TWO-LEVEL SCHWARZ METHODS FOR NONCONFORMING FINITE ELEMENTS AND DISCONTINUOUS COEFFICIENTS

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Abstract. Two-level domain decomposition methods are developed for a simple nonconforming approximation of second order elliptic problems. A bound is established for the condition number of these iterative methods, which grows only logarithmically with the number of degrees of freedom in each subregion. This bound holds for two and three dimensions and is independent of jumps in the value of the coefficients.

Key words. domain decomposition, elliptic problems, preconditioned conjugate gradients, nonconforming finite elements, Schwarz methods

AMS(MOS) subject classifications. 65F10, 65N30, 65N55

1. Introduction. The purpose of this paper is to develop a domain decomposition methods for second order elliptic partial differential equations approximated by a simple nonconforming finite element method, the nonconforming P_1 elements. We consider a variant of a two-level additive Schwarz method introduced in 1987 by Dryja and Widlund [5] for a conforming case. In these methods, a preconditioner is constructed from the restriction of the given elliptic problem to overlapping subregions into which the given region has been decomposed. In addition, in order to enhance the convergence rate, the preconditioner includes a coarse mesh component of relatively modest dimension. The construction of this component is the most interesting part of the work. Here we have been able to draw on earlier multilevel studies, cf. Brenner [1], Oswald [11], as well as on recent work by Dryja, Smith, and Widlund [4]. Our main result shows that the condition number of our iterative methods is bounded by $C(1 + log(H/h)$, where H and h are the mesh sizes of the global and local problems, respectively. We also note that this bound is independent of the variations of the coefficients across the subregion interfaces.

The *face based* and the Neumann-Neumann coarse spaces, that we are introducing, have the following characteristics. The nodal values are constant on each edge (or face) of the subregions and the values at the other nodes are given by a simple but nonstandard interpolation formula. Thus the value at any node in the interior of a subregion is a convex combination of three (or four) values given on the boundary, in case of triangular (or tetrahedral) substructures. We note that an important difference between nonconforming and the conforming case is that there are no nodes at the vertices (or wire basket) of the subregions.

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2. Differential and Finite Element Model Problems. To simplify the presentation, we assume that Ω is an open, bounded, polygonal region of diameter 1 in the plane, with boundary $\partial\Omega$. In a separate section, we extend all our results to the three dimensional case.

We introduce a partition of Ω as follows. In a first step, we divide the region Ω into nonoverlapping triangular substructures $\Omega_i, i = 1, \cdots, N$. Adopting common assumptions in finite element theory, cf. Ciarlet [2], all substructures are assumed shape regular, quasi uniform and not to have dead points, i.e. each interior edge is the intersection of the boundaries of two triangular regions. We can show that the theory also holds if we choose nontriangular substructures, where the boundary of each substructure is a composition of several curved edges, and each curved edge is the intersection of two substructures. Naturally, we need assumptions related to the quasi uniformity and nondegeneracy of this partition. Initially, we restrict our exposition to the case of triangular substructures since the main ideas are seen in this case. This partition induces a coarse mesh and we introduce a mesh parameter $H := \max\{H_1, \cdots, H_N\}$ where H_i is the diameter of Ω_i. We denote this triangulation by $\mathcal{T}^H$. Later, we extend the results to nontriangular substructures.

In a second step, we obtain the elements by subdividing the substructures into triangles in such a way that they are shape regular, and quasi uniform. We define a mesh parameter h as the diameter of the smallest element and denote this triangulation by $\mathcal{T}^h$. Similarly, we assume the triangulation $\mathcal{T}^h$ not to have any dead points.

We study the following selfadjoint second order elliptic problem:
Find $u \in W_0^1(\Omega)$, such that

$$a(u,v) = f(v), \; \forall\, v \in \; W_0^1(\Omega) \;, \tag{1}$$

where

$$a(u,v) = \int_\Omega a(x)\, \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx \;\; \text{for } f \in \; L^2 \,.$$

We assume that $a(x) \geq \alpha > 0$ and that it is a piecewise constant function with jumps occurring only across the substructure boundaries. This includes cases where there is a great variation in the value of the coefficient $a(x)$. We remark that there is no difficulty in extending the analysis and the results to the case where $a(x)$ does not vary greatly inside each substructure.

Definition 1. *The nonconforming P_1 element spaces (cf. Crouzeix and Raviart [3]) on the h-mesh and H-mesh is given by*

$V^h := \{v \,|\, v$ *linear in each triangle* $T \in \mathcal{T}^h$,
v continuous at the midpoints of the edges of $\mathcal{T}^h$, and
$v = 0$ at the midpoints of edges of $\mathcal{T}^h$ that belong to $\partial\Omega\}$,

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These spaces are nonconforming; in fact $V^H \not\subset V^h$ and $V^h \not\subset W_0^1(\Omega)$.

Let Σ be a region contained in Ω such that $\partial\Sigma$ does not cut through any element. Denote by $V^h_{|\bar{\Sigma}}$ and $\mathcal{T}^h{}_{|\bar{\Sigma}}$ the space V^h and the triangulation $\mathcal{T}^h$ restricted to $\bar{\Sigma}$, respectively.

Given $u \in V^h_{|\bar{\Sigma}}$, we define the discrete weighted energy semi norm by:

$$|u|^2_{W^1_{a,h}(\Sigma)} := a^h_\Sigma(u,u), \tag{2}$$

where

$$a^h_\Sigma(u,v) = \sum_{T \in \mathcal{T}^h{}_{|\bar{\Sigma}}} \int_T a(x)\ \nabla u \cdot \nabla u\ dx. \tag{3}$$

In a similar fashion, we define the inner product $a^H_\Omega(u,v)$ and the semi norm $|u|_{W^1_{a,H}(\Omega)}$ for $u,v \in\ V^H(\Omega)$. In order not to use unnecessary notation, we drop the subscript Ω when the integration is over Ω and the subscript a when $a = 1$.

The discrete problem associated with (1) is given by:
Find $u \in V^h$, such that

$$a^h(u,v) = f(v),\ \forall\ v \in\ V^h(\Omega). \tag{4}$$

Note that $|\cdot|_{W^1_{a,h}(\Omega)}$ is a norm, because if $|u|_{W^1_{a,h}(\Omega)} = 0$, then u is constant in each element. By the continuity at the midpoints of the edges and the zero boundary conditions, we obtain $u = 0$. Note also that f is a continuous linear form. Therefore, we can apply the Lax-Milgram theorem and find that there exists one and only one solution of the discrete equation (4).

We also define the weighted L^2 norm by:

$$\|u\|^2_{L^2_a(\Sigma)} := \int_\Sigma a(x)\ |u(x)|^2\ dx \quad \text{for } u\ \in\ (V^h + V^H + L^2_a)_{|\bar{\Sigma}}. \tag{5}$$

We introduce the following notation: $x \preceq y$, $f \succeq g$ and $u \asymp v$ meaning

$$x \le C\,y, \quad f \ge c\,g \quad \text{and} \quad c\,v \le u \le C\,v, \quad \text{respectively.}$$

Here C and c are positive constants independent of the variables appearing in the inequalities and the parameters related to meshes, spaces and, especially, the weight $a(x)$.

