An Additive Schwarz Method for the p-version Finite Element Method

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Abstract

The additive Schwarz method was originally proposed for the h-version finite element method for elliptic problems. In this paper, we apply it to the p-version, in which increased accuracy is achieved by increasing the degree of the elements while the mesh is fixed. We obtain a constant bound, independent of p, for the condition number of the iteration operator in two and three dimensions. The result holds for linear, self adjoint, second order elliptic problems and for quadrilateral elements.

1 Introduction.

In the p-version of the finite element method, the degree of the piecewise polynomial elements is increased in order to achieve the desired accuracy, while the mesh is fixed. This is in contrast to the standard h-version where fixed low order polynomial elements are used and the mesh is refined in order to obtain accuracy. For an overview and basic results about the p-version, see Babuška and Suri [**?**]. In this paper, we study a domain decomposition method using p-version finite elements in the framework provided by the additive Schwarz method (ASM). We consider linear, self-adjoint, second order elliptic problems and brick-shaped elements in the finite element discretization. We show that the condition number of the ASM iteration operator is bounded by a constant independent of p. The proof is similar to the one in Dryja and Widlund [**?**] and is based on Lions' partitioning lemma.

This paper is organized as follows. In Section 2, we define a model problem and introduce its discretization with the p-version finite element method. In Section 3, we review the basic framework of the ASM and apply it to our model problem with square elements in two dimensions. An ASM using brick-shaped elements in three dimensions is considered in Section 4. For an example of a method similar to ours, but using the h-version finite element method, see Bramble et al.,[**?**].

2 The Model Problem.

We consider a model problem for linear, self adjoint, second order elliptic problems, on a bounded Lipschitz region Ω. The discrete problem is given by the p-version finite element method. For simplicity, we first consider the following problem in R^n:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{aligned}$$

The standard variational formulation of this problem is :
Find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = f(v)\,, \qquad \forall\ v \in\ V\,,$$

where the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

defines a semi-norm $|u|_{H^1(\Omega)} = (a(u,u))^{1/2}$ in $H^1(\Omega)$, and a norm in $V = H_0^1(\Omega)$. Our analysis works equally well for any general self adjoint, continuous, coercive, bilinear form $b(\cdot,\cdot)$, since the H_0^1 norm and the one induced by $b(\cdot,\cdot)$ are equivalent:

$$c|u|^2_{H_0^1} \leq b(u,u) \leq C|u|^2_{H_0^1} .$$

A triangulation of the region Ω is introduced by dividing it into non-overlapping brick-like elements Ω_i , $i = 1, \cdots, N_e$. We suppose that the original region is a union of such elements and we denote the mesh size by H.

We define Q_p to be the set of polynomials of degree less then or equal to p in each variable, i.e. in two dimensions

$$Q_p = span\{x^i y^j : 0 \leq i,j \leq p\}$$

and we discretize the problem with continuous, piecewise, degree p polynomial finite elements:

$$V^p = \{\phi \in C^0(\Omega) : \phi|_{\Omega_i} \in Q_p,\ i = 1, \cdots, N_e\} .$$

Then the discrete problem takes the form :
Find $u_p \in V^p$ such that

$$a(u_p, v_p) = f(v_p) , \qquad \forall\ v_p \in\ V^p . \tag{1}$$

For simplicity, we will analyze square and brick-shaped elements, but, using affine mappings onto the reference square and cube, our analysis works also for general quadrilateral elements.

3 An additive Schwarz method using square elements in two dimensions.

The additive Schwarz method was originally developed for the standard h-version finite element method and we refer the reader to Dryja and Widlund [?] or Dryja [?] for more detailed discussions. We work now in dimension two and with square elements Ω_i .

Let N be the number of interior nodes. Our finite element space is represented as the sum of N+1 subspaces

$$V^p = V_0^p + V_1^p + \cdots + V_N^p .$$

The first space V_0^p serves the same purpose as the coarse space in the h-version. Here:
$V_0^p = V^1$, i.e. the space of continuous, piecewise Q_1 functions on the mesh defined by the elements Ω_i;
$V_i^p = V^p \bigcap H_0^1(\Omega_i')$ where Ω_i' is the $2H \times 2H$ open square centered at the i-th vertex. In other words Ω_i' is the interior of $\overline{\Omega}_{i_1} \bigcup \overline{\Omega}_{i_2} \bigcup \overline{\Omega}_{i_3} \bigcup \overline{\Omega}_{i_4}$, see figure 1 below. As in the h-version, the algorithm consists in solving, by an

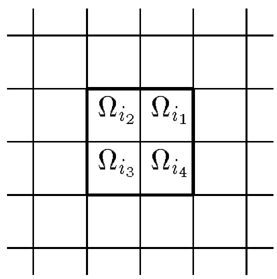

Figure 1: the substructure Ω_i

iterative method, the equation

$$P u_p = (P_0 + P_1 + ... + P_N) u_p = g_p' \,, \tag{2}$$

where the projections $P_i : V^p \to V_i^p$ are defined by

$$a(P_i v_p, \phi_p) = a(v_p, \phi_p) \,, \qquad \forall \phi_p \in V_i^p. \tag{3}$$

The following is the main result of this paper. The proof is first given for two dimensions and then extended to three in Section 4.

