the previous installment of this paper[12]. Using the efficient algorithms to find two vertex disjoint paths in an undirected graph (Cf. Ohtsuki[16], Seymour[18] and Shiloach[19]; also, Mishra and Tarjan[13]), it is relatively easy to devise an algorithm for bidirectional edges problem with time complexity of \( O(|E|^2 \cdot |V|) \).

In this and its companion paper[12], we devise an \( O(|E| \cdot |V|) \) time algorithm for bidirectional edges problem; the algorithm makes a novel use of bridges of a circuit in a general graph. We began in a prequel to this paper with a simple set of reduction processes which resulted in an \( O(|E| \cdot |V|^2) \) time algorithm; here, we introduce some additional machinery that further reduces the complexity to \( O(|E| \cdot |V|) \). The algorithms described here first appeared in [10] and [11].

The problem of finding all bidirectional edges arises naturally in the context of the simulation of an MOS transistor network, in which a transistor may operate as a unilateral or a bilateral device, depending on the voltages at its source and drain nodes, (Cf. Brand[2].) For efficient simulation, it is important to find the set of transistors that may operate as bilateral devices. Also, sometimes it is desired that information propagates in one direction only, and propagation in the wrong direction (resulting in a sneak path) can cause functional error. For a more detailed discussions of this problem, also consult the followings: Frank[6], Barzilai, Breece, Huisman, Iyengar and Silberman [1], Jouppi[7], Chen, Mathews and Newkirk[3], Brand[2], Lee, Gupta and Breuer[8], and Ciric[4].

The paper is organized as follows: Section 2 provides an overview of the algorithm, with a necessary recapitulation of the graph-theoretic terminology and the results developed in the companion paper[12]. Section 3 provides a new characterization of the “path-edges” of a Type.IV
Contents

1 Introduction 1
2 Overview 2
3 Characterization of Bidirectionality 5
4 Analyzing Augmented Graphs 12
5 Finding the Set of Admissible Edges 21
ABSTRACT

The “bidirectional edges problem” is to find an edge-labelling of an undirected network, $G = (V, E)$, with a source and a sink, such that an edge $[u, v] \in E$ is labelled $(u, v)$ or $(v, u)$ (or both) depending on the existence of a (simple) path from the source to sink that visits the vertices $u$ and $v$, in the order $u,v$ or $v,u$, respectively. In this paper, building upon the machinery developed in a prequel, we devise an efficient algorithm for this problem with a time complexity of $O(|E| \cdot |V|)$. The main technique exploits a clever partition of the graph into a set of paths and bridges which are then analyzed recursively.

Bidirectional edges problem arises naturally in the context of the simulation of an MOS transistor network, in which a transistor may operate as a unilateral or a bilateral device, depending on the voltages at its source and drain nodes. Here, it is required to detect the set of transistors that may operate as bilateral devices.

Key words.
Bridge, Cross-cut, Disjoint Paths, MOS Circuit, Pass Transistor, Sneak Path, complexity
Bidirectional Edges Problem: Part II

An Efficient Algorithm

B. Mishra\textsuperscript{1}

Department of Computer Science
Courant Institute of Mathematical Sciences
New York University
719 Broadway
New York, NY 10003

\textsuperscript{1}Supported in parts by National Science Foundation Grants DMS-8703458 and CCR-9002819.