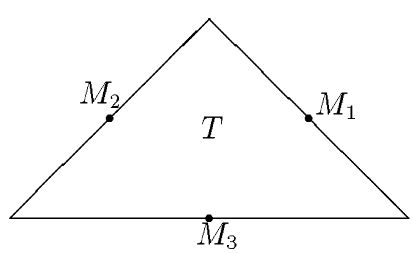


Fig. 1

Sometimes is more convenient to evaluate a norm of a finite element function in terms of the values of this function at the nodal points. By first working on a reference element and then using the assumption that the elements are shape regular, we obtain the following lemma:

LEMMA 1. *For* $u \in V^h_{|\Sigma}$,

$$\|u\|_{L^2_{a,h}(\Sigma)} \asymp h^2 \sum_{T \in \mathcal{T}^h{}_{|\Sigma}} a(T)\,(u^2(M_1) + u^2(M_2) + u^2(M_3)) \tag{6}$$

and

$$|u|_{W^1_{a,h}(\Sigma)} \asymp \sum_{T \in \mathcal{T}^h{}_{|\Sigma}} a(T)\,\{(u(M_1) - u(M_2))^2 + (u(M_2) - u(M_3))^2 + (u(M_3) - u(M_1))^2\}. \tag{7}$$

where M_1, M_2, M_3 *are the midpoints of the edges of the triangle* T *as in Fig. 1.*

An inverse inequality can be obtained by using only local properties. It is easy to see that for $u \in V^h$,

$$|u|_{W^1_{a,h}} \preceq h^{-1} \|u\|_{L^2_a}. \tag{8}$$

3. Additive Schwarz Schemes. We now describe the special additive Schwarz method introduced by Dryja and Widlund; see e.g. [6,7]. In this method, we cover Ω by overlapping subregions obtained by extending each substructure Ω_i to a larger region Ω_i'. We assume that the overlap is δ_i, where δ_i is the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$, and we denote by δ the minimum of the δ_i. We also assume that $\partial\Omega_i'$ does not cut through any element. We make the same construction for the substructures that meet the boundary except that we cut off the part of Ω_i' that is outside of Ω.

For each Ω_i', a P_1 nonconforming finite element subdivision is inherited from the h-mesh subdivision of Ω. The corresponding finite element space is defined by

$$V_i^h := \{v \,|\, v \in V^h, \text{ support of } v \subset \Omega_i'\}, \quad i = 1, \cdots, N. \tag{9}$$

The coarse space $V_0^h \subset V^h(\Omega)$ is given as the range of I_H^h (or $\tilde{I}_H^h$) where the *prolongation operator* I_H^h (or $\tilde{I}_H^h$) will be defined later.

Our finite element space is represented as a sum of $N+1$ subspaces

$$V^h = V_0^h + V_1^h + \cdots + V_N^h. \tag{10}$$

We introduce operators $P_i : V^h \to V_i^h$, $i = 0, \cdots, N$, by

$$a^h(P_i w, v) = a^h(w, v), \quad \forall\, v \in V_i^h, \tag{11}$$

and the operator $P : V^h \rightarrow V^h$, by

$$P = P_0 + P_1 + \cdots + P_N. \tag{12}$$

In matrix notation, P_0 is given by

$$P_0 = I_H^h ({I_H^h}^T K \, I_H^h)^{-1} {I_H^h}^T K \tag{13}$$

where K is the global stiffness matrix associated with $a_h(\cdot, \cdot)$.

We replace the problem (4) by

$$Pu = g, \quad g = \sum_{i=0}^{N} g_i \text{ where } g_i = P_i u. \tag{14}$$

By construction, (4) and (14) have the same solution. We point out that g_i can be computed, without knowledge of u, since we can find g_i by solving

$$a^h(g_i, v) = a^h(u, v) = f(v), \; \forall v \in V_i^h . \tag{15}$$

The operator P is positive definite and and symmetric with respect to $a^h(\cdot, \cdot)$. We can therefore solve (14) by a conjugate gradient method. In order to estimate the rate of convergence, we need to obtain upper and lower bounds for the spectrum of P. A lower bound is obtained by using the following lemma; cf. Zhang [13,12].

LEMMA 2. *Let P_i be the operators defined in equation (11) and let P be given by (12). Then*

$$a^h(P^{-1}v, v) = \min_{v = \sum v_i} \sum a^h(v_i, v_i), \; v_i \in V_i^h. \tag{16}$$

Therefore, if a representation $v = \sum v_i$ can be found such that

$$\sum_{i=0}^{N} a^h(v_i, v_i) \leq C_0^2 a^h(v, v), \quad \forall v \in V^h, \tag{17}$$

then

$$\lambda_{min}(P) \geq C_0^{-2}.$$

An upper bound on the spectrum is obtained by bounding

$$a^h(Pv, v) = a^h(P_0 v, v) + a^h(P_1 v, v) + \cdots + a^h(P_N v, v), \tag{18}$$

from above in terms of $a^h(v, v)$. Using Schwarz's inequality, the fact that the P_i are projections, and that the maximum number of regions that intersect at any point is uniformly bounded, it is easy to show that the spectrum of P is bounded above by

$$\max_{p \in \Omega} \{ \#(i : p \in \Omega_i') + 1 \}.$$

4. Properties of the P_1 nonconforming finite element space. We first define a two local equivalence maps in order to obtain some inequalities and local properties for our nonconforming space. Through these mappings, we can extend some results that are known for the piecewise linear conforming elements to our nonconforming case.

We use a bar to denote conforming spaces. Let $\bar{V}^{\frac{h}{2}}|_{\bar{\Omega}_i}$ be the conforming space of piecewise linear functions in $\bar{\Omega}_i$, where the $h/2$-mesh is obtained by joining midpoints of the edges of elements of $\mathcal{T}^h|_{\bar{\Omega}_i}$.

We define the *local equivalence map* $\mathcal{M}_i : V^h|_{\bar{\Omega}_i} \rightarrow \bar{V}^{\frac{h}{2}}|_{\bar{\Omega}_i}$, as follows:

ISOMORPHISM 1. *Given $u \in V^h|_{\bar{\Omega}_i}$, define $\bar{u} = \mathcal{M}_i u$ by the values of $\bar{u}$ at the three sets of points (cf. Fig. 2):*

i) If P is a midpoint of an edge of a triangle in $\mathcal{T}^h$, then

$$\bar{u}(P) := u(P).$$

ii) If P is a vertex of an element in $\mathcal{T}^h$ and belongs to the interior of Ω_i, and the T_j are the elements that have P as a vertex, then

$$\bar{u}(P) := \textit{mean of} \quad u|_{T_j}(P).$$

Here $u|_{T_j}(P)$, is the limit value of $u(x)$ when $x \in T_j$ approaches P.

iii) If Q is a vertex of $\mathcal{T}^h|_{\partial\Omega_i}$, and Q_l and Q_r the two midpoints of $\mathcal{T}^h|_{\partial\Omega_i}$ that are next neighbors of Q, then

$$\bar{u}(Q) := \frac{|Q_l Q|}{|Q_l Q_r|} u(Q_l) + \frac{|Q_r Q|}{|Q_l Q_r|} u(Q_r).$$

Here $|Q_r Q|$ is the length of the segment $Q_r Q$.

Case ii) is illustrated in Fig. 2, where

$$\bar{u}(P) = \frac{1}{6} \sum_{i=1}^{6} u|_{T_i}(P).$$

Case iii) is required in order to have property (21), which will be very important in our analysis.