Theorem 1 *The operator P of the additive algorithm defined by the spaces V_i^p satisfies the estimate $\kappa(P) \leq const.$ independent of p.*

Proof. The proof is similar to a result given in Dryja and Widlund [**?**] for the h-version. A constant upper bound for the spectrum of P is obtained by noting that for $i \geq 1$

$$a(P_i u_p, u_p) = a(P_i u_p, P_i u_p) = a_{\Omega_i'}(P_i u_p, P_i u_p) \leq a_{\Omega_i'}(u_p, u_p).$$

Each point is covered by no more than four subregions Ω_i' and the norm of P_0 is equal to one, therefore $\lambda_{max} \leq 5$.

A lower bound is obtained by using Lions' lemma, see Lions [**?**] for the case $N = 2$; a proof is also given in Widlund [**?**].

Lemma 1 *Let* $u_p = \sum_{i=0}^{N} u_{p,i}$, *where* $u_{p,i} \in V_i$, *be a representation of an element of* $V^p = V_0 + V_1 + \cdots + V_N$. *If the representation can be chosen so that*

$$\sum_{i=0}^{N} a(u_{p,i}, u_{p,i}) \leq C_0^2 a(u_p, u_p) , \qquad \forall u_p \in V^p ,$$

then

$$\lambda_{min}(P) \geq C_0^{-2} .$$

We have to define a partition of the finite element function u_p and obtain a good bound of C_0^2. We know from Strang [**?**] that there exists a linear map $\hat{I}_1 : V^p \rightarrow V^1$, which satisfies

$$\|u_p - \hat{I}_1 u_p\|_{L^2(\Omega)}^2 \leq C_1 H^2 |u_p|_{H^1(\Omega)}^2 \tag{4}$$

and

$$|u_p - \hat{I}_1 u_p|_{H^1(\Omega)}^2 \leq C_2 |u_p|_{H^1(\Omega)}^2 . \tag{5}$$

Let

$$u_{p,0} = \hat{I}_1 u_p , \qquad w_p = u_p - u_{p,0} .$$

In order to define $u_{p,i} \in V_i^p$, we consider a particular partition of unity $\{\theta_i\}$ consisting of the standard basis functions for $Q_1 (= V_1)$:

$$\theta_i \in V_1 , \qquad supp(\theta_i) = \Omega_i' , \qquad 0 \leq \theta_i \leq 1 , \qquad \sum_{i=1}^{N} \theta_i(x, y) = 1 .$$

Now $\theta_i w_p$ is an element of Q_{p+1} vanishing outside Ω_i'. Since in our partition we need an element of V_i^p , we interpolate back $\theta_i w_p$ into V_i^p . We define this interpolation operator I_p on one of the four elements Ω_{i_j} of Ω_i'; on the other three it is completely analogous. We transform this element into the reference square $[-1, 1] \times [-1, 1]$. θ_i is 1 at one vertex of Ω_i and 0 at the other three, so it has one of the four forms

$$\theta_i = \frac{1}{4}(x \pm 1)(y \pm 1) .$$

We define $u_{p,i} = I_p(\theta_i w_p)$ as the polynomial in Q_p interpolating $\theta_i w_p$ at the $(p+1)^2$ points (x_n, x_m), where the $x_n' s$ are the zeros of the polynomial

$$\mathcal{L}_{p+1}(x) = \int_{-1}^{x} L_p(s)\, ds\ . \tag{6}$$

Here $L_p(s)$ is the Legendre polynomial of degree p. This definition makes sense for $p \geq 1$ because $\mathcal{L}_{p+1}$ has p+1 distinct real zeros in $[-1,1]$. In fact $\mathcal{L}_{p+1}(\pm 1) = 0$ and $p-1$ roots interleave those of L_p, which, as is well known, has p distinct real zeros in $[-1,1]$. We define $\mathcal{L}_0 = 1$. We remark that even if this definition of the interpolation operator is local, we obtain an element in $V_i^p = V^p \bigcap H_0^1(\Omega_i')$. In fact, $u_{p,i}$ is continuous across element boundaries, because on the boundary of each element we have $p+1$ interpolation points and $u_{p,i}$ is a polynomial of degree p. Since I_p is a linear operator, we have

$$\sum_{i=1}^{N} u_{p,i} = u_p - u_{p,0}\ .$$

We note that

$$I_p|_{Q_p} = identity\ .$$

Since $|\cdot|_{H^1}$ is a semi-norm on Q_p, it is natural to introduce the quotient space $\hat{Q}_p = Q_p/Q_0$, on which $|\cdot|_{H^1}$ is a norm. Clearly $dim Q_p = (p+1)^2$, while $dim\hat{Q}_p = (p+1)^2 - 1$.
We now establish:

