Theorem 5.21 *Let U be a feasible augmented graph, with the blocks of B^{PQ}-bridges: $\mathbf{B}_1$, $\mathbf{B}_2$, ..., $\mathbf{B}_l$.*

Then U does not have a pair of interlacing bights, if for all i $(1 \leq i \leq l)$ $U_{\mathbf{B}_i}$ has no pair of interlacing bights.

PROOF.

Assume to the contrary. Then there exists a feasible U-Fragment with the blocks of overlapping B^{PQ}-bridges: $\mathbf{B}_1$, $\mathbf{B}_2$, ... $\mathbf{B}_l$, such that for all i $(1 \leq i \leq l)$ $U_{\mathbf{B}_i}$ has no pair of interlacing bights, but U has a pair of interlacing bights, A_1 and A_2. Let A_1 be between a_1 and a_2 and A_2, between b_1 and b_2, where a_1, a_2, b_1 and b_2 are four distinct vertices on J such that they are external vertices of attachment on J or one of s and t. Further assume that U has the *least* number of blocks of overlapping B^{PQ}-bridges. Assume that $\mathbf{B}_1$, $\mathbf{B}_2$, ..., $\mathbf{B}_l$ are so so ordered that if $i < j$ (for $1 \leq i,\ j \leq l$) then the vertices of attachment of $\mathbf{B}_i$ on P and Q are to the left of those of $\mathbf{B}_j$ on P and Q, respectively.

It then follows that of the four vertices a_1, a_2, b_1 and b_2, at least two lie on the subpath $J]t_P(\mathbf{B}_1);t_Q(\mathbf{B}_1)[$ and at least two lie on the subpath $J]s_P(\mathbf{B}_l);s_Q(\mathbf{B}_l)[$ and that $l = 2$.

But then it is easy to see that in this case either $U_{\mathbf{B}_1}$ or $U_{\mathbf{B}_2}$ has a pair of interlacing bights; this contradicts the assumption. ☐

of $U_{\mathbf{B}}$ obtained by contracting the set of residual paths, $\mathcal{L}$, be as in the step2 of the Algorithm ANALYZE-U- $\overline{U}$.

Theorem 5.18 *If $U'_{\mathbf{B}}$ contains a P-, Q-, PQ- or an ST-Cross-Cut pair then $U_{\mathbf{B}}$ has a P-, Q-, PQ- or an ST-Cross-Cut pair.*

PROOF.

Follows immediately from the following two facts:

1. If $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$ are the cross-cuts of $U'_{\mathbf{B}}$ then $U_{\mathbf{B}}$ has a pair of interlacing cross-cuts, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$.
2. If $L'_i[a; b]$ is a contraction of $L_i[a; b] \in \mathcal{L}$, and if $U'_{\mathbf{B}}$ has an external vertex of attachment at a, at b or on $L'_i]a; b[$ then $U_{\mathbf{B}}$ has an external vertex of attachment at a, at b or on $L_i]a; b[$, respectively. □

Theorem 5.19 *If $U'_{\mathbf{B}}$ does not have a pair of interlacing bights then $U_{\mathbf{B}}$ does not have a pair of interlacing bights.*

PROOF.

Let $U^0_{\mathbf{B}}$ $(= U_{\mathbf{B}})$, $U^1_{\mathbf{B}}$, ..., $U^m_{\mathbf{B}}$ $(= U'_{\mathbf{B}})$ be sequence of minors of the bridge-fragment $U_{\mathbf{B}}$, where $U^i_{\mathbf{B}}$ is obtained from $U^{i-1}_{\mathbf{B}}$ by contracting L_i (for $1 \le i \le m$). The theorem follows from the following claim:

Claim If $U^{i-1}_{\mathbf{B}}$ has a pair of interlacing bights then $U^i_{\mathbf{B}}$ has a pair of interlacing bights.

PROOF OF THE CLAIM. Let $L_i = L[a; b]$. Let J_0 be the cycle J of $U_{\mathbf{B}}$, and J_i, the cycle in $U^i_{\mathbf{B}}$ obtained by contracting the subpath L_i of J_{i-1} in $U^{i-1}_{\mathbf{B}}$. Assume that a_1, a_2, b_1 and b_2 are four distinct vertices of J_{i-1} such that there is a pair of interlacing bights, A_1 between a_1 and a_2 and A_2 between b_1 and b_2—here, a_1, a_2, b_1 and b_2 are either external vertices of attachment or one of s and t. Henceforth, assume that $|L[a; b]| \ge 3$, since otherwise $U^i_{\mathbf{B}} = U^{i-1}_{\mathbf{B}}$, and the claim holds trivially.

Consider the case when both A_1 and A_2 have common sections, M_1 and M_2, respectively, such that $L \cap M_i \ne \emptyset$ (for $i = 1, 2$).

Then M_1 and M_2 are common end sections of A_1 and A_2, respectively. Without loss of generality assume that $M_1 = A'_1[a_1; a]$ and $M_2 = A'_2[b_1; b]$. Then a_1 must be to the left of b_1 on $L]a; b[$; and $|\mathrm{EA}(L]a; b[)| \ge 2$. Let $[x; y]$ be the contraction of the subpath $L]a; b[$. Then $U^i_{\mathbf{B}}$ has external vertices of attachment at x and y.

Note that both x and y are external vertices of attachment of $U^i_{\mathbf{B}}$ and x and a_2 separate y and b_2 on J_i. The vertex-disjoint paths $[x, a] * A_1[a; a_2]$ and $[y, b] * A_2[b; b_2]$ define the appropriate interlacing bights.

Other cases can be handled in a similar fashion. □

§E Interlacing Cross-Cuts and Bights: Augmented Graph

Let U be a feasible augmented graph consisting of the cycle $J = \{P\} \cup \{Q\}$.

Theorem 5.20 *Let U be a feasible augmented graph, with the blocks of B^{PQ}-bridges: $\mathbf{B}_1$, $\mathbf{B}_2$, ..., $\mathbf{B}_l$.*

Then U has a P-, Q-, PQ- or an ST-cross-cut pair, if for some i $(1 \le i \le l)$ $U_{\mathbf{B}_i}$ has a P-, Q-, PQ- or an ST-cross-cut pair.

PROOF.

Obvious. □

(4) Since U is feasible, U an external vertex of attachment d, on $Q]s_Q; s^*_Q]$ or on $J[t^*_P; t^*_Q]$, where d is distinct from c. Assume that $d \in Q]s; s^*_Q]$. Then, if $\mathcal{C}_2$ is defined then we may assume that the bridges of $\mathcal{C}_2$ avoid the bridges of $\mathcal{C}_1$, since, otherwise, U has Q- or ST-cross-cuts. Hence B interlaces with a bridge of $\mathcal{C}_1$, but in this case, again U has Q- or ST-cross-cuts.

In the other case, i.e., $d \in J[t^*_P; t^*_Q]$, B interlaces with a bridge of $\mathcal{C}_1 \cup \mathcal{C}_2$, and U has P-, Q- or PQ-cross-cuts. $\square$

Lemma 5.16 *Let U and* $\mathbf{B}$ *be as in the Theorem 5.14. If U has an external vertex of attachment, c, on $P]s_P; t_P[$ and an external vertex of attachment, d, on $Q]s_Q; t_Q[$ then U has a PQ-, P-, Q- or an ST-cross-cut pair.*

PROOF.
We may assume that $\mathbf{B}$ has exactly one vertex of attachment, $s^*_P = t^*_P$, on $P]s; t[$ and has exactly one vertex of attachment, $s^*_Q = t^*_Q$, on $Q]s; t[$, since, otherwise, the statement follows from the previous Lemma 5.15. Since s_P is distinct from t_P, s_Q, distinct from t_Q, and since $\mathbf{B}$ is not a block of equivalent B^{PQ} 3-bridges, $\mathbf{B}$ has a vertex of attachment at s and a vertex of attachment at t. Since $\mathbf{B}$ is a single proper block of overlapping B^{PQ}-bridges, there must be a B^{PQ}-bridge, $B \in \mathbf{B}$, such that B has vertices of attachment at s and t and U has ST-cross-cuts. $\square$

Lemma 5.17 *Let U and* $\mathbf{B}$ *be as in the Theorem 5.14. If U has an external vertex of attachment, c, on $J[s_P; s_Q]$ and an external vertex of attachment, d, on $J[t_P; t_Q]$, and if* $\mathbf{B}$ *has more than one distinct vertices of attachment on $P]s; t[$ and more than one distinct vertices of attachment on $Q]s; t[$ then U has PQ-, P-, Q- or an ST-cross-cut pair.*

PROOF.
Since $\mathbf{B}$ is not a singleton set, and not a block of equivalent 3-bridges, and since $\mathbf{B}$ has more than one distinct vertices of attachment on $P]s; t[$ and more than one distinct vertices of attachment on $Q]s; t[$, it must have two bridges B_1 and $B_2 \in \mathbf{B}$ such that they provide two interlacing cross-cuts: $N_1[x_1; x_2]$ in B_1 and $N_2[y_1; y_2]$ in B_2 such that x_1 is to the left of y_1 on $P]s; t[$ and y_2 is to the left of x_1 on $Q]s; t[$. Hence U has PQ-cross-cuts. $\square$

PROOF OF THE THEOREM 5.14.

Since U is feasible either $s_P \neq t_P$ or $s_Q \neq t_Q$ (or both). Assume that $s_Q \neq t_Q$.

- CASE.1. *U has an external vertex of attachment on $Q]s_Q; t_Q[$.*

If $s_P = t_P$ then both s_Q and t_Q are distinct from s and t; and the theorem follows from Lemma 5.15. Hence assume that $s_P \neq t_P$. If U also has an external vertex of attachment on $P]s_P; t_P[$ then the theorem follows from the Lemma 5.16. On the other hand, if U has no external vertex of attachment on $P]s_P; t_P[$ then, since U is feasible, s^*_Q and t^*_Q are distinct, and the theorem follows from Lemma 5.15.

- CASE.2. *U has no external vertex of attachment on $Q]s_Q; t_Q[$.*

Since U is feasible, s^*_P and t^*_P are distinct. We may assume that U has no external vertex of attachment on $P]s_P; t_P[$ (otherwise, it is similar to the previous case.) Then s^*_Q and t^*_Q are distinct. Moreover, since U is feasible, not all the external vertices of attachment lie only on the subpath $J[s_Q; s_P]$, or only on the subpath $J[t_P; t_Q]$. But then the theorem follows from Lemma 5.17. $\square$

§D Interlacing Cross-Cuts and Bights: Minor of an Augmented Graph

Let $U_{\mathbf{B}}$ be an augmented graph with the single (possibly, degenerate) block of B^{PQ}-bridges $\mathbf{B}$ of $J = \{P\} \cup \{Q\}$ in $U_{\mathbf{B}}$. Let $\mathcal{L}$, the set of residual paths of J in $U_{\mathbf{B}}$ and $U'_{\mathbf{B}}$, the minor

Note that if Class.1 $= \emptyset$ then both Class.2 and Class.3 $\neq \emptyset$.

(1) *We may assume that* Class.1 $= \emptyset$.
Suppose not, *i.e.*, Class.1 $\neq \emptyset$. Then if Class.4 $\neq \emptyset$ then U has P-cross-cuts. If Class.2 $\neq \emptyset$ then a bridge of Class.2 overlaps with a bridge of Class.1 and U has P- or PQ-cross-cuts. If Class.3 $\neq \emptyset$ then again U has P- or PQ-cross-cuts.

Hence consider the case when Class.2 = Class.3 = Class.4 $= \emptyset$, *i.e.*,$\mathbf{B}$ = Class.1. Hence Class.1 is neither a singleton set; nor is it a block of equivalent B^{PQ} 3-bridges. Hence, either Class.1 has more than two vertices of attachment on P, in which case U has P-cross-cuts, or Class.1 has more than one vertex of attachment on Q, in which case U has PQ-cross-cuts.

(2) *We may assume that* Class.2 $\neq \emptyset$ *and* Class.3 $\neq \emptyset$ *and bridges of* Class.2 *avoid the bridges of* Class.3.
This follows from the facts that Class.1 $= \emptyset$ and if a bridge of Class.2 interlaces with a bridge of Class.3 then U has P- or PQ-cross-cuts. Hence the vertices of attachment of Class.2 are to the left of those of Class.3 on P and Q, respectively. Let x_2 and y_2 be the right-most vertices of attachment of Class.2 on P and Q, respectively; and similarly, x_3 and y_3, the left-most vertices of attachment of Class.3 on P and Q, respectively. Hence x_2 is to the left of x_3 on P and y_2, to the left of y_3 on Q.

(3) *We may further assume that* Class.4$\neq \emptyset$.

If Class.4 $= \emptyset$ then from (2) it follows that $\mathbf{B}$ is not a single block of B^{PQ}-bridges, which is a contradiction. Let v be an arbitrary vertex on the path $Q[y_2; y_3]$. If every bridge of Class.4 has all its vertices of attachment on $Q[s;t]$ either to the left of v or to the right of v on Q, then $\mathbf{B}$ is not a single block of B^{PQ}-bridges.

Hence Class.4 has a B^{PQ}-bridge that interlaces with a bridge of Class.2 and has a vertex of attachment on $Q]y_2; t]$. Similarly, Class.4 has a B^{PQ}-bridge that interlaces with a bridge of Class.3 and has a vertex of attachment on $Q[s; y_3[$.

Since U is feasible it has an external vertex of attachment d, distinct from c. If d lies on $J]y_2; c[$ or on $J]c; y_3[$ then U has P-, Q-, PQ or ST-cross-cuts.

Hence, assume that $d \in Q[y_2; y_3]$. Hence there is a bridge $B \in$ Class.4 with a vertex of attachment on $Q[s; d[$ and a vertex of attachment on $Q]d; t]$. If in addition, B has a vertex of attachment on $Q]s; d[$ and a vertex of attachment on $Q]d; t[$ then if we define the Classes (1, 2, 3 and 4) as above, then the Class.1$\neq \emptyset$ for d. Hence we can derive a contradiction as in (1). Otherwise it is easy to see that U has Q- or ST-cross-cuts.

- <u>CASE.2.</u>

By an argument similar to case1, we derive a contradiction, if $c \in P]s^*_Q; t^*_Q[$ (where $s^*_Q \neq t^*_Q$).

- <u>CASE.3.</u>

Hence assume that U has no external vertex of attachment on $P]s^*_P; t^*_P[$ or $Q]s^*_Q; t^*_Q[$, but U satisfies one of the conditions of the Lemma, say condition(1).

Without loss of generality assume that $c \in P]s_P; s^*_P]$ (where $s_P \neq s^*_P$). Let Class.1, Class.2, Class.3 and Class.4, with respect to c, be defined as before. In this case, Class.1 and Class.2 $= \emptyset$, but Class.3 $\cup$ Class.4 $\neq \emptyset$.

Let $\mathcal{C}_1 \subseteq \mathbf{B}$ be the set of bridges of $\mathbf{B}$ with a vertex of attachment at s. Notice that $\mathcal{C}_1$ is non-empty. Let x and y be the left- and right-most vertices of attachment of $\mathcal{C}_1$ on $P]s; t[$.

(1) If $\mathbf{B} \setminus \mathcal{C}_1$ has a bridge with a vertex of attachment on $P]s; y[$ then U has P-cross-cuts. Hence every bridge of $\mathbf{B} \setminus \mathcal{C}_1$ has all its vertices of attachment on $P[s;t]$ only on $P[y;t]$.

(2) If x and y are distinct then $\mathcal{C}_1$ is a singleton set. (Otherwise, U has P-cross-cuts.) Since $\mathbf{B}$ is a proper block, $\mathbf{B} \setminus \mathcal{C}_1$ has a bridge B such that B overlaps with the bridge of $\mathcal{C}_1$. Since the B^{PQ}-bridge, $B \in \mathbf{B} \setminus \mathcal{C}_1$, B has a vertex of attachment on $P]x; t[$.

(3) If $x = y$ then define $\mathcal{C}_2$ to be the set of bridges of $\mathbf{B}$ with exactly one vertex of attachment at x on $P]s; t[$. But in this case, since $\mathbf{B}$ has more than one distinct vertices of attachment on $P]s; t[$, there is a B^{PQ}-bridge $B \in \mathbf{B} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$ that overlaps with a bridge of $\mathcal{C}_1 \cup \mathcal{C}_2$. Since the B^{PQ}-bridge, $B \in \mathbf{B} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$, B has a vertex of attachment on $P]x; t[$.

constituent bights of J_1 in $\overline{U}'$ to a Jordan curve such that all but its end vertices lie in the face, F_1; each of its constituent bights of J_2 in U' to a Jordan curve such that all but its end vertices lie in the face, F_2; and each of its common sections to appropriate common sections in the Jordan curve $R(\pi)$ or $J(\pi)$. The claim follows from the following observations: every common section on $R(\pi)$ (or $J(\pi)$) is disjoint with every other common section on $R(\pi)$ (or $J(\pi)$, respectively), and do not contain a point of F_1, F_2 or EXT J; every Jordan curve corresponding to a bight of J_1 (respectively, J_2) can be mapped in F_1 (respectively, F_2) such that it does not meet any other Jordan curves in F_1 (respectively, F_2) and meets $R(\pi)$ or $J(\pi)$ at the appropriate ends of its common end sections.

Arguing in a manner similar to above, we can show that, since A_1' and A_2' are vertex-disjoint simple paths and since constituent bights of neither of them can interlace with one another in U' or $\overline{U}'$, it is, additionally, possible to map them such that $A_1'(\pi)$ and $A_2'(\pi)$ do not meet each other. Notice that $A_1'(\pi)$ and $A_2'(\pi)$ are Jordan curves with their end points $a_1(\pi)$ and $a_2(\pi)$, and $b_1(\pi)$ and $b_2(\pi)$ of $J(\pi)$, respectively, and with none of their points in EXT J. But this is impossible. ☐

§C Interlacing Cross-Cuts and Bights: Single Proper Block

Notice that if U is an infeasible augmented graph with a single block of B^{PQ}-bridges, it follows from the Theorem 3.1 that U does not have a pair of interlacing bights. Hence we may concentrate only on the case when U is a feasible augmented graph with a single (feasible) proper block of B^{PQ}-bridge, $\mathbf{B}$.

Theorem 5.14 *Let U be a feasible augmented graph with a single (feasible) proper block of B^{PQ}-bridges, $\mathbf{B}$. Notice that since U is feasible, $\mathbf{B}$ is not a block of equivalent 3-bridges. Then U has a PQ-, P-, Q- or an ST-cross-cut pair.* ☐

Lemma 5.15 *Let U and $\mathbf{B}$ be as in the Theorem 5.14. If*

1. *U has an external vertex of attachment, c, on $P]s_P;t_P[$ and $\mathbf{B}$ has more than one distinct vertices of attachment on $P]s;t[$, or*
2. *U has an external vertex of attachment, c, on $Q]s_Q;t_Q[$ and if $\mathbf{B}$ has more than one distinct vertices of attachment on $Q]s;t[$*

then U has PQ-, P-, Q- or an ST-cross-cut pair.

PROOF.
Assume to the contrary, i.e. U satisfies one of the conditions of the lemma, but does not have P-, Q-, PQ- or ST-cross-cuts.

- CASE.1.

First assume that $c \in P]s_P^*;t_P^*[$ (where $s_P^* \neq t_P^*$). Let us partition the bridges of $\mathbf{B}$ in to the following four classes:

1. Class.1 = $\{B \in \mathbf{B} \mid B$ has *at least one* vertex of attachment on $P]s;c[$ and *at least one* vertex of attachment on $P]c;t[.\}$
2. Class.2 = $\{B \in \mathbf{B} \mid B$ has *at least one* vertex of attachment on $P]s;c[$ and *no* vertex of attachment on $P]c;t[.\}$
3. Class.3 = $\{B \in \mathbf{B} \mid B$ has *no* vertex of attachment on $P]s;c[$ and *at least one* vertex of attachment on $P]c;t[.\}$
4. Class.4 = $\mathbf{B} \setminus$ (Class.1 $\cup$ Class.2 $\cup$ Class.3) = $\{B \in \mathbf{B} \mid B$ has *exactly one* vertex of attachment on $P]s;t[$ at $c.\}$

by the Claim.2. A_1 must be a bight of $\overline{U}'$. Hence if the path associated with A_1 contains a vertex of $V(H)$ it must lie on $R[s_Q;t_Q]$.

(2) Assume that the path associated with A_1 contains exactly one vertex $x \in V(H)$, where x lies on $R]s_Q;t_Q[$. Hence x must be $v_{\rm cut}$. Hence the path associated with A_2 does not contain $v_{\rm cut}$, and the end segment of A_2 starting at b_1, $A_2'[b_1;y_1]$ must terminate in s_Q or t_Q. In either case $\overline{U}'$ has a pair of interlacing bights.

