Differential Properties of Eigenvalues

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Abstract

We define and study a directional derivative for two functions of the spectrum of an analytic matrix valued function. These are the maximum real part and the maximum modulus of the spectrum. Results are first obtained for the roots of polynomials with analytic coefficients by way of Puiseux-Newton series. In this regard, the primary analytic tool is the so called Puiseux-Newton diagram. These results are then translated into the context of matrices. Precise results are obtained when the eigenvalues that achieve the maximum value for the function under consideration are all either nondefective or nonderogatory. In the defective derogatory cases a general lower bound for the directional derivative is given which, in particular, describes those directions in which the directional derivative attains an infinite value.

1 Introduction

In this study we consider the directional differentiability of two related functions of the spectrum of an analytic matrix function. Specifically, given an analytic (holomorphic) matrix valued mapping $A(z)$ from $\mathbb{C}^\nu$ to $\mathbb{C}^{n\times n}$, we are interested in the directional differentiability of the functions

$$\alpha(z) = \max\{\mathrm{Re}\,\lambda : \lambda \in \Sigma(z)\}, \tag{1}$$

and

$$\rho(z) = \max\{|\lambda| : \lambda \in \Sigma(z)\}, \tag{2}$$

where $\Sigma(z)$ is the spectrum of $A(z)$. The functions α and ρ are called the spectral abscissa and spectral radius maps, respectively, for the analytic matrix function $A(z)$. They are key functions in the analysis of various stability properties associated with

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the mapping $A(z)$. Both functions are, in general, nonlipschitzian. Thus nonstandard techniques are required in the analysis of their variational properties. In order to give some insight into the problem, let $\gamma: \mathbb{C} \mapsto \mathbb{C}^\nu$ be analytic and consider the eigenvalues of $\hat{A}(\zeta) = A(\gamma(\zeta))$ near zero. It is well known that these eigenvalues are given by so called Puiseux series, that is, series in fractional powers of ζ with the smallest such power being greater than or equal to $\frac{1}{n}$. Our results are a consequence of certain properties of these series. The derivation of these properties depends on a classical computational scheme due to Newton [11] known as the Puiseux-Newton diagram.

Our method of analysis requires the introduction of a notion of directional differentiability that depends on analyticity in the following way: For $w: \mathbb{C}^\nu \mapsto \mathbf{R}$, define $w^h(z;\cdot): \mathbb{C}^\nu \mapsto \mathbf{R} \cup \{\pm\infty\}$ by

$$w^h(z;d) = \inf_{\gamma \in \Gamma(z,d)} \liminf_{\epsilon \downarrow 0} \frac{w(\gamma(\epsilon)) - w(z)}{\epsilon}, \tag{3}$$

where

$$\Gamma(z,d) = \{\gamma: \mathbb{C} \mapsto \mathbb{C}^\nu | \gamma \text{ is analytic }, \ \gamma(0) = z, \text{ and } \gamma'(0) = d\}. \tag{4}$$

The superscript h in (3) is used to emphasize that $w^h(z;d)$ depends only on the holomorphic curves in $\mathbb{C}^\nu$ that pass through z. This notion of directional differentiability shares many of the properties of other such notions that have recently been developed in the literature on nonsmooth analysis [5,9,10,14]. Understanding the relationship between (3) and these other notions is important for the development of a calculus. However, we defer the discussion of these issues to a future work. At present we concentrate on the evaluation of (3) for the functions α and ρ.

Let $\mathcal{H}_G[\lambda]$ be the set of monic polynomials in λ whose coefficients are analytic mappings from $G \subset \mathbb{C}^\nu$ to $\mathbb{C}$. If $G = \mathbb{C}^\nu$, we simply write $\mathcal{H}[\lambda]$. Observe that $\Sigma(z)$ is by definition the set of roots of a polynomial in $\mathcal{H}[\lambda]$. For this reason we begin in Section 2 by studying the properties of α and ρ when it is assumed that

