Studies in Domain Decomposition: Multi-level Methods and the Biharmonic Dirichlet Problem

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Abstract

A class of multilevel methods for second order problems is considered in the additive Schwarz framework. It is established that, in the general case, the condition number of the iterative operator grows at most linearly with the number of levels. The bound is independent of the mesh sizes and the number of levels under a regularity assumption. This is an improvement of a result by Dryja and Widlund on a multilevel additive Schwarz algorithm, and the theory given by Bramble, Pasciak and Xu for the BPX algorithm.

Additive Schwarz and iterative substructuring algorithms for the biharmonic equation are also considered. These are domain decomposition methods which have previously been developed extensively for second order elliptic problems by Bramble, Pasciak and Schatz, Dryja and Widlund and others.

Optimal convergence properties are established for additive Schwarz algorithms for the biharmonic equation discretized by certain conforming finite elements. The number of iterations for the iterative substructuring methods grows only as the logarithm of the number of degrees of freedom associated with a typical subregion. It is also demonstrateed that it is possible to simplify the basic algorithms. This leads to a decrease of the cost but not of the rate of convergence of the iterative methods. In the analysis, new tools are developed to deal with Hermitian elements. Certain new inequalities for discrete norms for finite element spaces are also used.

Contents

Chapter 1

Introduction

1.1 An Overview

Finite element discretizations of elliptic problems, e.g. the biharmonic problem, often result in very large, sparse linear systems. Domain decomposition methods are very powerful iterative methods for solving such problems. The domain is decomposed into overlapping (resp. non-overlapping) subdomains; the algorithms are often referred to as Schwarz type (resp. iterative substructuring) methods. In each iteration step, a number of smaller subproblems, which correspond to the restriction of the original problem to subregions, or to a problem with a coarse mesh, are solved. The number of subproblems can be large and these methods are promising for parallel computation.

The Schwarz alternating method, the first domain decomposition algorithm, was proposed by H.A. Schwarz [42] in 1869. It involves solving problems on two subdomains sequentially. In [34], P. L. Lions interpreted the method in a variational framework. Many subdomain cases were also considered. Because of the sequential nature of the method, it is not ideal for parallel computations. In [24], Dryja and Widlund introduced the additive Schwarz method to remove this sequential behavior of the Schwarz alternating method making it possible to solve all the subproblems in parallel, see also Matsokin and Nepomnyaschikh [37] and Nepomnyaschikh [39]. The additive Schwarz method can be regarded as an iterative method for an equivalent system with a better condition number. Dryja and Widlund showed the optimal convergence properties of the algorithm for second order problems discretized by linear elements. Generalization to certain stationary and parabolic convection-diffusion problems have been carried out by Cai and Widlund; cf. [14,16,17].

Iterative substructuring methods for second order problems were developed by Bramble, Pasciak and Schatz [10,11], Bjørstad and Widlund [7], Widlund [47,46] and Dryja [23,22]. Techniques for analyzing some iterative substructuring methods using the additive Schwarz framework were developed in Dryja and Widlund [25]. They demonstrated that certain iterative substructuring methods can be viewed as additive Schwarz methods with a set of special overlapping subdomains. For discussions of the relationship between the two type schemes, see also Bjørstad and Widlund [8] and Chan and Goovaerts [19].

Multilevel methods, such as multigrid methods, are among the most efficient methods for linear equations arising from elliptic problems; cf. Hackbusch [32], McCormick [38] and the references therein. Recently, with the increasing interest in parallel computation, several new multilevel methods have been developed; cf. Yserentant [52], Bank, Dupont and Yserentant [1], Bramble, Pasciak and Xu [13], and Dryja and Widlund [28]. In this thesis, we give improved results for a class of multilevel methods by showing that the condition number of the iteration operator grows at most linearly with the number of levels in general, and is bounded by a constant independent of the mesh sizes and the number of levels if the elliptic problem is H^2–regular. This is an improvement on Dryja and Widlund's results on a multilevel additive Schwarz method as well as Bramble, Pasciak and Xu's results on the BPX algorithm.

We also study the additive Schwarz and iterative substructuring methods for the biharmonic Dirichlet problem. We construct additive Schwarz algorithms for the biharmonic equation using standard conforming finite element discretization. Some new tools are developed for the proofs of the optimal convergences of the algorithms. By using a *weak coupling* property of the degrees of freedom, some simplified versions are also derived. This leads to a decrease of the cost per iteration but not of the rate of convergence of the iterative methods. In the case of the iterative substructuring methods, we demonstrate that direct generalizations of some known iterative substructuring methods result in algorithms with a condition number which grows at least like $O((H/h)^2)$. Better algorithms are obtained by using certain vertex spaces. Some parallel multilevel algorithms are also constructed for the biharmonic problem. Some earlier works on the biharmonic problem can be found in Glowinski and Pironneau [30], Bjørstad [2], Widlund [45] and Chan, E and Sun [18].

The thesis is organized as follows. In the remainder of chapter 1, we review some basic Sobolev spaces. We also discuss some preliminary material for the biharmonic problem and some standard finite element discretizations. In chapter 2, we review some domain decomposition techniques in a form used in this thesis. In chapter 3, we discuss a class of multilevel methods, including a multilevel additive Schwarz method of Dryja and Widlund and the BPX algorithm of Bramble, Pasciak and Xu. We present our algorithms and analysis using the additive Schwarz framework. In chapter 4, we construct some interpolation and quasi-interpolation operators and establish approximation properties of these operators. We also construct discrete Sobolev norms and establish their equivalence to the corresponding continuous Sobolev norms. We also prove a theorem regarding the norm estimates of some standard interpolation operators. These results are used in chapters 5 and 6 to show the optimal and almost optimal convergence properties of the algorithms. In chapter 5, we construct additive Schwarz schemes for the biharmonic problem discretized by some standard finite elements and we establish the optimal convergence properties of the algorithms. By using the weak coupling property of the degrees of freedom of the finite elements, some computationally more efficient algorithms are also derived. We also consider a multilevel method. Numerical experiments with the algorithms are reported. In chapter 6, we discuss some generalization of the iterative substructuring methods for the biharmonic problem. We first demonstrate that direct generalizations of some algorithms designed for the second order problems result in algorithms with very large condition numbers. We then show that better algorithms can be obtained by adding certain vertex spaces to the space decomposition. Finally, we present numerical experiments to support our theoretical conclusions.

1.2 Some Sobolev Spaces

Let $\Omega \subset \mathcal{R}^d$ be a bounded Lipschitz domain and let $L^p(\Omega)$ be the Banach space of p-th power integrable functions.

We use the *multi-index notation* for partial derivatives. Let $\alpha \equiv (\alpha_1, \cdots, \alpha_d)$ be a non-negative multi-index representing the order of the partial derivatives, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_d$, and let

$$D^\alpha \stackrel{\text{def}}{=} \frac{\partial^\alpha}{\partial x^\alpha} \stackrel{\text{def}}{=} \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) \stackrel{\text{def}}{=} \{v |\ D^\alpha v \in L^p(\Omega),\ \text{if } |\alpha| \leq m\}$$

with a norm

$$\|v\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha v\|^p_{L^p(\Omega)}.$$

We also need the seminorm

$$|v|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha| = m} \|D^\alpha v\|^p_{L^p(\Omega)} \ .$$

The most useful space in this thesis is the case $p = 2$, which by convention is denoted by $H^m(\Omega)$,

$$H^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega).$$

To study the homogeneous Dirichlet problem, we need the Sobolev space

$$H_0^m(\Omega) \stackrel{\text{def}}{=} \{v | v \in H^m(\Omega), D^\alpha v|_{\partial\Omega} = 0, |\alpha| \leq m - 1\}.$$

This is the closure of the space of $C_0^\infty(\Omega)$ with respect to H^m–norm.

The following inequalities establish equivalences of certain norms in subspaces of $H^1(\Omega)$.

Lemma 1.2.1 (Friedrichs' inequality) *There exists a positive constant $C(\Omega)$, which depends only on the Lipschitz constant of the boundary of Ω, such that, for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) H_\Omega |u|_{H^1(\Omega)}.$$

Here H_Ω is the diameter of Ω.

Let $\{u\}_\Omega = \frac{\int_\Omega u}{|\Omega|}$ be the average value of the function u over Ω. We have

Lemma 1.2.2 (Poincaré's inequality) *There exists a positive constant $C(\Omega)$, which depends only on the Lipschitz constant of the boundary of Ω, such that, for all $u \in H^1(\Omega)$,*

$$||u - \{u\}_\Omega||_{H^1(\Omega)} \leq C(\Omega) H_\Omega |u|_{H^1(\Omega)}.$$

Here H_Ω is the diameter of Ω.

Proofs of Lemma 1.2.1 and Lemma 1.2.2 can be found in [40].

For applications to the biharmonic problem, the following generalization of Poincaré's inequality is useful.

Lemma 1.2.3 (Generalized Poincaré's Inequality) *If $\{D^\alpha u\}_\Omega = 0, |\alpha| \leq k$ then,*

$$||u||_{H^s(\Omega)} \leq C_\Omega H_\Omega^{k+1-s} |u|_{H^{k+1}(\Omega)} \quad \text{for } s \leq k+1,\, u \in H^{k+1}(\Omega),$$

and

$$|D^\alpha u|_\infty \leq C_\Omega H_\Omega^{k+1-|\alpha|-d/2} |u|_{H^{k+1}(\Omega)} \; (|\alpha| < k+1-d/2).$$

The second inequality also holds for the $W^{k+1,p}$–norm, with $d/2$ replaced by d/p.

Proof. The first inequality can be proved by applying Poincaré's inequality several times to $D^\alpha u$. The second follows from the embedding theorem:

$$H^{k+1}(\Omega) \hookrightarrow W^{s,\infty}(\Omega), \quad s < k+1-d/2;$$

the powers of H_Ω are obtained by a scaling argument. ∎

We also need the following lemma.

Lemma 1.2.4 *If $u \in H^{k+1}(\Omega)$, then there exists a polynomial $p \in \mathcal{P}_k$ such that*

$$\{D^\alpha(u-p)\}_\Omega = 0, \quad |\alpha| \leq k.$$

Proof. We establish that for any given $d_\alpha, |\alpha| \leq k$, there exists a polynomial $p \in \mathcal{P}_k$, such that

$$\int_\Omega D^\alpha p = d_\alpha.$$

The lemma then follows by setting $d_\alpha = \int_\Omega D^\alpha u$. Let

$$p = \sum_{|\beta| \le k} b_\beta x^\beta. \tag{1.1}$$

We define a partial order for the set of multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$:

$$\alpha \le \beta \quad \text{iff } \alpha_i \le \beta_i, \quad 1 \le i \le d.$$

We have

$$D^\alpha p = \sum_{|\beta| \le k} b_\beta D^\alpha x^\beta = \sum_{|\beta| \le k} b_\beta C_{\alpha,\beta} x^{\beta-\alpha}, \tag{1.2}$$

where $C_{\alpha,\beta} = \frac{\beta!}{(\beta-\alpha)!}$ for $\beta \ge \alpha$, and $C_{\alpha,\beta} = 0$ otherwise. We obtain a linear system

$$\sum_{|\beta| \le k} A_{\alpha,\beta} b_\beta = \int_\Omega D^\alpha p = d_\alpha, \tag{1.3}$$

with $A_{\alpha,\beta} = C_{\alpha,\beta} \int_\Omega x^{\beta-\alpha} dx$. It is easy to see that $A_{\alpha,\beta} = 0$ for $\beta \not\le \alpha$, and $A_{\alpha,\alpha} = \alpha! |\Omega|$. We can therefore find a permutation so that the matrix $A_{\alpha,\beta}$ is upper triangular with nonzero diagonals. Therefore there exists an unique solution b_β and thus a unique $p \in \mathcal{P}_k$ satisfying $\{D^\alpha(u-p)\}_\Omega = 0, \forall u \in H^{k+1}(\Omega)$. ∎

Corollary 1.2.5 *For the polynomial p constructed above, we have*

$$|u-p|_{H^s(\Omega)} \le c_\Omega H_\Omega^{k+1-s} |u|_{H^{k+1}(\Omega)} \quad \textit{for} \quad s \le k+1$$

Lemma 1.2.6 (Quotient space lemma) *Let*

$$\|\tilde{u}\|_{H^k/P_{k-1}} = \inf_{p \in P_{k-1}} \|u-p\|_{H^k}.$$

Then

$$|u|_{H^k} \le \|\tilde{u}\|_{H^k/P_{k-1}}.$$

For a bounded Lipschtz region Ω*, we have*

$$\|\tilde{u}\|_{H^k/P_{k-1}} \le C(\Omega) |u|_{H^k}$$

Proof. The lemma follows from Corollary 1.2.5 and Lemma 1.2.3. ∎

Lemma 1.2.7 (Bramble–Hilbert's lemma) *Let Ω be an open Lipschitz region in R^d. Let $k \geq 0$ be an integer and $p \in [0, \infty]$. Let f be a continuous linear form on $W^{k+1,p}(\Omega)$ with the property that*

$$f(p) = 0, \quad \forall p \in P_k(\Omega).$$

Then, there exists a constant $C(\Omega)$ such that

$$|f(v)| \leq C(\Omega) \|f\|^*_{W^{k+1,p}(\Omega)} |v|_{W^{k+1,p}(\Omega)}, \quad \forall v \in W^{k+1,p}(\Omega),$$

where $\|\cdot\|^*_{W^{k+1,p}(\Omega)}$ is the norm of the dual space of $W^{k+1,p}(\Omega)$.

Proof. For all $p \in P_k(\Omega)$, we have

$$|f(v)| = |f(v+p)| \leq \|f\|^*_{W^{k+1,p}(\Omega)} |v+p|_{W^{k+1,p}(\Omega)},$$

and thus

$$|f(v)| \leq \|f\|^*_{W^{k+1,p}(\Omega)} \inf_{p \in P_k(\Omega)} |v+p|_{W^{k+1,p}(\Omega)}.$$

The conclusion follows by the quotient space lemma. ∎

1.3 The Biharmonic Equation

Consider the biharmonic Dirichlet problem in a plane region

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = g_0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial n} = g_1 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.4}$$

There are also other types of boundary conditions which are of interest. For example, one can impose $u|_{\partial\Omega} = g_0$ and $\Delta u = g_2$; this corresponds to the mathematical model for a simply supported plate. That problem can be decomposed into two second order problems and is therefore relatively easy to solve.

Weak Formulation. We consider the variational form of the problem with homogeneous boundary condition: Find $u \in H^2_0(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H^2_0(\Omega), \tag{1.5}$$

where f is a bounded linear functional on $H_0^2(\Omega)$ and $a(u,v)$ is a symmetric, continuous, H_0^2-elliptic bilinear form. Two examples of such bilinear forms are

$$a(u,v) = \int_\Omega \Delta u \Delta v \; dx, \tag{1.6}$$

and

$$a(u,v) = \int_\Omega \{\Delta u \Delta v + (1-\sigma)(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2})\} \; dx, \tag{1.7}$$

The first one arises in *Fluid Dynamics*, and the second provides a variational formulation of the *Clamped Plate Problem*. Here $0 < \sigma < 1/2$ is *Poisson's coefficient* of the plate. By the Lax-Milgram Theorem, the boundness and ellipticity imply existence and uniqueness of the solution; cf. Lax and Milgram [33] and Ciarlet[21].

1.3.1 The Finite Element Formulation

The finite element formulation is obtained by replacing the infinite dimensional space $V = H_0^2(\Omega)$ with a finite dimensional subspace $V^h \subset V$.

We triangulate the domain Ω into non-overlapping regions called elements, generally triangles or rectangles. V^h is a space of piecewise polynomials with respect to the triangulation. The finite element solution $u_h \in V^h$ satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h. \tag{1.8}$$

We note that the finite element solution u_h is the $a(\cdot,\cdot)$–projection of the true solution u onto the finite element space V^h.

The norm derived from the bilinear form $a(\cdot,\cdot)$ defines a semi-norm in the Sobolev space $H^2(\Omega)$. It is a norm of the space $H_0^2(\Omega)$, and therefore a norm in its subspace V^h.

Let $\{\phi_i\}$ be the nodal basis for V^h. Then u_h can be represented as

$$u_h = \sum_i x_i \phi_i.$$

Thus we obtain a linear system for x, the degrees of freedom of u_h,

$$Kx = b$$

with the stiffness matrix given by

$$K_{i,j} = a(\phi_i, \phi_j)$$

and the load vector by

$$b_i = f(\phi_i).$$

The stiffness matrix K is symmetric, positive definite. After a proper scaling, its condition number $\kappa(K) = O(h^{-4})$. Since the system is usually very large, and the condition number of K is also quite large, solving the system can be very expensive. Many preconditioners have been designed for K. Among them, the additive Schwarz methods studied in this thesis seem to be particularly successful and promising.

1.3.2 Some Conforming Elements

For the biharmonic equation, the finite elements are all relatively complicated. In this thesis, we restrict ourselves to some standard conforming elements. In particular, we consider the Argyris triangle V_A^h, the Bell triangle V_B^h and the bicubic element V_Q^h. These elements are complicated but among the simplest conforming elements for the biharmonic equation.

The Argyris element consists of continuously differentiable functions, the restrictions of which to any triangle are in $\mathcal{P}_5$. The *degrees of freedom* for this element in a triangle with vertices $a_i, i = 1, 2, 3$, are given by

$$\{p(a_i), \frac{\partial^\alpha}{\partial x^\alpha} p(a_i), |\alpha| \leq 2, \ \ \frac{\partial}{\partial n_i} p(b_i)\},$$

where b_i is the midpoint of edge $\overline{a_j a_k}$, and n_i is the outward normal direction of $\overline{a_j a_k}$. The number of the degrees of freedom for each triangle is 21.

It is easy to see that, in general, the normal derivatives of an Argyris element is a polynomial of degree 4. Let $\mathcal{P}_B$ be the subspace of $\mathcal{P}_5$ formed by those polynomials of $\mathcal{P}_5$ whose normal derivatives along each side of a triangle are polynomials of degree 3 in t, the abscissa along an axis containing the side. We note that $\mathcal{P}_4 \subset \mathcal{P}_B \subset \mathcal{P}_5$. The Bell element consists of C^1 functions whose restrictions to a triangle are in $\mathcal{P}_B$. The degrees of freedom for the Bell element are given by

$$\{p(a_i), \frac{\partial}{\partial x}p(a_i), \frac{\partial}{\partial y}p(a_i), \frac{\partial^2}{\partial x^2}p(a_i), \frac{\partial^2}{\partial x \partial y}p(a_i), \frac{\partial^2}{\partial y^2}p(a_i)\}.$$

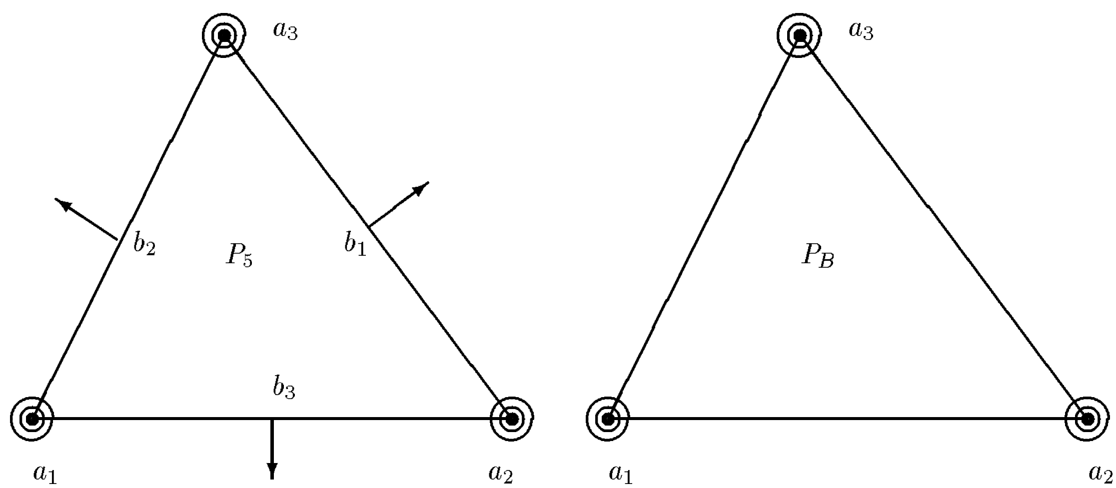

Figure 1.1: The Argyris and Bell triangles

If the domain is build from rectangles, we can also use the bicubic element, known as the Bogner-Fox-Schmit rectangle in the engineering literature. It is space of C^1 functions whose restrictions to a rectangle are in $Q_3 = \text{span}\{x^i y^j, i, j \leq 3\}$. The degrees of freedom of the bicubic element are given by

$$\{p(a_i), \ \frac{\partial p}{\partial x_1}(a_i), \ \frac{\partial p}{\partial x_2}(a_i), \ \frac{\partial^2 p}{\partial x_1 \partial x_2}(a_i)\}.$$

We note that nonconforming elements such as Morley's triangle, Adini's rectangle, etc., are also widely used in engineering computation. However, they will not be discussed here.

The approximation properties of these elements are well understood. We have

$$\|u - u_A\|_{2,\Omega} \leq Ch^4 |u|_{6,\Omega},$$

$$\|u - u_B\|_{2,\Omega} \leq Ch^3 |u|_{5,\Omega}.$$

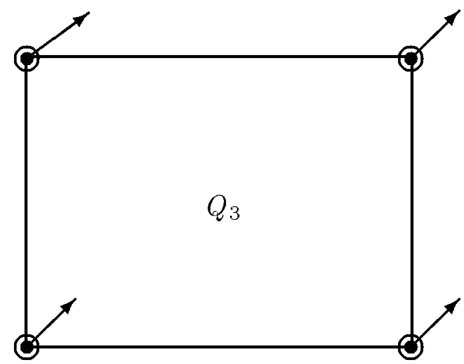

Figure 1.2: The bicubic element

$$\|u - u_{Q_3}\|_{2,\Omega} \leq Ch^2|u|_{4,\Omega},$$

For more details on the convergence and error estimates; see Ciarlet [20,21].

1.3.3 Properties of the Basis Functions

Only the following two properties of the basis functions will be used.

- The uniformity of the basis functions.

- The invariance of linear functions under interpolation.

We state the properties for the Bell element, the same results hold for the Argyris and bicubic elements. Let us denote the nodal basis functions of the Bell element by ϕ_i^α, $|\alpha| \leq 2$. The basis functions satisfy

$$\frac{\partial^\beta}{\partial x^\beta}\phi_i^\alpha(x_j) = \begin{cases} 1, & \text{if } j = i \text{ and } \beta = \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.3.1 *The basis functions ϕ_i^α of the Bell element are uniform to order 2 in the sense of Strang [44], i.e.*

$$|\phi_i^\alpha|_{W^{s,\infty}} \leq Ch^{|\alpha|-s}, s \leq 2, \tag{1.9}$$

and uniform to order 2 in the following weak sense,

$$|\phi_i^\alpha|_{H^s} \leq Ch^{1+|\alpha|-s}, s \leq 2. \tag{1.10}$$

The same conclusions hold for the Argyris element and the bicubic element.

If we write down the basis functions explicitly, it is straight forward to check that they are uniform to order 2.

We remark that for the additive Schwarz methods, we only need (1.10). For some basis functions, it is easy to check that (1.10) holds, but difficult to check whether (1.9) holds or not.

Another property of the basis functions is the *invariance of linear function under interpolation*. Let $\Pi_{V_B^h}$ be the standard interpolation operator to V_B^h. For p linear in a triangle, $\Pi_{V_B^h} p = p$. This implies that, restricted to a triangle,

$$p \in \text{span}\{\phi_i^\alpha; |\alpha| \leq 1\},$$

which gives us the following relationships for the basis functions of the Bell element:

$$\begin{array}{rcl} 1 & = & \sum_i \phi_i(x) \\ x & = & \sum_i x_i \phi_i(x) + \phi_i^x(x) \\ y & = & \sum_i y_i \phi_i(x) + \phi_i^y(x) \end{array}$$

Similar equations can be worked out for the Argyris elements and the bicubic elements.