(3) Hence if the path associated with A_1 contains a vertex $x \in V(H)$ then it must be one of s_Q or t_Q (say, s_Q). Let the end segment of A_2 starting with b_1, $A_2'[b_1;y_1]$ terminate in y_1 where y_1 lies on the subpath $Q]s_Q;t_Q[$. Hence $A_2'[b_1;y_1]$ must contain the $v_{\rm cut}$.

Since R is a path in a proper bridge, R must have an internal vertex, c. Let L be a path in B connecting the vertices c and $v_{\rm cut}$. An appropriate subpath of L meets R only in c' and $A_2'[b_1;y_1]$ only in y' and avoids the path associated with the bight A_1. (Follows from reducibility of B.) Hence the path $A_2'[b_1;y'] * L[y';c'] * R[c';t_Q]$ defines a bight of $\overline{U}'$ between b_1 and t_Q that interlaces with the bight A_1.

On the other hand, if the end segment of $A_2'[b_1;y_1]$ terminates in a vertex on the subpath $R[s_Q;t_Q]$ it is easy to show, by an argument as above, that $\overline{U}'$ has a pair of interlacing bights. In either case, it results in a contradiction. ☐

Theorem 5.13 *Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q;t_Q\rangle$. Let $R[s_Q;t_Q]$ be a path in B, connecting the vertices s_Q and t_Q, such that B is irreducible with respect to R. Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ be as in the definition 4.5. Assume that $|\mathcal{B}_3| = 0$; and the path R divides U into U', a U-Fragment on Q, and $\overline{U}'$ a $\overline{U}$-Fragment on Q, as in the definition 4.5.*

Then U does not have a pair of interlacing bights, if neither U' nor $\overline{U}'$ has a pair of interlacing bights.

PROOF.
In the following we use some simple ideas from 'point set topology.' (Cf. [5], [15] and [17].) Let π be a plane and $J(\pi)$ be *closed Jordan curve* in π: the closed Jordan curve $J(\pi)$ divides the rest of the plane into two connected open *residual domains*, the *interior domain* (INT J) and the *exterior domain* (EXT J).

Assume that a_1, a_2, b_1 and b_2 are four distinct vertices of J such that there is a pair of interlacing bights, A_1 between a_1 and a_2, and A_2 between b_1 and b_2—here, a_1, a_2, b_1 and b_2 are either external vertices of attachment or one of s and t. Let A_1' and A_2' be the paths associated with the bights A_1 and A_2, respectively. We derive a contradiction.

Let $J = P[s;t] \cup Q[s;t]$, $J_1 = P[s;t] \cup Q[s;s_Q] * R[s_Q;t_Q] * Q[t_Q;t]$ and $J_2 = Q[s_Q;t_Q] \cup R[s_Q;t_Q]$.

Let $J(\pi)$ be a closed Jordan curve in the plane π. The vertices on J are mapped onto a point set in $J(\pi)$ such that the mapping preserves their order in J. Let $s_Q(\pi)$ and $t_Q(\pi)$ be the points on $J(\pi)$ corresponding to the vertices s_Q and t_Q of J, respectively. There exists a Jordan curve in the plane π with its end points being $s_Q(\pi)$ and $t_Q(\pi)$ and with all other vertices of it in INT J; let $R(\pi)$ be such a Jordan curve. The vertices on R are mapped onto a point set in $R(\pi)$ such that the mapping preserves their order in R with respect to s_Q and t_Q. As a result the cycles J_1 and J_2 are mapped to the closed Jordan curves $J_1(\pi)$ and $J_2(\pi)$, uniquely defined by $R(\pi)$. Let the *faces* F_1 and F_2 be the residual domains INT J_1 and INT J_2, respectively.

Let $A'[a_1;a_2]$ be the path associated with a bight between a_1 and a_2 of J in U. Such a path can always be written as the concatenation of common sections on R, common sections on J, and bights between vertices of J_1 in $\overline{U}'$ or bights between vertices of J_2 in U'. By the assumption, it is impossible for two bights of J_1 in $\overline{U}'$ or two bights of J_2 in U', to interlace. Moreover, since A' is a simple path all its common sections are vertex-disjoint.

We claim that A' can be mapped to a Jordan curve $A'(\pi)$ with its end points being $a_1(\pi)$ and $a_2(\pi)$ such that none of the points on it lie in EXT J. This can be done by successively mapping each of its

Let A be a bight of U between two vertices a_1 and a_2, where a_1 and a_2 are external vertices of attachment of U or one of s and t. Let A' be the path associated with the bight A. If A' contains one or more vertices of $V(H)$ then the subpath of A', $A'[a_1;x_1] = u_0(= a_1)$, u_1, …, u_{n-1}, $u_n(= x_1)$, where $u_0, \ldots, u_{n-1} \notin V(H)$ and $u_n \in V(H)$, is called the *end segment* of A starting at a_1. The end segment of A starting at a_2 is defined similarly.

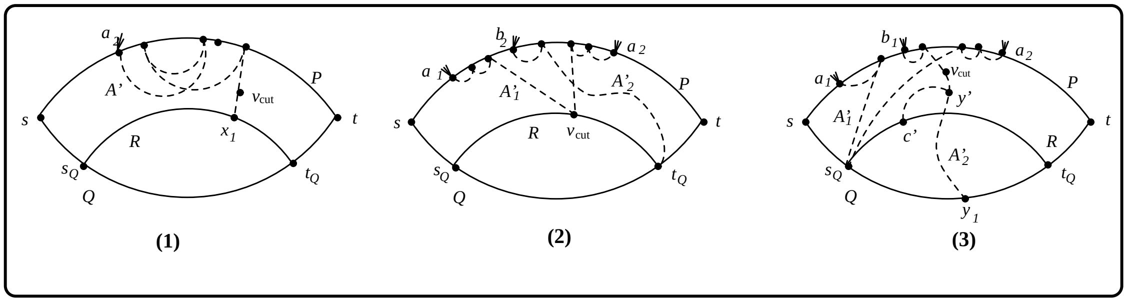


Figure 23: *Three Cases for Lemma 5.12.*

Claim.1. If the path associated with a bight A of U contains at least two distinct vertices of $V(H)$ then it contains at least two out of three vertices, v_{cut}, s_Q and t_Q.

PROOF OF THE CLAIM.1. Let A be a bight of U between two vertices a_1 and a_2. Let $A'[a_1;x_1]$ and $A'[a_2;x_2]$ be the end segments of A starting with a_1 and a_2, respectively. Since the path associated with A, A' contains two or more distinct vertices of $V(H)$, x_1 and x_2 are distinct and $A'[a_1;x_1]$ and $A'[a_2;x_2]$ are vertex-disjoint.

Notice that if x_1 is distinct from s_Q and t_Q then $A'[a_1;x_1]$ contains v_{cut}. This is due to following: Since $x_1 \in V(H)$ it lies on the path $R]s_Q;t_Q[$ or $Q]s_Q;t_Q[$. Since none of the other vertices of $A'[a_1;x_1]$ is in $V(H)$, there is a subpath of $A'[a_1;x_1]$ that connects a vertex of $P]s;t[$ with x_1, and otherwise avoids P, Q and R. But this subpath must contain v_{cut}. Similarly, if x_2 is distinct from s_Q and t_Q then $A'[a_2;x_2]$ contains v_{cut}.

The claim follows from the observation that $A'[a_1;x_1]$ and $A'[a_2;x_2]$ are vertex-disjoint, and hence, both cannot contain v_{cut}.

Claim.2. Let A be a bight of U. Then either A is also a bight of $\overline{U}'$, or the path associated with it contains at least two distinct vertices of $V(H)$.

PROOF OF THE CLAIM.2. Assume to the contrary, i.e., there is a bight A of U between a_1 and a_2 such that the path associated with it contains no more than one vertex of $V(H)$, but A is not a bight of $\overline{U}'$.

This is possible only if A contains only one vertex $x \in V(H)$ such that x lies on $Q]s_Q;t_Q[$. But, this implies that the end segments $A'[a_1;x]$ and $A'[a_2;x]$ both contain v_{cut}. But since $v_{\text{cut}} \in N(B)$ it is distinct from x, and this contradicts the fact that A', the path associated with the bight A, is simple.

Assume that a_1, a_2, b_1 and b_2 are four distinct vertices on J of U such that there is a pair of interlacing bights A_1 between a_1 and a_2, and A_2 between b_1 and b_2, where a_1, a_2, b_1 and b_2 are either external vertices of U or one of s and t.

Since $\overline{U}'$ does not have a pair of interlacing bights at least one of A_1 and A_2 (say, A_2) is not a bight of $\overline{U}'$. Let a_1 precede a_2 in the clockwise cyclic order of the vertices of J. Since a_1 and a_2 separate b_1 and b_2 on J, one of b_1 and b_2 (say b_1) lies on $J]a_1;a_2[$.

(1) We may assume that A_1 contains no more than one vertex of $V(H)$, since, otherwise, by the previous claims each of the paths associated with the bights A_1 and A_2 must contain at least two out of three vertices, v_{cut}, s_Q and t_Q, which is impossible, since these two paths are vertex-disjoint. Moreover,

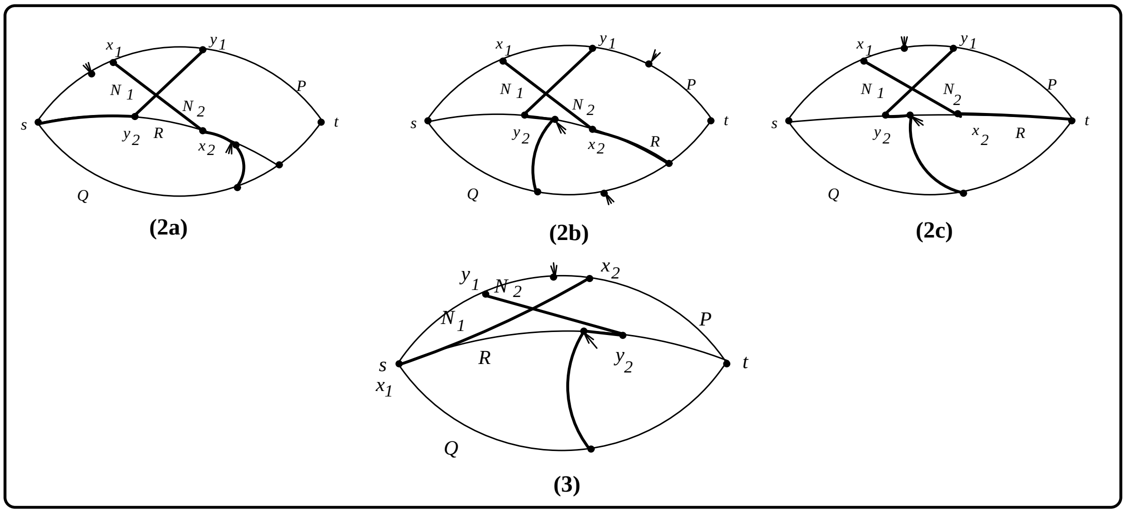


Figure 22: *Cases 2 and 3 of Lemma 5.11.*

(2) *$\overline{U}'$ has a PQ-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1 is to the left of y_1 on P.* Since U must be PQ- or P-feasible, $\left|\mathrm{UA}(\overline{U}')\right| > 0$ and since B is irreducible with respect to R, $\left|\mathrm{LA}(\overline{U}')\right| > 0$.

If both s_Q and t_Q are distinct from s and t then, clearly, U has weak PQ-cross-cuts. Hence assume that $s_Q = s$. If $\overline{U}'$ has a lower external vertex of attachment on $R]y_2;t_Q]$ and an upper external vertex of attachment on $P]s;y_1[$ then U has P-cross-cuts. If not, $\overline{U}'$ has a lower external vertex of attachment on $R[s;x_2[$ and an upper external vertex of attachment on $P]x_1;t[$. If t_Q is distinct from t then U has weak PQ-cross-cuts. On the other hand, if $t_Q = t$ then U has P-cross-cuts.

(3) *$\overline{U}'$ has a P-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in P[s;t]$, $y_1 \in P]x_1;x_2[$ and $y_2 \in R[s_Q;t_Q]$.*

If s_Q is distinct from s (or, symmetrically, t_Q distinct from t) then, since N_1 avoids $R]s_Q;t_Q]$, U has P-cross-cuts. If $s_Q = s$ and $t_Q = t$ then N_1 avoids $R]s_Q;t_Q[$. Moreover, since B is irreducible, $\overline{U}'$ has a lower external vertex of attachment $b' \in R]s_Q;t_Q[$ and there is a path $R_b[b';b]$ in $\mathcal{B}_2$ such that R_b meets $R]s_Q;t_Q[$ only in b' and meets $Q]s_Q;t_Q[$ only in b. Clearly, the cross-cuts, N_1 and $N_2[y_1;y_2] * R[y_2;b'] * R_b[b';b]$, form P-cross-cuts in U. ☐

§B Interlacing Bights: Single Bridge

Theorem 5.12 *Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q;t_Q\rangle$. Let $R[s_Q;t_Q]$ be a path in B, connecting the vertices s_Q and t_Q, such that B is reducible with respect to R. Let $\overline{U}'$ be its $\overline{U}$-Fragment, as in the definition 4.5.*

Then U does not have a pair of interlacing bights, if $\overline{U}'$ has no pair of interlacing bights.

PROOF.
First, we prove two useful claims. Since B is reducible with respect to R, there is a cut vertex, v_{cut}, in the nucleus of the bridge B, $N(B)$. Let H be the section graph induced by the edges of the subpaths $Q[s_Q;t_Q]$ and $R[s_Q;t_Q]$, *i.e.*,

$$H = \langle E(Q[s_Q;t_Q]) \cup E(R[s_Q;t_Q])\rangle.$$

PROOF.
(1) *$\overline{U}'$ has a Q- or an ST-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in R[s_Q;t_Q]$ and $y_1 \in P]s;t[$ and $y_2 \in R]x_1;x_2[$.*
By definition of Q- and ST-cross-cuts, there is a lower external vertex of attachment $c' \in R]x_1;x_2[$ and hence a path $R_c[c';c]$ in the U-Fragment U' such that R_c meets $R]s_Q;t_Q[$ only in c' and meets $Q]s_Q;t_Q[$ only in c.

Let us refer to the path $R[s_Q;x_1] * N_1[x_1;x_2] * R[x_2;t_Q]$ by M_1 and the path $R]x_1;x_2[$ by M_2.

By lemma 5.7, it suffices to show that U has a Y-Q pair. Since B is irreducible with respect to R, and since $|\mathcal{B}_3| = 0$, there are two vertex disjoint paths $R_a[a';a]$ and $R_b[b';b]$ where a', $b' \in R]s_Q;t_Q[$ and a, $b \in P]s;t[$. Let R'_a and R'_b be the appropriate subpaths of the disjoint paths R_a and R_b, respectively, such that R'_a and R'_b meet $P]s;t[$ only in a and b, respectively; R'_a meets only one of M_1 and M_2 only in a'' and R'_b meets only one of M_1 and M_2 only in b'', where a'' and $b'' \in V(M_1) \cup V(M_2) \setminus \{s_Q,t_Q\}$.

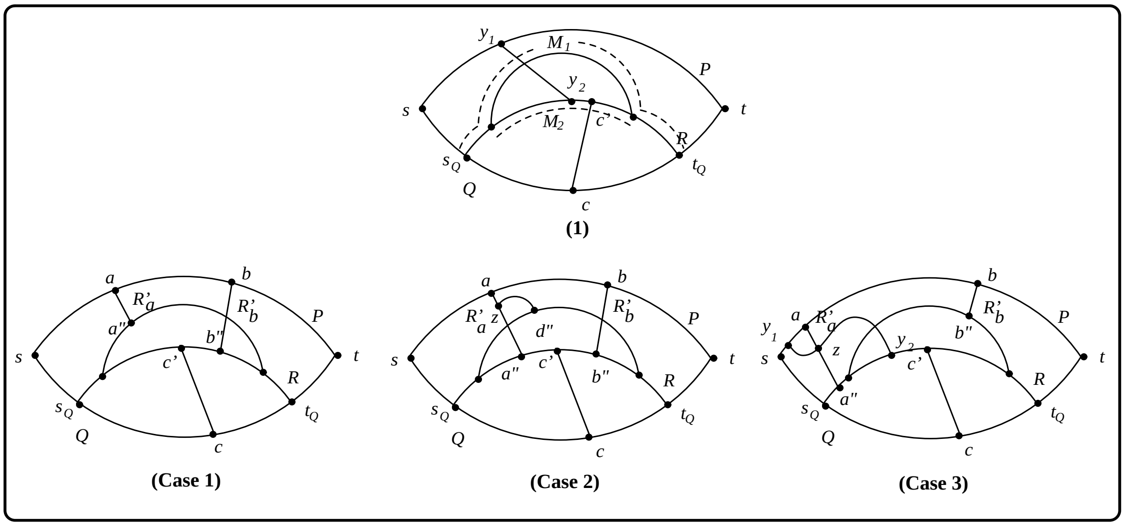


Figure 21: *Case 1 of Lemma 5.11.*

- <u>CASE.1.</u>
  $a'' \in V(M_1) \setminus \{s_Q,t_Q\}$ and $b'' \in V(M_2)$.

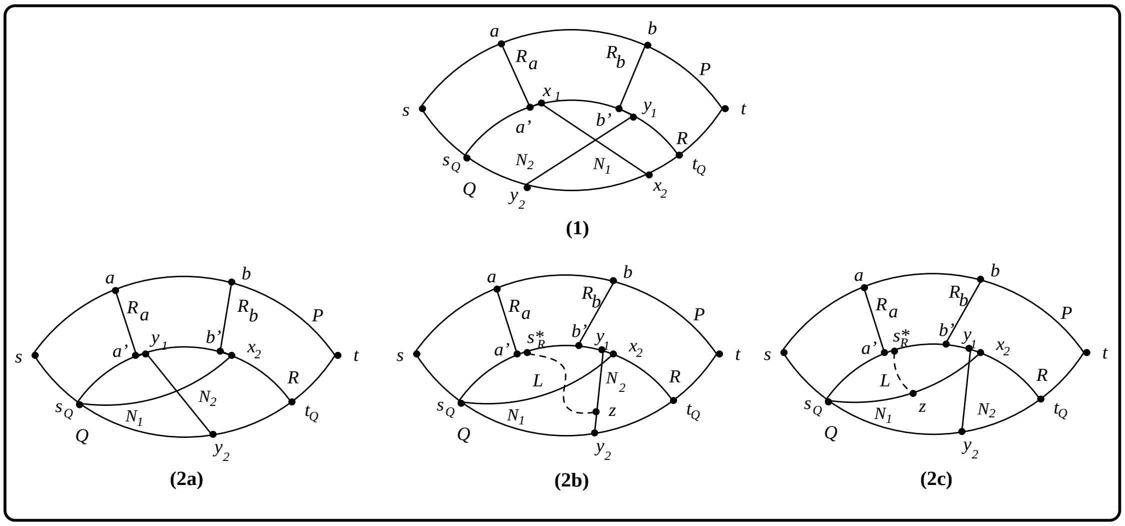
Then the cross-cut $N = R'_b[b;b''] * M_2[b'';c'] * R_c[c';c]$ and the Y-graph $Y = \{M_1[s_Q;a'']\} \cup \{R_a[a;a'']\} \cup \{M_1[a'';s_Q]\}$ form a Y-Q pair.

- <u>CASE.2.</u> *a'' and $b'' \in V(M_2)$.*

Since N_1 is a path in some $\mathcal{B}_1$ bridge, say B_1, B_1 is a proper bridge and N_1 has an internal vertex, d'. Let d be a vertex of attachment of B_1 on P. There is a path in B_1 whose ends are d and d' and which avoids P. A suitable subpath, $R'_d[d;d'']$, of it meets N_1 only in d'', an internal vertex of N_1, meets $P]s;t[$ in d and otherwise avoids P, R and N_1. If R'_d is vertex-disjoint with either of R'_a and R'_b, then this case reduces to an instance of the previous case. Suppose not. Then a suitable subpath of $R'_d[d;d'']$ meets N_1 only in d'', meets only one of R'_a and R'_b only in z and avoids the other. Without loss of generality assume that $z \in R'_a[a;a''[$. Then the pair of paths $R'_d[d'';z] * R'_a[z;a]$ and R'_b satisfies the condition of the previous case.