Lemma 2 *The interpolation operator* $I_p : \hat{Q}_{p+1}([-1,1]^2) \rightarrow \hat{Q}_p([-1,1]^2)$ *is uniformly bounded in the* $|\cdot|_{H^1}$ *norm, i.e.*

$$|I_p(f)|_{H^1} \leq const. |f|_{H^1}\ , \qquad \forall f \in \hat{Q}_{p+1}([-1,1]^2)\ .$$

Proof. If f is a function of x only, then $I_p f$ is a function of x only and

$$\frac{|I_p(f)|_{H^1}^2}{|f|_{H^1}^2} = \frac{\|\frac{\partial}{\partial x} I_p(f)\|_{L^2}^2}{\|\frac{\partial f}{\partial x}\|_{L^2}^2}.$$

Similarly, if f is a function of y only

$$\frac{|I_p(f)|_{H^1}^2}{|f|_{H^1}^2} = \frac{\|\frac{\partial}{\partial y} I_p(f)\|_{L^2}^2}{\|\frac{\partial f}{\partial y}\|_{L^2}^2}.$$

In general, if both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not vanish, it is easy to see that

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} = \frac{\|\frac{\partial}{\partial x}I_p(f)\|^2_{L^2} + \|\frac{\partial}{\partial y}I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2} + \|\frac{\partial f}{\partial y}\|^2_{L^2}} \leq$$

$$\leq \frac{\|\frac{\partial}{\partial x}I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial y}I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial y}\|^2_{L^2}} . \tag{7}$$

We consider the two terms separately. By symmetry, we need only to study the first term. We form a basis for $\hat{Q}_{p+1}$ using the polynomials

$$\phi_{0,j}(x,y) = \frac{1}{\sqrt{2}}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} , \qquad 1 \leq j \leq p+1,$$

$$\phi_{i,0}(x,y) = \frac{1}{\sqrt{2}}\frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}} , \qquad 1 \leq i \leq p+1,$$

and

$$\phi_{i,j}(x,y) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} , \qquad 1 \leq i,j \leq p+1. \tag{8}$$

We can disregard the space spanned by the $\phi_{0,j}$'s because every function f_0 in that space will not contribute to the x-term we are considering. In fact, if $f = f_0 + \tilde{f}$, we have

$$\|\frac{\partial}{\partial x}I_p f\|_{L^2} = \|\frac{\partial}{\partial x}I_p \tilde{f}\|_{L^2} \qquad \text{and} \qquad \|\frac{\partial}{\partial x}f\|_{L^2} = \|\frac{\partial}{\partial x}\tilde{f}\|_{L^2} .$$

In the resulting space $\tilde{Q}_{p+1}$ of dimension $p' = (p+2)(p+1)$, we choose the order

$$\phi_{10}, \phi_{11}, \cdots, \phi_{1p+1}, \phi_{20}, \phi_{21}, \cdots, \phi_{2p+1}, \cdots, \phi_{p+1,0}, \phi_{p+1,1}, \cdots, \phi_{p+1,p+1}.$$

If, for simplicity, we relabel the basis as $\{\phi_k(x,y) \, , \, k = 1, 2, \cdots, p'\}$, we have

$$f(x,y) = \sum_{k=1}^{p'} \alpha_k \phi_k(x,y) ,$$

$$I_p(f) = \sum_{k=1}^{p'} \alpha_k I_p(\phi_k), \qquad \text{with} \qquad I_p(\phi_k) = \begin{cases} 0 & \text{if } \phi_k \in \tilde{Q}_{p+1} - \tilde{Q}_p \\ \phi_k & \text{if } \phi_k \in \tilde{Q}_p . \end{cases}$$

The last relations follow from the fact that the coordinates of the interpolation nodes (x_n, x_m) are the zeros of $\mathcal{L}_{p+1}$ and $I_p|_{\tilde{Q}_p}$ is the identity. Hence

$$\|\frac{\partial f}{\partial x}\|^2_{L^2} = (\sum_{k=1}^{p'} \alpha_k \frac{\partial \phi_k}{\partial x}, \sum_{l=1}^{p'} \alpha_l \frac{\partial \phi_l}{\partial x})_{L^2} = \sum_{k,l} \alpha_k \alpha_l (\frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_l}{\partial x})_{L^2} =$$

$$= \alpha^T S_x \alpha \ ,$$

where $(S_x)_{k,l} = (\frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_l}{\partial x})_{L^2}$ is a symmetric, positive definite matrix of order p'. Similarly,

$$\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2} = \alpha^T B S_x B \alpha \ ,$$

where

$$B = \begin{pmatrix} I & & & & & & & \\ & 0 & & & & & & \\ & & I & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & I & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

is a projection matrix onto $\tilde{Q}_p$. Here the identity matrices are all of order $p+1$, the last zero block is of order $p+2$ and the other zeros are of order 1. Therefore,

$$\frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} = \frac{\alpha^T B S_x B \alpha}{\alpha^T S_x \alpha} \ .$$