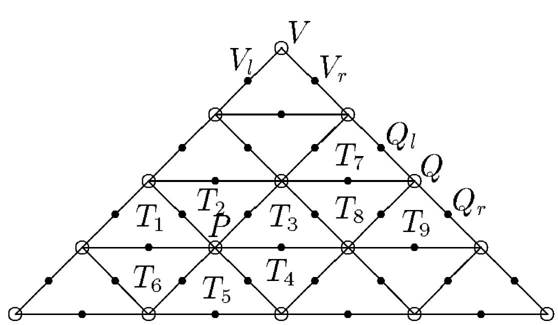


Fig. 2

LEMMA 3. *Given* $u \in V^h|_{\bar\Omega_i}$*, let* $\bar u \in \bar V^{\frac{h}{2}}|_{\bar\Omega_i}$ *given by* $\bar u = \mathcal{M}_i u$*. Then*

$$|\bar u|_{W^1_a(\Omega_i)} \asymp |u|_{W^1_{a,h}(\Omega_i)} \ , \tag{19}$$

$$\|\bar u\|_{L^2_a(\Omega_i)} \asymp \|u\|_{L^2_a(\Omega_i)} \ , \tag{20}$$

and

$$\int_{\partial\Omega_i} \bar u(s)\, ds = \int_{\partial\Omega_i} u(s)\, ds. \tag{21}$$

Here $|\cdot|_{W^1_a(\Omega_i)}$ *is the standard weighted energy semi norm for conforming functions.*

Proof. We first note that we have results similar to (6) and (7) for the conforming space $\bar V^{\frac{h}{2}}|_{\bar\Omega_i}$, where now M_1, M_2 and M_3 are the vertices of a triangle in $\mathcal{T}^{\frac{h}{2}}$. In order to prove (19), we compare (7) with the analogous formula for the piecewise linear conforming space.

For instance (see Fig. 2),

$$|\bar u(Q) - \bar u(Q_r)|^2 = \frac{|Q_l Q|}{|Q_l Q_r|}|u(Q_l) - u(Q_r)|^2.$$

The right hand side can be controlled by the energy semi norm of u restricted to the union of the triangles T_7, T_8 and T_9.

We also prove that if we take two next neighboring vertices of $\mathcal{T}^{\frac{h}{2}}$ in the interior of Ω_i, the energy semi norm can be bounded locally. If $a(x)$ does not vary a great deal, we can work with weighted semi norms. Using the fact that our arguments are local, it is easy to obtain the upper bound of (19).

The lower bound is easy to obtain since the degrees of freedom of V^h are contained in those of $\bar V^{\frac{h}{2}}$.

Similar arguments can also be used to obtain (20).

Finally, it is easy to see that (21) follows directly from iii) even if the refinement is not uniform. ◻

We define another local equivalence map $\mathcal{M}_i^E : V^h|_{\bar\Omega_i} \to \bar V^{\frac{h}{2}}|_{\bar\Omega_i}$, by:

ISOMORPHISM 2. *Given* $u \in V^h|_{\bar\Omega_i}$ *and an edge* E *of* $\partial\Omega_i$*, define* $\bar u = \mathcal{M}_i^E u$ *by the values of* $\bar u$ *at the three sets of points (cf. Fig. 2):*

- *i) Same as step i) of Isomorphism* 1.
- *ii) Same as step ii) of Isomorphism* 1.
- *iii) If* V *is a vertex* $\mathcal{T}^h|_{\partial\Omega_i}$ *and an end point of* E*, and* V_r *the midpoint of* $\mathcal{T}^h|_E$ *that is the next neighbor of* V*, then*

$$\bar u(V) := u(V_r).$$

- *iv) If* Q *is a vertex of* $\mathcal{T}^h|_{\partial\Omega_i}$ *and we are not in case iii), then*

$$\bar u(Q) := \frac{|Q_l Q|}{|Q_l Q_r|}u(Q_l) + \frac{|Q_r Q|}{|Q_l Q_r|}u(Q_r).$$

Using the same ideas as in Lemma 3, we can prove:

LEMMA 4. *Given $u \in V^h|_{\bar\Omega_i}$, let $\bar u \in \bar V^{\frac{h}{2}}|_{\bar\Omega_i}$ given by $\bar u = \mathcal{M}_i^E u$. Then*

$$|\bar u|_{W_a^1(\Omega_i)} \asymp |u|_{W_{a,h}^1(\Omega_i)} \,, \tag{22}$$

$$\|\bar u\|_{L_a^2(\Omega_i)} \asymp \|u\|_{L_a^2(\Omega_i)} \,, \tag{23}$$

and

$$\int_E \bar u(s)\, ds = \int_E u(s)\, ds. \tag{24}$$

5. The Interpolation Operator. Let $v \in V^h$ and let P_{ij} be the midpoint of the edge E_{ij} common to $\bar\Omega_i$ and $\bar\Omega_j$.

DEFINITION 2. *The Interpolation operator $I_h^H : V^h \to V^H$, is given by:*

$$(I_h^H v)(P_{ij}) := \frac{1}{|E_{ij}|} \int_{E_{ij}} v|_{\bar\Omega_i}(x)\ dx = \frac{1}{|E_{ij}|} \int_{E_{ij}} v|_{\bar\Omega_j}(x)\ dx. \tag{25}$$

The second equality follows from the fact that the mean of v on each edge of an element of $\mathcal{T}^h$ is equal to $v(M_1)$, where M_1 is the midpoint of the edge. It is important to note that the value of $(I_h^H v)(P_{ij})$ depends only on the values of v on the interface E_{ij}. This allows us to obtain stability properties that are independent of the differences of $a(x)$ across the substructure interfaces.

Before studying the stability properties of this operator, we need two lemmas for the piecewise linear conforming finite element space.

The following lemma is a Poincaré-Friedrichs inequality. The idea of the proof can be found in Ciarlet (Theorem 6.1) [2] and in Nečas (Chapter 2.7.2) [10].

LEMMA 5. *Let Γ be a subset of $\partial\Omega_i$, such that Γ and $\partial\Omega_i$ have measures of order H. Then,*

$$\|\bar u\|^2_{L^2(\Omega_i)} \preceq H^2 |\bar u|^2_{W^1(\Omega_i)} + \Big(\int_\Gamma \bar u(x)\, dx\Big)^2, \quad \forall \bar u \in W^1(\Omega_i). \tag{26}$$

As a consequence, if $\int_\Gamma \bar u(x)\, dx = 0$, we have the Poincaré inequality

$$\|\bar u\|_{L_a^2(\Omega_i)} \preceq H\, |\bar u|_{W_a^1(\Omega_i)}. \tag{27}$$

Proof. Consider initially a region Ω with diameter 1. If this is not the case, we use a linear change of variable to get the general result.

We prove first that the functional f, given by

$$f(\bar{u}) = \int_\Gamma \bar{u}(x)\,dx \ , \tag{28}$$

is continuous on the space $W^1(\Omega)$.

In fact

$$|f(\bar{u})| \preceq \|\bar{u}\|_{L^1(\Gamma)} \preceq \|\bar{u}\|_{L^2(\Gamma)} \preceq \|\bar{u}\|_{W^{\frac{1}{2}}(\Gamma)} \preceq \|\bar{u}\|_{W^1(\Omega)} \tag{29}$$

using the Cauchy-Schwarz inequality and a trace theorem.

We argue by contradiction assuming that (26) is false. Then, there exists a sequence $\{\bar{v}_l\}_{l=1}^{\infty}$ such that

$$\|\bar{v}_l\|_{W^1} = 1, \quad \forall\, l, \tag{30}$$

and

$$\lim_{l\to\infty}(|\bar{v}_l|_{W^1}^2 + (f(\bar{v}_l))^2) = 0. \tag{31}$$

Since the sequence $\{\bar{v}_l\}$ is bounded in $\|\cdot\|_{W^1}$, we can by Rellich's theorem find a subsequence, again denoted by $\{\bar{v}_l\}$, and a function $\bar{v} \in W^1$ such that

$$\lim_{l\to\infty}\|\bar{v}_l - \bar{v}\|_{L^2} = 0. \tag{32}$$

By using (31), we have

$$|\bar{v}|_{W^1}^2 = 0 \quad \text{and} \quad f(\bar{v}) = 0.$$

Therefore, $\bar{v} = 0$ and

$$\lim_{l\to\infty}\|\bar{v}_l - \bar{v}\|_{W^1} = 0\,, \tag{33}$$

which contradicts (30). □

The next lemma is a Poincaré-Friedrichs inequality for nonconforming P_1 elements. It is obtained by using Lemmas 3, 4 and 5.