The above equations show that in each triangle, a linear function can be constructed from only the basis functions associated with nodal values and the first derivatives, since the basis functions associated with second derivatives do not appear in the above equation. This fact is important in later chapters.

Chapter 2

Domain Decomposition Techniques

2.1 The Conjugate Gradient Method

Stationary schemes like the Jacobi, Gauss-Seidel, SOR, Block Jacobi and Block Gauss-Seidel methods can be viewed as *subspace correction* methods. The approximate solution is updated by solving problems associated with a set of subspaces of the solution space. These methods can be accelerated by using the conjugate gradient method or the Chebyshev semi-iterative method; we concentrate on the conjugate gradient method.

The conjugate gradient(CG) method is of fundamental importance for the domain decomposition methods. All algorithms studied in this thesis are variations of preconditioned conjugate gradient methods(PCG). Since CG and PCG are well known algorithms, we only give a brief description. For more complete treatment, see Golub and Van Loan [31]. We note that in the conjugate gradient algorithm, we only need to be able to apply the matrix A to a given vector; an explicit representation of the matrix is not needed. This is a very important for domain decomposition algorithms.

Conjugate Gradient Algorithm

Set $k = 0; x_0 = 0; r_0 = b$.
while $|r_k| \geq \epsilon |r_0|$
    $k = k + 1$
    if $k = 1$
        $p_1 = r_0$
    else
        $\beta_k = (r_{k-1}, r_{k-1})/(r_{k-2}, r_{k-2})$

$p_k = r_{k-1} + \beta_k p_{k-1}$
end
$\alpha_k = (r_k, r_k)/(p_k, Ap_k)$
$x_k = x_k + \alpha_k p^k$
$r_k = r_k - \alpha_k Ap^k$
end

The reduction in the energy norm of the error, after n steps of conjugate gradient iteration, is given by, see e.g. [31],

$$\|x - x_n\| \leq 2(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1})^n \|x - x_0\|.$$

Here $\kappa = \kappa(A)$ is the condition number of A given by

$$\kappa = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

The conjugate gradient method is therefore often an effective iterative algorithm to solve symmetric, positive definite systems

$$Ax = b.$$

The algorithm converges to a good approximate solution in relatively few iterations for a well conditioned matrix A. When A is not well conditioned, which is generally the case for discretizations of elliptic problems, we can introduce a preconditioner B and solve the preconditioned linear system

$$B^{-1}Ax = B^{-1}b.$$

by using the conjugate gradient method with the A-inner product. Domain decomposition is an effective way to construct such preconditioners. Other preconditioners are also used in practices, e.g. the diagonal scaling and incomplete Cholesky preconditioners.

2.2 Multiplicative Schwarz Schemes

The Schwarz alternating method is the first domain decomposition method, proposed by H. A. Schwarz in 1869, [42]. Schwarz established convergence using the *maximum principle*. In the 1930's, Sobolev extended the result to the partial differential equations of linear elasticity [43]. New theoretical tools for the analysis of multiplicative Schwarz schemes have been developed more recently. The continuous problem has been studied by

Lions [34]; see also Nepomnyaschikh [39]. For the finite element case, the two subspace case has been discussed in Bjørstad [3], Mandel and McCormick [35] and Bjørstad and Mandel [4]. Results for the case of more then two subspaces have been found by Widlund [48] and Mathew [36] and most successfully by Bramble, Pasciak, Wang, and Xu [12]. We review this theory and the results which can be used in analyzing multiplicative variants of the algorithms.

Let us consider the domain, shown in Figure 2.1, with $\Omega = \Omega_1 \cup \Omega_2$ on which we wish to solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

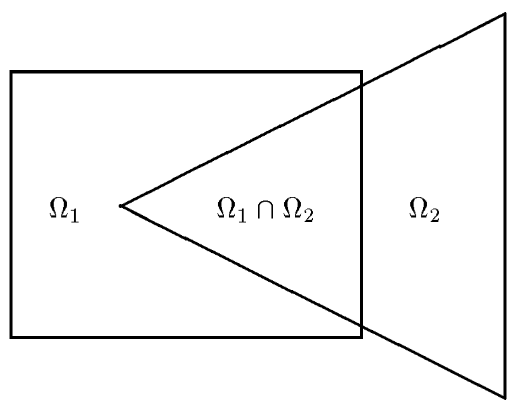

Figure 2.1: Schwarz alternating scheme

The Schwarz alternating procedure approximates the solution iteratively by solving problems on the each subdomain Ω_i.

$$\begin{cases} -\Delta u^{n+1/2} = f & \text{in } \Omega_1, \\ u^{n+1/2} = u^n & \text{on } \partial\Omega_1, \end{cases}$$

and

$$\begin{cases} -\Delta u^{n+1} = f & \text{in } \Omega_2, \\ u^{n+1} = u^{n+1/2} & \text{on } \partial\Omega_2. \end{cases}$$

Let

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \qquad \text{and} \qquad f(v) = \int_\Omega f v.$$

Then the algorithm can be written as

$$a(u^{n+1/2} - u^n, \phi) = f(\phi) - a(u^n, \phi), \qquad u^{n+1/2} - u^n \in H_0^1(\Omega_1), \forall \phi \in H_0^1(\Omega_1),$$

and

$$a(u^{n+1} - u^{n+1/2}, \phi) = f(\phi) - a(u^{n+1/2}, \phi), \quad u^{n+1} - u^{n+1/2} \in H_0^1(\Omega_2), \forall \phi \in H_0^1(\Omega_2).$$

In each half-step, the correction is thus the projection of the error onto the subspace $H_0^1(\Omega_1)$ or $H_0^1(\Omega_2)$. The projection $P_i : H_0^1(\Omega) \rightarrow H_0^1(\Omega_i)$, is defined by

$$a(P_i u, \phi) = a(u, \phi), \quad \forall \phi \in H_0^1(\Omega_i).$$

The error propagation operator for a complete step of the alternating Schwarz method is simply

$$(I - P_2)(I - P_1).$$

The alternating Schwarz method can be considered as a special example of an abstract multiplicative Schwarz scheme. Consider the general variational problem: Find $u \in V$ such that

$$a(u, \phi) = f(\phi), \quad \forall \phi \in V. \tag{2.1}$$

Let V_i be subspaces of V so that $V = V_1 + \cdots + V_N$. Associated with each subspace is the corresponding orthogonal projection operator $P_i : V \rightarrow V_i$. The abstract multiplicative Schwarz scheme proceeds by sequentially projecting the error onto the subspaces V_i. The error propagation operator for a complete step of the alternating Schwarz method is given by

$$(I - P_N) \cdots (I - P_1).$$

To include the case of approximate solver for the subproblems, it is sometimes necessary to consider a more general form

$$(I - T_N) \cdots (I - T_1),$$

where T_i are symmetric, positive semi-definite, linear operators with

$$\|T_i\|_a \leq \omega < 2.$$

Usually, the T_i are either the projections P_i or approximations thereof. Define E_i by

$$E_i = (I - T_i)(I - T_{i-1}) \cdots (I - T_1).$$

The effectiveness of the particular multiplicative algorithm is determined by the reduction in the error for one complete iteration step, i.e. the best γ in the inequality

$$|E_N^2 v|_a \leq \gamma |v|_a^2.$$

Estimates for γ are given in Bramble, Pasciak, Wang and Xu [12] and Xu [51].

Let

$$T = \sum_i T_i,$$

and let $\lambda_{\min}(T)$ be the minimum eigenvalue of T. A main result of Bramble, Pasciak, Wang, and Xu is given in

Theorem 2.2.1

$$|E_N v|_a^2 \leq \gamma |v|_a^2,$$

with

$$\gamma = 1 - \max\left(\frac{(2-\omega)\lambda_{\min}(T)}{(N + \omega^2 N(N-1)/2)}, \frac{(2-\omega)\lambda_{\min}(T)}{2(1 + \omega^2 N(N-1)/2)}\right).$$

To obtain stronger results, we use a coarse subspace $V_0 = V^H$ and make assumptions on the subspaces V_i. Let $\mathcal{E}$ be the matrix such that

$$|a(v_i, v_j)| \leq \epsilon_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2}, \qquad \forall v_i \in V_i, \forall v_j \in V_j, i, j = 1, \cdots N. \quad (2.2)$$

The inequalities (2.2) are strengthened Cauchy-Schwarz inequalities. Let $\rho(\mathcal{E})$ be the spectral radius of $\mathcal{E}$. We note that as the overlap between the spaces increases $\lambda_{\min}(T)$ will generally improve while the bound in (2.2) will degrade, i.e. the spectral radius $\rho(\mathcal{E})$ will increase.

Using the strengthened Cauchy inequality (2.2), the above result can be improved; cf. Xu [51].

Theorem 2.2.2 *Assume that (2.2) hold. Then*

$$|E_N v|_a^2 \le \gamma |v|_a^2,$$

with

$$\gamma = 1 - \frac{(2-\omega)\lambda_{\min}(T)}{2(1+\omega^2\rho(\mathcal{E})^2)}.$$

An important parameter in the above two theorems is $\lambda_{\min}(T)$. Estimates for $\lambda_{\min}(T)$ was obtained by Dryja and Widlund [24,25]. An important special case of this general result is when the T_i are projections P_i. In this case, $\lambda_{\min}(T)$ can be estimated by C_0^{-2} in the following assumption: For all $v \in V$, there exists a decomposition $v = \sum v_i$ with $v_i \in V_i$ such that

$$\sum_{i=1}^{N} a(v_i, v_i) \le C_0^2 a(v, v).$$

For the proof of this result, see Lemma 2.3.2 in the following section. In addition, we note, that since the P_i are orthogonal projections, ω is one.

We remark that it is also possible to treat one or several of the subspaces separately; cf. Bramble, Pasciak, Wang and Xu [12], Dryja [27] and Widlund [49].

2.3 Additive Schwarz Schemes

One possible problem with the multiplicative Schwarz methods is the sequential nature of the fractional steps of each iteration. For more than two subspaces, these schemes are also inherently nonsymmetric. Therefore, symmetric variants, which can almost double the number of fractional steps, have to be used if we wish to use the standard conjugate gradient method to accelerate the convergence. The additive Schwarz methods were designed by Dryja and Widlund to remove the inherent sequential behavior of the fractional steps and to preserve symmetry. Independent work on additive Schwarz methods can also be found in Matsokin and Nepomnyaschikh [37] and Nepomnyaschikh [39]. Optimal convergence properties of certain algorithms were established for second order self-adjoint elliptic problems, see [24,25]. Generalizations to nonsymmetric and indefinite cases have been made by Cai and Widlund; cf.[14,15,16,17]. We will show in chapter 5 that certain additive Schwarz methods for the biharmonic equation are also optimal.

Additive Schwarz methods can be viewed as iterative methods for the solution of an auxiliary linear problem that has the same solution as the original finite element problem.

We consider the finite dimensional variational problem: Find $u \in V$ such that

$$a(u, \phi) = f(\phi), \quad \forall \phi \in V. \tag{2.3}$$

Suppose V can be written as the sum of $N + 1$ subspaces

$$V = V_0 + V_1 + \cdots + V_N.$$

Let $P_i : V \to V_i, i = 0, \cdots, N$, be the orthogonal projections, with respect to $a(\cdot, \cdot)$, defined by

$$a(P_i u, \phi) = a(u, \phi), \quad \forall \phi \in V_i. \tag{2.4}$$

Let $P = \sum P_i$. The additive Schwarz method for solving equation (2.3) is introduced in terms of an auxiliary problem: Find $u \in V$ by solving

$$Pu \equiv \sum_i P_i u = g. \tag{2.5}$$

The right hand side g has to be chosen such that the auxiliary equation has the same solution as equation (2.3). For this to hold, the right-hand side must be equal to $g = \sum_{i=0}^{N} g_i$, where $g_i = P_i u_h$ can be computed by solving

$$a(g_i, v) \equiv a(P_i u_h, v) = a(u_h, v) = f(v), \forall v \in V_i.$$

Thus $P_i u$ can be found without knowing the solution of (2.3). Once the right-hand side g is known, we can use an iterative method, e.g. the conjugate gradient method, to solve equation (2.5). To define an additive Schwarz method, it is sufficient to define the space decomposition. The fundamental issue is to estimated the condition number of P.

The reason for going from problem (2.3) to problem (2.5) is that, by a suitable choice of the subspaces V_i, we can turn a large ill-conditioned system into a very well conditioned problem at the expense of solving many small independent linear systems.

The following lemmas, which establish the relations between the decomposition of u and the spectrum of P, allow us to develop bounds on the condition number of P.

Lemma 2.3.1 *Let V be a Hilbert space, V_i be subspaces of V and $V = \sum V_i$. Let P_{V_i} be the projections from V to V_i and $P = \sum_i P_{V_i}$. Then P is invertible and*

$$a(P^{-1}u, u) = \min_{\sum u_i = u} \sum_i a(u_i, u_i)$$

with $u_i \in V_i$ and the minimum is achieved at $u_i = P_i P^{-1} u$.

Proof. We use the properties of the projections and Cauchy-Schwarz's inequality to obtain,

$$\begin{aligned} a(P^{-1}u, u) &= \sum a(P^{-1}u, u_i) \\ &= \sum a(P_i P^{-1}u, u_i) \\ &\leq \sum |P_i P^{-1}u|_a |u_i|_a \\ &\leq (\sum |P_i P^{-1}u|_a^2)^{1/2} (\sum |u_i|_a^2)^{1/2} \\ &= a(P^{-1}u, u)^{1/2} (\sum |u_i|_a^2)^{1/2}. \end{aligned}$$

Thus,

$$a(P^{-1}u, u) \leq \sum a(u_i, u_i).$$

Equality holds if and only if $u_i = P_i P^{-1} u$. ∎

The following lemma is a direct consequence of Lemma 2.3.1.

Lemma 2.3.2 *Let V be a Hilbert space, V_i be subspaces of V and $V = \sum V_i$. Let P_{V_i} be the projections from V to V_i and $P = \sum_i P_{V_i}$. Then*

$$\lambda_{\min}^{-1}(P) = \lambda_{\max}(P^{-1}) = \max_u \frac{a(P^{-1}u, u)}{a(u, u)} = \max_u \min_{\sum u_i = u} \frac{\sum_i a(u_i, u_i)}{a(u, u)}$$

and

$$\lambda_{\max}^{-1}(P) = \lambda_{\min}(P^{-1}) = \min_u \frac{a(P^{-1}u, u)}{a(u, u)} = \min_u \min_{\sum u_i = u} \frac{\sum_i a(u_i, u_i)}{a(u, u)}.$$

Remark 2.3.1 If we can find a constant C_1 such that there exists a decomposition of $u = \sum_i u_i$ satisfying

$$\sum_i a(u_i, u_i) \leq C_1 a(u, u), \quad \forall u \in V,$$

then, it follows from Lemma 2.3.2 that $\lambda_{\min}(P) \geq C_1^{-1}$. This result, known as Lions' lemma, is very important in estimating the minimum eigenvalue of P; cf. Dryja and Widlund [24].

If we can find a constant C_2 such that $\forall u \in V$ and for any decomposition of $u = \sum_i u_i$, we have

$$\sum_i a(u_i, u_i) \geq C_2 a(u, u)$$

then, from the Lemma 2.3.2, we know that $\lambda_{\max}(P) \leq C_2^{-1}$.

For domain decomposition algorithms, the following lemma, gives a convenient way of estimating an upper bound of $\lambda(P)$.

Lemma 2.3.3 *Consider the undirected graph with a node for each subspace V_i, and an edge between node i and node j if and only if $V_i \cap V_j \neq 0$. Let N_c be the number of colors needed to color the graph so that no two nodes connected by an edge have the same color. Then*

$$\lambda_{\max}(P) \leq N_c.$$

Proof. All the subspaces of a particular color are disjoint; hence their corresponding projection operators are mutually orthogonal. Therefore the sum of the projection operators of a particular color is itself a projection operator. P then is the sum of N_c projection operators, each of norm one. ∎

An alternative way, as in the multiplicative case, is to use a strengthened Cauchy-Schwarz inequality. The proof of the following lemma is quite elementary.

Lemma 2.3.4 *Assume equation (2.2) holds, then*

$$\lambda_{\max}(P) \leq \rho(\mathcal{E}) + 1.$$

This is especially useful for the multilevel schemes.

Remark 2.3.2 As in the multiplicative case, we can also use approximate solvers T_i to replace P_i.

Remark 2.3.3 In the case of two subspaces, the relation between the additive and multiplicative versions for the same pair of subspaces is well understood; cf. [3],[4], and [35]. However little can be said for the case of more than two subspaces. The same two parameters $\rho(\mathcal{E})$ and $\lambda_{\min}(T)$ provide bounds for the convergence rates of both variants.

However, there is no general theory which explicitly relates the actual convergence rates of the two variants. The architectures of a particular parallel machine, etc. seems more likely to determine which variant to prefer rather than a strict mathematical analysis. We note that the multiplicative variant leads to a nonsymmetric operator. Therefore to accelerate it with the conjugate gradient method, we must introduce additional fractional steps to make it symmetric. We could also accelerate the nonsymmetric problem directly using GMRES, a conjugate gradient type method; cf. Eisenstat, Elman and Schultz [29] and Saad and Schultz [41].

Chapter 3

Second Order Problems

3.1 Second Order Equations

We consider second order uniformly elliptic equations of the following type

$$\begin{cases} -\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})u &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

with $\{a_{ij}\}$ symmetric, uniformly positive definite, bounded and piecewise smooth on a Lipschitz region $\Omega \subset R^d$, $d = 2$ or 3.

It is convenient to write (3.1) in a weak form: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v),\ \forall v \in H_0^1(\Omega), \tag{3.2}$$

with the bilinear form $a(u,v)$ and the functional $f(v)$ given by

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_j}\, dx, \quad f(v) = \int_\Omega f(x)v(x)\, dx.$$

The bilinear form $a(u,v)$ is *bounded* and *uniformly elliptic (coercive)*, i.e. there exist constants C_1 and C_2 such that

$$a(u,v) \le C_1 |u|_{H_0^1(\Omega)} |v|_{H_0^1(\Omega)} \qquad \text{and} \qquad a(u,u) \ge C_2 |u|^2_{H_0^1(\Omega)}$$

Existence and uniqueness follows from the Lax-Milgram's Theorem.

To simplify the presentation, we work with the model problem

$$\begin{cases} -\Delta u(x) &= f(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

We consider a finite element discretization of equation (3.2). We triangulate the domain Ω into non-overlapping elements τ_i (triangles or rectangles) and denote the triangulation by $\mathcal{T}^h = \{\tau_i\}$. We assume the triangulation is shape regular. The finite element space V^h is a subspace of $H_0^1(\Omega)$ consisting of continuous functions whose restrictions to each element τ_i are polynomials. Here, we will restrict ourselves to the simplest finite elements, in particular, we will use the linear element (for R^2 or R^3), the bi-linear element (for R^2) or the tri-linear element (for R^3). We will use the *nodal basis functions* $\{\phi_i(x)\}$ as the basis for V^h. The nodal basis functions satisfy

$$\phi_i(x_j) = \delta_{ij}, \quad \forall x_j \in \Lambda^h(\Omega).$$

To obtain the finite element problem, we replace the Sobolev space $H_0^1(\Omega)$ by the finite dimensional subspace V^h: Find $u_h \in V^h$ such that

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in V^h. \tag{3.4}$$

Using the nodal basis $\{\phi_i\}$, $u(x)$ can be represented as $u(x) = \sum_i x_i \phi_i(x)$. Thus, we obtain a linear system

$$Kx = b, \tag{3.5}$$

where $K = \{a(\phi_i, \phi_j)\}$ is the stiffness matrix and $b_i = f(\phi_i)$. K is a symmetric, positive definite, sparse matrix. It is known that its condition number $\kappa(A) = O(h^{-2})$. Thus K is ill conditioned for large problems.

3.2 Additive Schwarz Methods

3.2.1 The Algorithm

¿From chapter 2, we know that in order to define an additive Schwarz scheme, it is sufficient to define the space decomposition; different space decompositions give different algorithms.

Following Dryja and Widlund [24,25], we define two levels of triangulations. We start with a coarse triangulation $\mathcal{T}^H = \{\Omega_i\}$. Each Ω_i is then further divided into smaller elements to get a fine triangulation $\mathcal{T}^h = \{\tau_i\}$. Let H_i=diameter of Ω_i, $H = \max_i\{H_i\}$ h_i=diameter of τ_i, and $h = \max_i\{h_i\}$.

To get an overlapping decomposition of Ω, we extend each Ω_i to a larger region $\hat{\Omega}_i$, such that $cH_i \leq \text{dist}\{\partial\Omega_i, \partial\hat{\Omega}_i\} \leq CH_i$, and $\partial\hat{\Omega}_i$ align with the boundaries of element τ_j. We cut off the part of $\hat{\Omega}_i$ that is outside of Ω.

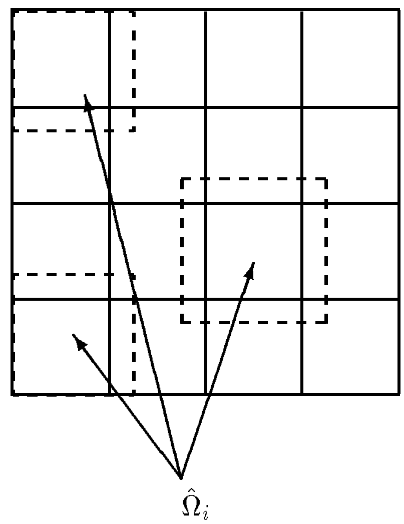

Figure 3.1: Two levels of triangulation

It is easy to see that $\{\hat{\Omega}_i\}$ forms a *finite covering* of domain Ω. Due to the generous overlap of $\{\hat{\Omega}_i\}$, we have a *partition of unity* $\{\theta_i\}$ satisfying

$$\sum_{i=1}^{N} \theta_i = 1 \text{ with } \theta_i \in W^{1,\infty}(\hat{\Omega}_i), 0 \leq \theta_i \leq 1, \text{ and } |\theta_i|_{W^{1,\infty}} \leq C/H_i.$$

Let V^h and V^H be the linear elements associated with the triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. Let $V_0 = V^H$ and $V_i = V^h(\hat{\Omega}_i) = V^h \cap H_0^1(\hat{\Omega}_i)$. We obtain a decomposition of the finite element space V^h

$$V^h = \sum_{i=0}^{N} V_i.$$

Let $P_{V_i} : V^h \to V_i$, be the H^1-projection defined by

$$a(P_{V_i} u, v) = a(u, v), \quad \forall v \in V_i.$$

The matrix form of P_{V_i}, after a permutation is

$$P_{V_i} = \begin{pmatrix} K_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} K = K_i^{+} K,$$

where K is the stiffness matrix associated with the domain Ω and K_i is the stiffness matrix associated with the subdomain $\hat{\Omega}_i$. The additive Schwarz operator, P is given by

$$P = \sum_{i=0}^{N} P_{V_i}.$$

The additive Schwarz algorithm for equation (3.4) is to solve an auxilary equation which is equivalent to equation (3.4).

Algorithm 3.2.1 *Find $u_h \in V^h$ by solving iteratively the equation*

$$Pu_h = g_h, \tag{3.6}$$

with $g_h = \sum_i g_i$ and the g_i given by the solutions for the following problems

$$a(g_i, \phi_h) = a(P_i u, \phi_h) = f(\phi_h), \forall \phi_h \in V_i. \tag{3.7}$$

We use the conjugate gradient method to solve $Pu_h = g_h$. We only need to compute Pv for a given $v \in V^h$ in each iteration, thus, the explicit representation of P is not needed. The computation is carried out in the following steps.

- Compute the right hand side g_h by first solving equations (3.7) and then setting $g = \sum g_i$.
- In each iteration, compute Pv_h for given v_h by first solving

  $$a(P_i v_h, \phi) = a(v_h, \phi), \quad \forall \phi \in V_i,$$

  and then setting $Pv_h = \sum P_i v_h$

Remark 3.2.1 We consider a special choice of subregions. Let $\hat{\Omega}_i = \text{supp}\{\phi_i\}$, and thus $V_i = \text{span}\{\phi_i\}$. Then the corresponding additive Schwarz method, without a coarse space, corresponds to applying the CG algorithm to $D^{-1}Kx = D^{-1}b$, where $D = diag(K)$. This is the Jacobi conjugate gradient method.