The proof of the lemma follows from a bound for the eigenvalues λ of the generalized eigenvalue problem

$$B S_x B \alpha = \lambda S_x \alpha \ . \qquad (9)$$

The structure of S_x can be made explicit by using the orthogonality of the Legendre polynomials and the formula

$$\int_{-1}^{x} L_n(s)\, ds = \frac{L_{n+1}(x) - L_{n-1}(x)}{2n+1} \ , \qquad n \geq 1 \ . \qquad (10)$$

In fact, if $\phi_k(x,y) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}$ and $\phi_l(x,y) = \frac{\mathcal{L}_n(x)}{\|L_{n-1}\|}\frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|}$, we find

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_k(x,y), \frac{\partial}{\partial x}\phi_l(x,y))_{L^2_{xy}} = (\frac{L_{i-1}(x)}{\|L_{i-1}\|}\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}, \frac{L_{n-1}(x)}{\|L_{n-1}\|}\frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|})_{L^2_{xy}} =$$

$$= (\frac{L_{i-1}(x)}{\|L_{i-1}\|}, \frac{L_{n-1}(x)}{\|L_{n-1}\|})_{L^2_x}(\frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_m(y)}{\|\mathcal{L}_m\|})_{L^2_y}.$$

This expression differs from zero iff

$$n = i \qquad \text{and} \qquad m = \begin{cases} j-2 \\ j \\ j+2 \end{cases},$$

and therefore each row of S_x has at most three nonzero elements. They are

$$(S_x)_{kl} = \begin{cases} (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j-2}}{\|\mathcal{L}_{j-2}\|})_{L^2_y} \\ (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}) = 1 \\ (\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j+2}}{\|\mathcal{L}_{j+2}\|})_{L^2_y} \end{cases}.$$

The only exceptions to this rule occur when one of the indexes is $0, 1$ or 2. In this case we cannot use formula (10), but we can use $\mathcal{L}_0 = 1 = L_0$ and $\mathcal{L}_1(x) = \int_{-1}^x ds = x + 1 = L_1 + L_0$. The exceptional elements are then

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,0}, \frac{\partial}{\partial x}\phi_{i,1}) = 1 \cdot (\frac{1}{\sqrt{2}}, \frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|})_{L^2_y} = \sqrt{\frac{3}{2}} = c_0\ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,0}, \frac{\partial}{\partial x}\phi_{i,2}) = 1 \cdot (\frac{1}{\sqrt{2}}, \frac{\mathcal{L}_2(y)}{\|\mathcal{L}_2\|})_{L^2_y} = -\sqrt{\frac{5}{6}} = b_0\ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,1}, \frac{\partial}{\partial x}\phi_{i,2}) = 1 \cdot (\frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|}, \frac{\mathcal{L}_2(y)}{\|\mathcal{L}_2\|})_{L^2_y} = -\sqrt{\frac{5}{8}} = c_1\ .$$

$$(S_x)_{kl} = (\frac{\partial}{\partial x}\phi_{i,1}, \frac{\partial}{\partial x}\phi_{i,3}) = 1 \cdot (\frac{\mathcal{L}_1(y)}{\|\mathcal{L}_1\|}, \frac{\mathcal{L}_3(y)}{\|\mathcal{L}_3\|})_{L^2_y} = -\sqrt{\frac{7}{40}} = b_1\ .$$

This shows that S_x has the structure

$$S_x = \begin{pmatrix} A_{p+2} & & & \\ & A_{p+2} & & \\ & & \ddots & \\ & & & A_{p+2} \end{pmatrix}.$$

Each block A_{p+2} is symmetric, pentadiagonal and of order $p+2$

$$A_{p+2} = \begin{pmatrix} 1 & c_0 & b_0 & & & & & & \\ c_0 & 1 & c_1 & b_1 & & & & & \\ b_0 & c_1 & 1 & 0 & b_2 & & & & \\ & b_1 & 0 & 1 & 0 & \dots & & & \\ & & b_2 & 0 & 1 & \dots & \dots & & \\ & & & \dots & \dots & \dots & & b_{p-2} & \\ & & & & \dots & & 1 & 0 & b_{p-1} \\ & & & & & b_{p-2} & 0 & 1 & 0 \\ & & & & & & b_{p-1} & 0 & 1 \end{pmatrix} .$$