LEMMA 6. *Let $u \in W^1_{a,h}(\Omega_i)$, where Ω_i is a triangular substructure of diameter $O(H)$. Let Γ be $\partial\Omega_i$ (or an edge of $\partial\Omega_i$). Then,*

$$\|u\|_{L^2(\Omega_i)}^2 \preceq H^2|u|_{W_h^1(\Omega_i)}^2 + (\int_\Gamma u(x)\,dx)^2, \quad \forall u \in W_h^1(\Omega_i). \tag{34}$$

As a consequence, if $\int_\Gamma u(x)\,dx = 0$, we have the Poincaré inequality

$$\|u\|_{L^2_{a,h}(\Omega_i)} \preceq H\,|u|_{W^1_{a,h}(\Omega_i)}. \tag{35}$$

The next lemma gives an example of an operator that is L_a^2- and W_a^1-stable.

LEMMA 7. *Let $\bar{u} \in W_a^1(\Omega_i)$, where Ω_i is a triangular substructure of diameter of $O(H)$. Define a linear function $\bar{u}_H$ in Ω_i by*

$$\bar{u}_H(P_{ij}) := \frac{1}{|E_{ij}|} \int_{E_{ij}} \bar{u}(x)\ dx\,, \quad j = 1,2,3\,, \tag{36}$$

where the E_{ij} are the edges of Ω_i, and P_{ij} is the midpoint of E_{ij}. Then,

$$|\bar{u}_H(P_{ij})|^2 \preceq \frac{1}{H^2}\|\bar{u}\|^2_{L^2(\Omega_i)} + |\bar{u}|^2_{W^1(\Omega_i)}, \tag{37}$$

$$|\bar{u}_H|_{W_a^1(\Omega_i)} \preceq |\bar{u}|_{W_a^1(\Omega_i)}, \tag{38}$$

and

$$\|\bar{u}_H - \bar{u}\|_{L_a^2(\Omega_i)} \preceq H|\bar{u}|_{W_a^1(\Omega_i)}. \tag{39}$$

Proof. Consider initially a region Ω with diameter of 1.

Using that $|E_{ij}| = \mathrm{O}(1)$, the Cauchy-Schwarz inequality and a trace theorem, we have

$$|\bar{u}_H(P_{ij})|^2 \asymp |\int_{E_{ij}} \bar{u}(x)\ dx|^2 \preceq \|\bar{u}\|^2_{L^2(E_{ij})}$$

$$\preceq \|\bar{u}\|^2_{W^{\frac{1}{2}}(E_{ij})} \preceq \|\bar{u}\|^2_{W^1(E_{ij})} \preceq \|\bar{u}\|^2_{L^2(E_{ij})} + |\bar{u}|^2_{W^1(E_{ij})}.$$

We obtain (37) by returning to a region of diameter H.

Note that for any constant c

$$|\bar{u}_H|^2_{W_H^1(\Omega_i)} \asymp \tag{40}$$

$$|\bar{u}_H(P_{i1}) - \bar{u}_H(P_{i2})|^2 + |\bar{u}_H(P_{i2}) - \bar{u}_H(P_{i3})|^2 + |\bar{u}_H(P_{i3}) - \bar{u}_H(P_{i1})|^2$$

$$\preceq \|\bar{u} - c\|^2_{W^1(\Omega_i)}.$$

By choosing $c = \bar{u}(P_{i1})$ and $\Gamma = E_{i1}$, we can apply Lemma 5 and obtain the W^1-stability (38).

We now prove the L^2-stability. Since $\bar{u} - \bar{u}_H$ has mean zero on $\partial\Omega_i$, we can apply the Poincaré inequality (27) and obtain

$$\|\bar{u} - \bar{u}_H\|_{L^2(\Omega_i)} \preceq H\ |\bar{u} - \bar{u}_H|_{H^1(\Omega_i)}.$$

Using the first part of this lemma, we obtain the L^2-stability (39). □

The next lemma shows that the interpolation operator I_h^H, defined by (25), is locally L_a^2- and W_a^1-stable.

LEMMA 8. *Let $u \in V^h(\Omega)$. Then $u_H = I_h^H u$ satisfies the following properties*

$$|u_H|_{W^1_{a,H}(\Omega_i)} \preceq |u|_{W^1_{a,h}(\Omega_i)} \,, \tag{42}$$

and

$$\|u_H - u\|_{L^2_a(\Omega_i)} \preceq H\,|u|_{W^1_{a,h}(\Omega_i)}\,, \quad i = 1, \cdots, N. \tag{43}$$

Proof. Let $u_H = I_h^H u$ and let $\bar{u} \in W^1(\Omega_i)$ be given by $\bar{u} = \mathcal{M}_i^{E_{i1}} u$ and let $\bar{u}_H(P_{i1})$ be given by (36). Using the properties (24) and (25), we have

$$u_H(P_{i1}) = \bar{u}_H(P_{i1}). \tag{44}$$

Therefore, by (44), (37) and Lemma 4, we have

$$|u_H(P_{i1})|^2 = |\bar{u}_H(P_{i1})|^2 \preceq \frac{1}{H^2}\|\bar{u}\|^2_{L^2(\Omega_i)} + |\bar{u}|^2_{W^1(\Omega_i)} \tag{45}$$

$$\preceq \frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)} + |u|^2_{W^1(\Omega_i)}.$$

We also obtain the same estimate for $|u_H(P_{i2})|$ and $|u_H(P_{i3})|$.

The rest of the proof is similar to that of Lemma 7. We now use the Poincaré inequality for nonconforming elements. □

6. The Prolongation Operator. In this section, we introduce several prolongation operators and establish that they are stable. The range of each of these operators will serve as a coarse space in our algorithms.

DEFINITION 3. *The Prolongation Operator $I_H^h : V^H \to V^h$, is given by:*

i) For all nodal points P of $\mathcal{T}^h$ that belongs to an edge E_{ij} common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, let $(I_H^h u_H)(P) := u_H(P_{ij})$, where P_{ij} is the midpoint of the edge E_{ij}.

ii) Given $I_H^h u_H$ at the nodal points of $\Gamma = \cup_i \partial\Omega_i$ from i), let $I_H^h u_H(\Omega)$ be the P_1-nonconforming harmonic extension inside each Ω_i.

It is easy to check that $u_h = I_H^h u_H \in V^h(\Omega)$. A disadvantage of step *ii)* is that we have to solve exactly a local Dirichlet problem for each substructure in order to obtain the harmonic extension. Other extensions can be used, which we call *approximate harmonic extensions.* They are given by simple explicit formulas and have the same L_a^2 and $W_{a,h}^1$ stability properties as the harmonic one.