- <u>CASE.3.</u> *a'' and $b'' \in V(M_1) \setminus \{s_Q;t_Q\}$.*

If $N_2[y_1;y_2]$ is vertex-disjoint with either of R'_a and R'_b, then this case reduces to an instance of the first case. Suppose not. Then a suitable subpath of N_2 meets M_2 only in y_1, meets only one of R'_a and R'_b only in z and avoids the other. Without loss of generality assume that $z \in R'_a[a;a''[$. Then the pair of paths $N_2[y_1;z] * R'_a[z;a]$ and R'_b satisfies the condition of the first case.



(1)

(2a) (2b) (2c)

Figure 20: *Two Cases for Lemma 5.10.*

(1) *U' has a PQ-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$.*
As a result of the fact that N_1 and N_2 form PQ-cross-cuts and the above claim, we may assume that $a' \in R]s_Q;y_1[$ and $b' \in R]x_1;t_Q[$. Then the cross-cut $N = N_2[y_2;y_1] * R[y_1;b'] * R_b[b';b]$ and the Y-Graph, $Y = \{R[s_Q;x_1]\} \cup \{R_a[a;a'] * R[a';x_1]\} \cup \{N_1[x_1;x_2]\}$ (if $a' \in R]x_1;y_1[$), or $Y = \{R[s_Q;a']\} \cup \{R_a[a;a']\} \cup \{R[a';x_1] * N_1[x_1;x_2]\}$ (if $a' \in R]s_Q;x_1]$), form a *Y-Q* pair.

(2) *U' has P-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in R[s_Q;t_Q]$.*
As a result of the fact that N_1 and N_2 form P-cross-cuts and the above claim, we may assume that at least one of a' and $b' \in R]x_1;x_2[$, say a'.

(2a) Assume that $b' \in R]y_1;t_Q[$ and x_2 is distinct from t_Q. Then the cross-cut $N = N_2[y_2;y_1] * R[y_1;a'] * R_a[a';a]$ and the Y-graph $Y = \{R[s_Q;x_1] * N_1[x_1;x_2] * R[x_2;b']\} \cup \{R_b[b';b]\} \cup \{R[b';t_Q]\}$ (if $b' \in R[x_2;t_Q[$), or $Y = \{R[s_Q;x_1] * N_1[x_1;x_2]\} \cup \{R_b[b;b'] * R[b';x_2]\} \cup \{R[t_Q;x_2]\}$ (if $b' \in R]y_1;x_2[$), form *Y-Q* pair. Similarly, if $b' \in R]s_Q;y_1[$ and x_1 is distinct from s_Q then U has a *Y-Q* pair.

(2b) From (2a), we see that if both x_1 and x_2 are both distinct from s_Q and t_Q then U has *Y-Q* cross-cuts.

(2c) Henceforth, assume that $x_1 = s_Q$ and hence x_2 is distinct from t_Q. Hence $b' \in R]s_Q;y_1]$, and since $b' \in R]x_1;x_2[$, also $a' \in R]s_Q;y_1]$. Moreover, if U' has an external vertex of attachment on $R]y_1;t_Q[$ then using the claim above we can show that U has a *Y-Q* pair. Hence all the external vertices of attachment of U' lie on $R]s_Q;y_1]$.

Let s^*_R be the left-most vertex of attachment of U' on R distinct from s. Hence, there is a bridge $B' \in \mathcal{B}_2$ with a vertex of attachment s^*_R on $R]s_Q;t_Q[$ and a vertex of attachment, y, on $Q]s_Q;t_Q[$. Let $L[s^*_R;y]$ be a cross-cut of U' between s^*_R and y. Since U' is feasible, not all the external vertex of attachment U' lie on $R[s_Q;s^*_R]$. If L avoids both N_1 and N_2 then the P-cross-cuts defined by N_1 and L, together with an application of the above claim, satisfies (2a).

Hence, assume that L does not avoid both N_1 and N_2. Then there is a vertex $z \in L]s^*_R;y]$ such that $L]s^*_R;z]$ meets N_1 or N_2 (but not both) in z. If L meets N_2 in z then the P-cross-cuts N_1 and $L[s^*_R;z] * N_2[z;y_2]$, together with an application of the above claim, satisfies (2a). On the other hand, if L meets N_1 in z then the P-cross-cuts $L[s^*_R;z] * N_1[z;x_2]$ and N_2 satisfies (2b). □

Lemma 5.11 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If $\overline{U}'$ contains a P-, Q-, PQ- or ST-cross-cut pair then U has a P- or a PQ-cross-cut pair.*

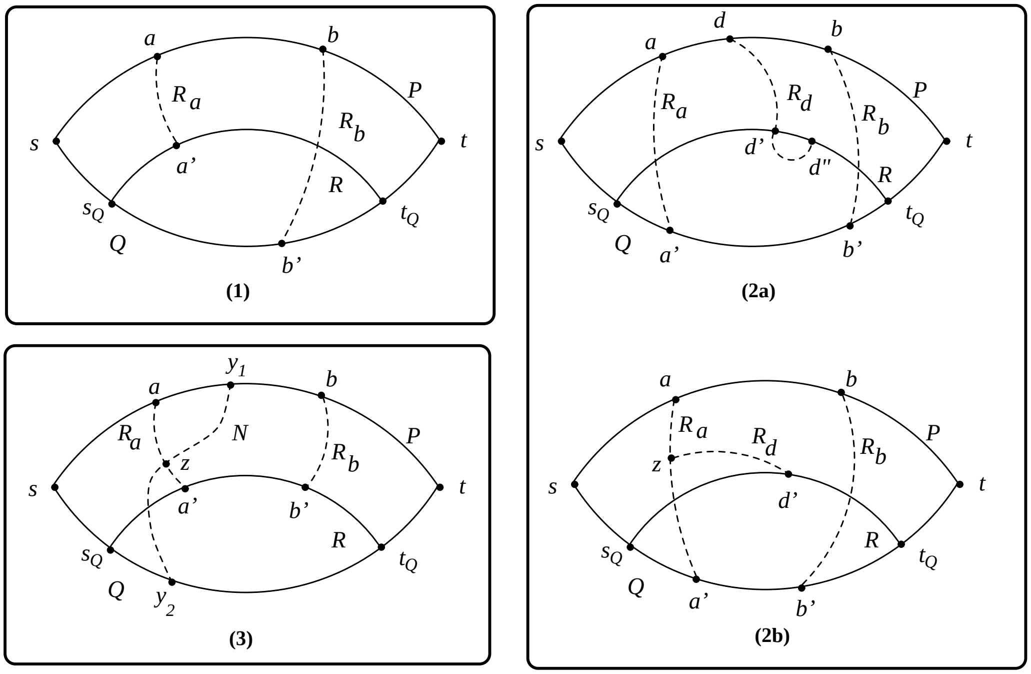


Figure 19: *Three Cases for Lemma 5.9.*

(3) a', $b' \in R]s_Q;t_Q[$.

Assume that $N[y_1;y_2]$ meets both R_a and R_b, since, otherwise, we can easily find a Y-Q-pair. Hence a suitable subpath of N meets Q only in y_2, meets only one of R_a and R_b in z, and avoids the other. Without loss of generality assume that $z \in R_a[a;a'[$. Then $R_a[a;z]{*}N[z;y_2]$ and $\{R_b[b;b']\} \cup \{R[s_Q;b']\} \cup \{R[b';t_Q]\}$ form a Y-Q-pair. $\square$

Lemma 5.10 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If U' contains a P-, Q-, PQ- or ST-cross-cut pair then U has a P-, Q-, PQ- or an ST-cross-cut pair.*

PROOF.

Since B is irreducible with respect to R, $|\mathrm{UA}(U')| > 0$. Moreover, since U' must be P-feasible, U' cannot have Q- or ST-cross-cuts.

Hence it suffices to show that if U' has PQ- or P-cross-cuts, then U has weak PQ-cross-cuts or Y-Q-cross-cuts. (Cf. Lemma 5.7.) Since B is irreducible with respect to R, and since $|\mathcal{B}_3| = 0$, there are two vertex disjoint paths $R_a[a';a]$ and $R_b[b';b]$ where a', $b' \in R]s_Q;t_Q[$ and a, $b \in P]s;t[$. (Cf. Lemma 4.3.) Without loss of generality, we may assume that a' is to the left of b'. The following claim will be found useful in the rest of the proof:

Claim Let c' be an arbitrary upper external vertex of attachment of U' distinct from a' and b'. Then there exists a path R_c in $\mathcal{B}_1$, joining c' to an internal vertex of P such that R_c avoids R_a or R_b or both.

PROOF OF THE CLAIM. Let $B' \in \mathcal{B}_1$ be a bridge such that c' is a vertex of attachment of B' on R and let c be a vertex of attachment of B' on $P]s;t[$. Then there is a path $R_c[c';c]$ in $\mathcal{B}_1$. If R_c avoids R_a or R_b then it satisfies the claim.

Hence, let R_c meet one or both of the paths R_a and R_b. It is possible to find a vertex $z \in R_c]c';c]$ such that $R_c]c';z]$ meets $R_a]a';a]$ or $R_b]b';b]$ (but not both) only in z. (This follows from the facts that R_a, R_b are vertex disjoint and a',b' and c' are distinct.) Without loss of generality, assume that $z \in R_a]a';a]$; then the path $R_c[c';z] * R_a[z;a]$ avoids R_b.

PQ-cross-cuts of $\overline{U}'$, it has an upper external vertex of attachment on $P]s;y_1[$ and U has P-cross-cuts. The case, when $t_Q = t$ and s_Q is distinct from s, is similar.

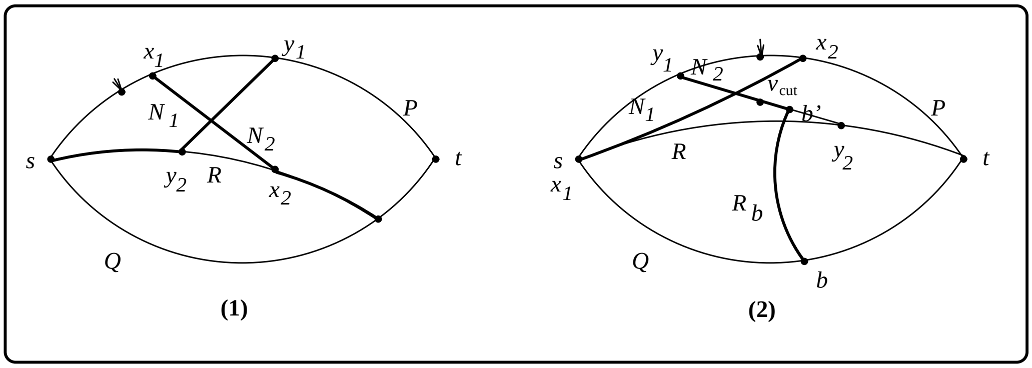


Figure 18: *Two Cases for Lemma 5.8.*

(2) *$\overline{U}'$ has a P-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in P[s;t]$, $y_1 \in P]x_1;x_2[$ and $y_2 \in R[s_Q;t_Q]$.*

If s_Q is distinct from s (or, symmetrically, t_Q distinct from t) then, since N_1 avoids $R]s_Q;t_Q]$, U has P-cross-cuts. If, on the other hand, $s_Q = s$ and $t_Q = t$ then, since B is reducible and since $y_2 \in R]s_Q;t_Q[$, the path N_2 must contain the cut vertex v_{cut} of $N(B)$. Let b be a vertex of attachment of B on $Q]s;t[= Q]s_Q;t_Q[$, say b. Let $R_b[b;v_{\text{cut}}]$ be a path in B joining b and v_{cut}; and let $R_b[b;b']$ be a suitable subpath of R_b such that R_b meets $Q]s_Q;t_Q[$ only in b and meets N_1 or N_2 (but not both) only in b'. If $R_b[b;b']$ meets N_1 in b' then there is a path from b to x_1 and to x_2 that does not contain the cut vertex, v_{cut}, where one of x_1 and $x_2 \in P]s;t[$—thus contradicting the assumption that B is reducible with respect to R. Hence $R_b[b;b']$ meets N_2 in b' and the cross-cuts N_1 and $N_2[y_1;b'] * R_b[b';b]$ form P-cross-cuts. ☐

Lemma 5.9 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If $|\mathcal{B}_3| \neq 0$ then U has a P-, Q-, PQ- or an ST-cross-cut pair.*

PROOF.

If $|\mathcal{B}_3| \neq 0$ then there is a cross-cut $N[y_1;y_2]$ of J between $y_1 \in P]s;t[$ and $y_2 \in Q]s_Q;t_Q[$ such that N avoids R.

We will show that U has a Y-Q-Pair (Cf. Lemma 5.7.) Since B is irreducible, there are two vertex disjoint paths R_a and R_b.

(1) *$a' \in Q]s_Q;t_Q[$ and $b' \in R]s_Q;t_Q[$.*

Then the cross-cut R_a and the Y-Graph $\{R_b[b;b']\} \cup \{R[s_Q;b']\} \cup \{R[t_Q;b']\}$ form a Y-Q pair.

(2) *a', $b' \in Q]s_Q;t_Q[$.*

Since R belongs to a proper bridge, B, R has an internal vertex d''. Let d be a vertex of attachment of B on $P]s;t[$. Then there is a path in B that joins d and d'', and a suitable subpath, $R_d[d;d']$, of it meets $R]s_Q;t_Q[$ only in d', and meets $P]s;t[$ only in d. Assume that R_d meets both R_a and R_b, since, otherwise, we can easily find a Y-Q pair. Hence, a suitable subpath of R_d meets $R]s_Q;t_Q[$ only in d', meets only one of R_a and R_b only in z, and avoids the other. Without loss of generality assume that $z \in R_a[a;a'[$, then the cross-cut R_b and the Y-graph $\{R_a[a;z] * R_d[z;d']\} \cup \{R[s_Q;d']\} \cup \{R[t_Q;d']\}$ form a Y-Q pair.

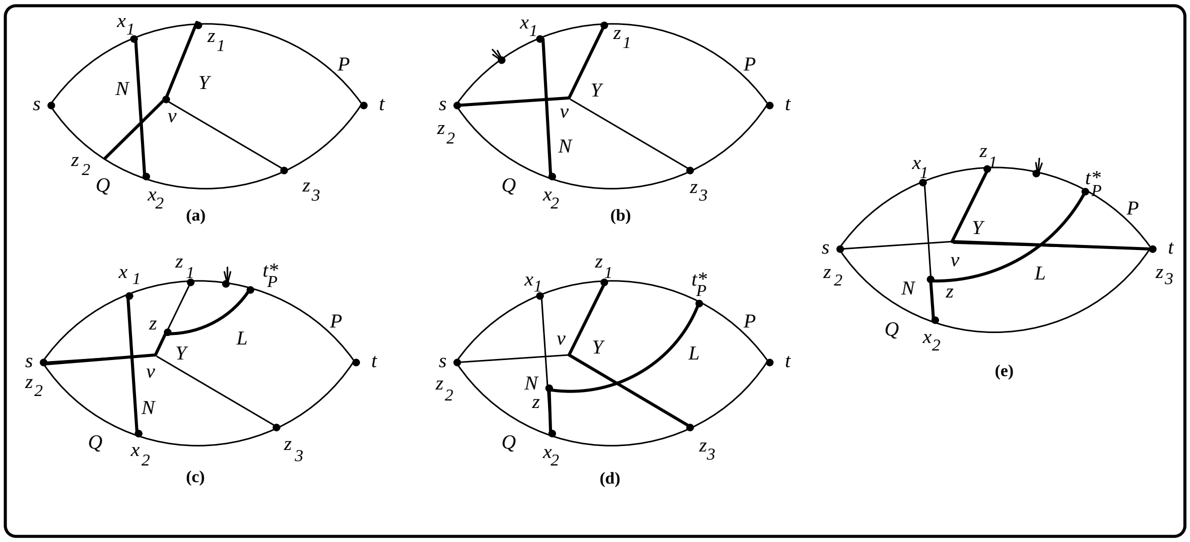


Figure 17: *Modifying a Y-Q Pair (Lemma 5.7).*

(2) *U has a Y-Q Pair, $N[x_1;x_2]$ and $Y=\{Y_1[z_1;v]\}\cup\{Y_2[z_2;v]\}\cup\{Y_3[z_3;v]\}$.*
Without loss of generality assume that x_1 is to the left of z_1. If z_2 is distinct from s then the cross-cuts N_1 and $Y_1[z_1;v]*Y_2[v;z_2]$ form weak PQ-cross-cuts, in which case the proof proceeds as in (1). However, if $z_2=s$ and if there is an external vertex of attachment on $P]s;z_1[$ then the vertex-disjoint cross-cuts N_1 and $Y_1[s;v]*Y_2[v;z_1]$ form P-cross-cuts.

Hence assume that $z_2=s$ and since B is P-feasible, all the external vertices of attachment lie on $J[z_1;t_Q]$. But since U is feasible, it has at least two distinct external vertices of attachment and not all its external vertices of attachment lie on $J[t^*_P;t_Q]$. This implies that U has an external vertex of attachment on $J[z_1;t^*_P[$ and an external vertex of attachment on $J]z_1;t_Q]$.

Let L be a path in B that joins t^*_P and v and avoids J. It is possible to find a vertex $z\in L]t^*_P;v]$ such that $L[t^*_P;z]$ meets N or Y (but not both) only in z, an internal vertex.

If z is an internal vertex of Y then the vertex-disjoint cross-cuts N and $Y[s;z]*L[z;t^*_P]$ form P-cross-cuts. On the other hand, if z is an internal vertex of N and if z_3 is distinct from t, then the cross-cuts $L[t^*_P;z]*N[z;x_2]$ and $Y_1[z_1;v]*Y_3[v;z_3]$ form weak PQ-cross-cuts; and the proof proceeds as in (1).

Hence, consider the case when z is an internal vertex of N and $z_3=t$. Observe that in this case $t_Q=t$ and U has an external vertex of attachment on $P]z_1;t[$. Clearly, the cross-cuts $L[t^*_P;z]*N[z;x_2]$ and $Y_1[z_1;v]*Y_3[v;t]$ form P-cross-cuts. □

Lemma 5.8 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.2. If $\overline{U}'$ contains a P-, Q-, PQ- or ST-cross-cut pair then U has a P-, Q-, PQ- or an ST-cross-cut pair.*
PROOF.
Notice that since B is reducible $\overline{U}'$ has no lower external vertex of attachment on $R]s_Q;t_Q[$ and hence no Q- or ST-cross-cuts.

(1) *$\overline{U}'$ has a PQ-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1 is to the left of y_1 on P.*
If both s_Q and t_Q are distinct from s and t then U has weak PQ-cross-cuts.

Notice that it is not possible that $s_Q=s$ and $t_Q=t$, since, otherwise, N_1 and N_2 are two vertex-disjoint paths in B such that they meet $P]s;t[$ only in x_1 and y_1, respectively, and meet $R]s_Q;t_Q[$ only in x_2 and y_2 and B is not reducible with respect to R.

Hence assume that $s_Q=s$ and t_Q is distinct from t. Hence $\overline{U}'$ has a lower external vertex of attachment at t_Q and has no lower external vertex of attachment on $R]s;t_Q[$. Since N_1 and N_2 form

Definition 5.7 *Y-Q* Pair.

Let U be a U- or a $\overline{U}$-Fragment with a cycle $J = \{P\} \cup \{Q\}$. Let $Y = \{Y_1[z_1; v]\} \cup \{Y_2[z_2; v]\} \cup \{Y_3[z_3; v]\}$ be a Y-Graph of J and let $N[x_1; x_2]$ be a cross-cut of J, vertex-disjoint with Y. Such a pair of subgraphs is said to be a *Y-Q pair* of U, if it satisfies the following two conditions:

1. $z_1 \in P]s; t[$ and $z_2, z_3 \in Q[s; t]$.
2. $x_1 \in P]s; t[$, $x_2 \in Q]z_2; z_3[$.

Lemma 5.7 *Let U be a feasible augmented graph with a single P-feasible B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q; t_Q \rangle$.*

If U has a weak PQ-cross-cut pair or a Y-Q pair, then U is guaranteed to have a PQ- or a P-cross-cut pair.