The elements c_0, c_1 and b_0, b_1 have been defined above and

$$b_j = \left(\frac{\mathcal{L}_j}{\|\mathcal{L}_j\|}, \frac{\mathcal{L}_{j+2}}{\|\mathcal{L}_{j+2}\|}\right)_{L^2_y} \qquad j \geq 2 .$$

By using (10) and $\|L_n\|^2_{L^2[-1,1]} = \frac{2}{2n+1}$, we can compute the b_j's explicitly .

$$\|\mathcal{L}_j\|^2 = \frac{4}{(2j-3)(2j-1)(2j+1)} , \qquad j \geq 2 , \qquad \|\mathcal{L}_1\| = \sqrt{\frac{8}{3}} ,$$

$$(\mathcal{L}_j, \mathcal{L}_{j+2}) = -\frac{2}{(2j-1)(2j+1)(2j+3)} , \qquad j \geq 1 , \qquad (\mathcal{L}_1, \mathcal{L}_2) = -\frac{2}{3}$$

and therefore

$$b_j = -\frac{1}{2}\sqrt{\frac{(2j-3)(2j+5)}{(2j-1)(2j+3)}} , \qquad j \geq 2 . \tag{11}$$

Our generalized eigenvalue problem (9) can now be written as

$$\begin{pmatrix} B_{p+2} & & & & \\ & B_{p+2} & & & \\ & & \ddots & & \\ & & & B_{p+2} & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{p+1} \end{pmatrix} =$$

$$= \lambda \begin{pmatrix} A_{p+2} & & & & \\ & A_{p+2} & & & \\ & & \ddots & & \\ & & & A_{p+2} & \\ & & & & A_{p+2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{p+1} \end{pmatrix} ,$$

where

$$B_{p+2} = \begin{pmatrix} A_{p+1} & \\ & 0 \end{pmatrix} .$$

The last zero is a scalar and the α_i's are vectors of length $p+2$. This is equivalent to $\alpha_{p+1} = 0$ and

$$\begin{pmatrix} A_{p+1} & \\ & 0 \end{pmatrix} \alpha_i = \lambda A_{p+2} \alpha_i \,, \qquad 1 \le i \le p \,. \tag{12}$$

But

$$A_{p+2} = \begin{pmatrix} A_{p+1} & b \\ b^T & 1 \end{pmatrix} ,$$

with $b^T = (0, \cdots, 0, b_{p-1}, 0) = b_{p-1} e^T$, where e is the $p - th$ column of the identity matrix of order $p+1$. Therefore, (12) is equivalent to

$$\begin{pmatrix} A_{p+1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} A_{p+1} & b \\ b^T & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} ,$$

i.e.

$$A_{p+1} v = \lambda (A_{p+1} - b_{p-1}^2 e e^T) v \,. \tag{13}$$

Since ee^T has rank 1, we see immediately that we have p eigenvalues equal to 1, corresponding to the eigenvectors v with $v_p = 0$. In order to find the only non-trivial eigenvalue, we apply the Shermann-Morrison formula (see Golub and Van Loan, [?], pg. 51)

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} u v^T A^{-1} \tag{14}$$

to $A_{p+1} - b_{p-1}^2 e e^T$ and obtain

$$(A_{p+1} - b_{p-1}^2 e e^T)^{-1} = A_{p+1}^{-1} + \frac{b_{p-1}^2}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1} e e^T A_{p+1}^{-1} \,.$$

With this formula, we have reduced (13) to the standard eigenvalue problem

$$Mv = (A_{p+1} - b_{p-1}^2 e e^T)^{-1} A_{p+1} v = \lambda v \,, \tag{15}$$

with

$$M = I + \frac{b_{p-1}^2}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1} e e^T \,.$$

Again we can see that the eigenspace corresponding to $\lambda = 1$ has dimension p by substituting $v \in$nullspace$(A_{p+1}^{-1}e)$ into (15). We obtain the non-trivial eigenvalue by substituting $v = A_{p+1}^{-1}e$ into (15):

$$Mv = A_{p+1}^{-1}e + \frac{b_{p-1}^2 e^T A_{p+1}^{-1} e}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} A_{p+1}^{-1}e = \lambda A_{p+1}^{-1}e = \lambda v \, ,$$

with

$$\lambda = 1 + \frac{b_{p-1}^2 e^T A_{p+1}^{-1} e}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} = \frac{1}{1 - b_{p-1}^2 e^T A_{p+1}^{-1} e} \, . \tag{16}$$