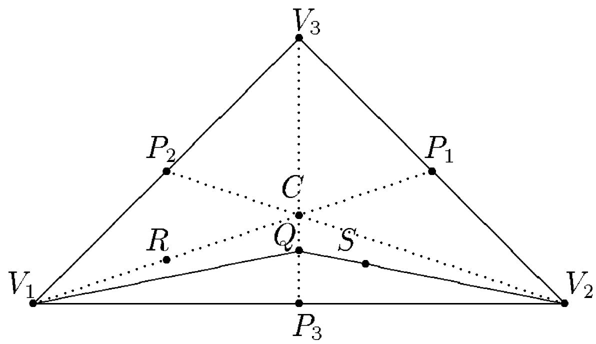


Fig. 3

Our first construction is a natural generalization of the partition of unity introduced by Dryja and Widlund in [6]; this partition of unity will provide the basis functions of our approximate extensions. Let P_j, $j = 1, 2, 3$, be the midpoints of the edges of Ω_i, and let V_j be the vertex of Ω_i that is opposite to P_j. Let C be the barycenter of the triangle Ω_i, i.e. the intersection of the line segments connecting V_j to P_j.

EXTENSION 1. *The construction of an approximate harmonic extension is defined by the following steps (see Fig. 3.):*

i) Let

$$\bar{u}(C) := \frac{1}{3}\{u_H(P_1) + u_H(P_2) + u_H(P_3)\}.$$

ii) For a point R that belongs to a line segment that connects C to a vertex V_j, let

$$\bar{u}(R) := \bar{u}(C).$$

iii) For a point Q that belongs to a line segments connecting C to P_j, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(P_j)$, i.e by

$$\bar{u}(Q) := \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(P_j).$$

Here $\lambda(Q)$ =distance(Q, P_j)/distance(C, P_j).

iv) For a point S that belongs to the line segment connecting the previous point Q to a vertex V_k, with $k \neq j$, let

$$\bar{u}(S) := \bar{u}(Q).$$

v) Finally, let $I_H^h u_H = I_h \bar{u}$, where I_h is the interpolation operator into the space V^h that preserves the values of a function at the midpoints of the edges of the elements.

Note that the function $\bar{u}$ just constructed is continuous except at the vertices V_j of Ω_i. The step *i)* can be viewed as emulating the mean value theorem for harmonic functions. However, near the vertices, $\bar{u}$ is a bad approximation of the harmonic extension. We know that the local behavior of the harmonic extension near a vertex V_j depends primarily on the boundary values in the vicinity of V_j. For instance, if $u_H(P_1) = 0, u_H(P_3) = 0$, and $u_H(P_2) = 1$, we should obtain $u_h \simeq 0$ near V_2; in addition, by using symmetry arguments, we should have $u_h \simeq 1/2$ for points near V_1 that lie on the bisector that passes through V_1 . With this in mind, we now construct an alternative approximate harmonic extension.

We change notation in order to be able to use Fig. 3. Let now C be the point where the three bisectors intersect.

EXTENSION 2. *The construction of the approximate harmonic extension is defined by (see Fig. 3):*

i) Same as Step i) of Extension 1.

ii) Define $\bar{u}(V_j) = \frac{1}{2}\sum_{l\neq j}\bar{u}(P_l)$. *For a point R that belongs to a line segment connecting C to V_j, define $\bar{u}(R)$ by linear interpolation between the values $\bar{u}(C)$ and $\bar{u}(V_j)$.*

iii) Same as Step iii) of Extension 1.

iv) For a point S that belongs to a line segment connecting the previous point Q to V_k, $k \neq j$, $\bar{u}(S)$ is defined by linear interpolation between the values $\bar{u}(Q)$ at Q and $f(Q,j,k)$ at V_k. Here,

$$f(Q,j,k) = \lambda(Q)\,\bar{u}(V_j) + (1-\lambda(Q))\,\bar{u}(P_k).$$

v) Same as Step v) of Extension 1.

A disadvantage of this extension is that we cannot just work in a reference triangle, since the angles are not preserved under a linear transformation. This is similar to the fact that under a linear transformation a harmonic function does not necessarily remain harmonic. We can construct other approximate harmonic extensions which combine the properties of the two extensions, given so far, and working, for instance, with the barycenter C as in Extension 2 and replacing the weight 1/2 in Step ii).

The next lemma shows that the extensions given above have quasi-optimal energy stability. Using ideas of Dryja and Widlund [6], we prove the following lemma.

LEMMA 9. *Let $u_H \in V^H(\Omega)$. Then*

$$|I_H^h u_H|_{W^1_{a,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}}\,|u_H|_{W^1_{a,H}(\Omega_i)} \qquad (46)$$

and

$$\|I_H^h u_H - u_H\|_{L^2_a(\Omega_i)} \preceq H\,|u_H|_{W^1_{a,H}(\Omega_i)}. \qquad (47)$$

Proof. Let $\theta_h^j \in V^h|_{\Omega_i}, j = 1,2,3$, be the approximate harmonic extensions constructed from the boundary values $\theta_h^j = 1$ at the h-mesh nodes on the edge E_{ij}, and $\theta_h^j = 0$ at the other boundary nodes of $\partial\Omega_i$. It easy to see that the θ_h^j form a basis of all approximate harmonic extensions that take constant values on the edges of the substructure. It is easy to show that if a point x belongs to the interior of an element of Ω_i, then $|\nabla\,\theta_h^j(x)|$ is bounded by C/r, where r is the minimum distance from x to any vertex of Ω_i. Note that any element that touches a vertex of Ω_i provides an order one contribution to the energy semi norm. To estimate the contribution to the energy semi norm from the rest of the substructure, we introduce polar coordinate systems centered at the vertices of Ω_i. Then,

$$|\theta_h^j|^2_{W_h^1(\Omega_i)} \preceq 1 + \int\int_h^H r^{-2}\, r\, dr\, d\varphi \preceq 1 + log(H/h). \tag{48}$$

Since the partition of unity θ_h^j forms a basis, it is easy to see that

$$|I_H^h u_H|^2_{W_h^1(\Omega_i)} \preceq \tag{49}$$

$$(1 + log(H/h))\left\{|u_H(P_1)|^2 + |u_H(P_2)|^2 + |u_H(P_3)|^2\right\}$$

and using ideas similar to that of Lemma 7, we have

$$|I_H^h u_H|^2_{W_h^1(\Omega_i)} \preceq (1 + log(H/h))\left\{|u_H(P_1) - u_H(P_2)|^2 + \right.$$

$$\left.|u_H(P_2) - u_H(P_3)|^2 + |u_H(P_3) - u_H(P_1)|^2\right\}$$

$$\asymp (1 + log(H/h))|u_H|^2_{W_h^1(\Omega_i)}.$$

By construction, it is easy to see that

$$|(I_H^h u_H)(x)| \leq \max_{i=1,2,3} |u_H(P_i)|.$$

Therefore

$$\|I_H^h u_H - u_H\|^2_{L^2(\Omega_i)} \preceq \sum_i H^2\,|u_H(P_i)|^2\,,$$

and by using (45) and (35), we obtain (47).

Since $a(x)$ varies little in each Ω_i, these arguments are also valid for the weighted norms and we obtain (46). □

Using Lemmas 6 and 9 and the triangular inequality, we have:

THEOREM 1. *Let $u \in V^h(\Omega)$. Then*

$$\|I_H^h I_h^H u - u\|_{L_a^2(\Omega_i)} \preceq H\, |u|_{W_{a,h}^1(\Omega_i)} \tag{50}$$

and

$$|I_H^h I_h^H u|_{W_{a,h}^1(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}}\, |u|_{W_{a,h}^1(\Omega_i)}. \tag{51}$$

REMARK 1. *It is easy to see that we do not need to use the fact that $u_H \in V_H(\Omega)$; we only need to calculate values $V_H(P_{ij})$ by formula (25) at the midpoint P_{ij} of the edge E_{ij}. The next step is to provide the constant value $V_H(P_{ij})$ to all nodes of the interface and perform an approximate harmonic extension.*

REMARK 2. *The extensions also can be constructed for nontriangular substructures. In a first step, we construct a partition of unity in Ω_i. This can be done by using ideas similar to those of the triangular case. By using the same technique as in the proof of Lemma 9, we can show that*

$$|I_H^h u_H|^2_{W_{a,h}^1(\Omega_i)} \preceq \tag{52}$$

$$(1 + log(H/h)) \sum_{j=1}^{N_e^i} a(\Omega_i)\, |u_H(P_{ij}) - u_H(P_{i(j-1)})|^2$$

where P_{ij} and $P_{i(j-1)}$ are neighboring midpoints of edges of $\partial\Omega_i$ and N_e^i is the number of edges of $\partial\Omega_i$. We obtain (50) by noting that each term of the sum is bounded by $|u|^2_{W_{a,h}^1(\Omega_i)}$.