3.2.2 Condition Number Estimate

Theorem 3.2.1 (Dryja and Widlund) *For the operator P defined above, there exists a constant C_1 such that*

$$C_1 a(v, v) \leq a(Pv, v) \leq (N_c + 1)a(v, v), \quad \forall v \in V^h.$$

Thus $\kappa(P) \leq (N_c+1)/C_1$, and the rate of convergence of the additive Schwarz algorithm is independent of the mesh size and the number of subdomains.

Proof. We first establish the upper bound. It is trivial to see that

$$a(Pu, u) \leq (N+1)a(u,u).$$

However we want to get an upper bound independent of the number of substructures as well as the mesh size h. For $u \in V^h$,

$$a(P_i u, u) = a(P_i u, P_i u) = a_{\hat{\Omega}_i}(P_i u, P_i u) \leq a_{\hat{\Omega}_i}(u,u).$$

Summing over i, using the finite covering property of $\{\hat{\Omega}_i\}$, we get

$$\sum_{i=1}^{N} a(P_i u, u) \leq N_c a(u,u).$$

We also note that $a(P_0 u, u) \leq a(u,u)$. Thus the upper bound follows with $C_2 = (N_c+1)$.

To prove the lower bound, for any $u \in V^h$, one needs to find a good partition $\{u_i\}$ of u as in Lions' Lemma. Let $Q^H : H_0^1(\Omega) \to V^H$ be the L_2 projection. It is known that Q^H has the following property

$$\|u - Q^H u\|_{L^2(\Omega)} \leq CH|u|_{H^1(\Omega)} \tag{3.8}$$

$$\|Q^H u\|_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)} \tag{3.9}$$

Let $\Pi^h : H_0^1(\Omega) \cap C(\Omega) \to V^h$, be the standard interpolation operator. Let $u_0 = u_H = Q^H u_h$, $w_h = u_h - u_H$, and $u_i = \Pi^h(\theta_i w_h)$. It is easy to see that $u_h = \sum_0^N u_i$. To see that we indeed get a good partition of u_h, we have to estimate the norms of u_i. It can be shown that

$$\begin{aligned} |\Pi^h(\theta_i w)|_{H^1(\hat{\Omega}_i)} &\leq C|\theta_i|_{L^\infty}|w|_{H^1(\hat{\Omega}_i)} + C|\theta_i|_{W^{1,\infty}}|w|_{L^2(\hat{\Omega}_i)} \\ &\leq C|w|_{H^1(\hat{\Omega}_i)} + C/H_i|w|_{L^2(\hat{\Omega}_i)} \end{aligned}$$

Using the approximation property of Q^H, we have

$$|u_i|_{H^1(\Omega)} = |\Pi^h(\theta_i w)|^2_{H^1(\hat{\Omega}_i)} \leq C|u|^2_{H^1(\hat{\Omega})}.$$

Summing over i, and using the finite covering property of $\{\hat{\Omega}_i\}$, we obtain

$$\sum_i |u_i|^2_{H^1(\Omega)} \leq C|u|^2_{H^1(\hat{\Omega})}.$$

The lower bound of P follows from Lions' Lemma. ∎

Remark 3.2.2 If we do not use the coarse space, the condition number of the operator will grows like H^{-2}. This fact was pointed out and proved by Widlund; cf. [47].

In practice, the maximum eigenvalue of P is between N_c and $N_c + 1$, and gets closer to N_c as H/h grows.

3.3 Multilevel Methods

In two level methods, we need to solve a coarse problem of size $O(1/H^2)$ and some local problems of size $(H/h)^2$. If h is small, we cannot have both $1/H$ and H/h small. Thus at least one of the subproblems is large. The computation can be made cheaper by recursively using the additive Schwarz method to solve the coarser problems.

We consider a class of multilevel algorithms including the multilevel additive Schwarz method considered in Dryja and Widlund [28] and the BPX algorithm studied in Bramble, Pasciak and Xu [13] and Xu [50]. We use the space decomposition and projection techniques, introduced by Dryja and Widlund, to study the algorithms. We improve the old results by showing that the condition number of the iteration operator grows at most linearly with the number of levels in general, and is bounded by a constant independent of mesh sizes and the number of levels under the H^2-regularity assumption of the elliptic operator. We note that similar results are already known for multigrid methods.

3.3.1 Description of the Multilevel Additive Schwarz Methods

We discuss a class of L-level additive Schwarz methods. We define a sequence of nested triangulations $\{\mathcal{T}^L_{l=1}\}$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau_i^1\}_{i=1}^{N_1}$, where τ_i^1 represent an individual triangle. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}(l = 2, \cdots, L)$ are defined by dividing each triangle in the triangulation $\mathcal{T}^{l-1}$ into several triangles, i.e.

$$\mathcal{T}^1 = \{\tau_i^1\}_1^{N_1} \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^2 = \{\tau_i^1\}_1^{N_2} \overset{\text{refinement}}{\Longrightarrow} \cdots \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^L = \{\tau_i^L\}_1^{N_L}.$$

We assume that all the triangulations are shape regular. Let $h_i^l = \text{diam}(\tau_i^l),\ h_l = \max_i\{h_i^l\}$ and $h = h_L$.

Let $V^l, l = 1, \cdots, L$, be the space of continuous piecewise linear element associated with the triangulation $\mathcal{T}^l$. The finite element solution, $u_h = P_{V^h} u \in V^h = V^L$, satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h = V^L. \tag{3.10}$$

We assume that there are $L-1$ sets of overlapping subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, 3, \cdots, L$. On each level, we have an overlapping decomposition

$$\Omega = \cup_{i=1}^{N_l} \hat{\Omega}_i^l.$$

We assume that the sets $\{\hat{\Omega}_i^l\}$ satisfy

Assumption 3.3.1 *The decomposition $\Omega = \cup_{i=1}^{N_l} \hat{\Omega}_i^l$ satisfies*

- $\partial \hat{\Omega}_i^l$ *aligns with the boundaries of level l triangles, i.e. $\hat{\Omega}_i^l$ is the union of level l triangles. Diameter$(\hat{\Omega}_i^l) = O(h_{l-1})$.*
- *On each level, the subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form a* finite covering *of Ω, with a covering constant N_c, i.e. we can color $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*
- *On each level, associated with $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a* partition of unity *$\{\theta_i^l\}$ satisfying*

$$\sum_i \theta_i^l = 1, \text{ with } \theta_i^l \in H_0^1(\hat{\Omega}_i^l) \cap C^0(\hat{\Omega}_i^l), 0 \leq \theta_i^l \leq 1 \text{ and } |\nabla \theta_i^l| \leq C/h_{l-1}.$$

The first property is very natural; it simply says that the restriction of the triangulation $\mathcal{T}^l$ to subdomain $\hat{\Omega}_i^l$ defines a triangulation for $\hat{\Omega}_i^l$ and the finite element problem on $\hat{\Omega}_i^l$ is well defined. The second condition is used to establish the upper bound of the spectrum of the additive Schwarz operator. The last condition is used for the lower bound of the spectrum.

One way of constructing subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, \cdots, L$, with the above properties is described in Dryja and Widlund [24,25]. Each triangle $\tau_i^{l-1}, i = 1, \cdots, N_l, l = 2, \cdots, L$, is extended to a larger region $\hat{\tau}_i^{l-1}$ so that $ch_i^{l-1} \leq \text{dist}(\partial\hat{\tau}_i^{l-1}, \partial\tau_i^{l-1}) \leq Ch_i^{l-1}$, aligning $\partial\hat{\tau}_i^{l-1}$ with the boundaries of level l triangles. We cut off the part of $\hat{\tau}_i^{l-1}$ that is outside

Ω. We use $\hat{\tau}_i^{l-1}$ as the subdomains $\hat{\Omega}_i^l$. Another way of constructing $\{\hat{\Omega}_i^l\}$ is given in section 3.3.3.

Let $N_1 = 1, V_1^1 = V^1$ and $V_i^l = V^l \cap H_0^1(\hat{\Omega}_i^l)$ for $i = 1, \cdots, N_l, l = 2, \cdots, L$. The finite element space $V^h = V^L$ is represented by

$$V^L = \sum_{l=1}^{L} V^l = \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l. \tag{3.11}$$

Let $P_i^l : V^h \to V_i^l$, be the projection defined by

$$a(P_i^l u, \phi) = a(u, \phi), \quad \forall \phi \in V_i^l.$$

The L-level additive Schwarz operator P is defined by

$$P = \sum_{l=0}^{L} \sum_{i=1}^{N_l} P_i^l. \tag{3.12}$$

Instead of solving the original finite element equation (3.10), we solve the following equivalent equation:

Algorithm 3.3.1 *Find $u_h \in V^L$ by solving iteratively the equation*

$$Pu_h = g_h$$

with $g_h = \sum_l \sum_i g_i^l$. Here the g_i^l are the solutions for the following finite element problems

$$a(g_i^l, \phi) = a(P_{V_i^l} u, \phi) = f(\phi), \quad \forall \phi_h \in V_i^l. \tag{3.13}$$

To find u_h, we first find the right hand side g_h, by solving (3.13), and we then use the conjugate gradient method to solve the system. In each iteration, we need to compute $P_i^l v_h$ for a given $v_h \in V^h$ by solving the equation

$$a(P_{V_i^l} v_h, \phi_h) = a(v_h, \phi_h) \quad \forall \phi_h \in V_i^l.$$

This is a finite element equation on $\hat{\Omega}_i^l$ with mesh size h_l, and $\dim(V_i^l) \approx c(h_{l-1}/h_l)^2$. Thus the size of all such problems can be very small.

3.3.2 Condition Number Estimate

When using the conjugate gradient method to solve a linear system, the crucial issue is the condition number of the iteration operator. Dryja and Widlund [28] established the following estimates for the spectrum of P, cf. theorem 3.2 in Dryja and Widlund [28].

Theorem 3.3.1 *For P defined above, the following inequalities hold*

$$C_1 L^{-1} a(u_h, u_h) \leq a(Pu_h, u_h) \leq C_2 L\, a(u_h, u_h) \ \forall u_h \in V^h. \tag{3.14}$$

Thus $\kappa(P) \leq C_2 C_1^{-1} L^2$, i.e. the condition number of P grows at most quadratically with the number of levels. Here the constants C_1 and C_2 are independent of the mesh sizes $\{h_l\}$ and L.

In this section, we improve the upper bound in (3.14) by eliminating the dependence on L. Using H^2-regularity, which holds for convex regions, we can also eliminate the dependence on L in the lower bound.

Theorem 3.3.2 *For the multilevel additive Schwarz operator P defined above, we have*

$$C_1 L^{-1} a(u_h, u_h) \leq a(Pu_h, u_h) \leq C_2\, a(u_h, u_h) \ \forall u_h \in V^h.$$

If the equation has H^2-regularity, then the lower bound can also be improved, and we have

$$C_1 a(u_h, u_h) \leq a(Pu_h, u_h) \leq C_2\, a(u_h, u_h) \ \forall u_h \in V^h.$$

All the constants are independent of $\{h_l\}$ and L.

Let $P^l : V^h \to V^l$, be the orthogonal projection. Then, for $u_h \in V^h$ and $1 \leq l \leq L$, we have

$$P^l u_h = \sum_{i=1}^{l} u^i, \text{ with } u^i = (P^i - P^{i-1}) u_h, \ \ u^1 = P^1 u_h. \tag{3.15}$$

In particular, $u_h = P^L u_h = \sum_{l=1}^{L} u^l$. Using the fact that $V^k \subset V^l$, for $k \leq l$, we obtain $P^k P^l = P^l P^k = P^k$. Thus $P^l - P^{l-1}$ is also a projection, and

$$(P^l - P^{l-1})(P^k - P^{k-1}) = \delta_{lk}(P^l - P^{l-1}).$$

Therefore $u_h = \sum_{i=0}^{L} u^i$ is an $a(\cdot,\cdot)$-orthogonal decomposition of u_h, and

$$a(u_h, u_h) = \sum_{l=1}^{L} a(u^l, u^l) \tag{3.16}$$

Since, on each level, the subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form a finite covering of Ω with a covering constant N_c, we can color the $\hat{\Omega}_i^l$ using only N_c colors, in such a way that all subdomains of the same color are disjoint. On each level, we can group the subdomains $\hat{\Omega}_i^l$ by color, obtaining N_c sets of subregions. Let $\Lambda_s, s = 1, \cdots, N_c$, be the sets of indices of these sets. Since the subdomains $\hat{\Omega}_i^l$ of the same color are disjoint, subspaces V_i^l of the same color are mutually orthogonal and $P_{i_1}^l P_{i_2}^l = 0$ for i_1, i_2 in the same Λ_s. This in turn implies that $P_{\Lambda_s}^l = \sum_{i\in\Lambda_s} P_i^l$ is a projection from V^h onto $V_{\Lambda_s}^l = \sum_{i\in\Lambda_s} V_i^l$. In particular, $(P_{\Lambda_s}^l)^2 = P_{\Lambda_s}^l$. We can therefore write

$$\sum_{i=1}^{N_l} P_i^l = \sum_{s=1}^{N_c} \sum_{i\in\Lambda_s} P_i^l \equiv \sum_{s=1}^{N_c} P_{\Lambda_s}^l,$$

where the $P_{\Lambda_s}^l$ are projections.

The next lemma, similar to the strengthened Cauchy inequality, plays an important role in obtaining an upper bound for $\lambda_{\max}(P)$.

Lemma 3.3.1 *For $k \le l$, and $u^k \in V^k$, we have*

$$a(P_{\Lambda_s}^l u^k, u^k)^{1/2} \le r_{k,l} a(u^k, u^k)^{1/2}$$

with $r_{k,k} \le 1, r_{k,k+1} \le 1$ and $r_{k,l} \le C r^{l-1-k}$, for $k < l-1$.

Proof. Since $P_{\Lambda_s}^l$ is a projection, the conclusions for $l = k$ and $l = k+1$ are trivial. For $l - 1 > k$, we decompose Λ_s into two disjoint sets Λ_I and Λ_B; $i \in \Lambda_I$ if the subdomain $\hat{\Omega}_i^l$ lies in the interior of a triangle τ^k, while $i \in \Lambda_B$ if $\hat{\Omega}_i^l$ intersects $\partial\tau_{i'}^k$ for some i'. We write $P_{\Lambda_s} = P_{\Lambda_I} + P_{\Lambda_B}$. We note that u^k is linear in each triangle τ_j^k and therefore harmonic in τ_j^k. For each $i \in \Lambda_I$, $P_i^l u^k \in H_0^1(\hat{\Omega}_i^l) \subset H_0^1(\tau_{i'}^k)$ for some i', and therefore, $a(P_{\Lambda_I}^l u^k, u^k) = 0$. Thus,

$$a(P_{\Lambda_s}^l u^k, u^k) = a(P_{\Lambda_B}^l u^k, u^k)$$

Let $S = \text{supp}\{P_{\Lambda_B} u^k\} = \cup_{i\in\Lambda_B} \hat{\Omega}_i^l$. Then

$$a(P_{\Lambda_B}^l u^k, u^k) = a_S(P_{\Lambda_B}^l u^k, u^k) \le a_S(u^k, u^k)$$

Since u^k is linear in τ_j^k, ∇u^k is constant in τ_j^k. Therefore

$$a_{S\cap\tau_j^k}(u^k,u^k) = \frac{\mathrm{mes}(S\cap\tau_j^k)}{\mathrm{mes}(\tau_j^k)} a_{\tau_j^k}(u^k,u^k) \le C\frac{h_{l-1}}{h_k} a_{\tau_j^k}(u^k,u^k) = Cr_{k,l}^2 a_{\tau_j^k}(u^k,u^k).$$

Summing over j, we obtain

$$a_S(u^k,u^k) \le Cr_{k,l}^2 a(u^k,u^k).$$

Therefore

$$a(P_{\Lambda_s}^l u^k,u^k) = a(P_{\Lambda_B}^l u^k,u^k) \le a_S(u^k,u^k) \le Cr_{k,l}^2 a(u^k,u^k).$$

∎

Proof of Theorem 3.3.2. We first establish the upper bound. Since $V_{\Lambda_s}^l = \sum_{i\in\Lambda_s} V_i^l \subset V^l$, we have $P_{\Lambda_s}^l P^l = P_{\Lambda_s}^l$. Thus,

$$a(P_{\Lambda_s}^l u_h,u_h) = a(P_{\Lambda_s}^l u_h, P_{\Lambda_s}^l u_h) = a(P_{\Lambda_s}^l P^l u_h, P_{\Lambda_s}^l P^l u_h).$$

Substituting (3.15) into the above equation, we obtain

$$\begin{aligned} a(P_{\Lambda_s^l} u_h,u_h) &= \sum_{k=1}^{l}\sum_{j=1}^{l} a(P_{\Lambda_s}^l u^k, P_{\Lambda_s}^l u^j) \le \sum_{k,j=1}^{l} |P_{\Lambda_s}^l u^k|_a |P_{\Lambda_s}^l u^j|_a \\ &= (\sum_{k=1}^{l} |P_{\Lambda_s}^l u^k|)^2 \le C(\sum_{k=1}^{l} r_{k,l}|u^k|_a)^2 \\ &\le C(\sum_{k=1}^{l} r_{k,l})(\sum_{k=1}^{l} r_{k,l} a(u^k,u^k)) \le C\frac{2}{1-r}\{\sum_{k=1}^{l} r_{k,l} a(u^k,u^k)\}. \end{aligned}$$

Summing over all colors ($1\le s\le N_c$), we get

$$\sum_{i=1}^{N_l} a(P_i^l u_h,u_h) = \sum_{s=1}^{N_c} a(P_{\Lambda_s}^l u_h,u_h) \le CN_c\frac{2}{1-r}(\sum_{k=1}^{l} r_{k,l} a(u^k,u^k)).$$

Summing over l and changing the order of the summation for k and l, we get

$$\begin{aligned} \sum_{l=1}^{L}\sum_{i=1}^{N_l} a(P_i^l u_h,u_h) &\le CN_c\frac{2}{1-r}(\sum_{k=1}^{L}\sum_{l=k}^{L} r_{k,l} a(u^k,u^k)) \\ &\le CN_c\frac{2}{1-r}\{\sum_{k=1}^{L} a(u^k,u^k)(\sum_{l=k}^{L} r_{k,l})\} \\ &\le CN_c(\frac{2}{1-r})^2 \sum_{k=1}^{L} a(u^k,u^k) \\ &= CN_c(\frac{2}{1-r})^2 a(u_h,u_h). \end{aligned}$$

In the last step, we have used the orthogonality property (3.16) of the u^k. This concludes the proof of the upper bound of $P = \sum_l \sum_i P_i^l$.

We now establish the lower bound. We note that, for the general case, it is given in Dryja and Widlund [28]. When the problem is H^2-regular, we can use Nitsche's trick to show that the $a(\cdot,\cdot)$-projection P^l satisfies the approximation property

$$\|P^l u - u\|_{L^2(\Omega)} \le C h_l |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,. \tag{3.17}$$

We use the orthogonal decomposition

$$u_h = P^L u_h = \sum_{l=1}^{L} u^l \equiv P^1 u_h + (P^2 - P^1)u_h + \cdots + (P^L - P^{L-1})u_h.$$

Since $u^l = (P^l - P^{l-1})u_h = (P^l - P^{l-1})^2 u_h = (I - P^{l-1})u^l$, we get, using (3.17),

$$\|u^l\|_{L^2(\Omega)} \le C\, h_{l-1} |u^l|_{H^1(\Omega)}. \tag{3.18}$$

We further decompose u^l as

$$u^l = \sum_{i=1}^{N_l} u_i^l\,, \text{ with } u_i^l \equiv \Pi^{h_l}(\theta_i^l u^l) \in V_i^l.$$

Here Π^{h_l} is the interpolation operator from V^h to V^{h_l} and $\{\theta_i^l\}$ a partition of unity as in Assumption 3.3.1. It can be shown that

$$\begin{aligned}
|u_i^l|^2_{H^1(\hat\Omega_i^l)} &= |\Pi^{h_l}(\theta_i u^l)|^2_{H^1(\hat\Omega_i^l)} \\
&\le C(|\theta_i|^2_{L^\infty(\Omega)} |u^l|^2_{H^1(\hat\Omega_i^l)} + |\theta_i|^2_{W^{1,\infty}(\Omega)} \|u_i^l\|^2_{L^2(\hat\Omega_i^l)}) \\
&\le C(|u^l|^2_{H^1(\hat\Omega_i^l)} + (1/h_{l-1}^2)\|u_i^l\|^2_{L^2(\hat\Omega_i^l)}).
\end{aligned}$$

Summing the above inequality over i, using the finite covering property of $\{\hat\Omega_i^l\}$ and inequality (3.18), we obtain

$$\begin{aligned}
\sum_i |u_i^l|^2_{H^1(\Omega)} &= \sum_i |u_i^l|^2_{H^1(\hat\Omega_i^l)} \le C \sum_i \{|u^l|^2_{H^1(\hat\Omega_i^l)} + 1/h_{l-1}^2 \|u^l\|^2_{L^2(\hat\Omega_i^l)}\} \\
&\le C\{|u^l|^2_{H^1(\Omega)} + 1/h_{l-1}^2 \|u^l\|^2_{L^2(\Omega)}\} \le C|u^l|^2_{H^1(\Omega)}.
\end{aligned}$$

Summing over l, $1 \le l \le L$, and using the orthogonality of u^l, we get

$$\sum_{l=1}^{L} \sum_i |u_i^l|^2_{H^1(\Omega)} \le C |u_h|^2_{H^1(\Omega)}.$$

The lower bound for P now follows from Lions' lemma. ∎

Remark 3.3.1 Although the proof of the upper bound is given for the model problem, it is easy to see that it works for any uniform elliptic operator. Since we can confine our study to one substructure at a time, we also see that the upper bound is independent of jumps in the coefficients between the substructures.

3.3.3 A Multilevel Diagonal Scaling

We begin this section by constructing a special decomposition of the domain Ω. We then show that this decomposition, and the corresponding decomposition of the finite element subspaces, satisfies Assumption 3.3.1. We then demonstrate that the algorithm is a multilevel diagonal scaling (MDS), a natural generalization of diagonal scaling. For problems with constant coefficients and with uniform triangulations, the multilevel diagonal scaling algorithm is identical, up to a constant multiple, to the BPX algorithm of Bramble, Pasciak and Xu [13]. In the general case, BPX with diagonal scaling results in MDS algorithm.

Let $\{\mathcal{T}^l\}_{l=1}^L$ be a nested sequence of triangulations, with $\mathcal{T}^{l+1}$ obtained from $\mathcal{T}^l$ by dividing the triangles (rectangles) of $\mathcal{T}^l$ into four triangles (rectangles). In three dimension, we make a similar construction. We consider the piecewise linear and bilinear elements or trilinear elements, respectively. As in the previous section, the finite element space associated with $\mathcal{T}^l$ is denoted by V^l, and $V^h = V^L$. Let ϕ_i^l be a nodal basis function of V^l, and associate with each ϕ_i^l a subdomain $\hat{\Omega}_i^l = \text{supp}\{\phi_i^l\}$. We choose $V_i^l = \text{span}\{\phi_i^l\} = V^l \cap H_0^1(\hat{\Omega}_i^l)$ and obtain the decomposition

$$V^L = \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l$$

and the projections $P_{V_i^l} : V^L \xrightarrow{H^1} V_i^l$. Using $P = \sum_{l=1}^{L} \sum_i P_{V_i^l}$, we define an additive Schwarz algorithm

Algorithm 3.3.2 (MDS) *Find the finite element solution $u_h \in V^h$ by solving iteratively the equation*

$$Pu_h = g_h$$

with an appropriate right hand side g_h.