$$\Sigma(z) = \{\lambda \in \mathbb{C} : P(z,\lambda) = 0\}, \tag{5}$$

where $P \in \mathcal{H}[\lambda]$. This approach has its limitations since in general we only obtain a lower bound on the value of the directional derivative (3). Nonetheless, this lower bound can be shown to be sharp in certain nondegenerate situations. Let $\mathcal{A}_0(z)$ and $\mathcal{R}_0(z)$ denote the elements of $\Sigma(z)$ that achieve the maximum value in (1) and (2), respectively. Then, in the context of matrices, the polynomial results are most meaningful when all of the eigenvalues in either $\mathcal{A}_0(z)$ or $\mathcal{R}_0(z)$ are nonderogatory, that is, their multiplicity in both the characteristic and minimal polynomial for $A(z)$ coincide. For a nonderogatory eigenvalue, the characteristic polynomial contains all of the essential information about the eigenvalue. This is the case in which our lower bound is most likely to be sharp. The opposite extreme is when all of the eigenvalues in either $\mathcal{A}_0(z)$ or $\mathcal{R}_0(z)$ are semisimple (nondefective), that is, when the multiplicity of these eigenvalues in the characteristic polynomial is greater than or equal to one and yet their multiplicity in the minimal polynomial is precisely one. In this case, the characteristic polynomial contains the least amount of information about the eigenvalues. It is in this latter case that the lower bounds that we obtain for the directional derivatives are most likely not sharp. On the other hand, one can precisely evaluate the ordinary directional derivative in the semisimple case. This is done in [12]. Thus the cases for which precise results are unknown are those where at least one of the eigenvalues in either $\mathcal{A}_0(z)$ or $\mathcal{R}_0(z)$ is both derogatory and defective. Nonetheless, the lower bounds that we establish do provide a great deal of information about when these directional derivatives attain the value $+\infty$.

In the discussion that follows certain statements are sensitive to the domain of the variable being discussed. Thus, in order to avoid confusion, we obey the following convention concerning the labeling of the variables ζ, ϵ, and z: ζ will always represent a complex scalar, ϵ a real scalar, and z a vector in $\mathbb{C}^\nu$.

2 Roots of Polynomials with Complex Coefficients

Let $P \in \mathcal{H}[\lambda]$. We begin by studying the differential properties of the function g at $z^0 \in \mathbb{C}^\nu$. From [2, pp. 376-381] there exists a neighborhood G of z^0 in $\mathbb{C}^\nu$ on which P has the unique representation

$$P = \prod_{\lambda_k \in \Sigma(z^0)} \mu_k, \tag{6}$$

where $\mu_k \in \mathcal{H}_G[\lambda]$ for each $\lambda_k \in \Sigma(z^0)$. The polynomials μ_k can be taken to have the form

$$\mu_k(\lambda, z) = (\lambda - \lambda_k)^{t_k} + c_{k1}(z)(\lambda - \lambda_k)^{t_k - 1} + \cdots + c_{kt_k}(z), \tag{7}$$

where

$$c_{kj}(z^0) = 0 \text{ for } j = 1, \ldots, t_k$$

and t_k is the multiplicity of the root λ_k. We begin with the case in which P has only a single root λ_0, with multiplicity t_0, i.e.

$$P(\lambda, z) = (\lambda - \lambda_0)^{t_0} + c_1(z)(\lambda - \lambda_0)^{t_0 - 1} + \cdots + c_{t_0}(z), \tag{8}$$

where

$$c_j(z^0) = 0 \text{ for } j = 1, \ldots, t_0 .$$

We will return to the general case at the end of this section.