7. The Neumann-Neumann Basis. In this section, we consider a Neumann-Neumann coarse space. This is the P_1 nonconforming version of a coarse space studied in Dryja and Widlund [8], and Mandel and Brezina [9]. However, here we use an approximate harmonic extension inside the substructures. We note that the coarse spaces considered by these authors differ only in how certain weights are chosen. Mandel and Brezina use weights that are convex combinations of the coefficient $a(x)$, while Dryja and Widlund use $a^{\frac{1}{2}}(x)$. Here we show that any convex combination of $a^{\beta}(x)$, for $\beta \geq 1/2$, leads to stability. We point out that the choice $\beta = 1/2$ can be viewed as a L^2-average, while $\beta = 1$ is an average in the L^1 sense.

We call the coarse space of the previous section, *face based.* There are some differences between Neumann-Neumann and face based coarse spaces. A Neumann-Neumann coarse space has one degree of freedom per substructure, while a face based uses one degree of freedom per edge. A Neumann-Neumann basis function associated with the substructure Ω_i, has support in Ω_i and its neighboring substructures, while a face based function basis, associated with an edge of a substructure, has support in just two substructures. The face based coarse space appears to be more stable since all the estimates, related to the jumps of the coefficients, are tight. In the lemmas that we have proved for the face based methods, all the stability results were derived in individual substructures, while in the Neumann-Neumann case, we need to work in an extended subdomain.

DEFINITION 4. *The Neumann-Neumann interpolation operator,* $I_{NN} : V^h \to V^h$*, as follows:*

i) For each substructure Ω_i*, calculate the mean value on* $\partial\Omega_i$*, i.e.*

$$m_i u := \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} u(s)\, ds.$$

Here $|\partial\Omega_i|$ *is the length size of* $\partial\Omega_i$*.*

ii) For all nodal points P *of* $\mathcal{T}^h$ *that belong to the edge* $E_{i,j}$*, let* $(I_{NN}u)(P) = (\tilde{I}_h^H u)(P_{ij})$*, where*

$$(\tilde{I}_h^H u)(P_{ij}) := \frac{a^\beta(\Omega_i)}{a^\beta(\Omega_i) + a^\beta(\Omega_j)}\, m_i u + \frac{a^\beta(\Omega_j)}{a^\beta(\Omega_i) + a^\beta(\Omega_j)}\, m_j u.$$

Here P_{ij} *is the midpoint of the edge* E_{ij}*.*

iii) Perform an approximate harmonic extension to define $I_{NN}u$ *inside the substructures.*

Note that we can also calculate $m_i u$ by:

$$m_i u = \sum_j \frac{|E_{ij}|}{|\partial\Omega_i|} (I_h^H u)(P_{ij}). \tag{53}$$

Therefore, there exists a linear transformation $I_H^H : V_H \to V_H$, such that $\tilde{I}_h^H u = I_H^H I_h^H u$. The next lemma establishes stability properties for I_H^H.

LEMMA 10. *Let* $u_H \in V^H(\Omega)$ *and* $\beta \geq 1/2$*. Then*

$$|I_H^H u_H|_{W^1_{a,H}(\Omega_i)} \leq C(\beta)\, |u_H|_{W^1_{a,H}(\Omega_i^{ext})}, \tag{54}$$

and

$$\|I_H^H u_H - u_H\|_{L^2_a(\Omega_i)} \leq C(\beta)\, H\, |u_H|_{W^1_{a,H}(\Omega_i^{ext})}. \tag{55}$$

Here the extended domain Ω_i^{ext} *is the union of* Ω_i *and the substructures that share an edge with* Ω_i*.*

Proof. Let us first prove the L^2_a stability. Note that (see Fig. 4.)

$$|u_H(P_{ij}) - (I_H^H u_H)(P_{ij})|^2 = |u_H(P_{ij}) - \frac{a^\beta(\Omega_i)\, m_i + a^\beta(\Omega_j)\, m_j}{a^\beta(\Omega_i) + a^\beta(\Omega_j)}|^2.$$

By using (53) and simple calculations, this quantity is equal to

$$\frac{1}{|a^\beta(\Omega_i) + a^\beta(\Omega_j)|^2} *$$

$$|a^\beta(\Omega_i)\, \{\frac{|E_{ik}|}{|\partial\Omega_i|}(u_H(P_{ij}) - u_H(P_{ik})) + \frac{|E_{il}|}{|\partial\Omega_i|}(u_H(P_{ij}) - u_H(P_{il}))\} +$$

$$a^\beta(\Omega_j)\, \{\frac{|E_{js}|}{|\partial\Omega_j|}(u_H(P_{ij}) - u_H(P_{js})) + \frac{|E_{jr}|}{|\partial\Omega_j|}(u_H(P_{ij}) - u_H(P_{jr}))\}|^2.$$

Using the shape regularity of the subdomains, it is easy to see that

$$a(\Omega_i)\, |u_H(P_{ij}) - (I_H^H u_H)(P_{ij})|^2 \preceq \tag{56}$$

$$\frac{a^{2\beta}(\Omega_i)}{|a^\beta(\Omega_i) + a^\beta(\Omega_j)|^2}\, |u_H|^2_{W^1_{a,H}(\Omega_i)} + \frac{a(\Omega_i)\, a^{2\beta-1}(\Omega_j)}{|a^\beta(\Omega_i) + a^\beta(\Omega_j)|^2}\, |u_H|^2_{W^1_{a,H}(\Omega_j)}$$

and using the fact that $\beta \geq 1/2$, we can bound this quantity by

$$\leq C(\beta)\, |u_H|^2_{W^1_{a,H}(\Omega_i \cup \Omega_j)}.$$

We obtain (55) by adding all the contributions (56) to the $L^2_a(\Omega_i)$ norm. We prove (54) by using the triangular inequality, an inverse inequality, and (55). □

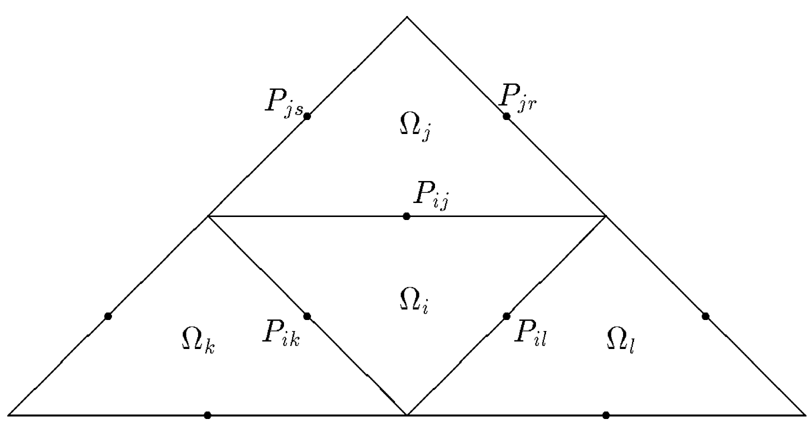


Fig. 4

THEOREM 2. *Let $u \in V^h(\Omega)$ and $\beta \geq 1/2$. Then*

$$\|I_{NN}u - u\|_{L^2_a(\Omega_i)} \leq C(\beta)\, H\, |u|_{W^1_{a,h}(\Omega_i^{ext})}\,, \tag{57}$$

and

$$|I_{NN}u|_{W^1_{a,h}(\Omega_i)} \leq C(\beta)(1 + log(H/h))^{\frac{1}{2}}\, |u|_{W^1_{a,h}(\Omega_i^{ext})}. \tag{58}$$