We define the degree of a vertex x_i as the number of edges directly connected to x_i, and the degree of a triangulation $\mathcal{T}^h$ as the maximum of the degrees of its vertices. It is easy to see that the overlapping subdomains $\{\hat{\Omega}_i^l\}$ satisfy Assumption 3.3.1. In particular, we see that on each level, $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form an finite covering of Ω with a covering constant less than or equal to degree($\mathcal{T}^l$)+1. We also see that on each level, $\{\hat{\Omega}_i^l\}$ provides relatively generous overlap. The nodal basis $\{\phi_i^l\}$ can be chosen as a partition of unity.

The optimal convergence properties of the algorithm follow from theorem 3.3.2, and as a consequence, we obtain an improved estimate for the BPX algorithm; see further discussion below.

To see that the above algorithm is in fact a generalization of the diagonal scaling method, we first consider a matrix representation of two simple algorithms.

Algorithm 1. In the two level additive Schwarz algorithm, the matrix form of the projections P_{V_i} is given by

$$P_{V_i} = \begin{pmatrix} K_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} K_h = \tilde{K}_i^+ K_h$$

after a permutation. Here K_i is the stiffness matrix associated with the subspace V_i. Let $\Pi_H^h : V^H \rightarrow V^h$ be the standard interpolation operator and Π_h^H be its adjoint. In matrix form, the two level additive Schwarz algorithm can then be written as a preconditioned system

$$B_h^{-1} K_h x = B_h^{-1} b,$$

where

$$B_h^{-1} = \Pi_H^h K_H^{-1} \Pi_h^H + \sum_{i=1}^{N} \tilde{K}_i^+.$$

Algorithm 2. With the special choice of the subregions $\hat{\Omega}_i = \text{supp}\{\phi_i\}$, and $V_i = \text{span}\{\phi_i\}$, the additive Schwarz algorithm corresponds to

$$D^{-1} K_h \vec{u} = D^{-1} b,$$

where $D = \text{diag}(K_h)$. This gives us the Jacobi conjugate gradient method.

In the multilevel case, let $\Pi_{l_1}^{l_2} : V^{l_1} \rightarrow V^{l_2}, (l_1 \geq l_2)$ be the standard interpolation (prolongation) operator, and let $\Pi_{l_2}^{l_1} : V^{l_2} \rightarrow V^{l_1} = (\Pi_{l_1}^{l_2})^t, (l_1 \geq l_2)$ be a local averaging operator, which is the adjoint operator of $\Pi_{l_1}^{l_2}$. Algorithm 3.3.2 can then be written as: Find the solution of $K_L x = b$ by solving the preconditioned system

$$B_L^{-1} K_L x = B_L^{-1} b,$$

where

$$B_L^{-1} = \Pi_1^L K_1^{-1} \Pi_L^1 + \Pi_2^L D_2^{-1} \Pi_L^2 + \cdots + \Pi_{L-1}^L D_{L-1}^{-1} \Pi_L^{L-1} + D_L^{-1}.$$

Here K_l is the stiffness matrix associated with V^l and $D_l = \mathrm{diag}(K_l)$. We note that K_1^{-1} can be replaced by any good preconditioner B_1 of K_1.

If we replace the matrices D_l by identity matrices, we obtain the BPX algorithm. However, since the diagonal elements contain information on the shapes of the triangles and the coefficients of the problems, we expect that the multilevel diagonal preconditioner will work better in practice for non-model problems, since it more closely reflects the properties of the problem.

The method described in this section is similar to the hierarchical basis method [52]. The work in each iteration is about $\frac{4}{3}$ of that of the hierarchical basis method. However, the condition number is much better than for the hierarchical basis method, and the method also works well in higher dimensions, at least for problems with smooth coefficients. A detailed comparison of the BPX algorithm and the hierarchical basis method is given in Yserentant [53].

3.3.4 Numerical Experiments

In this section, we report on some the numerical experiments with the multilevel additive Schwarz methods.

In these experiments, we only concern with the convergence properties of the algorithms. For implementations of the algorithms on a parallel computer, see Bjørstad, Moe and Skogen [5] and Bjørstad and Skogen [6]. They implemented multilevel additive Schwarz algorithms on a massively parallel, SIMD machine (MasPar MP-1). Approximate solvers for the subproblems are also discussed.

The experiments were carried out for Poisson's equation on a unit square with homogeneous Dirichlet boundary conditions

$$\left\{ \begin{array}{rcll} -\Delta u & = & f(x) & \text{in} \quad \Omega, \\ u & = & 0 & \text{in} \quad \partial\Omega. \end{array} \right. \tag{3.19}$$

total no. of elements	no. of subdomains of a lower level $N \times N$	ovlp ratio	no. of levels L	cond no. $\kappa(P)$	no. of iter. for $\epsilon = 10^{-6}$
8^2	2×2	1/2	3	7.2	11
16^2	2×2	1/2	4	9.3	17
32^2	2×2	1/2	5	10.7	20
64^2	2×2	1/2	6	11.7	21
9^2	3×3	1/3	2	4.6	9
27^2	3×3	1/3	3	7.1	16
81^2	3×3	1/3	4	8.4	19
243^2	3×3	1/3	5	9.5	21
27^2	3×3	1/3	2	4.8	8
81^2	3×3	1/3	2	4.7	7
16^2	4×4	1/4	2	5.1	13
64^2	4×4	1/4	3	7.3	17
256^2	4×4	1/4	4	8.4	20
64^2	4×4	1/4	2	5.3	8
25^2	5×5	1/5	2	5.7	14
125^2	5×5	1/5	3	7.6	17

Table 3.1: Multilevel Additive Schwarz Scheme, Using Bilinear Element

We divide the domain Ω into $N \times N$ square elements $\tau_{ij}^1, i,j = 1, \cdots, N$, and obtain a triangulation $\mathcal{T}^1 = \{\tau_{ij}^1\}$. We then divide each $\tau_{i,j}^1$ into $N \times N$ squares to obtain the triangulation $\mathcal{T}^2 = \{\tau_{ij}^2\}$, etc. The length of an edge of τ_{ij}^l is denoted by H_l and $H_l = (1/N)^l$. For $l = 2, \cdots, L$, we extend τ_{ij}^{l-1} to a larger square $\hat{\tau}_{ij}^{l-1}$. The overlap ratio

$$\text{overlap ratio} = \frac{\text{dist}\{\partial\hat{\tau}_{ij}^{l-1}, \partial\tau_{ij}^{l-1}\}}{H_{l-1}}$$

measures the width of $\hat{\tau}_{ij}^{l-1} \setminus \tau_{ij}^{l-1}$ in terms of H_{l-1}, the side of the square τ_{ij}^{l-1}. We use Ω as our domain for $l = 1$, and $\Omega_{ij}^l = \hat{\tau}_{ij}^{l-1}$ as our subdomains for $l = 2, \cdots, N$.

In these experiments, we take $N = 2, 3, 4$ or 5, and $\hat{\tau}_{ij}^{l-1} \setminus \tau_{ij}^{l-1}$ is one element wide(H_l), i.e. the overlap ratio is $1/N$. Therefore, we only need to solve very small linear systems of order 9, 16 25 and 36, respectively. We use the conjugate gradient method to solve the system $Pu_h = g_h$ iteratively. The last column of the table gives the number of iterations required to decrease the l_2 norm of the residual by a factor of $\epsilon = 10^{-6}$.

Chapter 4

Some Approximation Properties of Finite Elements

In this chapter, we discuss some properties of the conforming finite element for the biharmonic problems. We first discuss and prove some approximation properties of the finite elements. We then define some discrete norms for the finite element spaces and establish their equivalence to the corresponding continuous Sobolev norms or semi-norms. In section 4.4, we estimate the norm of the interpolant of a product of two functions. These results are used to prove the estimates for the convergence properties of the domain decomposition methods developed in chapters 5 and 6.

Let $\mathcal{T}^h = \{\tau_i\}$ and $\mathcal{T}^H = \{\Omega\}$ be the fine and coarse triangulations of the domain Ω, respectively. Let V_Q^h, V_A^h and V_B^h be the spaces of the bicubic, Argyris and Bell elements, respectively, associated with the fine triangulation. V_Q^H, V_A^H and V_B^H are similarly defined. For a vertex x_i, let $\tau(x_i)$ be the support of the basis functions associated with x_i, i.e. $\tau(x_i)$ is a region containing all elements with this vertex x_i. For a element τ, let $\hat{\tau}$ be the smallest region containing τ and all its neighboring elements, i.e. $\overline{\hat{\tau}} = \cup_{\overline{\tau}_j \cap \overline{\tau} \neq \emptyset} \overline{\tau}_j$.

We also need to consider some subspaces of V_Q^h, V_A^h and V_B^h. Let $\Phi_i^\alpha, |\alpha| = 0, 1,$ and Φ_i^{xy} be the basis functions of V_Q^H. Let

$$\tilde{V}_Q^H = \text{span}\{\Phi_i^\alpha, |\alpha| = 0, 1, i \in \Lambda^H\}.$$

Thus, $\tilde{V}_Q^H$ is the subspace of V_Q^H consisting of functions whose mixed second derivatives vanish at all vertices of the substructures. $\tilde{V}_A^H \subset V_A^H$ and $\tilde{V}_B^H \subset V_B^H$ are defined similarly. We use $\Lambda^h(D)$ and $\Lambda^H(D)$ to denote the sets of fine and coarse grid points in the set D, respectively.

4.1 Approximation by Quasi-Interpolation

In this section, we discuss and prove a stable approximation property of the finite element spaces V_Q^h, V_A^h and V_B^h. The results are based only on the two properties of the nodal basis functions stated in section 1.3.3. Thus if another basis satisfies the same assumptions, the same results hold.

We describe the results for the Bell element; the results for the Argyris and bicubic elements are similar.

We first construct a local linear operator. We note that for $\Omega \subset \mathcal{R}^2$, we have the inclusion $V_B^h \subset V_A^h \subset H^2(\Omega) \subset C^0(\Omega)$. Let $\phi_i^\alpha, |\alpha| \leq 2$ and let τ be an element. We define a local linear operator $I_B^\tau : H^2(\tau) \to \mathcal{P}_\mathcal{B}(\tau)$, by

$$I_B^\tau u \equiv \sum_i u(x_i)\phi_i(x) + \sum_i \sum_{|\alpha|=1} \{D^\alpha u\}_\tau \phi_i^\alpha(x), \quad x \in \tau. \tag{4.1}$$

Thus,

$$\begin{aligned}
(I_B^\tau u)(x_i) &= u(x_i), \\
(D^\alpha I_B^\tau u)(x_i) &= \{D^\alpha u\}_\tau,\ |\alpha| = 1, \\
(D^\alpha I_B^\tau)(x_i) &= 0,\ |\alpha| = 2\ .
\end{aligned}$$

We find that if p is a linear function in τ, then $I_B^\tau p = p$. The operator I_B^τ has the following local approximation properties.

Lemma 4.1.1 *Operator I_B^τ satisfies the following inequalities*

$$\begin{aligned}
|I_B^\tau u|_{H^2(\tau)} &\leq C|u|_{H^2(\tau)}, \\
|I_B^\tau u - u|_{H^s(\tau)} &\leq Ch^{2-s}|u|_{H^2(\tau)}, s = 0, 1.
\end{aligned}$$

Proof. By Lemma 1.2.4, we know that there exists a linear function $p(x)$ such that u can be written as $p + R$ with R satisfying

$$\{D^\alpha R\}_\tau = 0, |\alpha| = 0, 1.$$

The extended Poincaré's inequality gives

$$|R|_{H^s(\tau)} \leq C_\tau h_\tau^{2-s}|R|_{H^2(\tau)} = C_\tau h_\tau^{2-s}|u|_{H^2(\tau)}. \tag{4.2}$$

Using the linearity of I_B^τ, we have $I_B^\tau u = I_B^\tau p + I_B^\tau R = p + I_B^\tau R$. By the definition of I_B^τ and using the fact that $\{D^\alpha R\}_\tau = 0$ for $|\alpha| = 1$, we get

$$I_B^\tau R = \sum R(x_i)\phi_i(x).$$

By the triangle inequality

$$|I_B^\tau R|_{H^s(\tau)} \leq \sum |R(x_i)||\phi_i|_{H^s(\tau)}. \tag{4.3}$$

¿From Lemma 1.2.3, with $k = 1, |\alpha| = 0, \Omega = \tau$, $d = 2$, and (4.2), we have

$$|R(x_i)| \leq C_\tau h_\tau |R|_{H^2(\tau)} = C_\tau h_\tau |u|_{H^2(\tau)}.$$

The basis functions are uniform to order 2 in the weak sense, i.e.

$$|\phi_i^\alpha|_{H^s(\tau)} \leq Ch^{1+|\alpha|-s}, \text{ for } |\alpha| = 0, 1, |\beta| = 0, 1, 2,$$

especially $|\phi_i|_{H^s(\tau)} \leq Ch^{1-s}$. Substituting the above two inequalities into (4.3) gives

$$|I_B^\tau R|_{H^s(\tau)} \leq Ch_\tau |u|_{H^2(\tau)} h_\tau^{1-s} \leq Ch^{2-s}|u|_{H^2(\tau)}, s = 0, 1, 2. \tag{4.4}$$

The first part of the lemma follows from (4.4) with $s = 2$. By the triangle inequality and by using (4.2) and (4.4), we have

$$|I_B^\tau u - u|_{H^s} = |I_B^\tau R - R|_{H^s} \leq |I_B^\tau R|_{H^s} + |R|_{H^s} \leq Ch^{2-s}|u|_{H^2(\tau)}. \tag{4.5}$$

This proves the second part of the lemma. ∎

We now construct a global linear operator. Let $I_{\tilde{V}_B^h} : H^2(\Omega) \to \tilde{V}_B^h$, be the quasi-interpolation operator defined by

$$I_{\tilde{V}_B^h} u \equiv u_h = \sum_{i \in \Lambda^h(\Omega)} u(x_i)\phi_i(x) + \sum_{i \in \Lambda^h(\Omega)} \sum_{|\alpha|=1} \{D^\alpha u\}_{\tau(x_i)} \phi_i^\alpha(x).$$

Thus,

$$\begin{aligned} I_{\tilde{V}_B^h} u(x_i) &= u_h(x_i) = u(x_i), \\ D^\alpha I_{\tilde{V}_B^h} u(x_i) &= D^\alpha u_h(x_i) = \{D^\alpha u\}_{\tau(x_i)}, \ |\alpha| = 1, \\ D^\alpha I_{\tilde{V}_B^h} u(x_i) &= D^\alpha u_h(x_i) = 0, \ |\alpha| = 2\,, \end{aligned}$$

where $\overline{\tau}(x_i) = \cup_{x_i \in \overline{\tau}_j} \overline{\tau}_j$ defined as above. The quasi-interpolation operator $I_{\tilde{V}_A^h} : H^2(\Omega) \rightarrow \tilde{V}_A^h$, and $I_{\tilde{V}_Q^h} : H^2(\Omega) \rightarrow \tilde{V}_Q^h$, are defined similarly.

The quasi-interpolation operator $I_{\tilde{V}_B^h}$ has the following approximation properties.

Theorem 4.1.1 *Given a function $u \in H^2(\Omega)$, $u_h = I_{\tilde{V}_B^h} u \in \tilde{V}_B^h$ satisfies*

$$\begin{aligned} |u_h|_{H^2(\Omega)} &\leq C|u|_{H^2(\Omega)}, \\ |u_h - u|_{H^s(\Omega)} &\leq Ch^{2-s}|u|_{H^2(\Omega)}, s = 0, 1, 2. \end{aligned}$$

Here the constant C is independent of h.

Proof. We first consider one element τ. We note that, locally in an element τ, $I_{\tilde{V}_B^h}$ is defined similarly as I_B^τ, except that $I_{\tilde{V}_B^h}$ uses the average of the first derivatives over $\tau(x_i)$, while I_B^τ uses the average of the first derivatives over τ. Let

$$e_i^\alpha = \{D^\alpha u\}_{\tau(x_i)} - \{D^\alpha u\}_\tau, \quad |\alpha| = 1.$$

Then

$$\begin{aligned} |I_{\tilde{V}_B^h} u - I_B^\tau u|_{H^s(\tau)} &= |\sum_{|\alpha|=1} \sum_{i \in \Lambda^h(\tau)} e_i^\alpha \phi_i^\alpha(x)|_{H^s(\tau)} \\ &\leq \sum_{|\alpha|=1} \sum_{i \in \Lambda^h(\tau)} |e_i^\alpha| \, |\phi_i^\alpha(x)|_{H^s(\tau)}. \end{aligned} \tag{4.6}$$

¿From the fact that the basis functions $\phi_i^\alpha(x)$ are uniform to order 2 in the weak sense, we have

$$|\phi_i^\alpha(x)|_{H^s(\tau)} \leq C_s h^{d/2+1-s}, \text{ for } |\alpha| = 1, s = 1, 2.$$

Using Poincaré's inequality, we obtain

$$\begin{aligned} |e_i^\alpha| &= |\{D^\alpha u\}_{\tau(x_i)} - \{D^\alpha u\}_\tau| \\ &\leq Ch^{1-d/2}|D^\alpha u|_{H^1(\tau(x_i))} \leq Ch^{1-d/2}|u|_{H^2(\hat{\tau})}. \end{aligned}$$

Substituting this into (4.6), we get

$$|I_{\tilde{V}_B^h} u - I_B^\tau u|_{H^s(\tau)} \leq Ch^{2-s}|u|_{H^2(\hat{\tau})} \quad s = 1, 2.$$

Combining this inequality and (4.5) and using the triangle inequality, we have

$$|I_{\tilde{V}_B^h} u - u|_{H^s(\tau)} \leq |I_{\tilde{V}_B^h} u - I_B^\tau u|_{H^s(\tau)} + |u - I_B^\tau u|_{H^s(\tau)} \leq Ch^{2-s}|u|_{H^2(\hat{\tau})} \tag{4.7}$$

and

$$|I_{\tilde{V}_B^h} u|_{H^2(\tau)} \leq |I_{\tilde{V}_B^h} u - I_B^\tau u|_{H^2(\tau)} + |I_B^\tau u|_{H^2(\tau)} \leq C|u|_{H^2(\hat{\tau})}. \tag{4.8}$$

The theorem follows by summing (4.7) and (4.8) over all the triangles $\tau \subset \Omega$ and using the finite covering property of $\{\hat{\tau}\}$. ∎

Note that in Theorem 4.1.1, we made no conclusion about the boundary values of $I_{\tilde{V}_B^h} u$. However, if $u \in H_0^2(\Omega)$, we have for a boundary triangle τ ($\overline{\tau} \cap \partial\Omega \neq \emptyset$),

$$|\{u^\alpha\}_\tau - 0| \leq Ch|u|_{H^2(\Omega)}, \ |\alpha| = 1.$$

Using this fact and a simple modification at the boundary, we can show the following

Theorem 4.1.2 *Given a function $u \in H_0^2(\Omega)$, we can find $u_h \in \tilde{V}_B^h \cap H_0^2(\Omega)$ such that*

$$\begin{aligned} |u_h|_{H^2(\Omega)} &\leq C|u|_{H^2(\Omega)}, \\ |u_h - u|_{H^s(\Omega)} &\leq Ch^{2-s}|u|_{H^2(\Omega)}, s = 0, 1, 2. \end{aligned}$$

The conclusions of Theorem 4.1.1 and Theorem 4.1.2 also hold for the Argyris and bicubic elements, if we replace the subspace $\tilde{V}_B^h$ by $\tilde{V}_A^h$ and $\tilde{V}_Q^h$, and the smooth interpolation operator $I_{\tilde{V}_B^h}$ by $I_{\tilde{V}_A}^h$ and $I_{\tilde{V}_Q}^h$

4.2 Approximation by Interpolation

A proof of the following lemma can be found in Yserentant [52].

Lemma 4.2.1

$$\frac{1}{\pi\sigma^2}\int_{|x-x_0|\leq\sigma} |u(x)|dx \leq C(\log(R/\sigma) + 1/4)^{1/2}\|u\|_{H^1(S)},$$

where S denotes the disk of radius $R \geq \sigma > 0$ with center x_0 and $u \in H_0^1(S)$.

We know that for linear element

$$|u_h|_{L^\infty(D)} \leq C(1 + \log(H/h))^{1/2}\|u_h\|_{H^1(D)},$$

and that the right hand side can be replaced by the semi norm $|u_h|_{H^1(D)}$ if the mean value of u over D vanishes; cf. Bramble [9], Bramble, Pasciak and Schatz [10] and Yserentant [52]. Similar results for the bicubic element V_Q^h, the Argyris element V_A^h and the Bell element V_B^h can be established by applying Lemma 4.2.1 to ∇u.

Lemma 4.2.2 *For $u \in V^h$, we have*

$$|u|_{W^{1,\infty}(\Omega)} \leq C(1+\log(H/h))^{1/2}\|u_h\|_{H^2(\Omega)}.$$

The right hand side can be replaced by the semi-norm $|u_h|_{H^2(D)}$, if $\{u\}_D = \{u_x\}_D = \{u_y\}_D = 0$.