Our first objective is to understand the behavior of g along analytic curves in $\mathbb{C}^\nu$ passing through z^0. Let $\gamma\colon \mathbb{C} \mapsto \mathbb{C}^\nu$ be an analytic curve satisfying $\gamma(0) = z^0$. Compose each c_j with γ to obtain

$$P(\lambda, \gamma(\zeta)) = (\lambda - \lambda_0)^{t_0} + \beta_1(\zeta)(\lambda - \lambda_0)^{t_0 - 1} + \cdots + \beta_{t_0}(\zeta) = 0, \tag{9}$$

a polynomial equation in λ with analytic coefficients $\beta_j(\zeta) = c_j(\gamma(\zeta))$, satisfying

$$\beta_j(0) = 0, j = 1, \ldots, t_0.$$

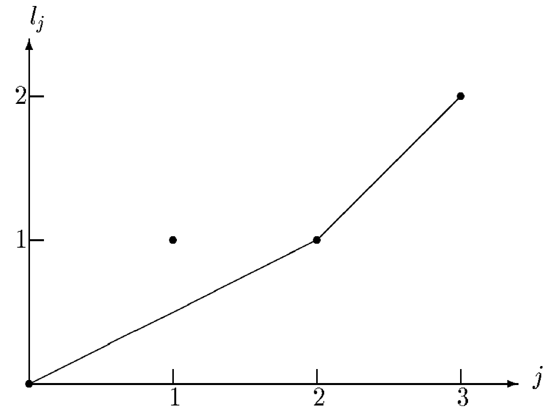

Figure 1

We may write

$$\beta_j(\zeta) = \beta_j^{(1)}\zeta + \beta_j^{(2)}\zeta^2 + \cdots.$$

where, for example,

$$\beta_j^{(1)} = c'_j(z^0)\gamma'(0). \tag{10}$$

In the discussion which follows we restrict ζ to a nontrivial real interval $[0, \epsilon_0]$ and write ϵ in place of ζ to emphasize this restriction.

As has already been noted, it is well known (e.g. [2], [7]) that the roots of (9) are described by series in fractional powers of ϵ. These series are commonly called Puiseux series, since it was Puiseux [13] who established their convergence; however they were derived formally by Newton two centuries earlier (also see [15, Chapter 1, Section 2] and [11, page 88] for examples and applications). We obtain the results we need by making use of a diagram devised by Newton for the purpose of calculating coefficients of Puiseux series.

Let $\hat{\beta}_j = \beta_j^{(\ell_j)}$ be the first nonzero value in the sequence $\{\beta_j^{(1)}, \beta_j^{(2)}, \ldots\}$. By definition, $\ell_j \geq 1, j = 1, \ldots, t_0$. If $\beta_j(\epsilon)$ is identically zero, take $\ell_j = \infty$; also, since the coefficient of $(\lambda - \lambda_0)^{t_0}$ in $P(\lambda, \epsilon)$ is one, take $\ell_0 = 0$, $\hat{\beta}_0 = 1$. Now plot the values ℓ_j versus j, and consider the lower boundary of the convex hull of the points plotted. Let s_j be the slope of the line on $[j, j+1]$ forming part of this boundary, $j = 0, \ldots, t_0 - 1$. Clearly $1/t_0 \leq s_0 \leq s_1 \leq \cdots \leq s_{t_0-1}$. Figure 1 shows the diagram for the following example (taken from [2]):

$$t_0 = 3; \lambda_0 = 0;\ \beta_1(\epsilon) = \epsilon;\ \beta_2(\epsilon) = -\epsilon - \epsilon^2;\ \beta_3(\epsilon) = \epsilon^2 + 2\epsilon^3.$$

We have $\ell_0 = 0,\ \ell_1 = 1,\ \ell_2 = 1,\ \ell_3 = 2$, and so $s_0 = s_1 = \frac{1}{2}, s_2 = 1$.

Now consider the following "Ansatz" argument. Suppose a root of (9) is to have the form

$$\lambda(\epsilon) - \lambda_0 = a\epsilon^p + \cdots \tag{11}$$

where a is nonzero and p is the smallest power of ϵ in the expansion for this root. Substituting (11) into (9), we need

$$\begin{aligned}(a^{t_0}\epsilon^{t_0 p} + \cdots) \quad &+ \quad (\hat{\beta}_1\epsilon^{\ell_1} + \cdots)(a^{t_0-1}\epsilon^{(t_0-1)p} + \cdots) + \cdots \\ &+ \quad (\hat{\beta}_{t_0-1}\epsilon^{\ell_{t_0-1}} + \cdots)(a\epsilon^p + \cdots) + (\hat{\beta}_{t_0}\epsilon^{\ell_{t_0}} + \cdots) = 0.\end{aligned}$$