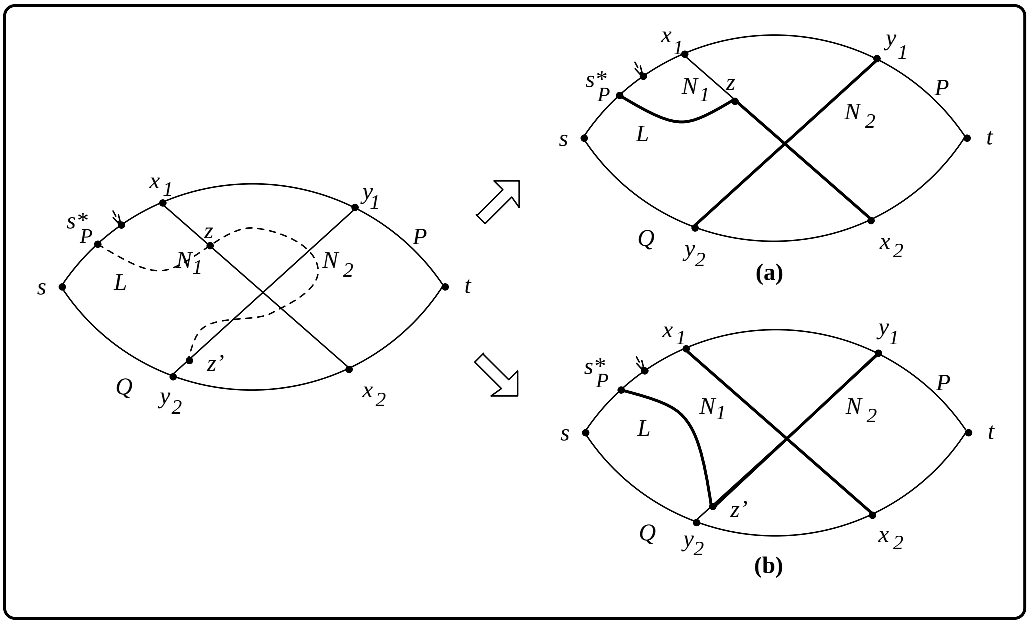


Figure 16: *Modifying a Weak PQ-Cross-Cut Pair (Lemma 5.7).*

PROOF.

(1) *U has a weak PQ-cross-cut pair, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$, where x_1 is to the left of y_1 on P.* Assume that N_1 and N_2 do *not* form PQ-cross-cuts, that is all the external vertices of attachment lie on $J[y_2; x_1]$ (the case when the external vertices of attachment lie on $J[y_1; x_2]$ is symmetric.) But since B is P-feasible, not all the external vertices of attachment of U lie on $J[s_Q; s^*_P]$. Hence U has an external vertex of attachment on $P]s^*_P; x_1]$. Since the cross-cuts N_1 and N_2 belong to B, B is a proper bridge and N_2 has an internal vertex, z'. Let L be a path in B that joins s^*_P and z' and avoids J. It is possible to find a vertex $z \in L]s^*_P; z']$ such that $L[s^*_P; z]$ meets N_1 or N_2 (but not both) only in z, an internal vertex of the appropriate cross-cut.

If z is an internal vertex of N_1 then the vertex-disjoint cross-cuts $L[s^*_P; z] * N_1[z; x_2]$ and N_2 form PQ-cross-cuts of U. If z is an internal vertex of N_2 then vertex-disjoint cross-cuts $L[s^*_P; z] * N_2[z; y_1]$ and N_1 form P-cross-cuts of U.

of attachment on $R[s;x_2[$ and an upper external vertex of attachment on $P]x_1;t[$. If t_Q is distinct from t then U has PQ-cross-cuts. On the other hand, if $t_Q = t$ then U has P-cross-cuts.

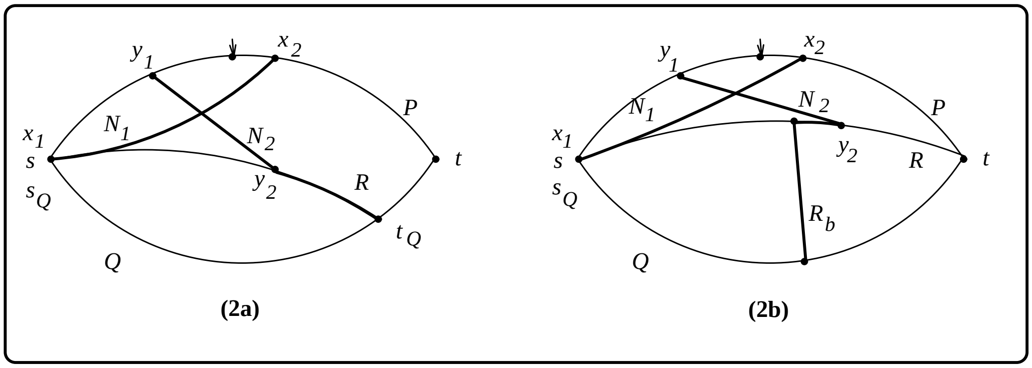
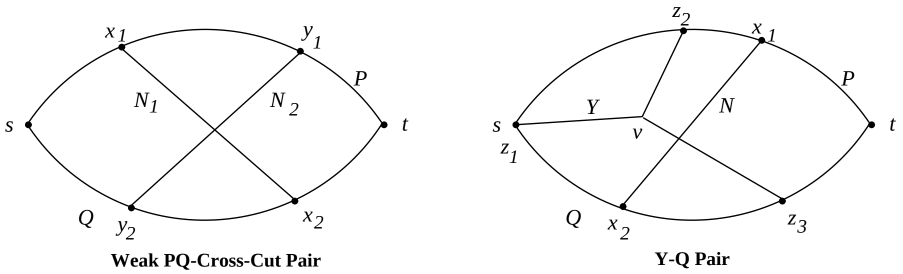


Figure 14: *Case(2) of Lemma 5.6.*

(2) *$\overline{U}'$ has a P-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in P[s;t]$, $y_1 \in P]x_1;x_2[$ and $y_2 \in R[s_Q;t_Q]$.*
If s_Q is distinct from s (or, symmetrically, t_Q distinct from t) then, since N_1 avoids $R]s_Q;t_Q]$, U has P-cross-cuts. If $s_Q = s$ and $t_Q = t$ then N_1 avoids $R]s_Q;t_Q[$. Moreover, since B is irreducible, $\overline{U}'$ has a lower external vertex of attachment $b' \in R]s_Q;t_Q[$ (by definition) and there is a path $R_b[b';b]$ in $\mathcal{B}_2$ such that R_b meets $R]s_Q;t_Q[$ only in b' and meets $Q]s_Q;t_Q[$ only in b. Clearly, the cross-cuts, N_1 and $N_2[y_1;y_2] * R[y_2;b'] * R_b[b';b]$, form P-cross-cuts in U. ☐

§A.2 Case(2): B is P-feasible.

Before presenting the proofs for this case, we present the following definitions and a technical Lemma.



Weak PQ-Cross-Cut Pair **Y-Q Pair**

Figure 15: *Weak PQ-Cross-Cut Pair and Y-Q Pair.*

Definition 5.6 WEAK PQ-CROSS-CUT PAIR.
Let U be a U- or a $\overline{U}$-Fragment with a cycle $J = \{P\} \cup \{Q\}$. A pair of interlacing vertex-disjoint cross-cuts $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$ is said to be a *weak PQ-cross-cut pair* of U, if x_1, $y_1 \in P]s;t[$ and x_2, $y_2 \in Q]s;t[$. ☐

(2) *U' has an ST-cross-cut pair, $N_1[s_Q;t_Q]$ and $N_2[y_1;y_2]$.*
Let a' be an upper external vertex of attachment and b' be a lower external vertex of attachment of U'; and let R_a be as above. Then the paths $L_1 = N_1[s_Q;t_Q]$ and $L_2 = R_a[a;a'] *R[a';y_1] *N_2[y_1;y_2]$ form either Q-cross-cuts or ST-cross-cuts of U.

(3) *U' has a P-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in R[s_Q;t_Q]$.*
Let $a' \in R]x_1;x_2[$ be an upper external vertex of attachment and b' be any lower external vertex of attachment and let R_a be as above. The paths $L_1 = R[s_Q;x_1] *N_1[x_1;x_2] *R[x_2;t_Q]$ and $L_2 = R_a[a;a'] *R[a';y_1] *N_2[y_1;y_2]$ form either Q-cross-cuts or ST-cross-cuts of U.

(4) *U' has a Q-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1, $x_2 \in Q[s_Q;t_Q]$.*
Let a' be any upper external vertex of attachment and $b' \in Q]x_1;x_2[$ be a lower external vertex of attachment and let R_a be as above. The paths $L_1 = N_1[x_1;x_2]$ and $L_2 = R_a[a;a'] * R[a';y_1] * N_2[y_1;y_2]$ form Q-cross-cuts of U. ☐

Lemma 5.6 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If $\overline{U}'$ contains a P-, Q-, PQ- or an ST-cross-cut pair then U has a P-, Q-, PQ- or an ST-cross-cut pair.*
PROOF.
Since B is PQ-feasible, it is easy to see that if $\overline{U}'$ has Q- or ST-cross-cuts, then U has Q- or ST-cross-cuts. Hence, we need to consider the remaining two cases only.

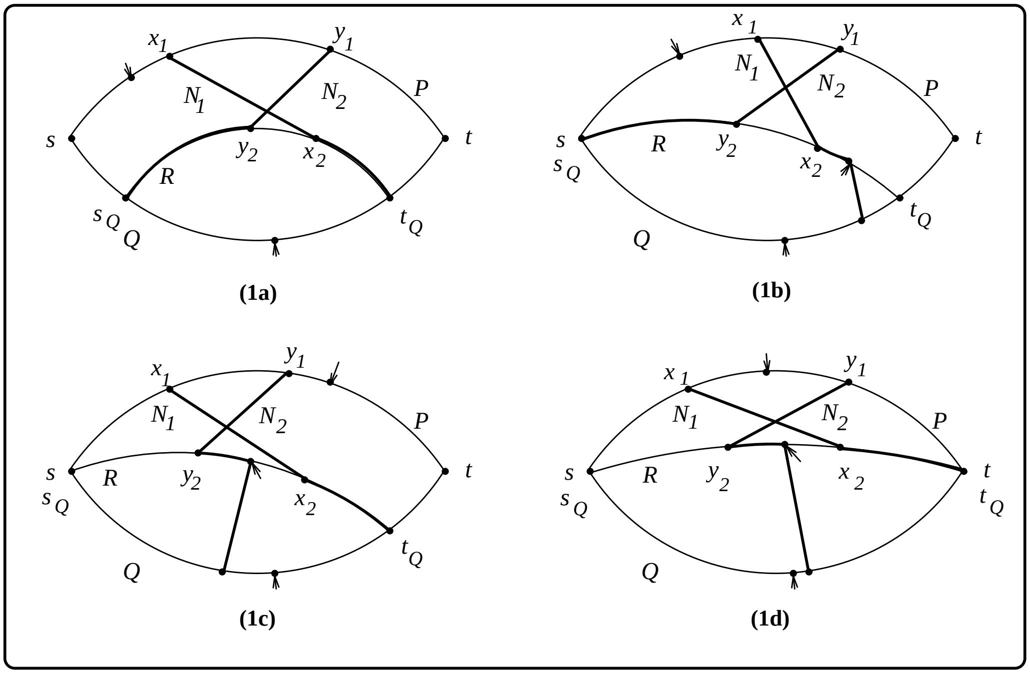


Figure 13: *Case(1) of Lemma 5.6.*

(1) *$\overline{U}'$ has a PQ-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$, where x_1 is to the left of y_1 on P.*
Since B of U has two distinct vertices of attachment x_1 and y_1 on P, s_P and t_P must be distinct and U is PQ-feasible. Hence, $\left|\mathrm{UA}(\overline{U}')\right| > 0$. Moreover, since B is irreducible with respect to R, $\left|\mathrm{LA}(\overline{U}')\right| > 0$ (by definition). If both s_Q and t_Q are distinct from s and t then, clearly, U has PQ-cross-cuts.

Hence assume that $s_Q = s$. If $\overline{U}'$ has a lower external vertex of attachment on $R]y_2;t_Q]$ and an upper external vertex of attachment on $P]s;y_1[$ then U has P-cross-cuts. If not, $\overline{U}'$ has a lower external vertex

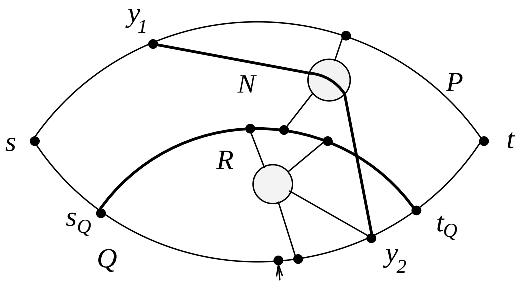


Figure 11: *Case: B is PQ-feasible and $|\mathcal{B}_3| \neq 0$.*

Lemma 5.5 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If U' contains a P-, Q-, PQ- or ST-cross-cut pair then U has a Q- or an ST-cross-cut pair.*

PROOF.

Since B is irreducible with respect to R, $|\text{UA}(U')| > 0$. Moreover, since B is PQ-feasible, and since U' is feasible, it must be a PQ-feasible U-Fragment of U on Q. Note that if $s_Q = s$ and $t_Q = t$, then $s_P = s_Q$ and $t_P = t_Q$ are distinct and by definition, U is PQ-feasible and has an upper external vertex of attachment on P.

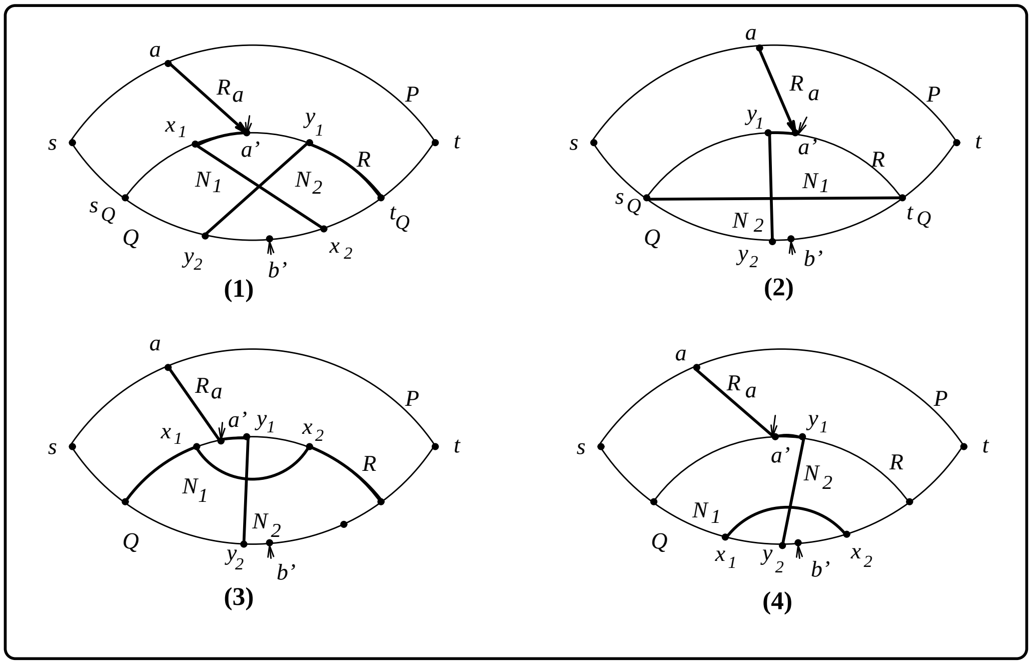


Figure 12: *Four Cases for Lemma 5.5.*

(1) *U' has a PQ-cross-cut pair, $N_1[x_1;x_2]$ and $N_2[y_1;y_2]$.*

Without loss of generality, assume that there exist external vertices of attachment $a' \in R]s_Q;y_1[$ and $b' \in Q]y_2;t_Q[$. Let $R_a[a;a']$ be a path in $\mathcal{B}_1$ such that R_a meets $P]s;t[$ only in a and meets $R]s_Q;t_Q[$ only in a'. The cross-cuts $L_1 = N_2[y_2;y_1] * R[y_1;t_Q]$ and $L_2 = R_a[a;a'] * R[a';x_1] * N_1[x_1;x_2]$ form Q-cross-cuts of U.

Appendix: Interlacing Cross-Cuts and Bights

In this appendix we present a proof for Theorem 4.2. The proof is by a complete induction on the graphs, where the well-ordering is the one induced by the lexico-graphic ordering of the signature of the graphs. This appendix is organized in a fashion closely resembling the structure of the algorithms ANALYZE-BLOCK and ANALYZE-U-$\overline{U}$.

§A Interlacing Cross-Cuts: Single Bridge

Notice that if U is an infeasible augmented graph, then it does not have a pair of interlacing bights. Hence we may assume that U is feasible.

Theorem 5.2 *Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q; t_Q\rangle$. Let $R[s_Q; t_Q]$ be a path in B, connecting the vertices s_Q and t_Q, such that B is reducible with respect to R. Let $\overline{U}'$ be its $\overline{U}$-Fragment, as in the definition 4.5.*

Then U has a P-, Q-, PQ- or a ST-cross-cut pair, if $\overline{U}'$ is a $\overline{U}$-Fragment containing a P-, Q-, PQ- or a ST-cross-cut pair. □

Theorem 5.3 *Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q; t_Q\rangle$. Let $R[s_Q; t_Q]$ be a path in B, connecting the vertices s_Q and t_Q, such that B is irreducible with respect to R. Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ be as in the definition 4.5. Notice that if $|\mathcal{B}_3| = 0$ then the path R divides U into U', a U-Fragment on Q, and $\overline{U}'$ a $\overline{U}$-Fragment on Q, as in the definition 4.5.*

Then U has a P-, Q-, PQ- or a ST-cross-cut pair, if one of the three following conditions is satisfied:

1. *$|\mathcal{B}_3| \neq 0$.*
2. *$|\mathcal{B}_3| = 0$, and U' is a feasible U-Fragment containing a P-, Q-, PQ- or a ST-cross-cut pair.*
3. *$|\mathcal{B}_3| = 0$, and $\overline{U}'$ is a $\overline{U}$-Fragment containing a P-, Q-, PQ- or a ST-cross-cut pair.* □

We have three cases to consider: (1) B is PQ-feasible (if $|\mathrm{LA}(U)| \neq 0$ and $|\mathrm{UA}(U)| \neq 0$), (2) B is P-feasible (if $|\mathrm{LA}(U)| = 0$), and (3) B is Q-feasible (if $|\mathrm{UA}(U)| = 0$), the last two cases being symmetric.

§A.1 Case(1): B is PQ-feasible.

First notice that if B is PQ-feasible, then B is irreducible with respect to path R; and the Theorem 5.2 is vacuously true. Hence for this case we only need to prove the Theorem 5.3.

Lemma 5.4 *Let U, B, R, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Theorem 5.3. If $|\mathcal{B}_3| \neq 0$ then U has a P-, Q-, PQ- or an ST-cross-cut pair.*

PROOF.

If $|\mathcal{B}_3| \neq 0$ then there is a cross-cut $N[y_1; y_2]$ of J between $y_1 \in P[s; t]$ and $y_2 \in Q[s_Q; t_Q]$ such that N avoids R. Since B is a PQ-feasible B^{PQ}-bridge, the cross-cuts R and N form either Q-cross-cuts or ST-cross-cuts of U. □

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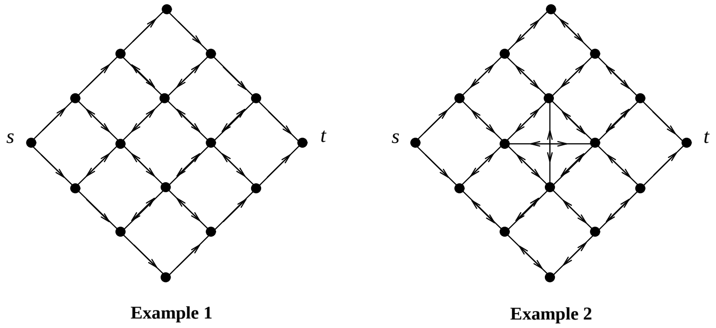

Figure 10: Labeling of the two example graphs.