Let V^h be one of the spaces V^h_Q, V^h_A or V^h_B, and let $\tilde{V}^H$ denote the relevant subspace $\tilde{V}^H_Q$, $\tilde{V}^H_A$ or $\tilde{V}^H_B$. Let Π_{V^H} and $\Pi_{\tilde{V}^H}$ be the standard interpolation operators from V^h to V^H and $\tilde{V}^H$, defined by

$$\Pi_{V^H} u_h = \sum_{|\alpha|\leq 2} \sum_{i\in\Lambda^H(\Omega)} \frac{\partial^\alpha u_h}{\partial x^\alpha} u_h(x_i)\Phi_i^\alpha(x), \quad x \in \Omega,$$

and

$$\Pi_{\tilde{V}^H} u_h = \sum_{|\alpha|\leq 1} \sum_{i\in\Lambda^H(\Omega)} \frac{\partial^\alpha u_h}{\partial x^\alpha} u_h(x_i)\Phi_i^\alpha(x), \quad x \in \Omega.$$

Lemma 4.2.3 *The interpolation operator Π_{V^H} has the following properties: If $u_h \in V^h$, and $u_H = \Pi_{V^H} u_h \in V^H$, then*

$$|u_h - u_H|^2_{H^s(\Omega)} \leq CH^{2(2-s)}(H/h)^2|u_h|^2_{H^2(\Omega)},\ s = 0, 1, 2,$$

and

$$|u_h - u_H|^2_{W^{s,\infty}(\Omega)} \leq CH^{2(1-s)}(H/h)^2|u_h|^2_{H^2(\Omega)},\ s = 0, 1$$

Proof. We give a proof for the Bell element. Let Ω_l be a substructure, and $x_i, i = 1,2,3,$ be its vertices. Let Φ_i^α be the basis functions for the Bell element at the vertex x_i. We have

$$u_H = \Pi_{V_B^H} u = \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\leq 2} D^\alpha u(x_i)\Phi_i^\alpha(x) \quad \forall x \in \Omega_l.$$

We want to estimate the difference $u_h - \Pi_{V^H} u_h$. Using the linear function invariance property of Π_{V^H}, we can assume, without loss of generality, that $\{D^\alpha u\}_{\Omega_l} = 0, |\alpha| = 0, 1$. By the triangle inequality, we have

$$\begin{aligned} |u_H|_{W^{s,\infty}(\Omega_l)} &\leq \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\leq 2} |D^\alpha u(x_i)||\Phi_i^\alpha(x)|_{W^{s,\infty}(\Omega_l)} \\ &\leq \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\leq 2} |u|_{W^{|\alpha|,\infty}(\Omega_l)}|\Phi_i^\alpha|_{W^{s,\infty}(\Omega_l)}, \end{aligned}$$

and

$$
\begin{aligned}
|u_H|_{H^s(\Omega_l)} &\leq \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\leq 2} |D^\alpha u(x_i)||\Phi_i^\alpha(x)|_{H^s(\Omega_l)} \\
&\leq \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\leq 2} |u|_{W^{|\alpha|,\infty}(\Omega_l)}|\Phi_i^\alpha|_{H^s(\Omega_l)}.
\end{aligned}
$$

Using the fact that the basis functions are uniform to order 2, i.e.

$$
|\Phi_i^\alpha(x)|_{W^{s,\infty}(\Omega_l)} \leq CH^{|\alpha|-s} \text{ and } |\Phi_i^\alpha(x)|_{H^s(\Omega_l)} \leq CH^{|\alpha|+1-s},
$$

we get

$$
|u_H|_{W^{s,\infty}(\Omega_l)} \leq C\left\{H^{-s}|u|_{L^\infty(\Omega_l)} + H^{1-s}|u|_{W^{1,\infty}(\Omega_l)} + H^{2-s}|u|_{W^{2,\infty}(\Omega_l)}\right\},
$$

and

$$
|u_H|_{H^s(\Omega_l)} \leq C\left\{H^{1-s}|u|_{L^2(\Omega_l)} + H^{2-s}|u|_{H^1(\Omega_l)} + H^{3-s}|u|_{H^2(\Omega_l)}\right\}.
$$

Since $\{D^\alpha u\}_{\Omega_l} = 0$, $|\alpha| = 0, 1$, we find that

$$
\begin{aligned}
|u|_{L^\infty} &\leq CH|u|_{H^2(\Omega_l)}, \\
|u|_{W^{1,\infty}(\Omega_l)} &\leq C(1+\log(H/h))^{1/2}|u|_{H^2(\Omega_l)}, \\
|u|_{W^{2,\infty}(\Omega_l)} &\leq Ch^{-1}|u|_{H^2(\Omega_l)}.
\end{aligned}
$$

The first inequality follows by the embedding Theorem, Poincaré's inequality, and a scaling argument. The second is just a consequence of Lemma 4.2.2. The third is the inverse inequality. Using these inequalities in the estimate for $|u_H|_{W^{s,\infty}(\Omega_l)}$ and $|u_H|_{H^s(\Omega_l)}$, we get

$$
\begin{aligned}
|u_H|_{W^{s,\infty}(\Omega_l)} &\leq C\left\{H^{1-s}|u|_{H^2(\Omega_l)} + H^{1-s}(1+\log(\frac{H}{h}))^{1/2}|u|_{H^2(\Omega_l)} + H^{1-s}(\frac{H}{h})|u|_{H^2(\Omega_l)}\right\} \\
&\leq CH^{1-s}(H/h)|u|_{H^2(\Omega_l)},
\end{aligned}
$$

and

$$
\begin{aligned}
|u_H|_{H^s(\Omega_l)} &\leq C\left\{H^{2-s}|u|_{H^2(\Omega_l)} + H^{2-s}(1+\log(\frac{H}{h}))^{1/2}|u|_{H^2(\Omega_l)} + H^{2-s}(\frac{H}{h})|u|_{H^2(\Omega_l)}\right\} \\
&\leq CH^{2-s}(H/h)|u|_{H^2(\Omega_l)}.
\end{aligned}
$$

By the triangle inequality, and the vanishing mean values property of u we have

$$
|u - \Pi_{V^H}u|^2_{W^{s,\infty}(\Omega_l)} \leq 2|u|^2_{W^{s,\infty}(\Omega_l)} + 2|\Pi_{V^H}u|^2_{W^{s,\infty}(\Omega_l)} \leq CH^{2-2s}(H/h)^2|u|^2_{H^2(\Omega_l)},
$$

and

$$|u - \Pi_{V^H} u|^2_{H^s(\Omega_l)} \le 2|u|^2_{H^s(\Omega_l)} + 2|\Pi_{V^H} u|^2_{H^s(\Omega_l)} \le CH^{4-2s}(H/h)^2|u|^2_{H^2(\Omega_l)}.$$

The conclusion for the Bell element follows by summing over all Ω_l. The results for the bicubic element and Argyris element can be proved similarly. ∎

Remark 4.2.1 It is easy to construct examples to demonstrate that the bound in the above theorem is sharp. As in the 3-d problems, interpolations for the Hermitian elements also have a very poor bound.

To get better bounds, we consider the interpolation interpolation operator $\Pi_{\tilde{V}^H}$. The approximation properties of $\Pi_{\tilde{V}^H}$ is summarized in the next lemma.

Lemma 4.2.4 *The interpolation operator $\Pi_{\tilde{V}^H}$, defined above, has the following properties:*
For $u_h \in V^h$, $u_H = \Pi_{\tilde{V}^H} u_h \in \tilde{V}^H$, we have

$$|u_h - u_H|^2_{H^s(\Omega_i)} \le CH^{2(2-s)}(1 + \log(H/h))|u_h|^2_{H^2(\Omega_i)},$$

and

$$|u_h - u_H|^2_{W^{s,\infty}(\Omega_i)} \le CH^{2(1-s)}(1 + \log(H/h))|u_h|^2_{H^2(\Omega_i)}.$$

Proof. We prove the result for the Bell element; the assertions for the bicubic and Argyris elements can be proved similarly. Let Ω_l be a substructure, and x_i, $i = 1, 2, 3$, be its vertices. Let Φ_i^α be the basis functions for the Bell element at the vertex x_i. We have

$$\Pi_{\tilde{V}^H} u = \sum_{i \in \Lambda^H(\Omega_l)} \sum_{|\alpha| \le 1} D^\alpha u(x_i) \Phi_i^\alpha(x), \quad \forall x \in \Omega_l.$$

Because of the linear function invariance property of $\Pi_{\tilde{V}}^H$, we can assume, without loss of generality, that $\{D^\alpha u\}_{\Omega_l} = 0, |\alpha| = 0, 1$. Note that the only difference between $\Pi_{\tilde{V}^H} u_h$ and $\Pi_{V^H} u_h$ is that $\Pi_{\tilde{V}^H} u_h$ does not contain the terms $D^\alpha u(x_i)\Phi_i^\alpha, |\alpha| = 2$. Therefore we can repeat the proof of Lemma 4.2.3 to obtain

$$|\Pi_{\tilde{V}^H} u|_{W^{s,\infty}(\Omega_l)} \quad \le \quad \sum_{i \in \Lambda^H(\Omega_l)} \sum_{|\alpha| \le 1} |u|_{W^{|\alpha|,\infty}(\Omega_l)} |\Phi_i^\alpha|_{W^{s,\infty}(\Omega_l)}$$

$$\begin{aligned} &\le CH^{-s}|u|_{L^\infty(\Omega_l)} + CH^{1-s}|u|_{W^{1,\infty}(\Omega_l)} \\ &\le CH^{1-s}(1+\log(H/h))^{1/2}|u|_{H^2(\Omega_l)}, \end{aligned}$$

and

$$\begin{aligned} |\Pi_{\tilde V^H} u|_{H^s(\Omega_l)} &\le \sum_{i\in\Lambda^H(\Omega_l)} \sum_{|\alpha|\le 1} |u|_{W^{|\alpha|,\infty}(\Omega_l)} |\phi_i^\alpha|_{H^s(\Omega_l)} \\ &\le C[H^{2-s}|u|_{H^2(\Omega_l)} + H^{2-s}(1+\log(H/h))^{1/2}|u|_{H^2(\Omega_l)} \\ &\le CH^{2-s}(1+\log(H/h))^{1/2}|u|_{H^2(\Omega_l)}. \end{aligned}$$

Using the triangle inequality and the vanishing mean value property of u, we have

$$\begin{aligned} |u_h - \Pi_{\tilde V^H} u_h|^2_{W^{s,\infty}(\Omega_l)} &\le |u|^2_{W^{s,\infty}(\Omega_l)} + |\Pi_{\tilde V^H} u_h|^2_{W^{s,\infty}(\Omega_l)} \\ &\le CH^{2-2s}(1+\log(H/h))|u|^2_{H^2(\Omega_l)}, \end{aligned}$$

and

$$\begin{aligned} |u_h - \Pi_{\tilde V^H} u_h|^2_{H^s(\Omega_l)} &\le |u|^2_{H^s(\Omega_l)} + |\Pi_{\tilde V^H} u_h|^2_{H^s(\Omega_l)} \\ &\le CH^{4-2s}(1+\log(H/h))|u|^2_{H^2(\Omega_l)}. \end{aligned}$$

The inequalities for operators $\Pi_{\tilde V^H_Q}$ and $\Pi_{\tilde V^H_A}$ can be established similarly. ∎

4.3 Some Discrete Norms

In this section, we define some discrete norms for the finite element spaces V^h_Q, V^h_A and V^h_B, and show that these discrete norms are equivalent to the corresponding continuous Sobolev norms. These discrete norms are used in establishing the optimal convergence property of the additive Schwarz method for the biharmonic equation when these finite elements are used. We note that similar ideas can be used to show the optimal property of the additive Schwarz methods for other conforming Lagrangian or Hermitian elements.

We considered the bicubic element V^h_Q, the Argyris element V^h_A and the Bell element V^h_B. We define the discrete norms in terms of the degrees of freedom of the finite element spaces.

Definition 4.3.1 *The following are discrete norms for the finite element spaces:*
(1) For $u \in V^h_Q$. Let τ be a rectangle with vertices $x_i, i = 1,2,3,4$. The discrete norms

over τ are given by

$$
\begin{aligned}
|u|^2_{Q,0} &\equiv h^2\sum_i |u(x_i)|^2 + h^4 \sum_{|\alpha|=1}\sum_i |D^\alpha u(x_i)|^2 + h^6 \sum_i |\frac{\partial^2 u}{\partial x \partial y}(x_i)|^2,\\
|u|^2_{Q,1} &\equiv \sum_{i\neq j} |u(x_i)-u(x_j)|^2 + h^2 \sum_{|\alpha|=1}\sum_i |D^\alpha u(x_i)|^2 + h^4 \sum_i |\frac{\partial^2 u}{\partial x \partial y}(x_i)|^2,\\
|u|^2_{Q,2} &\equiv \frac{1}{h^2}\sum_i |Lu(x_i)|^2 + \sum_{|\alpha|=1}\sum_{i\neq j} |D^\alpha u(x_i) - D^\alpha u(x_j)|^2 + h^2 \sum_i |\frac{\partial^2 u}{\partial x \partial y}(x_i)|^2,
\end{aligned}
$$

where

$$Lu\,(x_i) = u(x_i) - \{u(x_1) + (\nabla u)(x_1)\cdot (x_i - x_1)\}.$$

(2) For $u \in V_B^h$. Let τ be a triangle with vertices x_i. The discrete norms over τ are given by

$$
\begin{aligned}
|u|^2_{B,0} &\equiv h^2\sum_i |u(x_i)|^2 + h^4 \sum_{|\alpha|=1}\sum_i |D^\alpha u(x_i)|^2 + h^6 \sum_{|\alpha|=2}\sum_i |D^\alpha u(x_i)|^2,\\
|u|^2_{B,1} &\equiv \sum_{i\neq j} |u(x_i)-u(x_j)|^2 + h^2 \sum_{|\alpha|=1}\sum_i |D^\alpha u(x_i)|^2 + h^4 \sum_{|\alpha|=2}\sum_i |D^\alpha u(x_i)|^2,\\
|u|^2_{B,2} &\equiv \frac{1}{h^2}\sum_i |Lu(x_i)|^2 + \sum_{|\alpha|=1}\sum_{i\neq j} |D^\alpha u(x_i) - D^\alpha u(x_j)|^2 + h^2 \sum_{|\alpha|=2}\sum_i |D^\alpha u(x_i)|^2,
\end{aligned}
$$

(3) For $u \in V_A^h$. Let τ be a triangle with vertices x_i. The discrete norms over τ are given by

$$
\begin{aligned}
|u|^2_{A,0} &\equiv |u|^2_{B,0} + h^4 |D^{n_i}(y_i)|^2,\\
|u|^2_{A,1} &\equiv |u|^2_{B,1} + h^2 |D^{n_i}(y_i)|^2,\\
|u|^2_{A,2} &\equiv |u|^2_{B,2} + \sum_i |D^{n_i}u(x_i) - D^{n_i}u(y_i)|^2,
\end{aligned}
$$

where $y_i = (x_j + x_k)/2$ is the midpoint of the edge $\overline{x_j\, x_k}$.

The square of the global norms are defined by summing the square of the local norms over all $\tau \subset \Omega$.

Let $P(\tau)$ be a space of polynomials defined on τ. We say that two functionals f_1 and f_2 on $P(\tau)$ are equivalent, $f_1 \sim f_2$, if there exist two constants C_1 and C_2 independent of u in $P(\tau)$ such that

$$C_1 f_1(u) \le f_2(u) \le C_2 f_1(u). \tag{4.9}$$

If furthermore, C_1 and C_2 are independent of τ, then we say f_1 and f_2 are uniformly equivalent. We have the following theorem on uniform norm equivalency.

Theorem 4.3.1 *Let $\vec{u}$ be the vector consisting of the degrees of freedom of the finite element in a triangle τ. Then,*
for $u \in V_Q^h$

$$\begin{aligned} \|u\|^2_{L^2(\tau)} &\equiv (K_\tau^{(0)}\vec{u}, \vec{u}) \sim |u|^2_{Q,0} \\ |u|^2_{H^1(\tau)} &\equiv (K_\tau^{(1)}\vec{u}, \vec{u}) \sim |u|^2_{Q,1} \\ |u|^2_{H^2(\tau)} &\equiv (K_\tau^{(2)}\vec{u}, \vec{u}) \sim |u|^2_{Q,2}; \end{aligned}$$

for $u \in V_B^h$

$$\begin{aligned} \|u\|^2_{L^2(\tau)} &\equiv (K_\tau^{(0)}\vec{u}, \vec{u}) \sim |u|^2_{B,0} \\ |u|^2_{H^1(\tau)} &\equiv (K_\tau^{(1)}\vec{u}, \vec{u}) \sim |u|^2_{B,1} \\ |u|^2_{H^2(\tau)} &\equiv (K_\tau^{(2)}\vec{u}, \vec{u}) \sim |u|^2_{B,2}; \end{aligned}$$

for $u \in V_A^h$

$$\begin{aligned} \|u\|^2_{L^2(\tau)} &\equiv (K_\tau^{(0)}\vec{u}, \vec{u}) \sim |u|^2_{A,0} \\ |u|^2_{H^1(\tau)} &\equiv (K_\tau^{(1)}\vec{u}, \vec{u}) \sim |u|^2_{A,1} \\ |u|^2_{H^2(\tau)} &\equiv (K_\tau^{(2)}\vec{u}, \vec{u}) \sim |u|^2_{A,2}. \end{aligned}$$

The theorem still holds if we replace the norms on the left hand side by the norms over Ω and the right hand side by sums over all vertices $x_i \in \Omega$.

Proof. For a fixed τ, the left and right hand sides define quadratic forms, which have the same null space. Thus the two forms are equivalent, since they are defined on a finite dimensional space. However the equivalency constants C_1 and C_2 may depend on the triangle τ. We claim that C_1 and C_2 can be chosen so that they depend only on the minimum angle of triangle τ.

This is trivial for the bicubic elements, since it can be embedded in an affine family. Using a reference element and a mapping, it is therefore easy to see that the discrete norms are uniformly equivalent to the corresponding continuous norms. Argyris and Bell elements are not affine elements. To check the uniform norm equivalency for the Bell element and Argyris element, we note that similar triangles have the same equivalency constants. Therefore, it is enough to establish the result for triangles of unit size with a minimum angle bounded from below by $\theta_{\min}$. Let ΔABC be such a triangle, with $A = (0,0), B = (1,0)$. Then its third vertex C must lie in a compact region $D(\theta_{\min})$. We note that the continuous and discrete norms define two quadratic forms of the degrees of freedom:

$$|u|_c^2 = (K_c x, x) \quad \text{and} \quad |u|_d^2 = (K_d x, x),$$

and that K_c and K_d depend continuously on C. The eigenvalues $\lambda(K_c)$ and $\lambda(K_d)$ are continuous functions of the coefficients of the matrices, and therefore continuous functions of C. Let $\lambda_{\min}(K_c)$ and $\lambda_{\min}(K_d)$ be the minimum nonzero eigenvalues, and $\lambda_{\max}(K_c)$ and $\lambda_{\max}(K_d)$ be the maximum eigenvalues. Since C lies in the compact set $D(\theta_{\min})$, there exist constants C_1 and C_2, depending only on $D(\theta_{\min})$, thus only on $\theta_{\min}$, such that

$$C_1 \leq \lambda_{\min}(K_c) \leq \lambda_{\max}(K_c) \leq C_2,$$

and

$$C_1 \leq \lambda_{\min}(K_d) \leq \lambda_{\max}(K_d) \leq C_2.$$

Since the matrices K_c and K_d have the same null space, we have

$$(C_1/C_2)(K_c x, x) \leq (K_d x, x) \leq (C_2/C_1)(K_c x, x), \qquad \forall C \in D(\theta_{\min}).$$

Thus we have proved the norm equivalency for the class of triangles of unit size with minimum angles bounded from below by $\theta_{\min}$. The powers of h in the discrete norms can be obtained by shrinking ΔABC to a similar triangle of diameter h.

The global norm equivalency follows by summing over all elements. ∎

4.4 Estimates for the Interpolant of a Product of Two Functions

In the proofs of the estimates for the domain decomposition methods, we need to estimate the norm of the interpolant of the product of two functions, i.e. $|\Pi_{V^h}(\theta u)|_{H^2}$, where θ

is a smooth partition function and u is a finite element function. For simplification, we assume that $\theta(x) \in C^\infty(\Omega)$. Let

$$\Theta_0 = |\theta(x)|_{L^\infty(\tau)}, \quad \Theta_1 = |\theta(x)|_{W^{1,\infty}(\tau)}, \quad \Theta_2 = |\theta(x)|_{W^{2,\infty}(\tau)}. \tag{4.10}$$

Let $\Pi_{V_Q^h}, \Pi_{V_A^h}$ and $\Pi_{V_B^h}$ be the standard interpolation operator to the finite element spaces V_Q^h, V_A^h and V_B^h, respectively. We then have the following estimates

Lemma 4.4.1

$$|\Pi_{V_Q^h}(\theta u)|^2_{H^2(\tau)} \le C\left\{\Theta_0^2|\Pi_{V_Q^h}u|^2_{H^2(\tau)} + \Theta_1^2|\Pi_{V_Q^h}u|^2_{H^1(\tau)} + \Theta_2^2\|\Pi_{V_Q^h}u\|^2_{L^2(\tau)}\right\}, \quad \forall u \in V_Q^h. \tag{4.11}$$

$$|\Pi_{V_B^h}(\theta u)|^2_{H^2(\tau)} \le C\left\{\Theta_0^2|\Pi_{V_B^h}u|^2_{H^2(\tau)} + \Theta_1^2|\Pi_{V_B^h}u|^2_{H^1(\tau)} + \Theta_2^2\|\Pi_{V_B^h}u\|^2_{L^2(\tau)}\right\}, \quad \forall u \in V_B^h. \tag{4.12}$$

$$|\Pi_{V_A^h}(\theta u)|^2_{H^2(\tau)} \le C\left\{\Theta_0^2|\Pi_{V_A^h}u|^2_{H^2(\tau)} + \Theta_1^2|\Pi_{V_A^h}u|^2_{H^1(\tau)} + \Theta_2^2\|\Pi_{V_A^h}u\|^2_{L^2(\tau)}\right\}, \quad \forall u \in V_A^h. \tag{4.13}$$

All constants are independent of mesh size h.

Proof. We prove inequalities (4.12) and (4.13). The proof for (4.11) is similar. By Theorem 4.3.1, using the equivalent discrete norm, we have

$$\begin{aligned}
&|\Pi_{V_B^h}(\theta u)|^2_{H^2(\tau)} \\
&\sim \frac{1}{h^2}\sum_i |L(\theta u)(x_i)|^2 + \sum_{i\ne j}\sum_{|\alpha|=1} |D^\alpha(\theta u)(x_i) - D^\alpha(\theta u)(x_j)|^2 + h^2\sum_{|\alpha|=2}|D^\alpha(\theta u)(x_i)|^2 \\
&= T_1 + T_2 + T_3,
\end{aligned} \tag{4.14}$$

where

$$Lu(x_i) = u(x_i) - (u(x_1) - \nabla u(x_i)\cdot(x_i - x_1)).$$

Using the identity

$$L(\theta u)(x_i) = \theta(x_i)Lu(x_i) + u(x_i)L\theta(x_i) - (\theta(x_i) - \theta(x_1))(u(x_i) - u(x_1)),$$

we have

$$
\begin{aligned}
T_1 &= \frac{1}{h^2}\sum_i |\ \theta(x_i)Lu(x_i) + u(x_i)L\theta(x_i) + [\theta(x_i)-\theta(x_1)][u(x_i)-u(x_1)]\ |^2 \\
&\leq \frac{3}{h^2}\sum_i |\theta(x_i)Lu(x_i)|^2 + |u(x_i)L\theta(x_i)|^2 + |(\theta(x_i)-\theta(x_1))(u(x_i)-u(x_1))|^2 \\
&\sim 3\Theta_0^2|\Pi_{V_B^h}u|^2_{H^2(\tau)} + \frac{1}{h^2}\sum |L\theta(x_i)|^2|u(x_i)|^2 + \frac{1}{h^2}(\Theta_1 h)^2|\Pi_{V_B^h}u|^2_{H^1(\tau)}.
\end{aligned}
$$

L is a Taylor's expansion and we have $|L\theta(x_i)| \leq \Theta_2 h^2$. Since

$$
h^2\sum u(x_i)^2 \leq C\|\Pi_{V_B^h}u\|_{L^2(\tau)},
$$

we get an estimate for T_1,

$$
T_1 \leq C\{\Theta_0^2|\Pi_{V_B^h}u|^2_{H^2(\tau)} + \Theta_2^2\|\Pi_{V_B^h}u\|_{L^2(\tau)} + \Theta_1^2|\Pi_{V_B^h}u|^2_{H^1(\tau)}\}. \tag{4.15}
$$

Manipulating the terms in the expression of T_2, we have, by elementary calculus

$$
\begin{aligned}
T_2 &= \sum_{i\neq j}\sum_{|\alpha|=1} |\theta(x_i)u_\alpha(x_i) + u(x_i)\theta_\alpha(x_i) - \theta(x_j)u_\alpha(x_j) - u(x_j)\theta_\alpha(x_j)|^2 \\
&= \sum_{i\neq j}\sum_{|\alpha|=1} \{(u_\alpha(x_i)-u_\alpha(x_j))\theta(x_i) + (\theta(x_i)-\theta(x_j))u_\alpha(x_j) \\
&\qquad\qquad + (u(x_i)-u(x_j))\theta_\alpha(x_i) + (\theta_\alpha(x_i)-\theta_\alpha(x_j))u(x_j)\}^2 \\
&\leq 4\{\Theta_0^2\sum_{i\neq j}\sum_{|\alpha|=1} |(u_\alpha(x_i)-u_\alpha(x_j))|^2 + \Theta_1^2 h^2\sum |u_\alpha(x_j)|^2 \\
&\qquad + \Theta_1^2\sum_{i\neq j}|u(x_i)-u(x_j)|^2 + \Theta_2^2 h^2\sum |u(x_j)|^2\} \\
&\leq C\{\Theta_0^2|\Pi_{V_B^h}u|^2_{H^2(\tau)} + \Theta_1^2|\Pi_{V_B^h}u|^2_{H^1(\tau)} + \Theta_2^2\|\Pi_{V_B^h}u\|^2_{L^2(\tau)}\}.
\end{aligned} \tag{4.16}
$$

T_3 can be estimated as

$$
\begin{aligned}
T_3 &= h^2\sum_{|\alpha|=2}|D^\alpha(\theta u)(x_i)|^2 = h^2\sum_{|\alpha_1+\alpha_2|=2}|D^{\alpha_1}\theta D^{\alpha_2}u)(x_i)|^2 \\
&\leq \Theta_0^2 h^2\sum_{|\alpha|=2}|D^\alpha u(x_i)|^2 + \Theta_1^2 h^2\sum_{|\alpha|=1}|D^\alpha u(x_i)|^2 + \Theta_2^2 h^2\sum_i |u(x_i)|^2 \\
&\leq \Theta_0^2|\Pi_{V_B^h}u|^2_{H^2(\tau)} + \Theta_1^2|\Pi_{V_B^h}u|^2_{H^1(\tau)} + \Theta_2^2\|\Pi_{V_B^h}u\|^2_{L^2(\tau)}.
\end{aligned} \tag{4.17}
$$

Inequality (4.12) follows from the inequalities (4.14)–(4.17).