Proof. Using Lemmas 9, 10 and 8, we have

$$|I_{NN}u|_{W^1_{a,h}(\Omega_i)} \preceq (1+log(H/h))^{\frac{1}{2}}\,|I^H_H I^H_h u|_{W^1_{a,H}(\Omega_i)} \leq$$

$$C(\beta)\,(1+log(H/h))^{\frac{1}{2}}\,|I^H_h u|_{W^1_{a,H}(\Omega_i^{ext})} \leq$$

$$C(\beta)\,(1+log(H/h))^{\frac{1}{2}}\,|u|_{W^1_{a,h}(\Omega_i^{ext})}.$$

The L^2_a-stability is obtained by

$$\|I_{NN}u - u\|_{L^2_a(\Omega_i)} \leq \|I_{NN}u - I^H_H I^H_h u\|_{L^2_a(\Omega_i)} +$$

$$\|I^H_H I^H_h u - I^H_h u\|_{L^2_a(\Omega_i)} + \|I^H_h u - u\|_{L^2_a(\Omega_i)}\,,$$

and by using Lemmas 9, 10 and 8. □

REMARK 3. *We can also prove Theorem 2 for the case of nontriangular substructures; cf. Remarks* 1 *and* 2.

8. The Three Dimensional Case. We show in this section that the methods developed before can be extended to three dimensions.

For simplicity, we assume that Ω is a polyhedral region of diameter 1 in three dimensional space. As before, we introduce a nonoverlapping partition composed of tetrahedra Ω_i of diameter of order H. This defines a coarse space and a triangulation $\mathcal{T}^H$. We further subdivide the substructures into tetrahedra which results in a triangulation $\mathcal{T}^h$ and define the nonconforming P_1 finite element spaces V^h and V^H as in Definition 1. Here, the continuity is enforced at the barycenter of the faces of the triangulations.

The local equivalence maps are given by the following procedure. In each tetrahedral element of $\mathcal{T}^h$ (cf. Fig. 5.), we connect its centroid to the four vertices and to the barycenters of the four faces. We also connect each barycenter to the three vertices. In other words, we subdivide each tetrahedral element into twelve subtetrahedra. We denote this new triangulation by $\mathcal{T}^{\tilde{h}}$. The vertices of $\mathcal{T}^{\tilde{h}}$ are the vertices, barycenters, and centroids of the elements of $\mathcal{T}^h$.

Let $\bar{V}^{\tilde{h}}|_{\bar{\Omega}_i}$ be the conforming space of piecewise linear functions of the triangulation $\mathcal{T}^{\tilde{h}}|_{\bar{\Omega}_i}$.

We define the *local equivalence map* $\mathcal{M}_i : V^h|_{\bar{\Omega}_i} \rightarrow \bar{V}^{\tilde{h}}|_{\bar{\Omega}_i}$, as follows:

ISOMORPHISM 3. *Given* $u \in V^h|_{\bar{\Omega}_i}$, *define* $\bar{u} = \mathcal{M}_i u$ *by the values of* $\bar{u}$ *at the following sets of points:*

i) If P is a vertex of an element of $\mathcal{T}^{\tilde{h}}$ and belongs to the interior of Ω_i, and the K_j are the elements in $\mathcal{T}^{\tilde{h}}|_{\bar{\Omega}_i}$ that have P as a vertex, then

$$\bar{u}(P) := mean\ of\quad u|_{K_j}(P).$$

Here $u|_{K_j}(P)$ is the limit value of $u(x)$ when $x \in K_j$ approaches P.

ii) *If P is a barycenter of a triangle in $\mathcal{T}^h|_{\partial\Omega_i}$, then*

$$\bar{u}(P) := u(P).$$

iii) *If P is a vertex of a triangle in $\mathcal{T}^h|_{\partial\Omega_i}$ and T_j, $j = 1, \cdots, N_P$, are the triangles of $\mathcal{T}^h|_{\partial\Omega_i}$ that have P as a vertex, then*

$$\bar{u}(P) := \sum_{k=1}^{N_P} \frac{|T_k|}{|\cup_{j=1}^{N_P} T_j|} u(C_i).$$

Here C_i and $|T_i|$ are the barycenter and the area of the triangle T_i, respectively.

It is easy to check that the Lemma 3 holds, if we replace $\bar{V}^{h/2}|_{\bar{\Omega}_i}$ by $\bar{V}^{\tilde{h}}|_{\bar{\Omega}_i}$.

We define another local equivalence map $\mathcal{M}_i^F : V^h|_{\bar{\Omega}_i} \to \bar{V}^{\tilde{h}}|_{\bar{\Omega}_i}$, by:

ISOMORPHISM 4. *Given $u \in V^h|_{\bar{\Omega}_i}$ and a face F of $\partial\Omega_i$, define $\bar{u} = \mathcal{M}_i^F u$ by the values of $\bar{u}$ at the following sets of points:*

i) *Same as step i) of Isomorphism 3.*

ii) *Same as step ii) of Isomorphism 3.*

iii) *Let P be a vertex of a triangle in $\mathcal{T}^h|_{\partial\Omega_i}$ that belongs to ∂F, and let T_j, $j = 1, \cdots, N_P^F$, be the triangles of $\mathcal{T}^h|_F$ that have P as a vertex. Then*

$$\bar{u}(P) := \sum_{k=1}^{N_P^F} \frac{|T_k|}{|\cup_{j=1}^{N_P^F} T_j|} u(C_i).$$

iv) *Let P be a vertex of a triangle in $\mathcal{T}^h|_{\partial\Omega_i}$ that does not belong to ∂F, and let T_j, $j = 1, \cdots, N_P$, be the triangles of $\mathcal{T}^h|_F$ that have P as a vertex. Then*

$$\bar{u}(P) := \sum_{k=1}^{N_P} \frac{|T_k|}{|\cup_{j=1}^{N_P} T_j|} u(C_i).$$

It is easy to check that Lemma 4 holds, if we replace $\bar{V}^{h/2}|_{\bar{\Omega}_i}$ by $\bar{V}^{\tilde{h}}|_{\bar{\Omega}_i}$, and let the faces play the role previously played by the edges.

Let $v \in V^h$ and let C_{ij} be the barycenter of the face F_{ij} common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$.

DEFINITION 5. *The interpolation operator $I_h^H : V^h \to V^H$, is given by:*

$$(I_h^H v)(C_{ij}) := \frac{1}{|F_{ij}|} \int_{F_{ij}} v|_{\bar{\Omega}_i}(x)\ dx = \frac{1}{|F_{ij}|} \int_{F_{ij}} v|_{\bar{\Omega}_j}(x)\ dx\ ,$$

where $|F_{ij}|$ is the area of the face F_{ij}.

Using the same ideas as in two dimensions, we can prove lemmas analogous to Lemmas 5-8.