The terms involving the smallest powers of ϵ are among the terms

$$a^{t_0}\epsilon^{t_0 p}, \; \hat{\beta}_1 a^{t_0-1}\epsilon^{\ell_1+(t_0-1)p}, \ldots, \hat{\beta}_{t_0-1} a\epsilon^{\ell_{t_0-1}+p}, \; \hat{\beta}_{t_0}\epsilon^{\ell_{t_0}}. \tag{12}$$

For cancellation to take place, at least two terms with the same smallest power of ϵ must appear. Equivalently, p must equal one or more of the slopes $s_0, \ldots, s_{t_0-1}$ defined by the Puiseux-Newton diagram. The following discussion will apply to a particular choice of such p. Define b and g by $p = s_b = \ldots = s_{b+g-1}$, so that the line in the diagram with slope p passes from the point (b, ℓ_b) to the point $(b+g, \ell_{b+g})$. Cancellation of the coefficients of the terms with the smallest powers of ϵ in (12) requires a to be the root of a polynomial equation with degree g, with leading term $\hat{\beta}_b a^g$ and constant term $\hat{\beta}_{b+g}$, and with an additional intermediate nonzero term for each point (j, ℓ_j) lying on the line in the diagram with slope p, where $b < j < b+g$. Now let $p = q/f$, where q, f are relatively prime integers. By definition, p is an integral multiple of $1/g$, so g is an integral multiple of f, say $g = mf$. It is then clear from the diagram that of the $g-1$ abscissa values j between b and $b+g$, only every fth value is a candidate for the intersection of the line with a point with integer coordinates. Consequently the polynomial of degree g in a reduces to a polynomial of degree m in a^f, which we may denote by $Q(r)$. The conclusion is that the given value of p is associated with g roots with an expansion of the form (11), with a taking the values

$$r_h^{1/f}\omega^j, \quad h = 1, \ldots, m, \; j = 1, \ldots f \tag{13}$$

where the r_h are the m roots of $Q(r) = 0$, $r_h^{1/f}$ is the principal fth root of r_h and ω is the principal fth root of unity.

Completing the example given above, we see that the two values for p are $s_0 = s_1 = \frac{1}{2}$ and $s_2 = 1$. In the case $p = \frac{1}{2}$ we have $b = 0, g = 2, f = 2, m = 1$, with $Q(r) = r - 1$, so the possible values for a are ± 1, giving the Puiseux series

$$\lambda(\epsilon) - \lambda_0 = \pm\epsilon^{\frac{1}{2}} + \cdots.$$

In the case $p = 1$ we have $b = 2, g = 1, f = 1, m = 1$, with $Q(r) = r - 1$, so the only possible value for a is 1, giving the Puiseux series

$$\lambda(\epsilon) - \lambda_0 = \epsilon + \cdots.$$

The subsequent terms in the series may also be calculated by repeating the process.

The following result is the key to our analysis. It is a generalization of [3, Theorem 1]; see also [8]. The lemma provides information about the coefficients of (9) when it is assumed that roots of (9) lie in the half plane $\mathrm{Re}\,\overline{y}_0(z-\lambda_0) \le 0$ infinitely often as $\epsilon \downarrow 0$.

Lemma 1 *Consider the polynomial equation (9), with roots given by one or more Puiseux series of the form (11). Let* $y_0 \in \mathbb{C}$ *and suppose that there exists* $\epsilon_0 > 0$ *such that all the roots* $\lambda(\epsilon)$ *of (9) satisfy*

$$\mathrm{Re}\,\overline{y}_0(\lambda(\epsilon)-\lambda_0) \le \delta\epsilon + o(\epsilon) \tag{14}$$

for some sequence $\{\epsilon^k\}$ *with* $\epsilon^k \downarrow 0$. *Then*

$$\mathrm{Re}\,\overline{y}_0\beta_1^{(1)} \ge -t_0\delta, \tag{15}$$

$$\mathrm{Re}\,\overline{y}_0^2\beta_2^{(1)} \ge 0,\ \mathrm{Im}\,\overline{y}_0^2\beta_2^{(1)} = 0, \tag{16}$$

$$\beta_j^{(1)} = 0, \quad j = 3,\ldots,t_0. \tag{17}$$

Here (16) is understood to be vacuous if $t_0 = 1$.