In order to obtain an upper bound for λ, we need some properties of A_{p+1}^{-1} and the sequence of b_j's . Since from (11), $b_{p-1}^2 < \frac{1}{4}$, we find that

$$\lambda < \frac{1}{1 - \frac{1}{4} e^T A_{p+1}^{-1} e} \, .$$

What remains is to find a bound on $e^T A_{p+1}^{-1} e =$ the $p-$th diagonal element of A_{p+1}^{-1} . Let $a_p = det(A_p)$. By Cramer's rule,

$$e^T A_{p+1}^{-1} e = \frac{\hat{a}_p}{a_{p+1}} \, , \tag{17}$$

where the cofactor $\hat{a}_p$ is the determinant of the matrix obtained from A_{p+1} by deleting the p-th row and column. Applying Laplace's theorem for the expansion of determinants to a_{p+1} and $\hat{a}_p$, it is easy to prove the following recurrence relations:

$$a_{p+1} = a_p - b_{p-2}^2 (a_{p-2} - b_{p-3}^2 a_{p-3}) \, , \qquad p \geq 4, \tag{18}$$

$$\hat{a}_p = a_{p-1} - b_{p-2}^2 a_{p-2} \, , \qquad p \geq 4 \, . \tag{19}$$

(18) can be written as

$$\frac{a_p - a_{p+1}}{b_{p-2}^2} = a_{p-2} - b_{p-3}^2 a_{p-3} \, , \tag{20}$$

which shows that

$$a_p > a_{p+1} \qquad \text{iff} \qquad a_{p-2} > b_{p-3}^2 a_{p-3} \, . \tag{21}$$

Now, A_{p+1} is positive definite, because $\alpha^T A_{p+1}\alpha$ defines a L^2 norm of a function with components $\{\alpha_i\}$. Therefore A_{p+1}^{-1} is positive definite. Substituting (19) into (17) and using (20), we get

$$0 < eA_{p+1}^{-1}e = \frac{a_{p-1} - b_{p-2}^2 a_{p-2}}{a_{p+1}} = \frac{a_{p+1} - a_{p+2}}{b_{p-1}^2 a_{p+1}} = \frac{1}{b_{p-1}^2}(1 - \frac{a_{p+2}}{a_{p+1}}) \,. \quad (22)$$

This implies that $1 - \frac{a_{p+2}}{a_{p+1}} > 0$, i.e. $a_{p+1} > a_{p+2}$,$\forall p$. By (21) we then have $a_{p+2} > b_{p+1}^2 a_{p+1}$, i.e. $\frac{a_{p+2}}{a_{p+1}} > b_{p+1}^2$,$\forall p$. Hence

$$e^T A_{p+1}^{-1} e < \frac{1}{b_{p-1}^2}(1 - b_{p+1}^2) \,.$$

Since $\lim_{p\to\infty} b_p^2 = \frac{1}{4}$ and $b_p^2 < \frac{1}{4}$, for every $\epsilon > 0$ we have $b_p^2 > \frac{1}{4}(1-\epsilon)$ for p large enough. This gives us the desired bound

$$e^T A_{p+1}^{-1} e < \frac{4}{1-\epsilon}(1 - \frac{1}{4}(1-\epsilon)) = \frac{3+\epsilon}{1-\epsilon} = 3 + \epsilon'$$

for p large enough, and finally

$$\lambda < \frac{1}{1 - \frac{1}{4}(3+\epsilon')} = \frac{4}{1-\epsilon'} \,.$$

In other words, $\lambda < const.$ uniformly in p .
Numerical experiments in MATLAB show that a stronger result is actually true : $\lim_{p\to\infty} \lambda = 2$.
In conclusion, we have found a bound for the x-term in (7):

$$sup \frac{\|\frac{\partial}{\partial x} I_p(f)\|_{L^2}^2}{\|\frac{\partial f}{\partial x}\|_{L^2}^2} \leq const. \,.$$

Reasoning in the same way for the y-term, we find from formula (7)

$$\frac{|I_p(f)|_{H^1}^2}{|f|_{H^1}^2} \leq const.$$

□

We can now conclude the proof of the theorem by applying Lemma 2 to bound the H^1 norm of each component $u_{p,i} = I_p(\theta_i w_p)$ over a single element Ω_k.

$$|u_{p,i}|_{H^1(\Omega_k)}^2 \leq C|\theta_i w_p|_{H^1(\Omega_k)}^2 =$$

$$= C(\|\frac{\partial \theta_i}{\partial x} w_p + \theta_i \frac{\partial w_p}{\partial x}\|^2_{L^2(\Omega_k)} + \|\frac{\partial \theta_i}{\partial y} w_p + \theta_i \frac{\partial w_p}{\partial y}\|^2_{L^2(\Omega_k)}) \leq$$

$$\leq 2C(\|\frac{\partial \theta_i}{\partial x} w_p\|^2_{L^2(\Omega_k)} + \|\theta_i \frac{\partial w_p}{\partial x}\|^2_{L^2(\Omega_k)} + \|\frac{\partial \theta_i}{\partial y} w_p\|^2_{L^2(\Omega_k)} + \|\theta_i \frac{\partial w_p}{\partial y}\|^2_{L^2(\Omega_k)}) .$$