To establish inequality (4.13), we only need to estimate $\sum_i |D^{n_i}(\theta u)(y_i) - D^{n_i}(\theta u)(x_i)|^2$ and $h^2|D^{n_i}(\theta u)(y_i)|^2$. We estimate the first one; the estimate of the second is similar.

$$\begin{aligned}
&\sum_i |D^{n_i}(\theta u)(y_i) - D^{n_i}(\theta u)(x_i)|^2 \\
&\quad = \sum_i |\theta_{n_i}(y_i)u(y_i) + \theta(y_i)u_{n_i}(y_i) - \theta_{n_i}(x_i)u(x_i) + \theta(x_i)u_{n_i}(x_i)|^2 \\
&\quad = \sum_i |\theta_{n_i}(y_i)(u(y_i) - u(x_i)) + (\theta_{n_i}(y_i) - \theta_{n_i}(x_i))u(x_i) \\
&\qquad +(\theta(y_i) - \theta(x_i))u_{n_i}(y_i) + \theta(x_i)(u_{n_i}(y_i) - u_{n_i}(x_i))|^2 \\
&\quad = C\sum_i |\theta_{n_i}(y_i)(u(y_i) - u(x_i))|^2 + C\sum_i |(\theta_{n_i}(y_i) - \theta_{n_i}(x_i))u(x_i)|^2 \\
&\qquad +C\sum |(\theta(y_i) - \theta(x_i))u_{n_i}(y_i)|^2 + C\sum |\theta(x_i)(u_{n_i}(y_i) - u_{n_i}(x_i))|^2 \\
&\quad = T_1 + T_2 + T_3 + T_4. \qquad (4.18)
\end{aligned}$$

Using Theorem 4.3.1, each term can be estimated as follows:

$$\begin{aligned}
T_1 &\leq \Theta_1^2 \sum_i |u(y_i) - u(x_i)|^2 \\
T_2 &\leq \Theta_2^2 h^2 \sum_i u(x_i)^2 \leq C\Theta_2^2 |\Pi_{V_A^h} u|^2_{L^2(\tau)}. \qquad (4.19) \\
T_3 &\leq \Theta_1^2 h^2 \sum_i |u_{n_i}(y_i)|^2 \leq C\Theta_1^2 |\Pi_{V_A^h} u|^2_{H^1(\tau)}. \qquad (4.20) \\
T_4 &\leq \Theta_0^2 \sum_i |u_{n_i}(y_i) - u_{n_i}(x_i)|^2 \leq C\Theta_0^2 |\Pi_{V_A^h} u|^2_{H^2(\tau)}. \qquad (4.21)
\end{aligned}$$

Using the basis representation for functions in $\mathcal{P}_5$, we have

$$u(y_i) = \sum_j u(x_j)\phi_j(y_i) + \sum_j \sum_{|\alpha|\leq 2} u_\alpha(x_j)\phi_j^\alpha(y_i) + \sum_j u_{n_j}(y_j)\phi_j^{n_j}(y_i).$$

Recalling that $\sum_i^3 \phi_i(x) \equiv 1$, using the fact that $u \in \mathcal{P}_5$ and the fact that the basis functions for the Argyris element are uniform to order 2, we have

$$\begin{aligned}
&|u(y_i) - u(x_i)|^2 \\
&\quad = |\sum_j [u(x_j) - u(x_i)]\phi_j(y_i) + \sum_j \sum_{|\alpha|\leq 2} u_\alpha(x_j)\phi_j^\alpha(y_i) + \sum_j u_{n_j}(y_j)\phi_j^{n_j}(y_i)|^2 \\
&\quad \leq C\sum_j |u(x_j) - u(x_i)|^2 + \sum_j \sum_{|\alpha|\leq 2} h^{2|\alpha|}|u_\alpha(x_j)|^2 + \sum_j h^2|u_{n_j}(y_j)|^2 \\
&\quad \leq C|u|^2_{H^1(\tau)}.
\end{aligned}$$

Thus,

$$T_1 \leq \Theta_1^2 \sum_i |u(y_i) - u(x_i)|^2 \leq C\Theta_1^2 |u|^2_{H^1(\tau)}. \tag{4.22}$$

Inequality (4.13) now follows from (4.18)–(4.22). ∎

Chapter 5

Additive Schwarz Methods for the Biharmonic Problem

In this chapter, we study additive Schwarz methods for the biharmonic problem using the abstract framework of chapter 2. We use the bicubic element V_Q^h, the Argyris element V_A^h and the Bell element V_B^h.

Suppose that the finite element space V can be written as a sum of subspaces

$$V = V_0 + V_1 + \cdots + V_N,$$

where V_0 is a coarse subspace, and V_i subspaces associated with subregions $\hat{\Omega}_i$. Instead of solving the original finite element equation, the additive Schwarz method is introduced in terms of an auxiliary problem: Find $u \in V$ by solving iteratively the equation

$$Pu_h = (P_{V_0} + P_{V_1} + \cdots + P_{V_N})u_h = g_h$$

for some g_h.

The natural question is how to find decompositions of V and what properties of the decomposition give optimal algorithms.

As we pointed out earlier, the coarse problem is crucial in our algorithm. In the second order case, an obvious candidate for the coarse subspace is the space associated with $\mathcal{T}^H$. However for the biharmonic case, when Argyris and Bell's elements are used, the coarse finite element spaces V_A^H and V_B^H are not subspaces of the fine finite element spaces V_A^h and V_B^h. Therefore, new coarse subspaces have to be found. The discovery of the appropriate coarse subspaces results directly from an observation of a weak coupling property between the different degrees of freedom of the elements and some results on discrete Sobolev norms. This observation also leads to some simplified algorithms, which turns out to be useful even for the algorithms using the bicubic element.

In the iterative substructuring case, the situation is even worse. Even when we use the bicubic element and $V_Q^H \subset V_Q^h$ can be used as coarse subspace, the direct generalization of some algorithms for second order problems results in algorithms with condition numbers which grow at least like $(H/h)^2$. Better algorithms are obtained by adding certain vertex spaces to the space decomposition; cf. chapter 6.

The difficulty of proving the optimality of the algorithms depends on the presence of high order derivatives in the definition of the elements. The tools that work for second order equation and linear element cannot be used here.

5.1 The Bicubic Element

5.1.1 Basic Algorithms

We first triangulate the domain Ω into nonoverlapping rectangles $\Omega_i, i = 1, \cdots, N$, to obtain the coarse triangulation $\mathcal{T}^H = \{\Omega_i\}_1^N$. Then each rectangle Ω_i is further divided into smaller rectangles τ_i to obtain the fine triangulation $\mathcal{T}^h = \{\tau_i\}$. We also decompose Ω into overlapping subdomains $\hat{\Omega}_i, i = 1, \cdots, N$. We assume that the decomposition $\Omega = \cup_{i=1}^N \hat{\Omega}_i$ satisfies Assumption 3.3.1. Let $\Lambda^h(D)$ and $\Lambda^H(D)$ be the sets of the fine and coarse grid points in the set D, respectively.

We use two finite element spaces $V_0 = V_Q^H$ and V_Q^h, which are the bicubic element associated with the triangulations $\mathcal{T}^H$ and $\mathcal{T}^h$, respectively. In addition, we use the subspaces $V_i = V_Q^h(\hat{\Omega}_i) = V_Q^h \cap H_0^2(\hat{\Omega}_i)$.

Let $P_{V_i} : V_Q^h \rightarrow V_i$, be the $a(\cdot,\cdot)$–orthogonal projection and let

$$P = \sum_{i=0}^N P_{V_i}.$$

We have the following additive Schwarz algorithm

Algorithm 5.1.1 *Find $u_h \in V^h$ such that*

$$P u_h = g_h, \tag{5.1}$$

with $g_h = \sum_i g_i$ *and the* g_i *given by the solutions of*

$$a(g_i, \phi_h) = a(P_{V_i} u, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V_i. \tag{5.2}$$

To find u_h, we first find the right hand side g_h by solving equations (5.2). We then use the conjugate gradient method to solve the system. In each iteration, we need to compute Pv_h for some element $v_h \in V^h$. This is done in the following steps, which can be carried out in parallel.

- Compute $P_0 v_h$ by solving $K^H x = b$. This is a finite element problem in the subspace V_B^H. Here K^H is the stiffness matrix and $\dim(V_Q^H) = 4N$, where $N = |\Lambda^H(\Omega)|$, is the number of interior coarse grid points.

- Compute $P_i v_h$ by solving $K_i^h x = b$. This is a finite element problem in the subdomain $\hat{\Omega}_i$. Here K_i is the stiffness matrix and $\dim(V_i) = 4n_i$, where $n_i = |\Lambda^h(\hat{\Omega}_i)|$, is the number of fine grid points inside $\hat{\Omega}_i$.

5.1.2 Simplified Algorithms

It is of course desirable to reduce the work in each iteration without decreasing the rate of convergence. Let ϕ_i^α and Φ_i^α be the nodal basis functions of V_Q^h and V_Q^H, respectively. To describe our alternative algorithm, we need to introduce some new subspaces.

- The vertex space $V_{x_k} = \text{span}\{\phi_k^{xy}(x)\}$, for each vertex x_k. We note that for $\phi \in V_{x_k}$, $\tau(x_k) = \text{supp}(\phi)$ is a polygon of diameter $O(h)$.

- $\tilde{V}_i = \text{span}\{\phi_k^\alpha(x), |\alpha| \le 1, k \in \Lambda^h(\Omega_i)\}$. $\tilde{V}_i$ is a subspace of V_i consisting of functions with vanishing mixed second derivatives ($\frac{\partial^2 u}{\partial x \partial y}$) at all the vertices.

- $\tilde{V}_Q^H = \text{span}\{\Phi_i^\alpha, |\alpha| \le 1, i \in \Lambda^H(\Omega)\}$. $\tilde{V}_Q^H$ is a subspace of V_Q^H consisting of functions with vanishing mixed second derivative ($\frac{\partial^2 u}{\partial x \partial y}$) at all the vertices of the substructures.

Algorithm 5.1.2 *Find* $u_h \in V_Q^h$ *by solving*

$$P^{(2)} u_h \equiv (P_{\tilde{V}_Q^H} + \sum_i P_{\tilde{V}_i} + \sum_{k \in \Lambda^h} P_{V_{x_k}}) u_h = g_h \tag{5.3}$$

with the appropriate right hand side g_h.

The computation of g_h is similar to Algorithm 5.1.1. In each conjugate gradient step of Algorithm 5.1.2, we need to solve three sets of problems.

1. Compute $P_{\tilde{V}_Q^H} v_h$ by solving $\tilde{K}^H x = b$. This is a finite element problem in the subspace $\tilde{V}_Q^H$. Note that $\tilde{K}^H$ is a principal minor of K^H, and $\dim(\tilde{V}_Q^H) = \frac{3}{4}\dim(V_Q^H)$.

2. Compute $P_{\tilde{V}_i} v_h$ by solving $\tilde{K}_i^h x = b$. This is a finite element problem in the subdomain $\hat{\Omega}_i$. $\tilde{K}_i^h$ is a principal minor of K_i^h and $\dim(\tilde{V}_i) = \frac{3}{4}\dim(V_i)$.

3. Compute $P_{V_{x_k}} v_h$ by solving $K_{x_k}^h x = b$, with x and b scalars, where $K_{x_k}^h = a(\phi_k^{xy}, \phi_k^{xy})$ is the 1 by 1 stiffness matrix corresponding to the degree of freedom associated the mixed second derivative $(\frac{\partial^2 u}{\partial x \partial y})$ at the vertex x_k. These problems are very easy.

5.2 The Argyris and Bell Elements

We next consider the Bell and Argyris elements. First, we present the basic algorithms. We then describe the computationally more efficient algorithms.

5.2.1 Basic Algorithms

The triangulations $\mathcal{T}^h, \mathcal{T}^H$ and the subregions $\hat{\Omega}_i$ are defined as in the the second order case. We first divide the domain Ω into non-overlapping substructures $\Omega_i, i = 1, 2, \cdots N$. All the substructures Ω_i are further subdivided into elements τ_j^i. Thus we have two levels of triangulations, namely the coarse triangulation $\mathcal{T}^H = \{\Omega_i\}$ and the fine triangulation $\mathcal{T}^h = \{\tau_j\}$. We denote by h_i and H_i the diameters of element τ_i and substructure Ω_i, respectively, and let $h = \max_i h_i$ and $H = \max_i H_i$. We assume that all the substructures and elements are shape regular in the usual sense. We also extend each substructure to a larger region $\hat{\Omega}_i$. We assume that the distance between the boundaries $\partial\Omega_i$ and $\partial\hat{\Omega}_i$ is bounded from below by a fixed fraction of H_i and that $\partial\hat{\Omega}_i$ does not cut through any element. It is clear that $\{\hat{\Omega}_i\}$ satisfies Assumption 3.3.1.

Let V_B^h and V_B^H be the space of Bell elements with respect to $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. V_A^h and V_A^H are similarly defined. We present the algorithms for the Bell element. The algorithms for the Argyris element are similar.

In general, the second derivatives of $\Phi \in V_B^H$ at the edge nodes x_i have two different values except at the vertices of the substructures. Therefore, $V_B^H \not\subset V_B^h$. Thus, a new coarse space has to be found. An easy way of modifying V_B^H to achieve this goal is by replacing the basis functions of V_B^H. Note that, in a substructure Ω_j, basis functions Φ of V_B^H can be represented by the basis functions of V_B^h:

$$\Phi(x) = \sum_{x_i \in \Lambda^h(\Omega_j)} \sum_{|\alpha| \le 2} \Phi_\alpha(x_i)\phi_i^\alpha(x) + \sum_{x_i \in \Lambda^h(\partial\Omega_j)} \sum_{|\alpha| \le 2} \Phi_\alpha(x_i)\phi_i^\alpha(x) \quad x \in \overline{\Omega}_j,$$

which is now replaced by

$$\Psi(x) = \sum_{x_i \in \Lambda^h(\Omega_j)} \sum_{|\alpha| \le 2} \Phi_\alpha(x_i)\phi_i^\alpha(x) + \sum_{i \in \Lambda^h(\partial\Omega_j)} \sum_{|\alpha| < 2} \Phi_\alpha(x_i)\phi_i^\alpha(x).$$

Note that the second derivatives of Ψ vanish at the nodes on $\partial\Omega_i$. We define the coarse space U_B^H as

$$U_B^H = \text{span}\{\Psi_i^\alpha, |\alpha| \le 2,\ i \in \Lambda^H\}.$$

It is easy to see that we have the inclusion $U_B^H \subset V_B^h$. Let $V_0 = U_B^H$, $V_i = V_B^h \cap H_0^2(\hat{\Omega}_i)$. Then we obtain a space decomposition

$$V^h = \sum_{i=0}^{N} V_i$$

and the corresponding orthogonal projections P_{V_i}.

Algorithm 5.2.1 *Find $u_h \in V_B^h$ such that*

$$Pu_h = (P_{V_0} + P_{V_1} + \cdots + P_{V_N})u_h = g_h \tag{5.4}$$

with $g_h = \sum_0^N g_i$ and g_i given by the solutions of

$$a(g_i, \phi_h) = a(P_{V_i}u, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V_i$$

Once we have computed the right-hand side g_h, we use the conjugate gradient method to solve equation (5.4). In each iteration, we need to compute Pv_h for some element $v_h \in V^h$. This is done in the following steps

1. Compute $P_0 v_h$ by solving $K^H x = b$. This is a finite element problem in the subspace U_B^H. Here $K^H = K_{U_B^H} = \{a(\Psi_i, \Psi_j)\}$ is the stiffness matrix associate with U_B^H, using the modified basis $\{\Psi_i^\alpha\}$. We note that $\dim(V_0) = \dim(U_B^H) = \dim(V_B^H) \approx 6N$, where $N = |\Lambda^H(\Omega)|$ is the number of substructure vertices.

2. Compute $P_i v_h$ by solving $K_i^h x = b$. This is a finite element problem in the subdomain $\hat{\Omega}_i$. The number of unknowns $\approx 6\hat{n}_i$, where $\hat{n}_i = |\Lambda^h(\hat{\Omega}_i)|$ is the number of nodes inside $\hat{\Omega}_i$.

Remark 5.2.1 Notice that to solve the coarse problem, we need to form the matrix $K^H = K_{U_B^H} = \{a(\Psi_i, \Psi_j)\}$, where Ψ_i and Ψ_j are the modified basis functions. Although the stiffness matrix $K^H = K_{V_B^H} = \{a(\Phi_i, \Phi_j)\}$ can be computed explicitly in terms of the coordinates of the three vertices of Ω_l, the computation of $K_{U_B^H}$ for the modified basis is not straightforward. A standard way is to use numerical integration. An alternative is to replace $K_{U_B^H}$ by $K_{V_B^H}$. This is equivalent to using an inexact solver $P_0^{'}$ to replace P_0. It can be shown that $\{a(\Phi_i, \Phi_j)\}$ and $\{a(\Psi_i, \Psi_j)\}$ are spectrally equivalent. Thus we still have an optimal algorithm.

5.2.2 Simplified Algorithms

The simplified algorithms for Bell and Argyris element are quite similar to those for the bicubic element.

To describe these algorithms, we need to introduce the following subspaces.

- The vertex space $V_{x_k} = \text{span}\{\phi_k^\alpha(x), |\alpha| = 2\}$, for each vertex x_k.
- $\tilde{V}_i = \text{span}\{\phi_k^\alpha(x), |\alpha| \leq 1, k \in \Lambda^h(\Omega_i)\}$. Note that $\tilde{V}_i$ is a subspace of V_i consisting of functions with vanishing second derivatives at all the vertices of the elements.
- $\tilde{U}_B^H = \text{span}\{\Psi_i^\alpha, |\alpha| \leq 1, i \in \Lambda^H(\Omega)\}$. $\tilde{U}_B^H$ is a subspace of U_B^H consisting of the functions with vanishing second derivatives at all the vertices of the substructures.
- $\tilde{U}_A^H = \text{span}\{\Psi_i^\alpha, |\alpha| \leq 1, \Psi_{ij}^n(x), i \in \Lambda^H(\Omega)\}$. $\tilde{V}_A^H$ is a subspace of U_A^H consisting of functions with vanishing second derivatives at all the vertices of the substructures.

Algorithm 5.2.2 *Find $u_h \in V_B^h$ such that*

$$P^{(2)} u_h \equiv (P_{\tilde{U}_B^H} + \sum_i P_{\tilde{V}_i} + \sum_{k \in \Lambda^h} P_{V_{x_k}}) u_h = g_h \qquad (5.5)$$

with the appropriate right hand side g_h.

The computation of g_h is similar to Algorithm 5.2.1. In each iteration step of Algorithm 5.2.2, we need to solve three sets of problems:

1. Compute $P_{\tilde{U}_B^H} v_h$ by solving $\tilde{K}^H x = b$. This is a finite element problem in the subspace $\tilde{U}_B^H$. Note that $\tilde{K}^H$ is a principal minor of K^H and $\dim(\tilde{U}_B^H) \approx \frac{1}{2}\dim(U_B^H)$.

2. Compute $P_{\tilde{V}_i} v_h$ by solving $\tilde{K}_i^h x = b$. This is a finite element problem on the subdomain $\hat{\Omega}_i$. Here $\tilde{K}_i^h$ is a principal minor of K_i^h and $\dim(\tilde{V}_i) \approx \frac{1}{2}\dim(V_i)$.

3. Compute $P_{V_{x_k}} v_h$ at each interior vertex x_k by solving $K_{x_k}^h x = b$. Here $x, b \in \mathcal{R}^3$ and $K_{x_k}^h = \{a(\Psi_k^\alpha, \Psi_k^\beta)\}$ is a 3 by 3 matrix, Ψ_k^α and Ψ_k^β are the modified basis functions associated with the second derivatives at the vertex x_k. These problems are very easy.

5.3 Optimality of the Basic Algorithms

Our result, concerning the optimality of the basic additive Schwarz algorithm, is contained in the following theorem.

Theorem 5.3.1 *The condition numbers of the iteration operator P in Algorithms 5.1.1 and 5.2.1 are bounded by a constant independent of number of substructures and the mesh sizes. More precisely, we have*

$$C_1 a(u,u) \leq a(Pu,u) \leq C_2 a(u,u), \quad \forall u \in V^h,$$

Proof. The upper bound follows from the finite covering property of $\{\hat{\Omega}_i\}$ and the properties of the projections, with $C_2 = (N_c + 1)$. Here N_c is the finite covering constant.

To establish the lower bound, we need to construct a good partition $u = \sum_i u_i, \forall u \in V^h$ in order to use the Lions' Lemma.

For the bicubic elements, by Theorem 4.1.2, we know that $\forall u_h \in V_Q^h \subset H_0^2(\Omega)$, we can find a $u_H \in \tilde{V}_Q^H \cap H_0^2(\Omega)$ satisfying

$$|u_H|_{H^2(\Omega)} \leq C|u_h|_{H^2(\Omega)}, \tag{5.6}$$

and

$$|u_H - u_h|_{H^s(\Omega)} \leq CH^{2-s}|u_h|_{H^2(\Omega)}. \tag{5.7}$$

We define $u_0 = u_H$, $w_h = u_h - u_H$. Since $\tilde{V}_Q^H \subset V_Q^H$, we find $u_0 \in V_Q^H \subset V_Q^h$. Because of the generous overlap of $\hat{\Omega}_i$, we know that there exists a partition of unity $\{\theta_i\}$ with $\theta_i \in C_0^\infty(\hat{\Omega}_i)$, which satisfies

$$\sum_1^N \theta_i = 1,$$

and

$$\Theta_s \equiv \max_i |\theta_i|_{W^{s,\infty}(\hat{\Omega}_i)} \leq CH^{-s}, \ s \leq 2.$$

Let $\Pi^h = \Pi_{V_Q^h}$ be the standard interpolation operator to the finite element space V_Q^h. We define u_i by

$$u_i = \Pi^h(\theta_i w_h).$$

It is easy to check that u_i is well defined, and that $u_i \in V_i$. By the linearity of Π^h, we know that

$$u_h = \sum_{i=0}^N u_i, \quad u_i \in V_i.$$

¿From Lemma 4.4.1, we know

$$|u_i|^2_{H^2(\tau)} \leq C\left(\Theta_0^2|\Pi^h w_h|^2_{H^2(\tau)} + \Theta_1^2|\Pi^h w_h|^2_{H^1(\tau)} + \Theta_2^2|\Pi^h w_h|^2_{L^2(\tau)}\right).$$

Using the fact that $w_h = u_h - w_H \in V^h$, we obtain

$$|u_i|^2_{H^2(\tau)} \leq C\left(\Theta_0^2|w_h|^2_{H^2(\tau)} + \Theta_1^2|w_h|^2_{H^1(\tau)} + \Theta_2^2|w_h|^2_{L^2(\tau)}\right).$$

Summing over $\tau \in \hat{\Omega}_i$, we get

$$|u_i|^2_{H^2(\hat{\Omega}_i)} \leq C\left(\Theta_0^2|w_h|^2_{H^2(\hat{\Omega}_i)} + \Theta_1^2|w_h|^2_{H^1(\hat{\Omega}_i)} + \Theta_2^2|w_h|^2_{L^2(\hat{\Omega}_i)}\right).$$

The conclusion

$$\sum_{i=0}^N |u_i|^2_{H^2(\Omega)} \leq |u_h|^2_{H^2(\Omega)}$$

follows from the fact that $\Theta_s \leq CH^{-s}$, $|w_h|_{H^s(\Omega)} \leq H^{2-s}|u_h|_{H^2(\Omega)}$ and the finite covering property of $\hat{\Omega}_i$.

For the Argyris and Bell elements, we note, from the results on the discrete Sobolev norms, and the construction of U_B^H and $\tilde{U}_B^H$, that the modified basis functions Ψ_i^α have similar properties as the basis functions of V_B^H. In particular, Ψ_i^α have the two properties of studied in chapter 1. Thus, the stable approximation properties hold for $\tilde{U}_B^H$. The rest of the proof is similar to the case for the bicubic element. ∎

5.4 Optimality of the Simplified Algorithms

We have the following result on the optimality of the simplified algorithms.