Proof The coefficient $\beta_1(\epsilon)$ is the sum of the differences $\lambda_0 - \lambda(\epsilon)$ over the roots $\lambda(\epsilon)$ of (9); thus (15) follows from (14), letting $\epsilon \to 0$. The other results follow from the Puiseux-Newton diagram as follows. Consider the Puiseux series corresponding to $p = s_0$, the smallest possible value. In order for (14) to hold either

(i) $p \ge 1$ (e.g., if $f = 1$), or

(ii) $p = \frac{1}{2}$, $f = 2$, and $\mathrm{Re}\,\overline{y}_0 r_h^{\frac{1}{2}} = 0$ for $h = 1,\ldots,m$, where the r_h are the m roots of $Q(r)$, with $Q(r)$ taking the form

$$Q(r) = r^m + \beta_2^{(1)} r^{m-1} + \cdots + \beta_{2m}^{(m)}. \tag{18}$$

No other cases having $p < 1$ are possible due to the splitting of the roots as described in (13). In both cases (i) and (ii), $p \ge \frac{1}{2}$, so $\beta_j^{(1)} = 0$ for $j = 3,\ldots,t_0$ from the Puiseux-Newton diagram. In the case $p \ge 1$, we also have $\beta_2^{(1)} = 0$. In the case $f = 2$, observe that the condition $\mathrm{Re}\,\overline{y}_0 r_h^{\frac{1}{2}} = 0$ is equivalent to the two conditions $\mathrm{Re}\,\overline{y}_0^2 r_h \le 0$ and $\mathrm{Im}\,\overline{y}_0^2 r_h = 0$. Now, since $-\beta_2^{(1)}$ is the sum of the roots of $Q(r)$, (16) follows. □

We apply this result to the evaluation of $\alpha^h(z^0; d)$ where α is given by (1) with $\Sigma(z)$ and P defined in (5) and (8), respectively. First observe that we can replace the limit infimum in (3) by limit since the perturbed roots of the polynomial (9) are given by Puiseux series of the form (11) for some non-negative rational number p. Therefore, this limit always exists and can only take the value $+\infty$ if it is not finite. Consequently, $\alpha^h(z^0;\cdot)\colon \mathbb{C}^\nu \mapsto \mathbf{R} \cup \{+\infty\}$ and is given by

$$\alpha^h(z;d) = \inf_{\gamma\in\Gamma(z,d)} \lim_{\epsilon\downarrow 0} \frac{\alpha(\gamma(\epsilon)) - \alpha(z)}{\epsilon}. \tag{19}$$

We now state the main result of this section.

Theorem 2 *Let P be given by (8) and choose $d \in \mathbb{C}^\nu$. If any one of the conditions*

$$\text{Re } c_2'(z^0)d \geq 0, \ \text{Im } c_2'(z^0)d = 0, \tag{20}$$

$$c_j'(z^0)d = 0, \quad j = 3, \ldots, t_0 \tag{21}$$

is violated, then

$$\alpha^h(z^0; d) = +\infty;$$

otherwise

$$\alpha^h(z^0; d) \geq -\frac{1}{t_0} \text{Re } c_1'(z^0)d \ . \tag{22}$$

Moreover, if the rank of $c'(z^0)$ is t_0, where $c \colon \mathbb{C}^\nu \mapsto \mathbb{C}^{t_0}$ is given by

$$c(z) = \begin{bmatrix} c_1(z) \\ \vdots \\ c_{t_0}(z) \end{bmatrix},$$

then equality holds in (22) whenever (20) and (21) are satisfied.