On a square element Ω_k of side H , $|\frac{\partial \theta_i}{\partial x}|$ and $|\frac{\partial \theta_i}{\partial y}|$ are bounded by $1/H$ and, by construction, $\|\theta_i\|_{L^\infty} \leq 1$. Therefore

$$|u_{p,i}|^2_{H^1(\Omega_k)} \leq 2C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega_k)} + |w_p|^2_{H^1(\Omega_k)}) .$$

Since at most 4 components $u_{p,i}$ are nonzero for any element Ω_k, we obtain, when summing over i,

$$\sum_{i=1}^{N} |u_{p,i}|^2_{H^1(\Omega_k)} \leq 8C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega_k)} + |w_p|^2_{H^1(\Omega_k)}) ,$$

and summing over all the elements Ω_k

$$\sum_{i=1}^{N} |u_{p,i}|^2_{H^1(\Omega)} \leq 8C(\frac{2}{H^2}\|w_p\|^2_{L^2(\Omega)} + |w_p|^2_{H^1(\Omega)}) .$$

Using equations (4) and (5), we can conclude

$$\sum_{i=1}^{N} |u_{p,i}|^2_{H^1(\Omega)} \leq 8C(2C_1 + C_2)|u_p|^2_{H^1(\Omega)} = const\ |u_p|^2_{H^1(\Omega)} .$$

□

4 An Additive Schwarz Method using brick-shaped elements in three dimensions.

Theorem 1 can be extended to dimension three and brick-shaped elements and, by induction, to an arbitrary dimension. We define in this case

$$Q_p = span\{x^i y^j z^k \ : 0 \leq i, j, k \leq p\}$$

and we assume that the region $\Omega \subset R^3$ is the union of non-overlapping brick-shaped elements Ω_i of side H. If N is the number of interior nodes, we represent V^p as

$$V^p = V^p_0 + V^p_1 + \cdots + V^p_N ,$$

where $V_0^p = V^1$ and $V_i^p = V^p \cap H_0^1(\Omega_i')$. Now Ω_i' is the open cube of side $2H$ centered at the i-th interior node. We now prove the main theorem in dimension three.

Using the same notation as in the two dimensional case, we use Lions' lemma to bound the spectrum from below. The upper bound is obtained as before. The partition of the finite element function u_p, required by Lions' lemma, is constructed using again the results of Strang [**?**] to find $u_{p,0}$ and a piecewise linear partition of unity $\{\theta_i\}$. On the reference cube $[-1,1]^3$, θ_i can have one of the eight forms

$$\theta_i = \frac{1}{8}(x \pm 1)(y \pm 1)(z \pm 1)\,.$$

Recalling that $w_p = u_p - u_{p,0}$, we define $u_{p,i} = I_p(\theta_i w_p)$ as the polynomial in Q_p interpolating $\theta_i w_p$ at the $(p+1)^3$ points (x_l, x_m, x_n) , where the x_n's are the zeros of the integrals of the Legendre polynomials $\mathcal{L}_{p+1}$ defined in (6). Working again with the quotient spaces $\hat{Q}_p = Q_p/Q_0$, we can establish

Lemma 3 *The interpolation operator* $I_p : \hat{Q}_{p+1}([-1,1]^3) \to \hat{Q}_p([-1,1]^3)$ *is uniformly bounded in the* $|\cdot|_{H^1}$ *norm, i.e.*

$$|I_p(f)|_{H^1} \leq const.|f|_{H^1}\,, \qquad \forall f \in \hat{Q}_{p+1}([-1,1]^3)\,.$$

Proof. Since

$$\frac{|I_p(f)|^2_{H^1}}{|f|^2_{H^1}} \leq \frac{\|\frac{\partial}{\partial x} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial x}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial y} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial y}\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial z} I_p(f)\|^2_{L^2}}{\|\frac{\partial f}{\partial z}\|^2_{L^2}}\,, \tag{23}$$

we need only to consider one of the three terms, for example the x-term. As before, if in one term the denominator is zero, we can prove that (23) is still valid after dropping that term. We form a basis using the polynomials

$$\phi_{i,j,l}(x,y,z) = \frac{\mathcal{L}_i(x)}{\|L_{i-1}\|_{L^2}} \frac{\mathcal{L}_j(y)}{\|\mathcal{L}_j\|_{L^2}} \frac{\mathcal{L}_l(z)}{\|\mathcal{L}_l\|_{L^2}}\,, \qquad 1 \leq i,j,l \leq p+1\,.$$

and the same modification with the constant term $\frac{1}{\sqrt{2}}$ when one index is zero; cf.(8). We do not consider polynomials that do not depend on x, because they do not contribute to the x-term we are considering. We now want to order these remaining basis functions in such a way that we can use the results obtained in the two dimensional case. For every fixed l, we have a two dimensional subspace. We order the basis of this subspace as in the two dimensional case. We do this for $l = 0, 1, \cdots, p+1$. Relabeling the $\phi_{i,j,l}$'s as a one dimensional array, we have

$$\|\frac{\partial f}{\partial x}\|_{L^2}^2 = \alpha^T C_x \alpha \,, \qquad \|\frac{\partial}{\partial x} I_p(f)\|_{L^2}^2 = \alpha^T D C_x D \alpha \,.$$