The *prolongation operator* $I_H^h : V^H \to V^h$, is defined as in the two dimensional case. In a first step, we define $(I_H^h u_H)(P) := u_H(C_{ij})$ for all barycenters P of triangles in $\mathcal{T}^h|_{F_{ij}}$. Finally, we perform an P_1-nonconforming harmonic or approximate harmonic extension.

We describe the three dimensional version of Extension 1. This is a generalization of the partition of unity introduced by Dryja, Smith, and Widlund [4]. Let C_j, $j = 1, \cdots, 4$, be the barycenters of the faces F_j of $\partial\Omega_i$, and let V_j be the vertex of Ω_i that is opposite to C_j. Let C the centroid of Ω_i, i.e. the intersection of the line segments connecting the V_j to the C_j. Let E_{jk}, $k = 1, 2, 3$, be the edges of ∂F_j.

EXTENSION 3. *The construction of an approximate harmonic extension $I_H^h u_H$ is defined by the following steps (see Fig. 5.):*

i) Let

$$\bar{u}(C) := \frac{1}{4} \sum_{j=1}^{4} u_H(C_j).$$

ii) For a point Q that belongs to a line segment connecting C to C_j, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(C_j)$, i.e. by

$$\bar{u}(Q) := \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(C_j).$$

Here $\lambda(Q) =$distance$(Q, C_j)/$distance(C, C_j).

iii) For a point S that belongs to any of the three triangles defined by the previous Q, and the edges E_{jk}, $k = 1, \cdots, 3$, let

$$\bar{u}(S) := \bar{u}(Q).$$

iv) Finally, let $I_H^h u_H = I_h \bar{u}$, where I_h is the interpolation operator into the space V^h that preserves the values of a function at the barycenter of the faces of elements in $\mathcal{T}^h$.

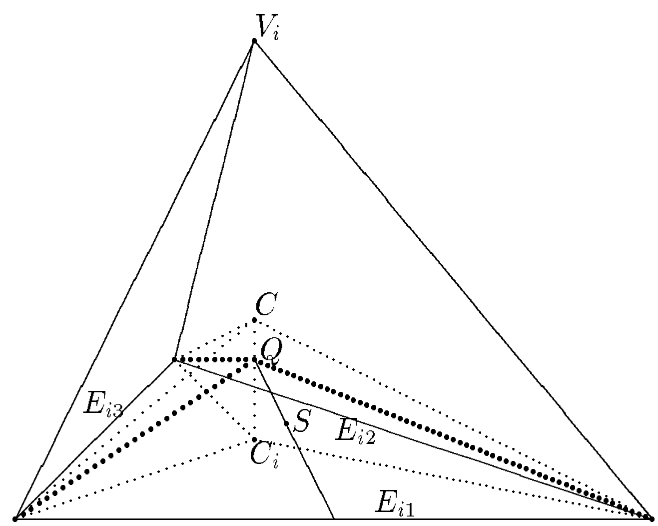
We can also construct an approximate harmonic extension similar to that of Extension 2. This gives a better approximate harmonic extension near the edges.

The prolongation operator I_H^h in three dimensions has the same stability properties as in the two dimensional case, i.e. Lemma 9 still holds.

The idea of the proof is the following. Consider the case where $u_H(\Omega_i)$ is given by $u_H(P_{i1}) = 1$ and $u_H(P_{i2}) = u_H(P_{i3}) = 0$. This gives the partition of the unity introduced by Dryja, Smith, and Widlund [4]. The energy semi norm of u_H is of order H.

Let $\theta_h^{i1} = I_H^h u_H(\Omega_i)$. We note that $|\nabla \theta_h^{i1}(x)|$ is bounded by C/r, where r is the distance to the nearest edge of Ω_i. The contribution to the energy semi norm from the union of the elements with at least one vertex on the edge of the substructure can be bounded by CH, using that the extension is given by a convex combination of the boundary values. To estimate the contribution to the energy from the rest of the substructure, we introduce cylindrical coordinates using the appropriate substructure edge as the z-axis. Integrating $|\nabla\theta_h^{i1}(x)|^2$ over this region, we find that is bounded by $C\,(1 + log(H/h))\,H$.

To prove Lemma 9 for a general u_H, we use the same ideas as for two dimensions. Similarly, we can extend the results to nontriangular substructures and to the Neumann-Neumann case.



$Fig.5$

9. Main Result. In this section, we consider the Schwarz method introduced in the previous sections and prove the following result.

THEOREM 3. *The operator P of the additive Schwarz algorithm, defined by the spaces V_0^h and V_i^h, satisfies:*

$$\kappa(P) \preceq (1 + log(\frac{H}{h}))\,(1 + \frac{H}{\delta}).$$

Here $\kappa(P)$ is the condition number of P. Therefore, if we use a generous overlapping, then

$$\kappa(P) \preceq 1 + log(\frac{H}{h}).$$

The proof of this theorem is essentially the same as in the case of a conforming space; see Dryja and Widlund [7].

Proof. As we have seen before, the upper bound is very easy to obtain. The lower bound is obtained by using Lemma 2. We partition the finite element function $u \in V_h$ as follows. We first choose $u_0 = I_H^h I_h^H u$ or $I_{NN}u$, i.e. apply a face based or

Neumann-Neumann interpolation operator. Let $w = u - u_0$. The other terms in the representation of u are defined by $u_i = I_h(\theta_i w), i = 1, \cdots, N$. Here I_h is the linear interpolation operator into the space V^h that preserves the values at the midpoints of the edges of the elements and $\{\theta_i\}$ is a partition of unity with $\theta_i \in C_0^\infty(\Omega_i')$ and $\sum \theta_i(x) = 1$.

For a relatively generous overlap of the subdomains, these functions can be chosen so that $\nabla\theta_i$ is bounded by C/H. By using the linearity of I_h, we can show that we have a correct partition of u. In order to estimate the semi norm of u_i, we work on one element K at a time. We obtain

$$|u_i|^2_{W^1_{a,h}(K)} \leq 2\,|\bar{\theta}_i w|^2_{W^1_{a,h}(K)} + 2\,|I_h((\theta_i - \bar{\theta}_i)w)|^2_{W^1_{a,h}(K)}$$

.

Here $\bar{\theta}_i$ is the average value of θ_i over K. It is easy to see, by using the inverse inequality (8), that

$$|I_h((\theta_i - \bar{\theta}_i)w)|^2_{W^1_{a,h}(K)} \preceq h^{-2}\,\|I_h((\theta_i - \bar{\theta}_i)w)\|^2_{L^2_a(K)}.$$

We can now use the fact that on K, θ_i differs from its average by at most $C\,h/H$. After summing over all elements of Ω_i', we arrive at the inequality

$$|u_i|^2_{W^1_{a,h}(\Omega_i')} \preceq |w|^2_{W^1_{a,h}(\Omega_i')} + H^{-2}\,\|w\|^2_{L^2_a(\Omega_i')}.$$

We sum over all i and use that each point in Ω is covered only a fixed number of times and obtain a uniform bound on C_0^2. We conclude the proof, by estimating the two terms of

$$|w|^2_{W^1_{a,h}(\Omega)} + H^{-2}\,\|w\|^2_{L^2_a(\Omega)}$$

by $|u|^2_{W^1_{a,h}(\Omega)}$. The bounds follow by using the stability results of Theorem 1 or 2.

For the case of small overlap, the proof is similar to that of the case of piecewise linear conforming space considered in Dryja and Widlund [7]. □

REMARK 4. *We can use the same technique and prove all our results for Q_1 nonconforming finite elements on rectangular elements.*

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