Theorem 5.4.1 *Algorithms 5.1.2 and 5.2.2 are optimal, i.e. the iteration operator* $P^{(2)}$ *satisfies*

$$C_1 a(u,u) \leq a(P^{(2)}u,u) \leq C_2 a(u,u),\ \forall u \in V_B^h,$$

where C_1 *and* C_2 *are independent of the number of substructures and the number of elements.*

Remark 5.4.1 The optimality of Algorithms 5.1.2 and 5.2.2 can essentially be proved by showing that the vertex spaces V_{x_k} are weakly coupled to the other subspaces. This property will also be important in developing iterative substructuring algorithms for the biharmonic equation.

Proof. We know that $\{\hat{\Omega}_i\}$ forms a finite covering of Ω with a covering constant N_c. It is easy to see that $\{\tau(x_k)\}_{k\in\Lambda^h}$ forms a finite covering of Ω with a covering constant degree($\mathcal{T}^h$) + 1. Thus $\{\hat{\Omega}_i, \tau(x_k)\}_{i,k}$ also forms a finite covering of Ω with a covering constant (degree($\mathcal{T}^h$) + 1) + N_c. The upper bound of P follows with $C_2 = (\text{degree}(\mathcal{T}^h) + 1) + N_c + 1$.

To establish the lower bound, we need to construct a good decomposition $u_h = \sum_i u_i$ in order to use the Lions' Lemma. By Theorem 4.1.2, we can find $u_H \in \tilde{V}_Q^H \cap H_0^2(\Omega)$ satisfying

$$|u_H|_{H^2(\Omega)} \leq C|u_h|_{H^2(\Omega)},$$

and

$$|u_H - u_h|_{H^s(\Omega)} \leq CH^{2-s}|u_h|_{H^2(\Omega)}.$$

Let $u_0 = u_H$, $w_h = u_h - u_H$ and

$$u_{x_k}(x) = \frac{\partial^2 w_h(x_k)}{\partial x \partial y}\phi_k^{xy}(x), \text{ for } x_k \in \Lambda^h(\Omega).$$

¿From the properties of the basis functions, it is clear that

$$|u_{x_k}(x)|_{H^s(\Omega)} \leq C|w_h(x)|_{H^s(\tau(x_k))}.$$

Summing over all $x_k \in \Lambda^h$, we have

$$\sum_{k\in\Lambda^h} |u_{x_k}|^2_{H^s(\Omega)} \le 3C^2 |w_h|^2_{H^s(\Omega)}.$$

Taking $s = 2$, and using the fact that $|w_h|_{H^2} \le C|u_h|_{H^2}$, we obtain

$$\sum_{k\in\Lambda^h} |u_{x_k}|^2_{H^2} \le C|u_h|^2_{H^2(\Omega)}. \tag{5.8}$$

Let $\hat{w}_h = w_h - \sum_k u_{x_k}$. Then $|\hat{w}_h|_{H^s} \le C|w_h|_{H^s}$. Let $u_i(x) = \Pi_{V_Q^h}(\theta_i \hat{w}_h)$. ¿From the linearity of $\Pi_{V_Q^h}$, we obtain

$$u_h = u_H + \sum_{k\in\Lambda^h} u_{x_k} + \sum_i u_i,$$

and

$$D^\alpha(u_h - u_H - \sum_{k\in\Lambda^h} u_{x_k})(x_k) = 0, \text{ for } |\alpha| = 2, k \in \Lambda^h.$$

Thus, $D^\alpha u_i(x_k) = 0$, for $k \in \Lambda^h$, which implies that $u_i \in \tilde{V}_i$. Using the same technique as in the proof of Theorem 5.3.1, we can prove that

$$\sum_i |u_i|^2_{H^2} \le C|\hat{w}_h|^2_{H^2}. \tag{5.9}$$

Combining (5.6), (5.8) and (5.9), we obtain

$$|u_H|^2_{H^2} + \sum_k |u_{x_k}|^2_{H^2} + \sum_i |u_i|^2_{H^2} \le C|u_h|^2_{H^2}.$$

The theorem now follows from Lions' Lemma. The conclusions for the Argyris and Bell elements can be established similarly. ∎

5.5 Numerical Results

In this section, we report on some numerical results with the additive Schwarz methods for the biharmonic equation. The computations were carried out on a *Sparc Station 1* and a *CONVEX C-1* machine. All the experiments are for the unit square domain $\Omega = [0,1]^2$.

We divide the domain Ω into $N_H \times N_H$ idential square subdomains Ω_{ij} and obtain a coarse triangulation $\mathcal{T}^H = \{\Omega_{ij}\}$. The length of sides of Ω_{ij} is $H = \frac{1}{N_H}$ and

$$\Omega_{ij} = ((i-1)H, iH) \times ((j-1)H, jH).$$

Each square Ω_{ij} is further divided into smaller square elements τ_{lk} of the same size and we obtain the fine triangulation $\mathcal{T}^h = \{\tau_{ij}\}$. Let n_h^2 be the number of elements. Then the length of sides of the elements is $h = \frac{1}{n_h}$ and

$$\tau_{ij} = ((i-1)h, ih) \times ((j-1)h, jh).$$

We extend each Ω_{ij} to a larger region $\hat{\Omega}_{ij}$.

$$\begin{aligned} \hat{\Omega}_{ij} &= ((i-1)H - rH, iH + rH) \times ((j-1)H - rH, jH + rH) && (5.10) \\ &= ((i-1)H - m_e h, iH + m_e h) \times ((j-1)H - m_e h, jH + m_e h). && (5.11) \end{aligned}$$

We cut off any part of $\hat{\Omega}_{ij}$ that is outside of Ω. Here r is the overlap ratio and m_e is the number of elements by which Ω_{ij} is extended in each direction. We note that rH has to be a multiple of h, since we want the $\partial\hat{\Omega}_{ij}$ to align with element boundaries.

For simplicity, we want all the subproblems to have the same size, i.e. all the submains $\hat{\Omega}_{ij}$ have the same size. For this to hold, we need to modify the boundary subdomains $\hat{\Omega}_{ij}$. Instead of extending Ω_{ij} by m_e elements, it is extended by $2m_e$ elements towards the interior. This is especially useful for computations on SIMD machines; cf. Bjørstad and Skogen [6]. For all the experiments of this section, we stop the iteration when the norm of the residual is reduced by a factor of $\epsilon = 10^{-4}$.

In the first set of experiments, we discretize the equation using the bicubic elements V_Q^h and we use Algorithm 5.1.1. The total number of degrees of freedom, $\dim(V_Q^h) = 4(n_h - 1)^2$. The results are summarized in table 5.1. The first column contains $4(n_h - 1)^2$, the total number of degrees of freedom. The second column contains N_H^2, the number of subdomains. In the third column, we give the overlap ratio r and the number of elements m_e by which $\hat{\Omega}_{ij}$ is extended from Ω_{ij}. We note that the real overlap ratio for boundary subdomains will be slightly different from that of the interior ones. The fourth, fifth and sixth columns give the minimum and minimum eigenvalues and the condition number of the iteration operator P, respectively. The last column contains the number of iteration required to reduce the norm of the residual by a factor of $\epsilon = 10^{-4}$.

We observe from table 5.1 that, in general, $\lambda_{\max}$ is between four and five, except in

total # unkns	# of subdom	ovlp ratio	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	# of iter.
$4 \times (8-1)^2$	2^2	1,1/2	0.99	5.0	5.0	8
$4 \times (16-1)^2$	2^2	1,1/4	0.51	4.3	8.3	10
$4 \times (16-1)^2$	4^2	1,1/4	0.65	4.5	7.0	11
$4 \times (16-1)^2$	4^2	2,1/2	1.00	9.0	9.0	12
$4 \times (32-1)^2$	4^2	2,1/4	0.65	4.5	6.9	10
$4 \times (32-1)^2$	8^2	1,1/4	0.62	4.5	7.3	11
$4 \times (64-1)^2$	8^2	2,1/4	0.62	4.5	7.2	10
$4 \times (64-1)^2$	16^2	1,1/4	0.61	4.5	7.3	11
$4 \times (80-1)^2$	8^2	2,1/5	0.48	4.3	9.0	11
$4 \times (80-1)^2$	16^2	1,1/5	0.47	4.3	9.1	12
$4 \times (100-1)^2$	10^2	2,1/5	0.48	4.3	9.0	11
$4 \times (100-1)^2$	20^2	1,1/5	0.47	4.3	9.1	12

Table 5.1: ASM Using the Bicubic Element V_Q^h.

the fourth row, where $\lambda_{\max} = 9$. The reason that $\lambda_{\max} = 9$ in the fourth row is that, in this case, we extend the Ω_{ij} too much and the covering constant $N_{cover} = 8$.

For comparison, we also compute the condition numbers of the additive Schwarz method without a coarse subspace. For a problem with 16 by 16 grid points, 4 by 4 subregions and overlap ratio $r = 1/4$, we have $\kappa(P) = 250$.

The condition numbers of the stiffness matrices K^h for these fourth order problems are extremely large. We give two examples here. For $n_h = 10$, we have $\kappa(K_{12,12}) \approx 1800$ and for $n_h = 20$ we have $\kappa(K_{20,20}) \approx 14000$.

In the next set of experiments, we consider Algorithm 5.1.2. We also use the bicubic element. The two triangulations $\mathcal{T}^h$, $\mathcal{T}^H$ and the subdomains $\hat{\Omega}_{ij}$ are defined as in the previous case and so are the overlap ratios r and m_e. The operator for Algorithm 5.1.2 is defined by

$$P^{(2)} = P_{\tilde{V}_Q^H} + \sum_i P_{V_i} + \sum P_{V_{x_k}}.$$

The V_Q^H is the bicubic element associated with the coarse triangulation $\mathcal{T}^H$, V_i the bicubic elements associated a subdomain Ω_{kl} and V_{x_k} the vertex spaces associated with a fine grid point $x_{kl} = (kh, lh)$.

total # unkns	# of subdom	ovlp ratio	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$
$4 \times (8-1)^2$	2^2	1/2	0.93	4.49	4.8
$4 \times (16-1)^2$	2^2	1/2	0.83	4.47	5.4
$4 \times (16-1)^2$	2^2	1/2	0.65	4.24	6.4
$4 \times (32-1)^2$	2^2	1/4	0.83	4.45	5.3
$4 \times (32-1)^2$	4^2	1/4	0.74	4.43	6.0
$4 \times (32-1)^2$	8^2	1/4	0.60	4.43	7.4
$4 \times (64-1)^2$	8^2	1/4	0.64	4.42	6.9
$4 \times (64-1)^2$	16^2	1/4	0.60	4.41	7.4

Table 5.2: ASM Using the Bicubic Element V_Q^h: The Simplified Version

We note that we can use different weights for different projections. In our experiments, we use

$$P^{(2)} = P_{\tilde{V}_Q^H} + \sum_i P_{\tilde{V}_i} + 2 \sum P_{V_{x_k}}.$$

The results are summarized in table 5.2. We observe that, for the same triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$, the condition number $\kappa(P^{(2)})$ is approximately of the same size as $\kappa(P)$ of Algorithm 5.1.1, in some cases, even smaller.

In a third set of experiments, we consider the Bell element discretization. We use Algorithm 5.2.1 to solve the linear system. Since we need triangular elements, the two triangulations $\mathcal{T}^h$, $\mathcal{T}^H$ are defined by dividing the squares obtained from the previous cases into two triangles. For simplicity we use square subdomains $\hat{\Omega}_{ij}$ as in the previous cases. The results are summarized in table 5.3.

We recall that, in this case, the coarse problem

$$a(P_{U_B^H} u, \Phi) = a(u, \Phi), \quad \forall \Phi \in U_B^H,$$

corresponds to the linear system

$$K_{U_B^H} x = b.$$

Here U_B^H is the modified coarse space. In our experiments, we do not form matrix $K_{U_B^H}$. Instead, we use $K_{V_B^H}$, the stiffness matrix for the coarse Bell elements. This is equivalent to solving $K_{U_B^H} x = b$ by an approximate solver. Since $K_{U_B^H}$ and $K_{V_B^H}$ are spectrally

total # unkns	# of subdom	ovlp ratio	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	# of iter.
$6 \times (16-1)^2$	2^2	2,1/2	1.0	6.0	6.0	9
$6 \times (16-1)^2$	2^2	1,1/4	0.33	4.3	13.0	12
$6 \times (16-1)^2$	4^2	1,1/4	0.40	4.6	11.3	14
$6 \times (32-1)^2$	4^2	2,1/4	0.44	4.5	10.3	12
$6 \times (32-1)^2$	8^2	1,1/4	0.35	4.6	12.8	15
$6 \times (64-1)^2$	8^2	2,1/4	0.38	4.5	11.9	13
$6 \times (64-1)^2$	16^2	1,1/4	0.35	4.6	13.2	14
$6 \times (80-1)^2$	10^2	2,1/5	0.41	5.4	12.8	13

Table 5.3: ASM Using the Bell Element V_B^h

equivalent, we still have an algorithm with optimal convergence properties. The numerical results are summarized in table 5.3. Since we do not solve the coarse problem exactly, $\lambda_{\max}(P)$ does not always lies in between 4 and 5.

5.6 Multilevel Methods for the Biharmonic Problem

In this section, we study multilevel additive Schwarz methods for the biharmonic equation. Although, all the multilevel methods studied in chapter 3 can be modified for the biharmonic problem, we only consider a special case, which in matrix form corresponds to a *multilevel block diagonal scaling* (**MBDS**). To simplify the presentation, we use the bicubic elements.

We first define a sequence of nested rectangular triangulations $\{\mathcal{T}^l\}_{l=1}^L$ as in case of second order problems, cf. chapter 3. We use τ_i^l to denote the elements in $\mathcal{T}^l$. The level l grid points are denoted by Λ^l, and the basis function by $\phi_{i,\alpha}^l, i \in \Lambda^l$.

Let $V^l = V_Q^{h_l}$ be the bicubic element associated with $\mathcal{T}^l$. The finite element solution $u_h \in V^L$ satisfies

$$a(u_h, \phi_h) = f(\phi_h), \;\; \phi_h \in V^h = V^L. \tag{5.12}$$

Let $\Omega_i^l = \mathrm{supp}\{\phi_{i,\alpha}^l\}$ be the support of an individual basis function and let $V_i^l = \mathrm{span}\{\phi_{i,\alpha}^l\}$ be the span of the level l basis functions at the grid point x_i. We note that for bicubic elements $\dim\{V_i^l\} = 4$. On each level, we have an overlapping decomposition of the domain

$$\Omega = \cup_i \Omega_i^l.$$

This decomposition satisfies Assumption 3.3.1. In particular, the decomposition has the finite covering property and there exists a partition of unity $\{\theta_i^l\}$ associate with $\{\Omega_i^l\}$. We use the space decomposition

$$V^h = V^L = V^H + \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l.$$

The operator of the L-level additive Schwarz algorithm is given by

$$P = \sum_{l=1}^{L} \sum_{i=1}^{N_l} P_i^l \stackrel{\text{def}}{=} \sum_{l=1}^{L} \sum_{i=1}^{N_l} P_{V_i^l}$$

with $P_{V_i^l} : V^h \to V_i^l$, the $a(\cdot,\cdot)$-orthogonal projections.

Algorithm 5.6.1 (MBDS Algorithm) *Find $u_h \in V^L$ by solving*

$$Pu_h = g_h \tag{5.13}$$

with an appropriate right hand side g_h.

In matrix form, equation (5.13) can be written as:

$$B^{-1} K_L x = B^{-1} b$$

where

$$B^{-1} = D_L^{-1} + \Pi_{L-1} D_{L-1}^{-1} \Pi_{L-1}^t + \cdots + \Pi_1 K_1^{-1} \Pi_1^t.$$

Here K_l is the stiffness matrix associated with $\mathcal{T}^l$, $D_l = \text{diag}\{K_l\}$, Π_l a prolongation operator and Π_l^t a restriction operator. The matrix K_1^{-1} can be replaced by any good preconditioner of K_1.

Theorem 5.6.1 *The multilevel additive Schwarz operator P satisfies*

$$C_1 L^{-1} a(u,u) \le a(Pu,u) \le CLa(u,u).$$

Thus

$$\kappa(B^{-1}K) \le CL^2$$

Remark 5.6.1 For second order problems, the corresponding algorithm is a multilevel diagonal scaling method, which is equivalent to the BPX algorithm of Bramble, Pasciak and Xu [13]. We note that in the second order case, we obtain a constant upper bound for the operator P; cf. chapter 3 and Zhang [54]. It is also possible to strengthen the result of Theorem 5.6.1.

Chapter 6

Iterative Substructuring Algorithms

6.1 Introduction and Overview

In this chapter, we study iterative substructuring methods for the biharmonic problem. We first review some methods for second order problems, using the additive Schwarz framework introduced by Dryja and Widlund [25]. We then generalize some iterative substructuring schemes to the biharmonic equation. We demonstrate that direct generalizations of the iterative substructuring methods designed for second order problems give slow algorithms. Better algorithms are obtained by adding certain vertex spaces to the decomposition of the finite element spaces.

We recall that iterative substructuring methods are domain decomposition methods using non-overlapping subregions; cf. Bramble, Pasciak and Schatz [10,11], Widlund [47]. Techniques for analyzing some iterative substructuring methods in the additive Schwarz framework were developed by Dryja and Widlund [25,26], where it was shown that the algorithm of [10] can be viewed as an additive Schwarz method with a set of specially chosen subdomains.

As always, we use two triangulations $\mathcal{T}^H = \{\Omega_i\}$ and $\mathcal{T}^h = \{\tau_i\}$. Let $\Gamma_{ij} = \overline{x_i x_j}$ be a common edge of the two neighboring substructures Ω_i and Ω_j and let $\Omega_{i,j} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j$. The subdomains $\{\Omega_{ij}\}$ play the role of the overlapping subdomains $\{\hat{\Omega}_i\}$ in the decomposition of domain. Using $\{\Omega_{ij}\}$, we obtain a decomposition of the domain Ω.

$$\overline{\Omega} = \cup \overline{\Omega}_{ij}.$$

Let V^h, V^H be the linear element associated with $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. Associated with each edge Γ_{ij}, we have a subspace V_{ij}, defined by

$$V_{ij} = V^h \cap H_0^1(\Omega_{ij}).$$

Using the space decomposition

$$V^h = V^H + \sum_{ij} V_{ij},$$

and the corresponding projections

$$P = P_{V^H} + \sum P_{V_{ij}},$$

we obtain

Algorithm 6.1.1 *Find $u_h \in V^h$ by solving*

$$Pu_h = g_h$$

with an appropriate right hand side g_h.

Let $V_{ij}^{\text{ha}} \subset V_{ij}$ be space of discrete harmonic functions and let $V_i = V^h \cap H_0^1(\Omega_i)$. Then we have an orthogonal decomposition of V_{ij}:

$$V_{ij} = V_i + V_j + V_{ij}^{\text{ha}}.$$

Thus, $P_{V_{ij}} = P_{V_i} + P_{V_j} + P_{V_{ij}^{\text{ha}}}$. Computing $P_{V_i} v_h$ corresponds to solving a local finite element problem in Ω_i. In matrix form, $P_{V_{ij}^{\text{ha}}}$ is closely related to the Schur complement S_{ij}. Let J be the square root of the discrete Laplacian operator along the edge Γ_{ij}. If we replace S_{ij} by J in our preconditioner, we obtain the BPS algorithm considered in Bramble, Pasciak and Schatz [10].

The condition number of the BPS algorithm grows like $(1 + log^2(H/h))$, see Bramble, Pasciak and Schatz [10], or Dryja and Widlund [25] for a proof using the additive Schwarz framework.

We remark that the BPS algorithm and the algorithms developed in this chapter are iterative substructuring methods. We do not solve subproblems on overlapping subdomains Ω_{ij}. Instead, we solve subproblems on non-overlapping subdomains Ω_i, and subproblems related to the interface Γ_{ij}. Estimates for the convergence rate of iterative substructuring algorithms are independent of the jumps in the coefficients of the elliptic problems.

6.2 Iterative Substructuring Methods for the Biharmonic Problem

We present the algorithms and the analysis only for the bicubic elements. The algorithms and analysis for the Argyris and Bell elements are similar. We first look at a direct generalization of the BPS algorithm and show that the condition number grows at least like $O((H/h)^2)$; this is also verified by numerical experiments. Modifications are needed to get well conditioned operators.

As in the additive Schwarz algorithms cases, we use V_Q^h and V_Q^H to denote the bicubic elements associated with the fine and coarse triangulations, respectively. $\tilde{V}_Q^H$ is also defined as in the previous chapter. As in the second order case, we need the subspaces V_{ij}, defined by

$$V_{ij} = V_Q^h \cap H_0^2(\Omega_{ij}).$$

We describe several possible decompositions of the finite element space. Each of them results in an algorithm. Our first algorithm resembles one of the algorithms of Bramble, Pasciak and Schatz for second order elliptic problems.

The finite element space V_Q^h can be represented as:

$$V_Q^h = V_Q^H + \sum V_{ij}. \tag{6.1}$$

Using this space decomposition and corresponding projections, we have

Algorithm 6.2.1 *Find $u_h \in V_Q^h$ such that*

$$P^{(1)} u_h \equiv (P_{V_Q^H} + \sum P_{V_{ij}}) u_h = g_h$$

with an appropriate right hand side g_h.

We recall that the condition number of the BPS algorithm grows like $(1 + \log^2(H/h))$. However, for the biharmonic operator, the condition number of $P^{(1)}$ grows as fast as $(1 + \log(H/h))(H/h)^2$; cf. table 6.1 and the algorithm is not very practical. If we use the decomposition 6.1, we obtain $u_h = u_H + \sum_{ij} u_{ij}$. We note that the only choice for u_H is the interpolant $\Pi_{V_Q^H} u_h$. Because of the appearance of the second derivatives in the interpolation operator $\Pi_{V_Q^H}$, we get a poor bound when we try to bound $|\Pi_{V_Q^H} u_h|_{H^2}$ by

$|u_h|_{H^2}$ and this is reflected in the poor bound in the condition number of $P^{(1)}$. This is a new phenomenon for Hermitian type elements.

One way to overcome this difficulty is to add certain vertex subspaces to the space decomposition 6.1. We then have more freedom to choose u_H and can avoid the interpolation operator $\Pi_{V_Q^H}$. For each grid point $x_k \in \Lambda^h(\Omega)$, we define a vertex space V_{x_i} by

$$V_{x_i} = \text{span}\{\phi_i^{xy}\},$$

where ϕ_i^{xy} is the basis function of V_Q^h associated with the mixed second derivative. We then have a space decomposition

$$V_Q^h = V_Q^H + \sum V_{ij} + \sum_{k \in \Lambda^H(\Omega)} V_{x_k}.$$

Using the corresponding projections, we obtain

Algorithm 6.2.2 *Find $u_h \in V_Q^h$ such that*

$$P^{(2)} u_h = (P_{V_Q^H} + \sum P_{V_{ij}} + \sum_{k \in \Lambda^H(\Omega)} P_{V_{x_k}}) u_h = g_h$$

with an appropriate right hand side g_h.

In each iteration of this algorithm, we have three sets of subproblems.

1. Compute $P_{V_Q^H} v_h$ by solving $K_{V_Q^H} x = b$. This is a finite element problem in V_Q^H. Here $K_{V_Q^H}$ is the stiffness matrix, and $\dim(V_Q^H) = 4N$, where N is the number of interior coarse grid points.

2. Compute $P_{V_{ij}} v_h$ by solving $K_{V_{ij}} x = b$ exactly, or by using a preconditioner. This is a finite element problem in the subdomain Ω_{ij}. Techniques for constructing preconditioners for the problem on the union of two subdomains can be used.