We are interested in an upper bound for the eigenvalues of the generalized eigenvalue problem

$$D C_x D \alpha = \lambda C_x \alpha \,. \tag{24}$$

Here the stiffness matrix C_x, of order $p' = (p+2)^2(p+1)$, has the structure

$$C_x = \begin{pmatrix} S_x & c_0 S_x & b_0 & & & & \\ c_0 S_x & S_x & c_1 S_x & b_1 S_x & & & \\ b_0 S_x & c_1 S_x & S_x & 0 & b_2 S_x & & \\ & b_1 S_x & 0 & S_x & 0 & \ddots & \\ & & b_2 S_x & 0 & S_x & \ddots & b_{p-1} S_x \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & b_{p-1} S_x & 0 & S_x \end{pmatrix}$$

and the interpolation matrix D, of the same size, has the structure

$$D = \begin{pmatrix} B & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \ddots & & \\ & & & & B & \\ & & & & & 0 \end{pmatrix} .$$

The matrices S_x and B have been defined in Section 3. Clearly, $D C_x D$ is the matrix

$$\begin{pmatrix} \tilde{S}_x & c_0 \tilde{S}_x & b_0 \tilde{S}_x & & & & \\ c_0 \tilde{S}_x & \tilde{S}_x & c_1 \tilde{S}_x & b_1 \tilde{S}_x & & & \\ b_0 \tilde{S}_x & c_1 \tilde{S}_x & \tilde{S}_x & 0 & b_2 \tilde{S}_x & & \\ & b_1 \tilde{S}_x & 0 & \tilde{S}_x & 0 & \ddots & \\ & & b_2 \tilde{S}_x & 0 & \tilde{S}_x & \ddots & b_{p-1} \tilde{S}_x \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & b_{p-1} \tilde{S}_x & 0 & \tilde{S}_x \end{pmatrix} ,$$

where $\tilde{S}_x = BS_xB$. Therefore equation (24) gives us the block equations

$$\tilde{S}_x(\alpha_1 + c_0\alpha_2 + b_0\alpha_3) = \lambda S_x(\alpha_1 + c_0\alpha_2 + b_0\alpha_3)\,,$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{S}_x(b_{i-2}\alpha_{i-2} + \alpha_i + b_i\alpha_{i+2}) = \lambda S_x(b_{i-2}\alpha_{i-2} + \alpha_i + b_i\alpha_{i+2})\,,$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{S}_x(b_{p-1}\alpha_{p-1} + \alpha_{p+1}) = \lambda S_x(b_{p-1}\alpha_{p-1} + \alpha_{p+1})\,.$$

These are all generalized eigenvalue problems of the form

$$\tilde{S}_x v = \lambda S_x v\,, \tag{25}$$

where v is a linear combination of some α_i . But this is the same generalized eigenvalue problem considered in the two dimensional case; see eq. (9). We can then apply the two dimensional result and conclude that the eigenvalues λ are bounded by a constant independent of p. Reasoning in the same way for the terms in y and z, we complete the proof of lemma 3.

□

In order to complete the proof of theorem 2, we just repeat some of the arguments given in the two dimensional case. We first apply lemma 3 to bound the H^1 norm of each component $u_{p,i} = I_p(\theta_i w_p)$ over a single element Ω_k. We then sum over i, recalling that at most 8 components $u_{p,i}$ are nonzero for any element Ω_k and finally we sum over all elements. We conclude the proof using equations (4) and (5).

□

Remark. The result can be extended to any dimension by induction. The only nontrivial part is the proof of the lemma about the interpolation operator. The induction step from dimension n to $n+1$ is essentially analogous to the arguments in the proof of Lemma 3. We consider one term at a time and order the basis in the following way: first fix the $(n+1)$-th index to be equal to 1 and order the resulting subspace in the same way as in the case of n variables; then fix the $(n+1)$-th index to be equal to 2 and repeat the process, until the $(n+1)$-th index is equal to $p+1$. With this choice, the stiffness and projection matrices have a block structure that allow us to reduce the $(n+1)$ dimensional generalized eigenvalue problem to one of dimension n.

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