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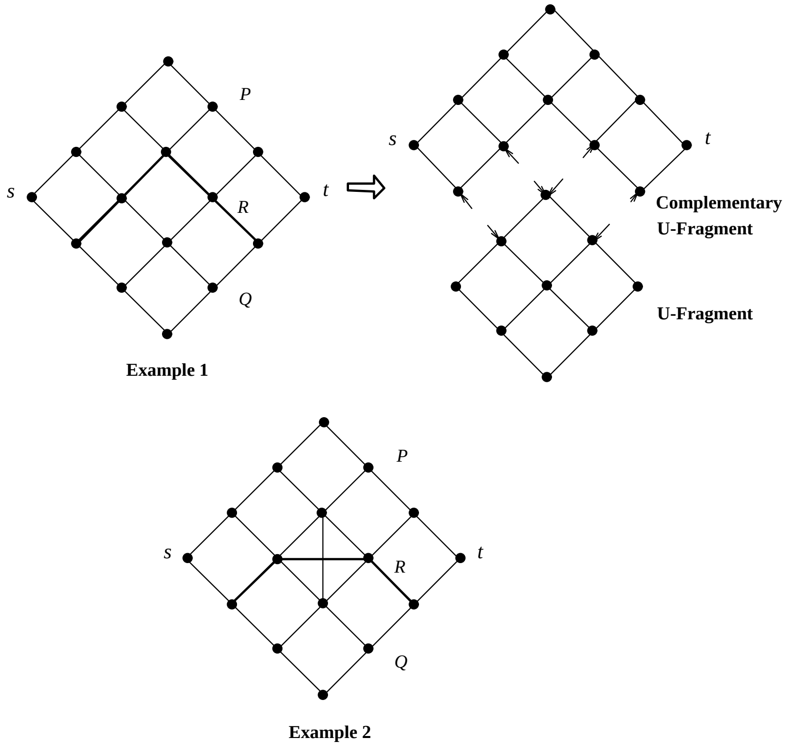

Example 1

Example 2

Figure 9: Two example graphs.

In the first example, since $\mathcal{B}_3 = \emptyset$, we determine its U- and $\overline{U}$-fragments and analyze them further to notice that the B^{PQ}-bridge in the first example does not have P-, Q-, PQ- or ST-cross-cuts and the edges of $P[s;t]$ (and similarly, edges of $Q[s;t]$) are all unidirectional. Only the edges of the B^{PQ}-bridge are bidirectinal.

In the second example, since $\mathcal{B}_3 \neq \emptyset$, we see that all the edges of $P]s;t[$ are admissible. By a similar anlysis, we see that all the edges of $Q]s;t[$ are admissible. As all the edges of the B^{PQ}-bridge are bidirectinal, we see that only the edges directly incident on s and t are unidirectional.

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```
                RECUR:
                if $|\mathcal{B}_3| \neq 0$ then
                    $\mathrm{AS}_2 := E(P[s_P;t_P])$;
                else
                    Let $U'$ and $\overline{U}'$ be its U- and $\overline{U}$-Fragment, respectively, as in the
                    Definition 4.5. Let $\mathrm{AE}(U' : P)$ be the set of admissible edges of $U'$
                    obtained by calling FIND-AE-U-$\overline{U}((U' : P))$;

                    $\mathrm{AS}_2^1 := \mathrm{AE}(U' : P)$;

                    if ANALYZE-U-($\overline{U}'$) returns 'YES' then
                        $\mathrm{AS}_2^2 := E(P[s_P;t_P])$;
                    else
                        $\mathrm{AS}_2^2 := \emptyset$;
                    end {if} ;
                    $\mathrm{AS}_2 := \mathrm{AS}_2^1 \cup \mathrm{AS}_2^2$;
                end {if} ;

                $\mathrm{AE}(U_{\mathbf{B}} : P) := \mathrm{FE}(U_{\mathbf{B}} : P) \cap E(\mathrm{AS}_2)$;

                return $\mathrm{AE}(U_{\mathbf{B}} : P)$;
    end {case} ;

end{FIND-AE-BLOCK.} □
```

Remark 5.4 As before, we see that the algorithm reduces the augmented graph U to U' and $\overline{U}'$ in $O(|E|)$ time; and $U' \prec U$ and $\overline{U}' \prec U$. □

Theorem 5.1 *Let U be an augmented graph with the path P associated with it. Then the Algorithm* FIND-AE-U-Ū *computes* $\mathrm{AE}(U : P)$*, the set of Admissible Edges of* $(U : P)$ *in time* $O(|E| \cdot |V|)$*.*

PROOF.

The proof of the time-complexity of the algorithm is similar to that of ANALYZE-U-Ū.

The rest of the proof is by a simple case by case analysis, which shows that our algorithm returns exactly those edges $e \in \mathrm{AE}(U : P)$ for which ANALYZE-U-Ū$(u(U : e))$ returns 'YES'. We omit a detailed proof. (See [11].) □

Example 5.5 Consider the example graphs that were shown in the companion paper[12] (also, see figure 9.) Since the graphs posses mirror symmetric, we might simply consider only one of the paths, say P. First note that both the graphs are TYPE.IV.

Proceeding with the algorithms described in this section, we first find a path $R[s_Q;t_Q]$ connecting the two extreme vertices of attachment of the bridge on $Q]s;t[$. Note that both the graphs are irreducible with respect to R.

```
Algorithm FIND-AE-BLOCK(($U_{\mathbf{B}} : P$)):

  step1. FEASIBILITY-TEST: If $\mathrm{FE}(U_{\mathbf{B}} : P) = \emptyset$ then return $\mathrm{AE}(U_{\mathbf{B}} : P) := \emptyset$. Otherwise go to
      the next step.

  step2.

      case
          $|\mathbf{B}| > 1$:
              $\mathrm{AE}(U_{\mathbf{B}} : P) := FE(U_{\mathbf{B}} : P)$;
              return $\mathrm{AE}(U_{\mathbf{B}} : P)$;
          $|\mathbf{B}| = 1$:
              Let $\mathbf{B} = \{B\}$;
              if the pair of vertices associated with $B$ is $\langle s_Q, t_Q \rangle$ then
                  Divide: Find a path $R$ in $B$ connecting the vertices $s_Q$ and $t_Q$. Modify $R$
                  such that the bridges with vertices of attachment solely on $R$ avoid other
                  bridges. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Definition 4.5.

                  Recur:
                  if $B$ is reducible with respect to $R$ then
                      Let $\overline{U}'$ be the $\overline{U}$-Fragment as in the Definition 4.5. Let $\mathrm{AE}(\overline{U}' : P)$
                      be the set of admissible edges of $\overline{U}'$ obtained by calling FIND-AE-
                      U-$\overline{U}$(($\overline{U}' : P$));

                      $\mathrm{AS}_1 := \mathrm{AE}(\overline{U}' : P)$;
                  else (* $B$ is irreducible with respect to $R$. *)
                      if $|\mathcal{B}_3| \neq 0$ then
                          $\mathrm{AS}_1 := E(P[s; t])$;
                      else
                          Let $U'$ and $\overline{U}'$ be its $U$- and $\overline{U}$-Fragment, respectively, as
                          in the Definition 4.5.

                          if ANALYZE-U-($U'$) returns 'YES' then
                              $\mathrm{AS}_1^1 := E(P[s; t])$;
                          else
                              $\mathrm{AS}_1^1 := \emptyset$;
                          end {if} ;
                          Let $\mathrm{AE}(\overline{U}' : P)$ be the set of admissible edges of $\overline{U}'$ ob-
                          tained by calling FIND-AE-U-$\overline{U}$(($\overline{U}' : P$));
                          $\mathrm{AS}_1^2 := \mathrm{AE}(\overline{U}' : P)$;
                          $\mathrm{AS}_1 := \mathrm{AS}_1^1 \cup \mathrm{AS}_1^2$;
                      end {if} ;
                  end {if} ;
                  $\mathrm{AE}(U_{\mathbf{B}} : P) := \mathrm{FE}(U_{\mathbf{B}} : P) \cap \mathrm{AS}_1$;
              else if the pair of vertices associated with $B$ is $\langle s_P, t_P \rangle$ then
                  Divide: Find a path $R$ in $B$ connecting the vertices $s_P$ and $t_P$. Modify $R$
                  such that the bridges with vertices of attachment solely on $R$ avoid other
                  bridges. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the definition 4.5.
```

Now we are ready to describe the algorithm that computes all the admissible edges.

Algorithm FIND-AE-U-$\overline{\mathrm{U}}$($(U : P)$):

step1. If $\mathrm{FE}(U : P) = \emptyset$ then return $\mathrm{AE}(U : P) := \emptyset$.

step2. Let $\mathbf{B}_1$, $\mathbf{B}_2$, …, $\mathbf{B}_l$ be the set of blocks of B^{PQ}-bridges of U. Let $\mathbf{B}$ stand for an arbitrary block of B^{PQ}-bridges $\mathbf{B}_i$ (for $i = 1, 2, \ldots, l$.) Let $U_{\mathbf{B}} = \{P\} \cup \{Q\} \cup \{\mathbf{B}\}$ be the subgraph of U whose external vertices of attachment are same as those of U on $Q]s;t[$. Let $J = \{P\} \cup \{Q\}$. The graph minor $U'_{\mathbf{B}}$ of $U_{\mathbf{B}}$ is obtained as follows:

Let v_1, …, v_{p-1} and w_1, …, w_{q-1} be the vertices of attachment of the set of B^{PQ}-bridges of $\mathbf{B}$ on $P]s;t[$ and $Q]s;t[$, ordered in their left-to-right order, respectively. Let $\mathcal{L} = \{P[v_0(=s); v_1], \ldots, P[v_{p-1}; v_p(=t)], Q[w_0(=s); w_1], \ldots, Q[w_{q-1}; w_q(=t)]\}$ be the set of residual paths of J.

Let $L[a;b] \in \mathcal{L}$ be a residual path. If $|L[a;b]| \geq 3$ then contract the subpath $L]a;b[= L[x;y]$ to a single edge, $[x,y]$. Let $\mathrm{EA}(L]a;b[)$ be the set of external vertices of attachment of U on the subpath $L]a;b[$.

1. If $|\mathrm{EA}(L]a;b[)| \geq 2$ then $U'_{\mathbf{B}}$ has external vertices of attachment at x and y.
2. If $|\mathrm{EA}(L]a;b[)| = 1$ then $U'_{\mathbf{B}}$ has external vertices of attachment at x or y, the choice being arbitrary.
3. If $|\mathrm{EA}(L]a;b[)| = 0$ then $U'_{\mathbf{B}}$ has no external vertices of attachment at x or y.

Each of the edges of the cycle, J', of $U'_{\mathbf{B}}$ is a pseudo-edge.

step3. For each block $\mathbf{B}$, find $\mathrm{AE}(U'_{\mathbf{B}} : P')$, the set of admissible edges of $U'_{\mathbf{B}}$, by calling FIND-AE-BLOCK($(U'_{\mathbf{B}} : P')$). Let

$$\mathrm{AE}_1 := \{e \in E(P) : \text{for some } i, \exists e' \in \mathrm{AE}(U'_{\mathbf{B}_i} : P'_i) \text{ such that } e' \text{ is a contraction of a subpath of } P \text{ containing } e.\}$$

$$\mathrm{AE}(U : P) := \mathrm{FE}(U : P) \cap \mathrm{AE}_1;$$

return $\mathrm{AE}(U : P)$.

end{FIND-U-$\overline{\mathrm{U}}$.} □

Remark 5.3 Note that since all of the non-recursive steps can be done in time $O(|E| + |pE|)$ time, we see that the algorithm reduces the augmented graph U to $U'_{\mathbf{B}_1}, U'_{\mathbf{B}_2}, \ldots, U'_{\mathbf{B}_l}$ in $O(|E|)$ time; and $U'_{\mathbf{B}_i} \prec U$, for $1 \leq i \leq l$. □

5 Finding the Set of Admissible Edges

However, it is easy to see that a straight-forward application of the algorithm ANALYZE-U-$\bar{\text{U}}$ will result in an $O(|E| \cdot |V|^2)$ algorithm. Instead, if we simply compute the set of edges, AE, of $P[s;t]$ and $Q[s;t]$ in G such that $e \in$ AE if and only if ANALYZE-U-$\bar{\text{U}}(u(G:e))$ returns 'YES' then these are exactly the bidirectional edges of P and Q in G. The set AE can be computed efficiently in time $O(|E| \cdot |V|)$. The algorithm is developed in the rest of this section.

Remark 5.1 It is easy to see that $\mathrm{FE}(U : P)$ can be computed in time $O(|E|)$. The set $\mathrm{FE}(U : P)$ can be completely described as follows:

If U has less than four distinct vertices of attachment then $\mathrm{FE}(U : P) = \emptyset$, otherwise $\mathrm{FE}(U : P) = \mathrm{FE}_1 \cap \mathrm{FE}_2$, where FE_1 and FE_2 are as follows:

1. If $s_Q \neq t_Q$ and U has no external vertex of attachment on $Q]s_Q;t_Q[$ then

$$\mathrm{FE}_1 = E\left(\mathrm{FS}_1^1 \cup \mathrm{FS}_1^2 \cup \mathrm{FS}_1^3\right),$$

   otherwise $\mathrm{FE}_1 = E(P[s;t])$. FS_1^1, FS_1^2 and FS_1^3 are as follows:
   (a) If s_P^* and t_P^* are distinct then $\mathrm{FS}_1^1 = P[s_P^*;t_P^*]$ else $\mathrm{FS}_1^1 = \emptyset$.
   (b) If s_P^* and t_P^* are distinct and U has an external vertex of attachment on $Q[t_Q;t]$ then $\mathrm{FS}_1^2 = P[s;t_P^*]$ else $\mathrm{FS}_1^2 = \emptyset$.
   (c) If s_P^* and t_P^* are distinct and U has an external vertex of attachment on $Q[s;s_Q]$ then $\mathrm{FS}_1^3 = P[s_P^*;t]$ else $\mathrm{FS}_1^3 = \emptyset$.
2. If $s_P \neq t_P$ then

$$\mathrm{FE}_2 = E\left(\mathrm{FS}_2^1 \cup \mathrm{FS}_2^2 \cup \mathrm{FS}_2^3\right),$$

   otherwise $\mathrm{FE}_2 = E(P[s;t])$. FS_2^1, FS_2^2 and FS_2^3 are as follows:
   (a) $\mathrm{FS}_2^1 = P[s_P;t_P]$.
   (b) If s_Q^* and t_Q^* are distinct and U has an external vertex of attachment on $Q]s_Q^*;t]$ then $\mathrm{FS}_2^2 = P[s;s_P]$ else $\mathrm{FS}_2^2 = \emptyset$.
   (c) If s_Q^* and t_Q^* are distinct and U has an external vertex of attachment on $Q[s;t_Q^*[$ then $\mathrm{FS}_2^3 = P[t_P;t]$ else $\mathrm{FS}_2^3 = \emptyset$. □

Definition 5.2 THE VERTICES ASSOCIATED WITH A B^{PQ}-BRIDGE OF $(U : P)$.

Let U be an augmented graph, with a path P associated with it and with a single B^{PQ}-bridge, B. If $\mathrm{FE}(U : P) \neq \emptyset$, we associate a pair of left- and right-most vertices (either $\langle s_P, t_P \rangle$ or $\langle s_Q, t_Q \rangle$) with the bridge B as follows:

1. $s_P = t_P$ *and* s_Q *and* t_Q *are distinct*: The pair of vertices associated with B is $\langle s_Q, t_Q \rangle$.
2. $s_Q = t_Q$ *and* s_P *and* t_P *are distinct*: The pair of vertices associated with B is $\langle s_P, t_P \rangle$.
3. s_P *and* t_P *as well as* s_Q *and* t_Q *are distinct*: The pair of vertices associated with B is $\langle s_Q, t_Q \rangle$. □

Let $u,\, v \in V(J)$. Then the distance between u and v is defined to be

$$distance(u, v) = \text{MIN}\left(|J[u; v]|\,, |J[v; u]|\right).$$

Let $e = [u, v] \in E(J)$ and $w \in V(J)$. Then, similarly, the distance of e from w is defined as

$$distance(e, w) = \text{MIN}\left(distance(u, w), distance(v, w)\right).$$

We partition the pseudo-edges, $pE(H_j)$, of H_j into the following two classes: $pE^{(1)}(H_j)$ and $pE^{(2)}(H_j) = pE(H_j) \setminus pE^{(1)}(H_j)$, where

$$pE^{(1)}(H_j) = \Big\{ e \in pE(H_j) \;\;:\;\; \begin{array}{l} e \in E(J) \text{ and there exists a vertex } w\text{, a vertex of attachment} \\ \text{of a } B^{PQ}\text{-bridge or one of } s \text{ and } t\text{, such that} \\ distance(e, w) \leq 2 \Big\}. \end{array}$$

(a) We define a function g_1 that maps a pseudo-edge, $e' = [u, v] \in pE^{(1)}(H_j)$ to a graph edge, $e \in gE(H_j)$. If H_j has a B^{PQ}-bridge, B with a vertex of attachment at w' such that $distance(e', w') \leq 2$ then $g_1(e') = e$, where e is a graph edge of B incident at w'; otherwise, $g_1(e') =$ undefined. Since g_1 maps at most eight distinct pseudo-edges of $pE^{(1)}(H_j)$ to one graph edge of $gE(H_j)$, and since g_1 is not defined for at most eight edges, we have $\left|pE^{(1)}(H_j)\right| \leq 8 \cdot |gE(H_j)| + 8$.

(b) We define a function g_2 that maps a pseudo-edge, $e' = [u, v] \in pE^{(2)}(H_j)$ to a graph edge, $e \in gE(H)$. In this case, H_j must be a U- or a $\overline{U}$-fragment of the augmented graph H, and there is a path R in H such that an edge $e \in E(R)$ corresponds to the edge e'. Moreover, every such edge is a graph-edge. Define $g_2(e') = e$. Since g_2 maps at most two distinct pseudo-edges of $\bigcup_{j=1}^{m} pE^{(2)}(H_j)$ to one graph edge of $gE(H)$, we have $\sum_{j=1}^{m} \left|pE^{(2)}(H_j)\right| \leq 2 \cdot |gE(H)|$.

Hence

$$\sum_{j=1}^{m} |pE|\,(H_j) \leq \sum_{j=1}^{m} \Big[8 \cdot gE(H_j) + 8\Big] + 2 \cdot gE(H) \leq 26 \cdot gE(H),$$

since each graph edge of H occurs in one graph H_j and since,

$$m \leq \text{MAX}(2, |gE(H)|) \leq 2 \cdot |gE(H)|\,.$$

Now, by our previous observations, we know that the algorithms spend the following amount of time in stage i:

$$O\left(\sum_{j=1}^{n} \Big[|gE(U_j)| + |pE(U_j|\Big] + \sum_{j=1}^{n} \sum_{k=1}^{m_j} \Big[|gE(U_{j,k})| + |pE(U_{j,k})|\Big] \right) = O\left(|E|\right),$$

where the graphs of stage i be $\{U_1, \ldots, U_n\}$ and the graphs of stage $i+1$ be $\{U_{1,1}, \ldots, U_{1,m_1}, \ldots, U_{n,1}, \ldots, U_{n,m_n}\}$. As noted earlier, as there are at most $O(|V|)$ stages, the algorithm has the desired $O(|E| \cdot |V|)$ time complexity. $\Box$

The correctness of the algorithm will follow from the following theorem. The proof of the theorem is rather technical and is presented in several parts in the appendix. The proof is by a complete induction on the following well-ordering on augmented graphs.

Definition 4.9 Signature of an Augmented Graph.

Let U be an augmented graph, consisting of the cycle $J = \{P\} \cup \{Q\}$ and a set of B^{PQ}-bridges, $\mathcal{B}$. To U, we assign a pair of positive integers $\langle i_i, i_2 \rangle$, called its *signature*, where

$$i_1 = \sum_{B \in \mathcal{B}} |V(N(B))| \leq |V(U)|,$$

and $i_2 = 1$ or 0, depending, respectively, on whether U is an arbitrary augmented graph, or an augmented graph with exactly one block of B^{PQ}-bridges.

We say augmented graphs $U_1 \prec U_2$, if

$$\mathit{signature}(U_1) <_{\text{lex}} \mathit{signature}(U_2).$$

This defines a well-ordering among the augmented graphs. Let $U_0 \succ U_1 \succ \cdots \succ U_n$ be a decreasing chain of graphs ordered by the above ordering. Hence $n \leq 2 \cdot |V(U_0)|$. □

Theorem 4.2 *Let U be an augmented graph.*

1. *If the algorithm* Analyze-U-$\bar{\text{U}}(U)$ *returns* 'yes', *then U is feasible and has a P-, Q-, PQ- or an ST-cross-cut pair.*
2. *If the algorithm* Analyze-U-$\bar{\text{U}}(U)$ *returns* 'no' *then U does not have a pair of interlacing bights.* □

Corollary 4.3 *An edge e on $P[s;t]$ is bidirectional if and only if* Analyze-U-$\overline{U}(u(G : e))$ *returns* 'yes'. □

§4.3 Complexity Analysis.

Theorem 4.4 *Let U be a U-Fragment or a $\bar{U}$-Fragment. The algorithm* Analyze-U-$\bar{\text{U}}(U)$ *terminates in $O(|E| \cdot |V|)$ time.*

proof.

Let the pseudo-edges of a graph U be denoted by $pE(U)$ and the graph-edges of U, by $gE(U)$. We define a *set of graphs of a stage* of the algorithm as follows: graphs of stage $1 = \{U\}$. Let graphs of stage i be $\{U_1, \ldots, U_n\}$. Suppose, we apply the appropriate algorithm (one of Analyze-U-$\bar{\text{U}}$ and Analyze-Block) to U_j to reduce it to a set of graphs $\{U_{j,1}, \ldots, U_{j,m_j}\}$ such that $U_{j,k} \prec U_j$. Then, the graphs of stage $i+1$ be $\{U_{1,1}, \ldots, U_{1,m_1}, \ldots, U_{n,1}, \ldots, U_{n,m_n}\}$.

It is easy to see that the total number of stages is $\leq 2 \cdot |V(U)|$. If we now show that the total amount of time spent on the graphs of stage i is $O(|E(U)|)$ (for all i), then we have exhibited an $O(|E| \cdot |V|)$ time complexity for the complete algorithm.

We start with the following observation: Let H be a graph in stage $i-1$, and $\{H_1, \ldots, H_m\}$ be the graphs of stage i, obtained from H. It can be readily checked that all the pseudo-edges of H_j lie on the cycle J.