Proof Suppose $+\infty > \delta > \alpha^h(z^0; d)$. Then there is a $\gamma \in \Gamma(z^0, d)$ such that

$$\lim_{\epsilon \downarrow 0} \frac{\alpha(\gamma(\epsilon)) - \alpha(z^0)}{\epsilon} < \delta, \tag{23}$$

or equivalently,

$$\alpha(\gamma(\epsilon)) - \alpha(z^0) < \delta\epsilon \text{ for } \epsilon \in [0, \epsilon_0],$$

for some $\epsilon_0 > 0$. Let $\beta_j^{(1)} = c_j'(z^0)d$ for $j = 1, 2, \cdots, t_0$ as in (10). By invoking Lemma 1 with $y_0 = 1$, we see that (20) and (21) must be satisfied. Thus, if any one of these conditions is violated, we must have $\alpha^h(z^0; d) = +\infty$. By letting $\delta \downarrow \alpha^h(z^0; d)$, we also obtain from Lemma 1 the inequality

$$\alpha^h(z^0; d) \geq -\frac{1}{t_0} \text{Re } \beta_1^{(1)}.$$

Let us now suppose that the rank of $c'(z^0)$ is t_0. We need to establish equality in (22). Clearly, we need only consider the case in which (20) and (21) hold for $y_0 = 1$. In this case, equality follows if we can exhibit a curve $\gamma \in \Gamma(z^0, d)$ such that

$$\lim_{\epsilon \downarrow 0} \frac{\alpha(\gamma(\epsilon)) - \alpha(z^0)}{\epsilon} = -\frac{1}{t_0} \text{Re } \beta_1^{(1)}. \tag{24}$$

Consider the coefficients of the powers of $(\lambda - \lambda_0)$ in the polynomial

$$((\lambda - \lambda_0) + \frac{\beta_1^{(1)}}{t_0}\zeta)^{t_0 - 2}(\lambda - \lambda_0 + i(\beta_2^{(1)}\zeta)^{\frac{1}{2}} + \frac{\beta_1^{(1)}}{t_0}\zeta)(\lambda - \lambda_0 - i(\beta_2^{(1)}\zeta)^{\frac{1}{2}} + \frac{\beta_1^{(1)}}{t_0}\zeta)$$

$$
\begin{aligned}
&= (\lambda-\lambda_0)^{t_0} + \beta_1^{(1)}\zeta(\lambda-\lambda_0)^{t_0-1} + (\beta_2^{(1)}\zeta + O(\zeta^2))(\lambda-\lambda_0)^{t_0-2} + \dots, \quad (25)\\
&\equiv (\lambda-\lambda_0)^{t_0} + v_1(\zeta)(\lambda-\lambda_0)^{t_0-1} + v_2(\zeta)(\lambda-\lambda_0)^{t_0-2} + \dots,
\end{aligned}
$$

where $i = \sqrt{-1}$. Note that these coefficients satisfy (16) and (17) with $y_0 = 1$. Also note that if γ can be chosen from (4) so that (9) has these coefficients, then (24) is satisfied for this curve and the proof is complete. We now show that this can indeed be done.

Define $F\colon \mathbb{C}^{\nu+1} \mapsto \mathbb{C}^{t_0}$ by

$$F(z,\zeta) = c(z) - v(\zeta),$$

where $v\colon \mathbb{C} \mapsto \mathbb{C}^{t_0}$ is the curve whose component functions are the coefficients of the powers of $(\lambda-\lambda_0)$ in the polynomial (25):

$$v(\zeta) = [v_1(\zeta), v_2(\zeta), \dots, v_t(\zeta)]^T = \zeta c'(z^0)\gamma'(0) + O(\zeta^2), \tag{26}$$

where the second equality follows from the definition of v. Let $I \subset \{1,\dots,\nu\}$ be such that the matrix $c_I'(z^0)$ is nonsingular and set $J = \{1,\dots,\nu\}\setminus I$. By the implicit function theorem [6], there is a neighborhood $\bar{G} \subset \mathbb{C}^{\nu-t_0+1}$ of $((z^0)_J, 0)$ and an analytic mapping $\hat{\gamma}\colon \bar{G} \mapsto \mathbb{C}^{t_0}$ such that