3. Compute $P_{V_{x_k}} v_h$ for each coarse grid point. There are $N = |\Lambda^H(\Omega)|$ such problems. For each x_k, $P_{V_{x_k}} v_h$ is computed by solving $K_{x_k}^h x = b$, where x and b are scalars and $K_{x_k}^h = a(\phi_k^{xy}, \phi_k^{xy})$ is the 1 by 1 stiffness matrix corresponding to the degree of freedom associated the mixed second derivative at the vertex x_k. Solving such problems are trivial.

We want to further reduce the cost per iteration, without decreasing the rate of convergence. Using the fact that the degrees of freedom associated with the second order derivatives are only weakly coupled to the other degrees of freedom, we can modify Algorithm 6.2.2 slightly, to get a computationally more efficient algorithm. Let

$$\tilde{V}_{ij} = \text{span}\{\phi_k^\alpha, |\alpha| = 0, 1\},$$

where $\phi_k^\alpha, |\alpha| = 0, 1$, are the basis functions of V_Q^h associated with the interior vertices of Ω_{ij}. We note that $\tilde{V}_{ij}$ is a subspace of V_{ij}, with vanishing mixed second derivatives at all the fine grid points $x_k \in \Lambda^h(\Omega)$. Using $\tilde{V}_{ij}$ and $\tilde{V}_Q^H$ instead of V_{ij} and V_Q^H, we obtain the following space decomposition,

$$V_Q^h = \tilde{V}_Q^H + \sum \tilde{V}_{ij} + \sum_{k \in \Lambda^h(\Omega)} V_{x_k}.$$

Using the sum of the corresponding projections

$$P^{(3)} = P_{\tilde{V}_Q^H} + \sum P_{\tilde{V}_{ij}} + \sum_{k \in \Lambda^h(\Omega)} P_{V_{x_k}}.$$

We obtain

Algorithm 6.2.3 *Find $u_h \in V_Q^h$ such that*

$$P^{(3)} u_h = (P_{\tilde{V}_Q^H} + \sum P_{\tilde{V}_{ij}} + \sum_{x_k \in \Lambda^h(\Omega)} P_{V_{x_k}}) u_h = g_h$$

with an appropriate right hand side g_h.

We note that, in the modified algorithm, we reduce the size of the the coase problem and the size of the subproblems associated with Ω_{ij}. This reduces the amount of work per iteration considerably. We need to solve the following subproblems in each iteration.

1. Compute $P_{\tilde{V}_Q^H} v_h$ by solving $K_{\tilde{V}_Q^H} x = b$. This is a finite element problem in the space $\tilde{V}_Q^H$. We note that $K_{\tilde{V}_Q^H}$ is a principal minor of $K_{V_Q^H}$ and $\dim(\tilde{V}_Q^H) = \frac{3}{4}\dim(V_Q^H)$.
2. Compute $P_{\tilde{V}_{ij}} v_h$ by solving $K_{\tilde{V}_{ij}^h} x = b$. This is a finite element problem in the subdomain Ω_{ij} using the space $\tilde{V}_{ij}$. This task is similar to computing $P_{V_{ij}} v_h$. We note however that $K_{\tilde{V}_{ij}}$ is a principal minor of $K_{V_{ij}}$ and $\dim(\tilde{V}_{ij}) = \frac{3}{4}\dim(V_{ij})$.

3. Compute $P_{V_{x_k}} v_h$ by solving $K_{V_{x_k}} x = b$; cf. Algorithm 6.2.2. There are n such problems, where n is the total number of interior fine grid points. Each subproblem itself is similar to the those of Algorithm 6.2.2. Although we have more subproblems for this algorithm, each of them is easy to solve. The total amount of work is proportional to the number of unknowns.

6.3 Condition Number Estimates

We have the following theorem for the operators defined above.

Theorem 6.3.1 *The condition numbers of the operators $P^{(i)}, i = 1, 2, 3$, for the additive algorithms defined above, satisfy the following estimates.*

$$\kappa(P^{(1)}) \leq C(H/h)^2(1 + \log(H/h)) \tag{6.2}$$

$$\kappa(P^{(i)}) \leq C(1 + \log(H/h))^2, i = 2, 3. \tag{6.3}$$

The constant C is independent of h and H.

Before we prove the theorem, let us discuss the decompositions of the domain and the corresponding partition of unity for the iterative substructuring methods constructed in this chapter.

Let x_{ij}^1 and x_{ij}^2 be the two end points of Γ_{ij}, let τ be an element of $\mathcal{T}^h$ and let

$$r(\tau) = \min\{\mathrm{dist}(\tau, x_{ij}^1), \mathrm{dist}(\tau, x_{ij}^2)\}.$$

We do not use a the subscript for $r(\tau)$, since we always work on only one substructure Ω_{ij} at a time.

First we consider the decomposition

$$\Omega = \cup_{ij} \overline{\Omega}_{ij}.$$

Associated with this decomposition of the domain, there is a partion of unity $\{\theta_{ij}\}$. The properties of $\{\theta_{ij}\}$ are summarized in the following lemma.

Lemma 6.3.1 *There exists a partition of unity $\{\theta_{ij}\}$, which satisfies*

$$\begin{aligned} \sum_{ij} \theta_{ij}(x) &= 1, \text{ for } x \in \Omega, \\ \theta_{ij} &\in C_0^\infty(\bar{\Omega}_{ij} - \{x_{ij}^1, x_{ij}^2\}) \\ |\theta_{ij}|_{W^{s,\infty}(\tau)} &\leq Cr(\tau)^{-s}. \end{aligned} \tag{6.4}$$

If we use additional subdomains, we have a better partition of unity in the sense that we can get a better bound for the $W^{s,\infty}$–norm of the partition functions. We consider the following decomposition of domain Ω:

$$\overline{\Omega} = \cup_{ij}\overline{\Omega}_{ij} \cup_{k\in\Lambda^H(\Omega)} \tau(x_k).$$

Corresponding to this decomposition of Ω, we have a partition of unity $\{\theta_{ij}, \theta_k\}$. The properties of this partition of unity are summarized in the following lemma.

Lemma 6.3.2 *The partition of unity $\{\theta_{ij}, \theta_k\}(i, j, k \in \Lambda^H)$ has the following properties:*

$$\begin{aligned} \sum_{ij} \theta_{ij}(x) + \sum_{k\in\Lambda^H} \theta_k(x) &= 1, \text{ for } x \in \Omega, \\ \theta_{ij} \in C_0^\infty(\bar{\Omega}_{ij}) \quad \text{and} & \quad \theta_k \in C_0^\infty(\overline{\tau}(_k)) \\ |\theta_{ij}|_{W^{s,\infty}(\tau)} &\leq C(\min\{r(\tau)^{-s}, h^{-s}\}), \quad s = 1, 2, \\ |\theta_k|_{W^{s,\infty}(\tau(x_k))} &\leq Ch^{-s}, \quad s = 1, 2. \end{aligned} \tag{6.5}$$

The following lemma gives bounds of the spectrum of $P^{(1)}$.

Lemma 6.3.3 *The operator $P^{(1)}$ satisfies the estimates:*

$$C_1((1 + \log(H/h))(H/h)^2)^{-1}a(u_h, u_h) \leq a(P^{(1)}u_h, u_h) \leq C_2a(u_h, u_h), \quad \forall u_h \in V_Q^h.$$

The constants C_1 and C_2 are independent of h and H.

Proof. The upper bound follows from the finite covering property of $\{\Omega_{ij}\}$ and properties of the projections, with $C_2 \leq N_c + 1$. Here $N_c = 4$ for rectangular elements and 3 for triangular elements.

To prove the lower bound, we need to get a partition for $u_h \in V_Q^h$. Let $u_H = \Pi_{V_Q^H} u_h$ and $w_h = u_h - u_H$. Then

$$D^\alpha w_h(x_j) = 0, \ |\alpha| = 0, 1, \quad \frac{\partial^2 w_h}{\partial x \partial y}(x_j) = 0, \quad \forall x_j \in \Lambda^H(\Omega). \tag{6.6}$$

Let θ_{ij} be the partition of unity given in Lemma 6.3.1. We note that each θ_{ij} is smooth except at the two substructure vertices common to Ω_i and Ω_j, and that θ_{ij} is uniformly bounded. Using equation (6.6), we can show that

$$D^{\alpha}(\theta_{ij}w_h)(x_j) = 0,\ |\alpha| = 0, 1, \quad \frac{\partial^2(\theta_{ij}w_h)}{\partial x \partial y}(x_j) = 0, \quad \forall x_j \in \Lambda^H(\Omega).$$

It is not difficult to see that the interpolant $\Pi_{V_Q^h}(\theta_{ij}w_h)$ is well defined and that $\Pi_{V_Q^h}(\theta_{ij}w_h) \in V_{ij}$. By the linearity of the interpolation operator $\Pi_{V_Q^h}$, we have,

$$\begin{aligned} u_h &= u_H + w_h = u_H + \sum(\theta_{ij}w_h) \\ &= u_H + \sum \Pi_{V_Q^h}(\theta_{ij}w_h) \equiv u_H + \sum u_{ij}, \end{aligned}$$

where $u_H \in V_Q^H$ and $u_{ij} \in V_{ij}$. We now estimate $|u_H|_{H^2}$ and $|u_{ij}|_{H^2}$. By Lemma 4.2.3, we have

$$|u_H|^2_{H^2(\Omega_i)} \le C(H/h)^2 |u_h|^2_{H^2(\Omega_i)}.$$

¿From Lemma 4.4.1, we obtain

$$\begin{aligned} |u_{ij}|^2_{H^2(\Omega_{ij})} &= \sum_{\tau \subset \Omega_{ij}} |u_{ij}|^2_{H^2(\tau)} = \sum_{\tau \subset \Omega_{ij}} |\Pi_{V_Q^h}(\theta_{ij}w_h)|^2_{H^2(\tau)} \\ &\le C \left\{ \sum_{\tau \subset \Omega_{ij}} \Theta_0^2(\tau)|w_h|^2_{H^2(\tau)} + \sum_{\tau \subset \Omega_{ij}} \Theta_1^2(\tau)|w_h|^2_{H^1(\tau)} + \sum_{\tau \subset \Omega_{ij}} \Theta_2^2(\tau)|w_h|^2_{L^2(\tau)} \right\} \\ &= T_1 + T_2 + T_3, \end{aligned}$$

where $\Theta_s(\tau) = |\theta_{ij}|_{W^{s,\infty}(\tau)}$. Using the fact that $|\theta_{ij}| \le 1$, we get

$$T_1 \le C \sum_{\tau \subset \Omega_{ij}} |w_h|^2_{H^2(\tau)} = C|w_h|^2_{H^2(\Omega_{ij})}.$$

We have the estimates

$$\sum_{\tau \subset \Omega_{ij}} h^2/r(\tau)^2 \le C \sum_{l=1}^{[H/h]} 1/l \le C(1 + \log(H/h)), \tag{6.7}$$

Thus,

$$\begin{aligned} T_2 &= C \sum_{\tau \subset \Omega_{ij}} \Theta_1^2(\tau)|w_h|^2_{H^1(\tau)} \\ &\le C \sum_{\tau \subset \Omega_{ij}} \Theta_1^2(\tau)h^2|w_h|^2_{W^{1,\infty}(\Omega_{ij})} \\ &\le C \sum_{\tau \subset \Omega_{ij}} \frac{h^2}{r(\tau)^2}|w_h|^2_{W^{1,\infty}(\Omega_{ij})} \\ &\le C(1 + \log(H/h))|w_h|^2_{W^{1,\infty}(\Omega_{ij})}. \end{aligned}$$

We note that, since w_h vanishes at the substructure vertices,

$$|w_h|^2_{L^\infty(\tau)} \le Cr^2 |w_h|^2_{W^{1,\infty}(\Omega_{ij})}.$$

Thus,

$$|w_h|^2_{L^2(\tau)} \le Ch^2 |w_h|^2_{L^\infty(\tau)} \le Ch^2 r^2 |w_h|^2_{W^{1,\infty}(\Omega_{ij})}.$$

Therefore,

$$\begin{aligned} T_3 &= C \sum_{\tau \subset \Omega_{ij}} \Theta_2^2(\tau) |w_h|^2_{L^2(\tau)} \le C \sum_{\tau \subset \Omega_{ij}} \frac{1}{r^4} h^2 r^2 |w_h|^2_{W^{1,\infty}(\Omega_{ij})} \\ &= C \sum_{\tau \subset \Omega_{ij}} \frac{h^2}{r^2} |w_h|^2_{W^{1,\infty}(\Omega_{ij})} \le C(1 + \log(H/h)) |w_h|^2_{W^{1,\infty}(\Omega_{ij})}. \end{aligned}$$

Thus,

$$|u_{ij}|^2_{H^2(\Omega_{ij})} = T_1 + T_2 + T_3 \le C|w_h|^2_{H^2(\Omega_{ij})} + C(1 + \log(H/h))|w_h|^2_{W^{1,\infty}(\Omega_{ij})}.$$

¿From Lemma 4.2.3, we know that

$$|w_h|^2_{W^{1,\infty}(\Omega_{ij})} \le C(H/h)^2 |u_h|^2_{H^2(\Omega_{ij})},$$

and

$$|w_h|^2_{H^2(\Omega_{ij})} \le C(H/h)^2 |u_h|^2_{H^2(\Omega_{ij})}.$$

Thus,

$$|u_{ij}|^2_{H^2(\Omega_{ij})} \le C((1 + \log(H/h))(H/h)^2 |u_h|^2_{H^2(\Omega_{ij})}.$$

Summing over all Ω_{ij} and taking the finite covering property of Ω_{ij} into account, we get

$$|u_H|^2_{H^2(\Omega)} + \sum |u_{ij}|^2_{H^2(\Omega_{ij})} \le C(1 + \log(H/h))(H/h)^2 |u_h|^2_{H^2(\Omega)}.$$

The lower bound now follows from Lions' lemma. ∎

Remark 6.3.1 For any decomposition $u_h = u_H + \sum_{ij} u_{ij}$, the coarse function u_H has to be the interpolant $\Pi_{V_Q^H} u_h$. It is easy to construct a function u_h such that

$$|\Pi_{V_Q^H} u_h|^2_{H^2(\Omega)} \ge C(H/h)^2 |u_h|^2_{H^2(\Omega)}.$$

Thus, it follows from Lemma 2.3.2 that $\lambda_{\min}(P^{(1)}) \le C(H/h)^{-2}$. This shows that the estimate in Lemma 6.3.3 is almost sharp. This bad bound for the condition number of $P^{(1)}$ indicates that Algorithm 6.2.1 is not practical.

We next give bounds for the spectrum of $P^{(2)}$.

Lemma 6.3.4 *The operator $P^{(2)}$ satisfies the estimates:*

$$C_1(1+\log(H/h))^{-2}a(u_h,u_h) \le a(P^{(2)}u_h,u_h) \le C_2 a(u_h,u_h), \quad \forall u_h \in V_Q^h$$

Proof. As in the proof of the previous lemma, the upper bound follows by the finite covering property of $\{\Omega_{ij}, \tau_{x_k}\}$ with $C_2 = N_c + 1$.

We now establish the lower bound. Let $u_H = \Pi_{\tilde{V}_Q^H} u_h \in \tilde{V}_Q^H$ and $w_h = u_h - u_h$. By the definition of $\Pi_{\tilde{V}_Q^H} u_h$, we know that

$$\frac{\partial^\alpha u_H}{\partial x^\alpha}(x_k) = \frac{\partial^\alpha u_h}{\partial x^\alpha}(x_k), \quad |\alpha| \le 1 \quad \text{and} \quad \frac{\partial^2 u_H}{\partial x \partial y}(x_k) = 0.$$

This implies

$$\frac{\partial^\alpha w_h}{\partial x^\alpha}(x_k) = 0, \quad |\alpha| \le 1 \text{ and } \frac{\partial^2 w_h}{\partial x \partial y}(x_k) = \frac{\partial^2 u_h}{\partial x \partial y}(x_k).$$

Let

$$u_{x_k} = \Pi_{V_Q^h}(\theta_k w_h) \text{ and } u_{ij} = \Pi_{V_Q^h}(\theta_{ij} w_h).$$

Then,

$$u_{ij}(x) = \frac{\partial u_{ij}}{\partial x}(x) = \frac{\partial u_{ij}}{\partial y}(x) = \frac{\partial^2 u_{ij}}{\partial x \partial y}(x) = 0, \quad \forall x \in \partial\Omega_{ij}$$

and

$$u_{x_k}(x_k) = \frac{\partial u_{x_k}}{\partial x}(x_k) = \frac{\partial u_{x_k}}{\partial y}(x_k) = 0, \quad \frac{\partial^2 u_{x_k}}{\partial x \partial y}(x_k) = \frac{\partial^2 u_h}{\partial x \partial y}(x_k), \quad \forall x_k \in \Lambda^H(\Omega).$$

Therefore,

$$u_{x_k} = \frac{\partial^2 u_h}{\partial x \partial y}(x_k)\phi_k^{xy} \in V_{x_k}, \quad \forall k \in \Lambda^H(\Omega) \quad \text{and} \quad u_{ij} \in V_{ij}.$$

By the linearity of $\Pi_{V_Q^h}$, we obtain

$$\begin{aligned} u_h &= u_H + w_h = u_h + \sum_{ij} \theta_{ij} w_h + \sum_{k \in \Lambda^H(\Omega)} \theta_k w_h \\ &= u_h + \sum_{ij} \Pi_{V_Q^h}(\theta_{ij} w_h) + \sum_{k \in \Lambda^H(\Omega)} \Pi_{V_Q^h}(\theta_k w_h) \\ &= u_h + \sum_{ij} u_{ij} + \sum_{k \in \Lambda^H(\Omega)} u_{x_k}. \end{aligned}$$

We now estimate $|u_{ij}|_{H^2}$ and $|u_{x_k}|_{H^2}$. As in the proof of the last lemma, we have

$$\begin{aligned}
|u_{ij}|^2_{H^2(\Omega_{ij})} &= \sum_{\tau\subset\Omega_{ij}} |u_{ij}|^2_{H^2(\tau)} = \sum_{\tau\subset\Omega_{ij}} |\Pi_{V_Q^h}(\theta_{ij}w_h)|^2_{H^2(\tau)} \\
&\le C\left\{\sum_{\tau}\Theta_0^2(\tau)|w_h|^2_{H^2(\tau)} + \sum_{\tau\subset\Omega}\Theta_1^2(\tau)|w_h|^2_{H^1(\tau)} + \sum_{\tau\subset\Omega}\Theta_2^2(\tau)|w_h|^2_{L^2(\tau)}\right\} \\
&= T_1 + T_2 + T_3 \\
&\le C|w_h|^2_{H^2(\Omega_{ij})} + C(1+\log(H/h))|w_h|^2_{W^{1,\infty}(\Omega_{ij})} \\
&\le C(1+\log(H/h))^2|u_h|^2_{H^2(\Omega_{ij})}.
\end{aligned}$$

In the last inequality, we have used the following inequalities from Lemma 4.2.4.

$$|w_h|^2_{W^{1,\infty}(\Omega_{ij})} \le C(1+\log(H/h))|u_h|^2_{H^2(\Omega_{ij})},$$

and

$$|w_h|^2_{H^2(\Omega_{ij})} \le C(1+\log(H/h))|u_h|^2_{H^2(\Omega_{ij})}.$$

By Lemma 4.4.1 and the fact that $|\theta_k|_{W^{s,\infty}(\tau(x_k))} \le Ch^{-s}$, we have

$$\begin{aligned}
|u_{x_k}|^2_{H^2(\Omega)} &= |u_{x_k}|^2_{H^2(\tau(x_k))} = |\Pi_{V_Q^h}(\theta_k w_h)|^2_{H^2(\tau(x_k))} \\
&\le C\{\, h^{-4}|w_h|^2_{L^2(\tau(x_k))} + h^{-2}|w_h|^2_{H^1(\tau(x_k))} + |w_h|^2_{H^2(\tau(x_k))}\,\} \\
&\le C|w_h|^2_{H^2(\tau(x_k))}.
\end{aligned}$$

In the last step, we have used the fact that $D^\alpha w_h(x_k) = 0, |\alpha| = 0, 1$, which implies that

$$|w_h|_{L^2(\tau(x_k))} \le Ch^2|w_h|_{H^2(\tau(x_k))} \quad \text{and} \quad |w_h|_{H^1(\tau(x_k))} \le Ch|w_h|_{H^2(\tau(x_k))}.$$

Combining the estimates for u_{ij} and u_{x_k}, we obtain

$$|u_H|^2_{H^2(\Omega)} + \sum_{ij}|u_{ij}|^2_{H^2(\Omega_{ij})} + \sum |u_k|^2_{H^2(\Omega)} \le (1+\log(H/h)^2)|u_h|^2_{H^2(\Omega)},$$

which by Lions' lemma results in the required lower bound of the spectrum of $P^{(2)}$. ∎

Lemma 6.3.5 *The operator $P^{(3)}$ satisfies the estimates:*

$$C_1(1+\log(H/h))^{-2}a(u_h,u_h) \le a(P^{(3)}u_h,u_h) \le C_2 a(u_h,u_h), \quad \forall u_h \in V_Q^h.$$

Proof. The proof is quite similar to that of Lemma 6.3.4 ∎

6.4 Numerical Results

In this section, we report on some numerical results with the iterative substructuring methods for the biharmonic equation. We only consider the bicubic element and domain $\Omega = [0,1]^2$. As for the additive Schwarz cases, we divide the Ω into N_H by N_H square subdomains Ω_{ij}. Each Ω_{ij} is further divided into elements. Let n_h^2 be the total number of elements. Then, the total number of degrees of freedom is $4(n_h - 1)^2$. Let $H = \frac{1}{N_H}$ and $h = \frac{1}{n_h}$. We stop the iteration when the norm of the residual is reduced by a factor of $\epsilon = 10^{-4}$.

In the first set of experiments, we consider the iterative substructuring methods without vertex spaces; cf. algorithm 6.2.1. To study the dependence of the condition number of P on $(H/h)^2$, the number of elements inside a substructure, we restrict our experiments to the two by two substructure case. ¿From table 6.1, we can clearly see that the minimum eigenvalue of P decreases at least quadratically with H/h and that the maximum eigenvalue of P remains almost fixed, as predicted by our theory. We note, however, that although the condition number of P grows very fast, the number of iteration grows slower than expected. The reason might be that convergence rate of the conjugate gradient methods depends not only on the condition number, but also on the distribution of the eigenvalues of the iteration operator. In these special two by two subdomain cases, we believe that only a few eigenvalues of P decrease fast, while the rest of the eigenvalues remain in a fixed interval. For a finite difference discretization, a similar phenomenon has been observed by Chan, E and Sun [18]. They also gives a rigorous proof. We believe however that this is not true for problems with many substructures.

In the next set of experiments, we consider the iterative substructuring methods, with vertex spaces, i.e. Algorithm 6.2.2. For a comparison with last set of experiments, we also restrict our experiments to two by two substructure cases. ¿From table 6.2, we can see that the maximum eigenvalue is uniformly bounded by three. In fact the maximum eigenvalue decreases slightly as H/h increases. The minimum eigenvalue of P appears to decrease logarithmically with H/h, the number of elements in each substructure.

total # unkns	# of subdom	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	# of iter.
$4\times(4-1)^2$	2^2	0.99E-01	2.1	21	9
$4\times(8-1)^2$	2^2	0.22E-01	2.1	95	11
$4\times(16-1)^2$	2^2	0.53E-02	2.1	400	13
$4\times(32-1)^2$	2^2	0.13E-02	2.1	1600	14

Table 6.1: ISM Using the Bicubic Element V_Q^h, without Vertex Spaces

total # unkns	# of subdom	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	# of iter.
$4\times(4-1)^2$	2^2	0.374	2.1	5.7	8
$4\times(8-1)^2$	2^2	0.222	2.1	9.6	8
$4\times(16-1)^2$	2^2	0.146	2.1	14.5	8
$4\times(32-1)^2$	2^2	0.103	2.1	20.5	9
$4\times(64-1)^2$	2^2	0.0766	2.1	27.6	9
$4\times(100-1)^2$	2^2	0.0645	2.1	32.6	9

Table 6.2: ISM Using the Bicubic Element, with Vertex Spaces

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