```
Algorithm ANALYZE-BLOCK($U_{\mathbf{B}}$):

 step1. FEASIBILITY-TEST: If $U_{\mathbf{B}}$ is infeasible, return 'NO'; otherwise go to the next step.

 step2.

    case
        $|\mathbf{B}| > 1$:
            return 'YES';
        $|\mathbf{B}| = 1$:
            Let $\mathbf{B} = \{B\}$;
            Divide: Let $\langle s', t' \rangle$ be the pair of vertices associated with $B$. Find a path $R$ in $B$
            connecting the vertices $s'$ and $t'$ such that the bridges with vertices of attachment
            solely on $R$ avoid other bridges. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be as in the Definition 4.5.

            Recur:
            if $B$ is reducible with respect to $R$ then
                Let $\overline{U}'$ be the $\overline{U}$-Fragment as in Definition 4.5.
                if ANALYZE-U-($\overline{U}'$) returns 'YES' then
                    return 'YES'
                end {if} ;
            else (* $B$ is irreducible with respect to $R$. *)
                if $|\mathcal{B}_3| \neq 0$ then
                    return 'YES'
                else (* $|\mathcal{B}_3| = 0$ *)
                    Let $U'$ and $\overline{U}'$ be its $U$- and $\overline{U}$-Fragment, respectively, as in Defi-
                    nition 4.5.

                    if ANALYZE-U-($U'$) returns 'YES' cor
                              ANALYZE-U-($\overline{U}'$) returns 'YES' then
                        return 'YES'
                    end {if} ;
                end {if} ;
            end {if} ;
            return 'NO'
    end {case}

end{ANALYZE-BLOCK.} □
```

Remark 4.8 It is easily seen that the algorithm reduces the graph U to graphs U' and $\overline{U}'$ in $O(|E|)$ time. Furthermore, since the B^{PQ}-bridge, B, of U is proper, and since the path R must contain at least one vertex of $N(B)$,

$$\sum_{B' \in \mathcal{B}_i} |V(N(B'))| < |V(N(B))|, \qquad i = 1, 2. \quad \Box$$

Algorithm ANALYZE-U-$\overline{\mathrm{U}}$(U):

step1. If U is infeasible; return 'NO'.

step2. Let $\mathbf{B}_1$, $\mathbf{B}_2$, …, $\mathbf{B}_l$ be the set of blocks of B^{PQ}-bridges of U. Let $\mathbf{B}$ stand for an arbitrary block of B^{PQ}-bridges $\mathbf{B}_i$ (for $i = 1, 2, \ldots, l$.) Let $U_{\mathbf{B}} = \{P\} \cup \{Q\} \cup \{\mathbf{B}\}$ be the subgraph of U whose external vertices of attachment are same as those of U on $P]s;t[$ and on $Q]s;t[$. Let $J = \{P\} \cup \{Q\}$. The graph minor $U'_{\mathbf{B}}$ of $U_{\mathbf{B}}$ is obtained as follows:

Let v_1, …, v_{p-1} and w_1, …, w_{q-1} be the vertices of attachment of the set of B^{PQ}-bridges of $\mathbf{B}$ on $P]s;t[$ and $Q]s;t[$, ordered in their left-to-right order, respectively. Let $\mathcal{L} = \{P[v_0(= s); v_1], \ldots, P[v_{p-1}; v_p(= t)], Q[w_0(= s); w_1], \ldots, Q[w_{q-1}; w_q(= t)]\}$ be the set of residual paths of J.

Let $L[a;b] \in \mathcal{L}$ be a residual path. If $|L[a;b]| \geq 3$ then contract the subpath $L]a;b[= L[x;y]$ to a single edge, $[x,y]$. Let $\mathrm{EA}(L]a;b[)$ be the set of external vertices of attachment of U on the subpath $L]a;b[$.

1. If $|\mathrm{EA}(L]a;b[)| \geq 2$ then $U'_{\mathbf{B}}$ has external vertices of attachment at x and y.
2. If $|\mathrm{EA}(L]a;b[)| = 1$ then $U'_{\mathbf{B}}$ has external vertices of attachment at x or y, the choice being arbitrary.
3. If $|\mathrm{EA}(L]a;b[)| = 0$ then $U'_{\mathbf{B}}$ has no external vertices of attachment at x or y.

Each of the edges of the cycle, J', of $U'_{\mathbf{B}}$ is a *pseudo-edge*.

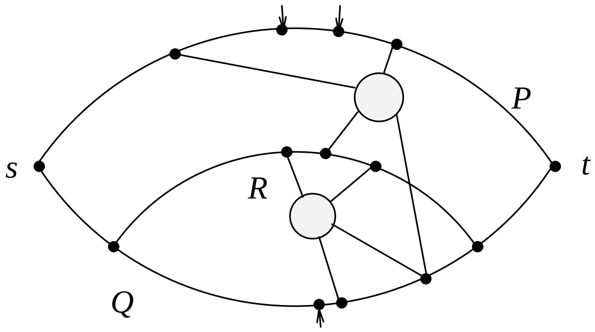
step3. For each block $\mathbf{B}$, analyze $U'_{\mathbf{B}}$ by calling ANALYZE-BLOCK($U'_{\mathbf{B}}$). If the answer is 'YES' for any of the blocks of $\mathcal{B}$, return 'YES'; otherwise, return 'NO'.

end{ANALYZE-U-$\overline{\mathrm{U}}$.} ☐

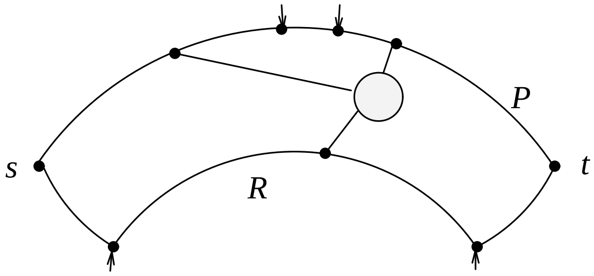
Remark 4.7 Since the step1 and step2 take $O(|E|)$ time, the algorithm reduces the augmented graph U to $U'_{\mathbf{B}_1}$, $U'_{\mathbf{B}_2}$, …, $U'_{\mathbf{B}_l}$ in $O(|E|)$ time. Also, note that, for each $1 \leq i \leq l$, since

$$\sum_{B \in \mathbf{B}_i} |V(N(B))| \leq \sum_{B \in \mathcal{B}} |V(N(B))|,$$

and in this sense, the non-recursive part of the algorithm does not increase the "complexity" of the original problem. This simplification is formalized through the notion of the "signature" of a graph subsequently. ☐



Augmented Graph (reducible)



Complementary U-Fragment

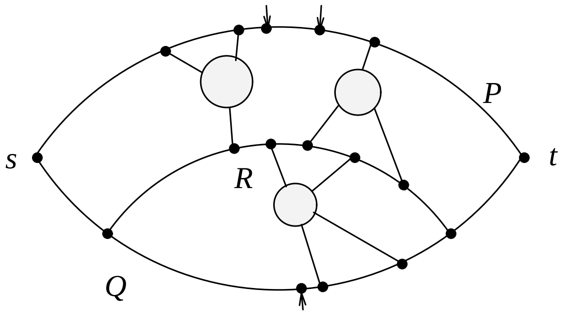
Figure 8: $\overline{U}$*-Fragment: B is reducible.*

of U and $\overline{U}'$, corresponding to the edges of $R[s_Q; t_Q]$ of U are considered to be *pseudo-edges*. □

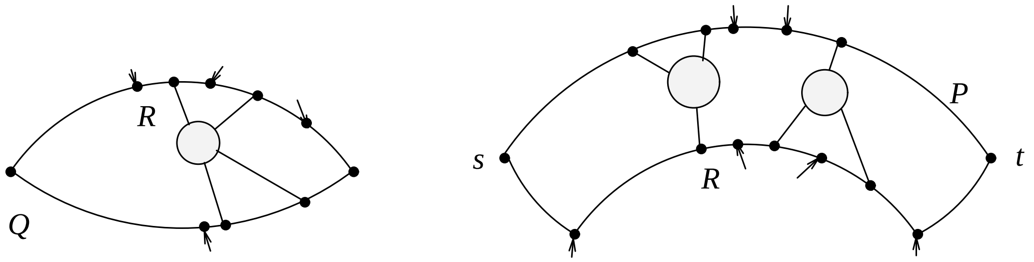
The following proposition can be easily verified:

Proposition 4.1 *Let* U *and* $\overline{U}$ *be, respectively, the* U*- and* $\overline{U}$*-Fragments of an augmented graph* U'*. Then both* U *and* $\overline{U}$ *are themselves augmented graphs.* □

§**4.2 Analyzing a U-Fragment** Let U be an augmented graph. In this section, we present an algorithm to analyze an augmented graph and determine if it has a P-, Q-, PQ- or an ST-cross-cut pair. It consists of two mutually recursive algorithms ANALYZE-U-$\overline{\text{U}}$, and ANALYZE-BLOCK.



Augmented Graph (irreducible)



U-Fragment **Complementary U-Fragment**

Figure 7: *U-Fragment and $\overline{U}$-Fragment: B is irreducible.*

attachment of $\mathcal{B}_2$ on $R]s_Q;t_Q[$ and (iii) the vertex s_Q (if distinct from s) and the vertex t_Q (if distinct from t). The vertices s_Q and t_Q of $\overline{U}'$ are marked, l'_Q and r'_Q, respectively. $l'_P = l_P$ and $r'_P = r_P$.

- *B is reducible with respect to R.* (See Figure 8.)

Let $\overline{U}'$ be the *maximal* nonseparable subgraph of U containing the vertices s and t, but without the path $Q]s_Q;t_Q[$ and any of the $\mathcal{B}_2$ or $\mathcal{B}_4$ bridges. Notice that $\overline{U}'$ contains the cycle $J = \{P[s;t]\} \cup \{Q[s;s_Q] * R[s_Q;t_Q] * Q[t_Q;t]\}$ and all its bridges have attachment on both P and R. The subgraph $\overline{U}'$ is called a *$\bar{U}$-Fragment of U on Q*. The set of *external vertices of attachment*, *EA* consist of (i) the external vertices of attachment of U on $J]s_Q;t_Q[$, and (ii) the vertex s_Q (if distinct from s) and the vertex t_Q (if distinct from t). The vertices s_Q and t_Q of $\overline{U}'$ are marked, l'_Q and r'_Q, respectively. $l'_P = l_P$ and $r'_P = r_P$. ☐

Remark 4.6 (1) Given a U- or a $\overline{U}$-Fragment, U, with a single B^{PQ}-bridge, B, in linear time, we can either divide U into a U-Fragment, U' and a $\overline{U}$-Fragment, $\overline{U}'$ (if B is irreducible with respect to R) or reduce U to a $\overline{U}$-Fragment, $\overline{U}'$ (if B is reducible with respect to R.)

(2) Let U and U' be two distinct U- or $\overline{U}$-Fragments. In order to distinguish between the paths and vertices of U' from those of U, we use the primed versions for U': for instance, P', Q', s'_P, t'_P, s'_U, t'_U, etc. refer to those of U' where as P, Q, s_P, t_P, s_U, t_U, etc. refer to those of U. In what follows, either U is a U- or a $\overline{U}$-Fragment of U', or *vice versa.*

(3) If U and $\overline{U}'$ are the U- and $\overline{U}$-Fragments of an augmented graph, U, then the edges

second kind. By our assumption the total number of such disjoint ties and hence the separation number λ must not exceed one. But, since B is a bridge, $\lambda(U^-; E_1, E_2)$ must be at least one; let $v_{\text{cut}} \in V(F_1 \cap F_2)$, where $\langle F_1, F_2\rangle$ is the cutting pair corresponding to this λ.

Furthermore, $v_{\text{cut}} \in N(B)$. This is a consequence of the following reasoning: Since U is feasible, and has no external vertex of attachment on $Q]s_Q; t_Q[$, s_P^* and t_P^* must be distinct, i.e., B has at least two vertices of attachment on $P]s; t[$. If $v_{\text{cut}} \in V(P]s; t[)$ or $\in V(Q]s_Q; t_Q[)$ then there is a vertex of attachment $x \in V(P]s; t[)$, where x is distinct from v_{cut}. Hence there is a path from x to a vertex of $R]s_Q; t_Q[$ in B; but such a path avoids v_{cut}, resulting in a contradiction.

As a result, it is easy to see that in order to determine if a bridge B is irreducible with respect to a path R, we only have to find if the separation number $\lambda(U^-; E_1, E_2) > 1$. This can be done in $O(|E|)$ time. □

Notation 4.4 It will be customary to associate following named vertices with an augmented graph. An augmented graph U has two distinguished vertices l_P and r_P on $P[s; t]$, and two distinguished vertices l_Q and r_Q on $Q[s; t]$, such that all the vertices of attachment of all of its B^{PQ}-bridges lie on $P[l_P; r_P]$ and $Q[l_Q; r_Q]$ Also, the left-most and right-most upper external vertices of attachment on P are denoted by s_U and t_U and the left-most and right-most lower external vertices of attachment on Q by s_L and t_L, respectively. □

Definition 4.5 U-FRAGMENT AND COMPLEMENTARY U-FRAGMENT.

Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q, t_Q\rangle$. Let $R[s_Q; t_Q]$ be a path in B, connecting s_Q and t_Q, and let $J' = \{P\} \cup \{Q\} \cup \{R\}$ be a subgraph of U. As a result B will be decomposed into the following sets of bridges of J' in U:

1. *The set of bridges, $\mathcal{B}_1$, with vertices of attachment on P and R.*
2. *The set of bridges, $\mathcal{B}_2$, with vertices of attachment on Q and R.*
3. *The set of bridges, $\mathcal{B}_3$, with vertices of attachment on P, Q and R.*
4. *The set of bridges, $\mathcal{B}_4$, with vertices of attachment solely on R.*

Assume that the path R is such that the bridges of $\mathcal{B}_4$ avoid those of $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$.

Note that a path R' in B connecting s_Q and t_Q can be found in $O(|E|)$ time and then, using an algorithm similar to the ambitus-finding algorithm (see Tarjan and Mishra[13]), R' can be modified to R in $O(|E|)$ time such that bridges of $\mathcal{B}_4$ avoid those of $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$.

- *B is irreducible with respect to R.* (See Figure 7.)

If $|\mathcal{B}_3| \neq 0$ then the U- and $\overline{U}$-Fragment of B are undefined; otherwise, they are as follows:

The subgraph $U' = \{R[s_Q; t_Q]\} \cup \{Q[s_Q; t_Q]\} \cup \mathcal{B}_2$ of U, is called a *U-Fragment of U on Q*. The set of *external vertices of attachment*, *EA*, of U' consists of (i) the external vertices of attachment of U on $Q]s_Q; t_Q[$, and (ii) the vertices of attachment of $\mathcal{B}_1$ on $R]s_Q; t_Q[$. $l'_P = l'_Q = s$ and $r'_P = r'_Q = t$.

The subgraph $\overline{U}' = \{Q[s; s_Q] * R[s_Q; t_Q] * Q[t_Q; t]\} \cup \{P\} \cup \mathcal{B}_1$ of U, is called a *complementary U-Fragment* (or simply $\overline{U}$-Fragment) *of U on Q*. The set of *external vertices of attachment*, *EA*, of $\overline{U}'$ consists of (i) the external vertices of attachment of U on $J]s_Q; t_Q[$, (ii) the vertices of vertices s_Q and t_Q. B is said to be *reducible* with respect to R, if (i) U has no external vertex of attachment on $Q]s_Q;t_Q[$ and (ii) has a vertex, $v_{\text{cut}} \in N(B)$, (called the *cut vertex*) in the nucleus of B, $N(B)$, satisfying the following condition:

> *If N is an arbitrary path in B that connects a vertex of $P]s;t[$ with a vertex of $Q]s_Q;t_Q[$ or $R]s_Q;t_Q[$ and meets P, Q and R only in its end vertices then N contains v_{cut}.*

Otherwise, B is said to be *irreducible* with respect to R. $\square$

When U has a single reducible B^{PQ}-bridge, the analysis of U is carried on by analyzing only a subgraph $\overline{U}'$ of U ($\overline{U}$-fragment). Otherwise, the divide-and-conquer algorithm may need to explore at most two subgraphs to determine if it has the required structure. In the former case, we say the graph U *reduces* to the subgraph $\overline{U}'$.

Remark 4.3 Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, as in the preceding definition. Let $R[s_Q;t_Q]$ be a path in B connecting s_Q and t_Q–the vertices associated with B. Whether B is reducible with respect to R can easily be determined in $O(|E|)$ time as a consequence of the following observation.

It can be seen that B is irreducible with respect to R iff the following holds:

> *There are at least two vertex-disjoint paths $R_a[a;a']$ and $R_b[b;b']$ in B such that (i) R_a and R_b meet $P]s;t[$ only in a and b, respectively; (ii) R_a meets only one of $Q]s_Q;t_Q[$ and $R]s_Q;t_Q[$ only in a'; and (iii) R_b meets only one of $Q]s_Q;t_Q[$ and $R]s_Q;t_Q[$ only in b'.*

Suppose that there are *no* two vertex-disjoint paths R_a and R_b as in the above statement. We shall show that B has a cut vertex $v_{\text{cut}} \in N(B)$, and B is reducible with respect to R.

Let U^- be the subgraph of U obtained by deleting the residual subpaths $J[s_Q;s]$ and $J[t;t_Q]$. Let $E_1 = E(P]s;t[)$ and $E_2 = E(R]s_Q;t_Q[) \cup E(Q]s_Q;t_Q[)$ be disjoint subsets of $E(U^-)$. Let $\lambda = \lambda(U^-;E_1,E_2)$ be the separation number of E_1 and E_2 in U^- (Cf. pp. 43, Tutte(1984)[20] [2]). By a version of Menger's Theorem, due to Nash-William and Tutte (Theorem II.36[20], also see [9] and [14]), there exists a set of λ disjoint ties between between E_1 and E_2. Since the vertices of $P[s_P;t_P]$ are disjoint with those of $R]s_Q;t_Q[$ and $Q]s_Q;t_Q[$, these ties must be ties of the

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[2] We consider a graph G in which two disjoint subsets E_1 and E_2 of $E(G)$ are specified. Following Tutte, we define a *cutting pair* of E_1 and E_2 in G as an ordered pair $\langle F_1, F_2\rangle$ of edge-disjoint subgraphs of G such that

$$E_1 \subseteq E(F_1), \qquad E_2 \subseteq E(F_2), \qquad \text{and} \qquad E_1 \cup E_2 = G.$$

We define the *order* of a cutting pair $\langle F_1, F_2\rangle$ of E_1 and E_2 as $|V(F_1 \cap F_2)|$, the number of common vertices of F_1 and F_2. We now define $\lambda(G;E_1,E_2)$ as the least number of vertices required to separate E_1 and E_2. Thus

$$\lambda(G;E_1,E_2) = \textsc{min}_{\langle F_1,F_2\rangle} |V(F_1 \cap F_2)|,$$

where the minimum is taken over all cutting pairs $\langle F_1, F_2\rangle$ of E_1 and E_2 in G.

There are two kinds of *tie* between E_1 and E_2. A *tie of the first kind* is a vertex graph contained in both $\langle E_1\rangle$ and $\langle E_2\rangle$. A *tie of the second kind* is a path in G with one end in $\langle E_1\rangle$ but not $\langle E_2\rangle$, with its other end in $\langle E_2\rangle$ but not $\langle E_1\rangle$, and with no edge or internal vertex in either $\langle E_1\rangle$ or $\langle E_2\rangle$.

`end`{LABEL-TYPE-IV.} □

In order to prove an $O(|E|\cdot|V|)$ time-complexity of the algorithm, we need to show that each substep of `step3` takes $O(|E|\cdot|V|)$ time. In the next two sections, we devise a divide-and-conquer algorithm for this purpose.

4 Analyzing Augmented Graphs

Now we are ready to study an algorithm to determine if an augmented graph is feasible and if so, if it has one of the requisite disjoint cross-cut pairs. The techniques developed to analyze an augmented graph can then be easily modified to devise an algorithm to determine the set of admissible edges.

We start by introducing some further new concepts: Of particular interest to us, will be the following special kinds of augmented subgraphs: (i) *U-Fragment* and (ii) its accompanying *$\overline{U}$-Fragment.* These augmented subgraphs allow us to develop an $O(|E|\cdot|V|)$ time algorithm to detect if a feasible augmented graph has a *P-cross-cut pair*, *Q-cross-cut pair*, *PQ-cross-cut pair* or *ST-cross-cut pair.*

§4.1 U-Fragment and Complementary U-Fragment. Consider an augmented graph U with a set of B^{PQ}-bridges. If U is feasible and contains a single block of B^{PQ}-bridges, $\mathbf{B}$, then the block $\mathbf{B}$ is said to be a *feasible block of B^{PQ}-bridges.* Similarly, if U is feasible and contains a a single B^{PQ}-bridge, B. Then the bridge B is said to be a *feasible B^{PQ}-bridge.*

Also, there will be occasions where we distinguish among the feasible augmented graphs by referring to them as: *PQ-feasible* (if $|\mathrm{LA}(U)| \neq 0$ and $|\mathrm{UA}(U)| \neq 0$), *P-feasible* (if $|\mathrm{LA}(U)| = 0$), and *Q-feasible* (if $|\mathrm{UA}(U)| = 0$).

Definition 4.1 THE PAIR OF VERTICES ASSOCIATED WITH A B^{PQ}-BRIDGE.

Let U be a feasible augmented graph, with a single (feasible) B^{PQ}-bridges B. We associate a pair of left- and right-most vertices (either $\langle s_P, t_P\rangle$ or $\langle s_Q, t_Q\rangle$) with the bridge B as follows:

1. *$s_P = t_P$ and s_Q and t_Q are distinct*: The pair of vertices associated with B is $\langle s_Q, t_Q\rangle$.
2. *$s_Q = t_Q$ and s_P and t_P are distinct*: The pair of vertices associated with B is $\langle s_P, t_P\rangle$.
3. *s_P and t_P as well as s_Q and t_Q are distinct:*
    (a) *U is PQ-feasible*: The pair of vertices associated with B is $\langle s_P, t_P\rangle$ or $\langle s_Q, t_Q\rangle$, the choice being arbitrary.
    (b) *U is P-feasible*: the pair of vertices associated with B is $\langle s_Q, t_Q\rangle$.
    (c) *U is Q-feasible*: the pair of vertices associated with B is $\langle s_P, t_P\rangle$. □

Definition 4.2 REDUCIBLE AND IRREDUCIBLE B^{PQ}-BRIDGE (WITH RESPECT TO R).

Let U be a feasible augmented graph with a single (feasible) B^{PQ}-bridge, B, such that the pair of vertices associated with B is $\langle s_Q, t_Q\rangle$. Let $R[s_Q; t_Q]$ be a path in B connecting the

Let U be an augmented graph. We associate one of the paths P and Q with U such that if P (respectively Q) is associated with U then U has no external vertex of attachment on P (respectively, Q). We represent an augmented graph with the associated path, P or Q, by $(U : P)$ and $(U : Q)$, respectively.

Definition 3.5 SET OF FEASIBLE EDGES.

Let U be an augmented graph with the path P associated with it. A *set of feasible edges* of $(U : P)$, $FE(U : P)$, is a set of distinct edges of $P[s;t]$, $\{e_1, e_2, \ldots, e_k\}$, such that activation of an edge $e = [u,v] \in P[s;t]$ results in a feasible augmented graph $u(U : e)$ if and only if

$$e \in \mathrm{FE}(U : P).$$

The set of feasible edges of $(U : Q)$ is defined in an identical manner. □

Clearly, if $e \in P[s;t] \setminus FE(U : P)$, it is unidirectional.

Definition 3.6 SET OF ADMISSIBLE EDGES.

Let U be an augmented graph with the path P associated with it. A *set of admissible edges* of $(U : P)$, $AE(U : P)$, is a set of distinct edges of $P[s;t]$, $\{e_1, e_2, \ldots, e_k\}$, such that

1. Activation of an edge $e = [u,v] \in P[s;t]$ results in a (feasible) augmented graph $u(U : e)$ with a P-, Q-, PQ- or an ST-cross-cut pair, if $e \in \mathrm{AE}(U : P)$.
2. Activation of an edge $e = [u,v] \in P[s;t]$ results in an augmented graph $u(U : e)$ with no pair interlacing bights, if $e \notin \mathrm{AE}(U : P)$.

The set of admissible edges of $(U : Q)$ is defined in an identical manner. □

Finally, using the notion of admissible edges, the LABEL-TYPE-IV algorithm may be expressed as follows:

§3.4 Labeling the edges of a Type.IV Graph Let G be TYPE.IV graph with the paths P and Q.