$$F(\hat{\gamma}(z_J,\zeta), z_J, \zeta) = 0,$$

for all $(z_J,\zeta) \in \bar{G}$ with

$$\hat{\gamma}((z^0)_J, 0) = (z^0)_I.$$

Furthermore,

$$\hat{\gamma}'((z^0)_J, 0) = -c_I'(z^0)^{-1}[c_J'(z^0), -v'(0)] \tag{27}$$

Define $\gamma\colon \mathbb{C} \mapsto \mathbb{C}^\nu$ by

$$\gamma_J(\zeta) = (z^0)_J + \zeta d_J, \tag{28}$$

and

$$\gamma_I(\zeta) = \hat{\gamma}(\gamma_J(\zeta), \zeta). \tag{29}$$

Then for all ζ sufficiently small $c(\gamma(\zeta)) = v(\zeta)$. Consequently, with this choice of γ, the polynomial (9) is precisely the polynomial (25). Moreover, from (26)–(29) we have

$$\gamma_J'(0) = d_J$$

and

$$\gamma_I'(0) = \hat{\gamma}'(\gamma_J(0), 0)\begin{bmatrix} d_J \\ 1 \end{bmatrix} = d_I,$$

so that $\gamma'(0) = d$. Thus, $\gamma \in \Gamma(z^0, d)$. This concludes the proof of the theorem. □

Let us now return to the general case and recall the factorization (6). Our results in this case follow directly from the special case (8).

Theorem 3 *Let α be given by (1) with Σ and P as defined in (5) and (6), respectively. Choose $d \in \mathbb{C}^\nu$. If for some $\lambda_k \in \mathcal{A}_0(z^0) = \{\lambda_k \in \Sigma(z^0) : \mathrm{Re}\,\lambda_k = \alpha(z^0)\}$ any one of the conditions*

$$\mathrm{Re}\; c'_{k2}(z^0)d \geq 0,\;\; \mathrm{Im}\; c'_{k2}(z^0)d = 0, \tag{30}$$

$$c'_{kj}(z^0)d = 0,\;\; j = 3, \ldots, t_k, \tag{31}$$

is violated, then

$$\alpha^h(z^0; d) = +\infty\;; \tag{32}$$

otherwise,

$$\alpha^h(z^0; d) \geq \max\{-\frac{\mathrm{Re}\; c'_{k1}(z^0)d}{t_k} : \lambda_k \in \mathcal{A}_0(z^0)\}. \tag{33}$$

Moreover, if the vectors $\{c'_{kj}(z^0) : \lambda_k \in \mathcal{A}_1(z^0, d),\; j = 1, \ldots, t_k\}$ are linearly independent, where

$$\mathcal{A}_1(z^0, d) = \{\lambda_k \in \mathcal{A}_0(z^0) : \frac{\mathrm{Re}\; c'_{k1}(z^0)d}{t_k} = \min\{\frac{\mathrm{Re}\; c'_{\ell 1}(z^0)d}{t_\ell} : \lambda_\ell \in \mathcal{A}_0(z^0)\}\},$$

then equality holds in (33).

Proof The proof is almost identical to that of Theorem 2. The primary difference is that now all of the roots in $\mathcal{A}_0(z^0)$ contribute to the value of $\alpha^h(z^0; d)$. In order to see this observe that the inequality (23) implies that

$$\mathrm{Re}\;(\lambda(\gamma(\epsilon)) - \lambda_k) < \delta\epsilon \text{ for } \epsilon \in [0, \epsilon_0] \text{ and } \lambda_k \in \mathcal{A}_0(z^0),$$

for some $\epsilon_0 > 0$. Consequently, (33) again follows from Theorem 1.