```
Algorithm LABEL-TYPE-IV($G$):

step1. Label every edge $[s,u]$ incident on $s$, $\langle s,u \rangle$ and every edge $[u,t]$ incident on $t$, $\langle u,t \rangle$.

step2. Label every edge $e$ of the $B^{PQ}$-bridge not incident on $s$ or $t$, bidirectional.

step3. Let $\mathrm{AE}(G : P) \subseteq E(P[s;t])$ and $\mathrm{AE}(G : Q) \subseteq E(Q[s;t])$ be the sets of admissible edges
       of the augmented graph $G$, when the paths associated with $G$ are $P$ and $Q$, respectively.

    1. Label every edge $e = [u,v] \in \mathrm{AE}(G : P)$ bidirectional and every edge $e = [u,v] \in$
       $E(P[s;t]) \setminus \mathrm{AE}(G : P)$, $\langle u,v \rangle$, if $u$ is to the left of $v$ on $P$.
    2. Label every edge $e = [u,v] \in \mathrm{AE}(G : Q)$ bidirectional and every edge $e = [u,v] \in$
       $E(Q[s;t]) \setminus \mathrm{AE}(G : Q)$, $\langle u,v \rangle$, if $u$ is to the left of $v$ on $Q$.
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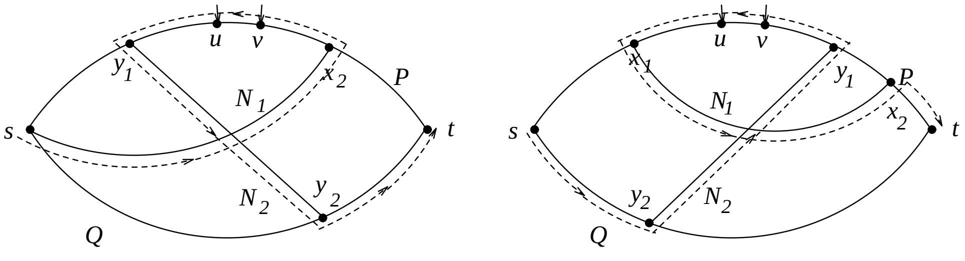


(a) **(b)**

Two Main Cases for P-Cross-Cut Pair

Figure 6: *Case.b: P-Cross-Cut Pair.*

t, without loss of generality, assume that x_2 is distinct from t. Since U has an external vertex of attachment $c \in P]x_1; x_2[$, c must be one of u and v; and hence, $e \in P[x_1; x_2]$.

If $e \in P[y_1; x_2]$, the the simple paths $P[s; t]$ and $P[s; x_1]* \; N_1[x_1; x_2]* \; P[x_2; y_1]* \; N_2[y_1; y_2]*$ $Q[y_2; t]$ traverse e in either direction, and e is bidirectional. (Figure 6(a).)

Next, if $e \in P[x_1; y_1]$, then we treat this case differently, depending on whether x_1 is distinct from s or not. If $x_1 \neq s$, then the simple paths $P[s; t]$ and $Q[s; y_2]* \; N_2[y_2; y_1]* \; P[y_1; x_1]*$ $N_1[x_1; x_2]* \; P[x_2; t]$ traverse e in either direction, and e is bidirectional.(Figure 6(b).)

If, on the other hand, $x_1 = s$, we shall see that the cross-cuts can be so modified that this case reduces to one of the earlier cases. Clearly, $e \in P[s^*_P; y_1]$ and B has a vertex of attachment at s^*_P and a vertex of attachment at $y \in Q]s; t[$. Let $L[s^*_P; y]$ be a cross-cut of J between s^*_P and y. If L avoids both N_1 and N_2 then the P-cross-cut pair defined by N_1 and L satisfies the condition shown in Figure 6(a).

Hence assume that L does not avoid both N_1 and N_2. Then there is a vertex $z \in L]s^*_P; y[$ such that $L[s^*_P; z]$ meets N_1 or N_2 (but not both) in z. If L meets N_2 in z, then the P-cross-cut pair N_1 and $L[s^*_P; z] * N_2[z; y_2]$ satisfies the first condition. On the other hand, if L meets N_1 in z then the P-cross-cut pair $L[s^*_P; z] * N_1[z; x_2]$ and N_2 satisfies the condition shown in Figure 6(b).

(2) We show that there is no simple path $A'[s; t]$ in G such that A' traverses e in the order v and u. Assume to the contrary, then the subpaths $A'[s; v]$ and $A'[u; t]$ are vertex disjoint paths in U. Notice that u and v are distinct from s and t, and hence, are external vertices of attachment of U introduced as a result of the activation of the edge e. Moreover, since $A'[s; v]$ does not contain u or t, it is a not a subpath of J and hence a path associated with a bight, A_1 of U between s and v. By a similar argument, $A'[u; t]$ is a path associated with a bight, A_2 of U between u and t. But since u is to the left of v on P, s and v separate u and t on J; and A_1 and A_2 form a pair of interlacing bights. This however is a contradiction. ☐

§3.3 Feasible and Admissible Edges. Now, based on the characterization of "bidirectionality" developed in the preceding section, we are ready to introduce the concept of *feasible and admissible edges*.

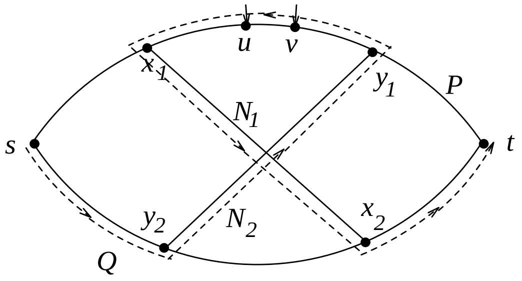


Figure 5: *Case.a: PQ-Cross-Cut Pair.*

is, a_1, a_2 b_1 and b_2 are four distinct vertices on J such that there is a pair of interlacing bights A_1 between a_1 and a_2, and A_2 between b_1 and b_2, where a_1, a_2, b_1 and b_2 are either external vertices or one of s and t. Hence U has at least two distinct external vertices of attachment and the bridges of of $\mathcal{B}$ have at least four distinct vertices of attachment.

Since U is infeasible, we may assume that U has no external vertex of attachment on $Q]s_Q; t_Q[$, say. First, if $s_P^* = t_P^*$, then each of the paths associated with A_1 and A_2 must contain at least two of the three vertices $s_P^*(= t_P^*)$, s_Q and t_Q, contradicting the vertex-disjointness of A_1 and A_2. Hence, we assume that $s_P^* \neq t_P^*$ and that all the external vertices of attachment lie only on $J[s_Q; s_P^*]$. Then a_1 and a_2 lie on $J[s_Q; s_P^*]$, and the path associated with A_1 must contain the vertices s_Q and s_P^*, and every path associated with any other bight A_2 must meet A_1—again a contradiction. □

§3.2 Bidirectionality of an Edge

Theorem 3.2 *Let $e = [u, v]$ be an arbitrary edge on the path $P[s; t]$ or $Q[s; t]$ of a* Type.IV *graph, G; and let $U = u(G : e)$ be the augmented graph obtained from G by the activation of the edge e.*

1. *If U is feasible and has a P-, Q-, PQ- or ST-cross-cut pair then e is a bidirectional edge.*
2. *If U has no pair of interlacing bights then e is a unidirectional edge.*

PROOF.
Without loss of generality assume that $e \in P[s; t]$ and u is to the left of v on P.

(1) Since U is feasible, and since U has no external vertex of attachment on Q, $e \in P[s_P^*; t_P^*]$. Clearly U can have only P- or PQ-cross-cut pairs.

(Case.a) *U has PQ-cross-cut pair, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$, where x_1 is to the left of y_1 on $P]s; t[$.* Since not all the external vertices of attachment lie only on the subpath $J[y_2; x_1]$ or only on the subpath $J[y_1; x_2]$, we have $e \in P[x_1; y_1]$. But since the simple paths $P[s; t]$ and $Q[s; y_2] * N_2[y_2; y_1] * P[y_1; x_1] * N_1[x_1; x_2] * Q[x_2; t]$ traverse e in either direction, e is bidirectional. (Figure 5.)

(Case.b) *U has a P-cross-cut pair, $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$ where x_1 and $x_2 \in P[s; t]$, $y_1 \in P]x_1; x_2[$ and $y_2 \in Q]s; t[$.* Since at least one of the vertices x_1 and x_2 is distinct from s and

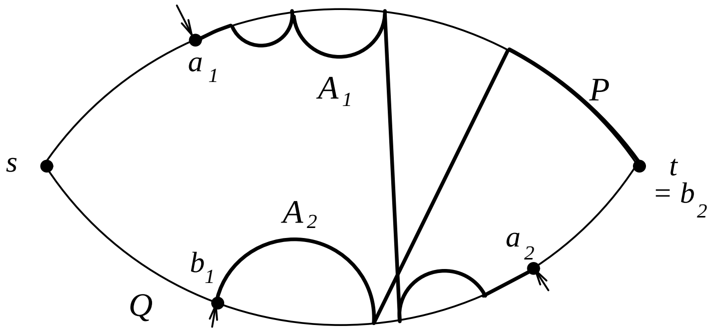


Figure 4: *Interlacing Bights.*

3. There is a lower external vertex of attachment $c \in Q]x_1; x_2[$. ☐

Definition 3.3 BIGHTS AND INTERLACING BIGHTS.

Let the augmented graph U and the cycle $J = \{P\} \cup \{Q\}$ be as defined earlier. Let a and b be two distinct vertices on J such that they are either external vertices of attachment or one of s and t. Let A be a path in G connecting the vertices a and b, where A is not a subpath of J.

This path can be dissected uniquely into an alternating sequence of (possibly empty) *common sections* (subpaths of J) and *cross-cuts* of J. A suitable subpath of A, $A[a'; b']$, meets J in its end vertices a' and b' such that the *common end sections* $A[a; a']$ and $A[b'; b]$ are subpaths of J. We say $A[a'; b']$ is a *bight* of J between the vertices a and b; and $A[a; b] = J[a; a'] * A[a'; b'] * J[b'; b]$ the path associated with the bight.

Let a_1, a_2, b_1 and b_2 be four distinct vertices on J such that they are external vertices of attachment or one of s and t. Let $A_1[a'_1; a'_2]$ and $A_2[b'_1; b'_2]$ be the bights of J between a_1 and a_2, and b_1 and b_2, respectively. We say the bights A_1 and A_2 *interlace*, if the associated paths are vertex-disjoint and a_1 and a_2 separate b_1 and b_2 in the cycle J. (Figure 4.) ☐

Definition 3.4 FEASIBLE AUGMENTED GRAPH.

Let U be an augmented graph with the set of B^{PQ}-bridges, $\mathcal{B}$, such that U has at least two distinct external vertices of attachment and the bridges of $\mathcal{B}$ have at least four distinct vertices of attachment. U is said to be *feasible* if it satisfies the following two conditions:

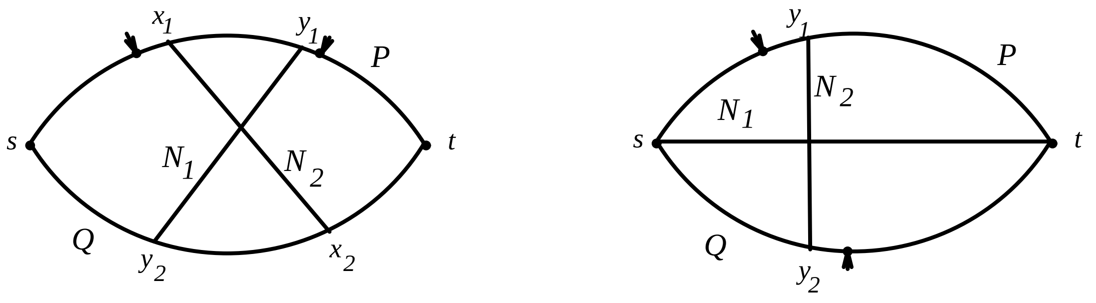
1. If U has no external vertex of attachment on $Q]s_Q; t_Q[$ (where $s_Q \neq t_Q$) then (i) s_P^* and t_P^* are distinct and (ii) not all its external vertices lie only on $J[s_Q; s_P^*]$ or only on $J[t_P^*; t_Q]$.
2. If U has no external vertex of attachment on $P]s_P; t_P[$ (where $s_P \neq t_P$) then (i) s_Q^* and t_Q^* are distinct and (ii) not all its external vertices lie only on $J[s_Q^*; s_P]$ or only on $J[t_P; t_Q^*]$.

Otherwise, U is said to be *infeasible*. ☐

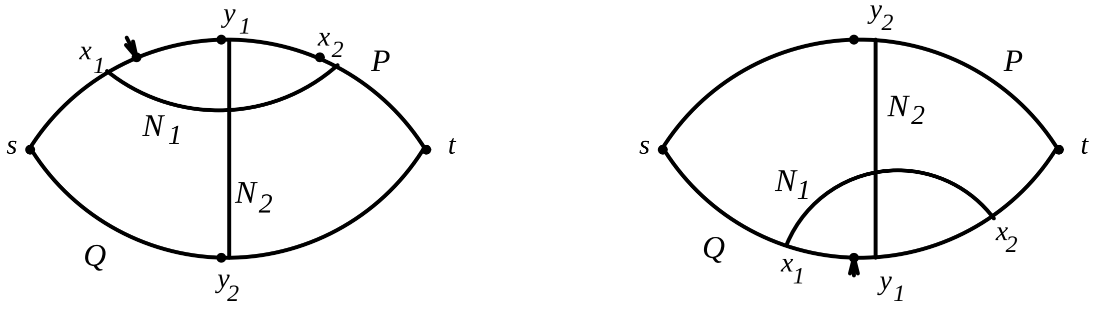
Lemma 3.1 *Let U be an augmented graph. If U is infeasible then U does not have a pair of interlacing bights.*

PROOF.

Let U be an augmented graph with the set of B^{PQ}-bridges, $\mathcal{B}$. Assume to the contrary—that



PQ-Cross-Cut Pair **ST-Cross-Cut Pair**



P-Cross-Cut Pair **Q-Cross-Cut Pair**

Figure 3: *Interlacing Cross-Cuts.*

1. x_1, $y_1 \in P]s;t[$ and x_2, $y_2 \in Q]s;t[$, where x_1 is to the left of y_1 on P.
2. Not all the external vertices of attachment lie only on the subpath $J[y_2;x_1]$ or only on the subpath $J[y_1;x_2]$.

- ST-CROSS-CUT PAIR, if it satisfies the following two conditions:
  1. $x_1 = s$, $x_2 = t$, $y_1 \in P]s;t[$ and $y_2 \in Q]s;t[$.
  2. There are an upper external vertex of attachment on $P]s;t[$ and a lower external vertex of attachment on $Q]s;t[$.
- P-CROSS-CUT PAIR, if it satisfies the following three conditions:
  1. x_1, $x_2 \in P[s;t]$ and at least one of them is distinct from s and t.
  2. $y_1 \in P]s;t[$ and $y_2 \in Q]s;t[$.
  3. There is an upper external vertex of attachment $c \in P]x_1;x_2[$.
- Q-CROSS-CUT PAIR, if it satisfies the following three conditions:
  1. x_1, $x_2 \in Q[s;t]$ and at least one of them is distinct from s and t.
  2. $y_1 \in Q]s;t[$ and $y_2 \in P]s;t[$.

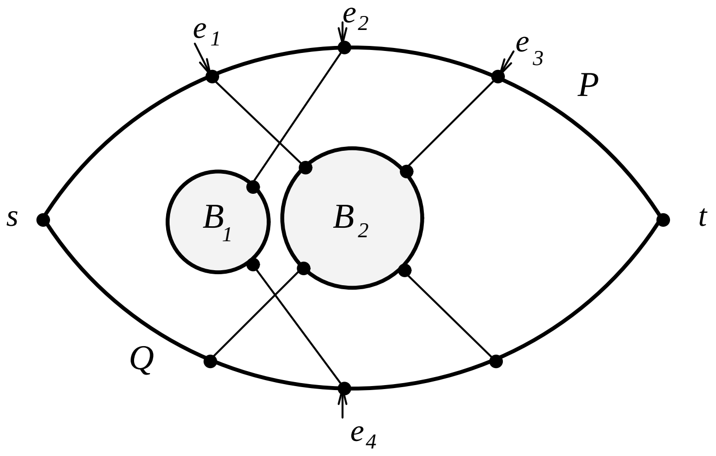


Figure 2: *An Augmented Graph.*

§**3.1**

Definition 3.1 Let U be a graph, consisting of the cycle $J = \{P\} \cup \{Q\}$ and a set (possibly, empty) of B^{PQ}-bridges, $\mathcal{B}$. Some (possibly none) of its vertices on $P]s;t[$ may be labeled as *upper external vertices of attachment*, *UA*, and some (possibly none) of its vertices on $Q]s;t[$, as *lower external vertices of attachment*, *LA*. The set $\{\text{UA}\} \cup \{\text{LA}\}$ is its set of *external vertices of attachment*, *EA*. We refer to U as an *Augmented Graph.* (See Figure 2: UA $= \{e_1, e_2, e_3\}$ and LA $= \{e_4\}$.) ☐

Note that every TYPE.IV graph with no external vertex of attachment is trivially an augmented graph.

Usually, a non-trivial augmented graph may be created by activation of an edge as follows:

Let $e = [u, v]$ be an arbitrary edge on the path $P[s;t]$. Let u be to the left of v on $P[s;t]$. We *activate* the edge e by introducing:

1. An external vertex of attachment at u, if u is distinct from s.
2. An external vertex of attachment at v, if v is distinct from t.

We say that the resulting graph, an augmented graph, is obtained from G by the activation of the edge e, and represent it by $u(G : e)$.

In general, the external vertices will be created in the process of recursive analysis of the graph and represent conduits through which certain vertex disjoint paths may enter and exit the augmented graph.

Definition 3.2 INTERLACING CROSS-CUTS.

Let U be an augmented graph with a cycle $J = \{P\} \cup \{Q\}$ and let $N_1[x_1; x_2]$ and $N_2[y_1; y_2]$ be a pair of interlacing vertex-disjoint cross-cuts such that not all the vertices x_1, x_2, y_1 and y_2 lie only on $P[s;t]$ or lie only on $Q[s;t]$. Such a pair of cross-cuts (Figure 3) is said to be a:

- *PQ*-CROSS-CUT PAIR, if it satisfies the following two conditions:

common steps and retain enough informations from one labeling to another future labeling, we could improve the efficiency of the over-all algorithm.

Here, we will see how to exploit the intuitions noted above in order to devise an algorithm that can label the edges of P and Q of a TYPE.IV graph correctly in time $O(|E| \cdot |V|)$. Such an algorithm would be adequate for our stated goal of finding an $O(|E| \cdot |V|)$ time algorithm for the general "bidirectional edges problem."

Towards this goal, we proceed by forming a characterization of the bidirectionality of an edge of the path $P[s;t]$ in terms two properties: *feasibility property* and *admissibility property*, the later being a stronger condition. These properties lead to the definitions of two subsets of the edges $E(P[s;t])$: the *feasible edges*, $\mathrm{FE}(G:P)$ and the *admissible edges*, $\mathrm{AE}(G:P)$.

$$E(P[s;t]) \supseteq \mathrm{FE}(G:P) \supseteq \mathrm{AE}(G:P).$$

The feasible edges are relatively easy to compute; but the admissible edges cause more complications as they require one to detect if the graph has certain pairs of interlacing vertex-disjoint cross-cuts (with respect to the cycle $J = \{P\} \cup \{Q\}$). However, since the bidirectional edges on P and Q of a TYPE.IV graph are simply

$$\mathrm{AE}(G:P) \cup \mathrm{AE}(G:Q),$$

it suffices to show how to compute $\mathrm{AE}(G:P)$ in time $O(|E| \cdot |V|)$.

§2.3 Some Notations Let G, s, t, $P[s;t]$, $Q[s;t]$ and $J = \{P\} \cup \{Q\}$ be as before. If B is a bridge of the cycle J (and similarly, for $\mathbf{B}$, a block of bridges) with at least one vertex of attachment on $P]s;t[$, then the left- and the right-most vertices of attachment of B on $P[s;t]$ are referred to by $s_P(B)$ and $t_P(B)$ (and, in case of a block of bridges, $\mathbf{B}$, $s_P(\mathbf{B})$ and $t_P(\mathbf{B})$), and the left-most and the right-most vertices of attachment of B on $P]s;t[$ are referred to by $s^*_P(B)$ and $t^*_P(B)$ (and, in case of a block of bridges, $\mathbf{B}$, $s^*_P(\mathbf{B})$ and $t^*_P(\mathbf{B})$).

If, on the other hand, B is a bridge of the cycle J with at least one vertex of attachment on $Q]s;t[$, then $s_Q(B)$, $t_Q(B)$, $s_Q(\mathbf{B})$, $t_Q(\mathbf{B})$, etc. are defined in an identical manner.

If the bridge or the block of bridges under consideration is clear from the context then we simply write s_P, s_Q, s^*_P, s^*_Q, t_P, t_Q, t^*_P and t^*_Q.

3 Characterization of Bidirectionality

Let G be a TYPE.IV graph consisting of the cycle of $J = \{P\} \cup \{Q\}$, and a single B^{PQ}-bridge, B. In this section, we provide a new characterization for an edge $e \in E(P)$ to be bidirectional.

Before presenting the main theorem, we introduce the following notions:(i) an *augmented graph*, (ii) and the conditions under which such a graph is *feasible*. The main admissibility criterion is based on the condition whether a feasible augmented graph has certain pairs of interlacing vertex-disjoint cross-cuts. Such pairs of cross-cuts are classified into four categories: *P-cross-cut pair*, *Q-cross-cut pair*, *PQ-cross-cut pair* and *ST-cross-cut pair.*

Finally, we introduce the concepts of (i) a *Set of Feasible Edges* and (ii) a *Set of Admissible Edges* of an Augmented Graph with an associated path.

- B^Q-BRIDGES: The set of bridges with *no* vertex of attachment on $P]s;t[$ and *at least one* vertex of attachment on $Q]s;t[$.

If a bridge B of $J = P \cup Q$ in G is not a B^{PQ}-, B^P- or B^Q-bridge then it has only s or t as vertices of attachment. ☐

Example 2.4 In the figure 1, we show B^{PQ}-, B^P- and B^Q- bridges of the paths P and Q. Bridges B_1, B_2 and B_3 are B^{PQ}-bridges; B_4 is a B^P-bridge and B_5, a B^Q-bridge.

Definition 2.5 AMBITUS.

Let J, P and Q be as in the previous definition. Then J is called an *ambitus* if every B^P- or B^Q-bridge avoids every B^{PQ}-bridge. ☐

See Mishra and Tarjan[13] for a linear time algorithm to compute an ambitus.

Let $\mathcal{B} = B_1, \ldots, B_k$ be the bridges of J in G. A non-empty subset of bridges $\mathbf{B} \subseteq \mathcal{B}$ is called a *block* of bridges if it satisfies the following two conditions: (1) If $B_i \in \mathbf{B}$ and B_i and B_j overlap, then $B_j \in \mathbf{B}$. (2) No non-empty proper subset of $\mathbf{B}$ satisfies the above condition.

We say $\mathbf{B}$ is *proper*, if it contains more than one bridge of J in G, otherwise, it is *degenerate*.

§**2.2** Henceforth, we assume that the graph G is a TYPE.IV graph satisfying the following conditions:

Definition 2.6 TYPE.IV GRAPHS.

A TYPE.IV *graph* G is a nonseparable graph G consisting of a cycle J containing the vertices s and t and exactly one B^{PQ}-bridge, such that if the vertices s and t together with their incident edges are deleted from G then the resulting derived subgraph is also nonseparable. ☐

The following observation follows from the algorithmic results of [12]:

Theorem 2.1 *Suppose we have an algorithm that correctly labels the edges of $P[s;t]$ and $Q[s;t]$ of a* TYPE.IV *graph $G = (E, V)$ in time $O(T(|E|, |V|)) \geq O(|E| \cdot |V|)$, where $T(\cdot, \cdot)$ is a monotonically-nondecreasing convex function in both its arguments,* i.e.,

$$T(x, \cdot) + T(y, \cdot) \leq T(x+y, \cdot) \qquad \text{and} \qquad T(\cdot, x) + T(\cdot, y) \leq T(\cdot, x+y),$$

where $x \geq 0$ and $y \geq 0$.

Then there is a set of mutually recursive algorithms that correctly labels the edges of an undirected connected strict graph $G = (E, V)$ in time $O(T(|E|, |V|))$. ☐

We also observed that using well-known algorithms for "two vertex disjoint paths problem," it is easy to label the edges of P and Q of a TYPE.IV graph correctly in time $O(|E| \cdot |V|^2)$; this insight leads to an over-all $O(|E| \cdot |V|^2)$ time algorithm for an arbitrary undirected graph.

Essentially, such an algorithm "examines" each edge of the path P (and Q) individually, in order to find appropriate paths in the graph G that would yield a correct labeling. However, many of the steps performed by the algorithm to label an edge of P are needlessly replicated when a "near-by" edge of P is examined subsequently. Clearly, if we can economize on the

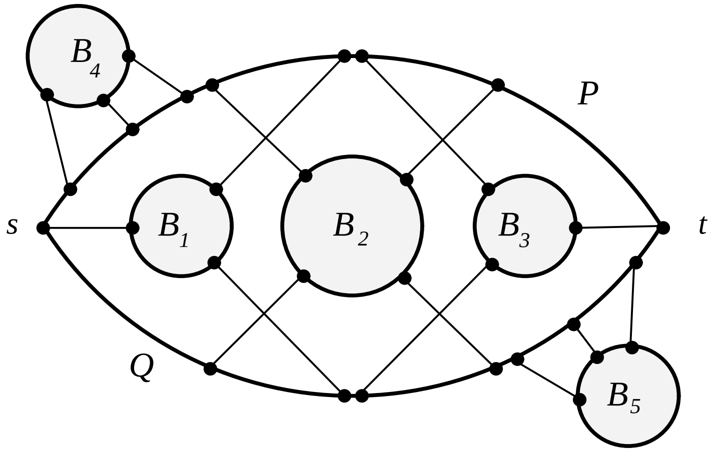


Figure 1: B^{PQ}-, B^{P}- *and* B^{Q}*-bridges of* P *and* Q.

edges; B is a bridge of J in G. The component C of G^{-} is the *nucleus* of B (denoted, $N(B)$). Such a bridge is called *proper*; if a bridge B does not have a nucleus (i.e., B is an edge), it is *degenerate*.

If J is a cycle of the graph G, then a path N in G avoiding J but having its two ends x and y in J is called a *cross-cut* of J between x and y. If B is a bridge of the cycle J in G, then the vertices of attachment of B dissect J into subpaths called the *residual paths* of B in J.

Definition 2.2 RELATIONS BETWEEN BRIDGES.

Let B_1 and B_2 be two distinct bridges of a cycle J of G.

- We say B_1 *avoids* B_2 if and only if one of the following two conditions is satisfied:

1. $|W(G, B_1)| \leq 1$ or $|W(G, B_2)| \leq 1$.
2. All the vertices of attachment of B_1 are contained in a single residual path L of B_2.

- If B_1 and B_2 do not avoid one another we say that they *overlap*.
- If there exist two vertices of attachment x_1 and x_2 of B_1 and two vertices of attachment y_1 and y_2 of B_2, all four distinct, such that x_1 and x_2 separate y_1 and y_2 in the cycle J, then we say that they *interlace*. ☐

Definition 2.3 BRIDGES WITH RESPECT TO THE PATHS.

Let G be an undirected graph with two distinguished vertices s and t with two internally vertex disjoint paths $P[s;t]$ and $Q[s;t]$, which meet each other only in their end vertices, s and t; $J = \{P\} \cup \{Q\}$ is a cycle in G. We consider three different classes of bridges with respect to J:

- B^{PQ}-BRIDGES: The set of bridges with *at least one* vertex of attachment on $P]s;t[$ and *at least one* vertex of attachment on $Q]s;t[$.
- B^{P}-BRIDGES: The set of bridges with *at least one* vertex of attachment on $P]s;t[$ and *no* vertex of attachment on $Q]s;t[$.

graph, and shows how this characterization can be used to devise an efficient algorithm. In the last two sections (4 and 5), the algorithms are developed, followed by arguments for their correctness and an analysis of their time-complexity. A key technical theorem is proven in the appendix.

2 Overview

We begin by recalling some of the key notations and ideas developed in the companion paper[12] and next, present an overview of an efficient algorithm for the "bidirectional edges problem." In this paper, standard graph theoretic terminology is used without explicit definitions here; readers unfamiliar with the terminology may consult [11,12] or [13].

Consider an undirected graph $G = (V, E)$ consisting of a finite set V of *vertices* and a set E of pairs of vertices, called *edges*. A *path* in G from u to v (u, $v \in V$) in G is a sequence of vertices in V ($u =$) u_0, u_1, …, u_k ($= v$) such that $\langle u_i, u_{i+1}\rangle \in E$ for $0 \leq i < k$. (Sometimes denoted by $u \xrightarrow{*} v$.) The vertices u and v are called the *ends* of the path P. All other vertices of the path (i.e., u_i's for $0 < i < k$) are the *internal vertices* of the path.

If P is a path from u to v, $u = u_0, u_1, \ldots, u_k = v$, and $0 \leq i \leq j \leq k$ then the subpath from u_i to u_j, including both u_i and u_j is represented by $P[u_i; u_j]$; the subpath excluding u_i but including u_j, by $P]u_i; u_j]$; the subpath including u_i but excluding u_j, by $P[u_i; u_j[$ and the subpath excluding both u_i and u_j, by $P]u_i; u_j[$. If $P_1 = u_0, u_1, \ldots, u_i$ and $P_2 = u_i, u_{i+1}, \ldots, u_k$ are two paths then the *concatenation* of P_1 and P_2 is $P_1 * P_2 = u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_k$.

The path is *simple* if $u_0, \ldots, u_k$ are distinct (except possibly $u_0 = u_k$) and the path is a *cycle* if $u_0 = u_k$. By convention there is a path of no edges from every vertex to itself (*null path*), but a cycle must contain at least two edges. Two simple paths P_1 and P_2 are said to be *vertex disjoint*, if the vertices of P_1 and P_2 are mutually distinct; *internally vertex disjoint*, if the internal vertices of P_1 and P_2 are mutually distinct.

§**2.1** We recall the following definitions.

Definition 2.1 BRIDGES.[Tutte]

Let J be a fixed subgraph of G. For a subgraph G_1 of G, a *vertex of attachment* of G_1 in G is a vertex of G_1 that is incident in G with some edge not belonging to G_1; subgraph G_1 is said to be *J-detached* in G, if all its vertices of attachment are in J. We define a *bridge* of J in G as any subgraph B that satisfies the following three conditions:

- B is not a subgraph of J.
- B is J-detached in G.
- No proper subgraph of B satisfies both (1) and (2).

The set of vertices of attachment of a bridge B of a subgraph J in G is denoted by $W(G, B) = \{v_0, v_1, \ldots, v_{k-1}\}$. □

Let G^- be the graph derived from G by deleting the vertices of J and all their incident edges. Let C be any component of G^-. Let B be the subgraph of G obtained from C by adjoining to it each edge of G having one end in C and one in J, and adjoining also the ends in J of all such

1 Introduction

Let $G = (V, E)$ be a finite undirected graph with two distinguished vertices, the *source*, s, and the *sink*, t. We call an edge $e = [u, v]$ of G 'bidirectional', if there are two simple paths connecting s and t and traversing e in either order—u, v and v, u. Similarly, we call an edge $e = [u, v]$ of G 'unidirectional', if every simple path connecting s and t and containing e, traverses e only in one order, say u, v; additionally, e is labeled $\langle u, v\rangle$. The "bidirectional edges problem" is to find all the 'bidirectional' and 'unidirectional' edges of G, together with the labelings of the 'unidirectional' edges.

The notions of 'unidirectional' and 'bidirectional' edges can be formalized in terms of the *labeling function*, ℓ, that maps each undirected edge $[u, v]$ to a subset of $\{\langle u, v\rangle, \langle v, u\rangle\}$.

Definition 1.1 The *edge-labeling function*, ℓ, is defined as follows:

$$\ell([u,v]) \ni \begin{cases} \langle u,v\rangle, & \text{iff there is a simple path} \\ & \qquad (s =)\ w_0, \ldots, w_i, w_{i+1}, \ldots, w_n\ (= t) \\ & \qquad \text{such that } w_i = u \text{ and } w_{i+1} = v; \\ \langle v,u\rangle, & \text{iff there is a simple path} \\ & \qquad (s =)\ w_0, \ldots, w_i, w_{i+1}, \ldots, w_n\ (= t) \\ & \qquad \text{such that } w_i = v \text{ and } w_{i+1} = u. \end{cases}$$

Clearly, an edge $e = [u, v]$ is bidirectional, if $\ell([u, v]) = \{\langle u, v\rangle, \langle v, u\rangle\}$; and unidirectional, if $\ell([u, v]) = \{\langle u, v\rangle\}$ or $\{\langle v, u\rangle\}$. □

The relation between bidirectional edges problem and the classical two vertex disjoint paths problem is elucidated in the previous installment of this paper[12]. Using the efficient algorithms to find two vertex disjoint paths in an undirected graph (Cf. Ohtsuki[16], Seymour[18] and Shiloach[19]; also, Mishra and Tarjan[13]), it is relatively easy to devise an algorithm for bidirectional edges problem with time complexity of $O(|E|^2 \cdot |V|)$.

In this and its companion paper[12], we devise an $O(|E| \cdot |V|)$ time algorithm for bidirectional edges problem; the algorithm makes a novel use of *bridges* of a circuit in a *general* graph. We began in a prequel to this paper with a simple set of reduction processes which resulted in an $O(|E| \cdot |V|^2)$ time algorithm; here, we introduce some additional machinery that further reduces the complexity to $O(|E| \cdot |V|)$. The algorithms described here first appeared in [10] and [11].

The problem of finding all bidirectional edges arises naturally in the context of the simulation of an MOS transistor network, in which a transistor may operate as a unilateral or a bilateral device, depending on the voltages at its source and drain nodes. (Cf. Brand[2].) For efficient simulation, it is important to find the set of transistors that may operate as bilateral devices. Also, sometimes it is desired that information propagates in one direction only, and propagation in the wrong direction (resulting in a *sneak path*) can cause functional error. For a more detailed discussions of this problem, also consult the followings: Frank[6], Barzilai, Breece, Huisman, Iyengar and Silberman [1], Jouppi[7], Chen, Mathews and Newkirk[3], Brand[2], Lee, Gupta and Breuer[8], and Cirit[4].

The paper is organized as follows: Section 2 provides an overview of the algorithm, with a necessary recapitulation of the graph-theoretic terminology and the results developed in the companion paper[12]. Section 3 provides a new characterization of the "path-edges" of a TYPE.IV

Contents

ABSTRACT

The "bidirectional edges problem" *is to find an edge-labelling of an undirected network,* $G = (V, E)$*, with a source and a sink, such that an edge* $[u, v] \in E$ *is labelled* $\langle u, v \rangle$ *or* $\langle v, u \rangle$ *(or both) depending on the existence of a (simple) path from the source to sink that visits the vertices* u *and* v*, in the order* u,v *or* v,u*, respectively. In this paper, building upon the machinery developed in a prequel, we devise an efficient algorithm for this problem with a time complexity of* $O(|E| \cdot |V|)$*. The main technique exploits a clever partition of the graph into a set of paths and bridges which are then analyzed recursively.*

Bidirectional edges problem arises naturally in the context of the simulation of an MOS transistor network, in which a transistor may operate as a unilateral or a bilateral device, depending on the voltages at its source and drain nodes. Here, it is required to detect the set of transistors that may operate as bilateral devices.

KEY WORDS.
Bridge, Cross-cut, Disjoint Paths, MOS Circuit, Pass Transistor, Sneak Path, complexity

Bidirectional Edges Problem: Part II

An Efficient Algorithm

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