In order to establish equality in (33), it is again sufficient to exhibit the existence of a curve $\gamma \in \Gamma(z^0, d)$ such that

$$\lim_{\epsilon \downarrow 0} \frac{\alpha(\gamma(\epsilon)) - \alpha(z)}{\epsilon} = \max\{-\frac{\mathrm{Re}\; c'_{k1}(z^0)d}{t_k} : \lambda_k \in \mathcal{A}_0(z^0)\}. \tag{34}$$

We generate such a curve precisely as in the proof of Theorem 2 except now we must choose the curve from $\Gamma(z^0, d)$ so as to match the coefficients in (25) for each $\lambda_k \in \mathcal{A}_1(z^0, d)$. Just as before, it is the linear independence of the gradients $\{c'_{k1}(z^0) : \lambda_k \in \mathcal{A}_1(z^0, d)\}$ which guarantees that this can be done via the implicit function theorem. Moreover, it is clear that we need only match the coefficients for $\lambda_k \in \mathcal{A}_1(z^0, d)$ since these are the dominant first order terms. □

3 Eigenvalues of Complex Matrices

Let $\mathcal{H}[\mathbb{C}^\nu, \mathbb{C}^{n\times n}]$ denote the set of mappings from $\mathbb{C}^\nu$ to $\mathbb{C}^{n\times n}$ each of whose components is an analytic map from $\mathbb{C}^\nu$ to $\mathbb{C}$. Let $A \in \mathcal{H}[\mathbb{C}^\nu, \mathbb{C}^{n\times n}]$. In this section we again study the differential properties of the function α given in (1) with the multifunction $\Sigma(z)$ defined to be the spectrum of $A(z)$, that is, $\Sigma(z)$ is given by (5) where

$$P(\lambda, z) = \det[\lambda I - A(z)] \tag{35}$$

is the characteristic polynomial for $A(z)$. One can now apply the results of the previous section to obtain differential information about α since $P \in \mathcal{H}[\lambda]$. However, by itself, this result is not completely satisfactory since it does not describe the relationship between A and the terms $c'_{kj}(z^0)d$ appearing in (33). In this section we describe this relationship, making use of results from [3] which in turn depend on work of Arnold [1]. Since our description depends on the Jordan decomposition of $A^{(0)} = A(z^0)$, we need to introduce the notation necessary for this discussion.

Suppose $A^{(0)}$ is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_\eta$, having (algebraic) multiplicities $t_1, \ldots, t_\eta$, respectively. Let the Jordan form of $A^{(0)}$ be given by

$$A^{(0)} = SJS^{-1}$$

where

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_\eta \end{bmatrix},$$

$$J_k = \begin{bmatrix} J_{k1} & & \\ & \ddots & \\ & & J_{km_k} \end{bmatrix},$$

and the Jordan block

$$J_{kl} = \begin{bmatrix} \lambda_k & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot \ \ 1 \\ & & & \lambda_k \end{bmatrix}$$

has dimension n_{kl}. We have

$$n_{k1} + \cdots + n_{km_k} = t_k, \quad k = 1, \ldots, \eta.$$

If $m_k = 1$, λ_k is said to be a nonderogatory eigenvalue, while if $m_k = t_k$, i.e. $n_{k1} = \cdots = n_{km_k} = 1$, λ_k is said to be semisimple (nondefective).

Definition 4 *Define the* jth generalized trace *of a square matrix A, denoted by*

$$\mathrm{tr}^{(j)} A,$$

as the sum of the elements on the diagonal of A which is $j-1$ positions below the main diagonal. Thus one obtains the ordinary trace in the case $j = 1$ and the bottom left element of the matrix in the case that j is the dimension of the matrix. If j exceeds the dimension of A, take $\mathrm{tr}^{(j)} A = 0$.

Theorem 5 *Let $A \in \mathcal{H}[\mathbb{C}^\nu, \mathbb{C}^{n\times n}]$ and choose $z^0 \in \mathbb{C}^\nu$. Suppose that $A^{(0)} = A(z^0)$ has Jordan form as described above. Define*

$$A_q^{(1)} = \frac{\partial A}{\partial z_q}(z^0)\ ,\ \textit{for } q = 1,\ldots,\nu.$$

For each $q = 1,\ldots,\nu$ partition $S^{-1}A_q^{(1)}S$ conformally with the partition of J and denote its diagonal block corresponding to J_k by B_{qk}, $k = 1,\ldots,\eta$, with each B_{qk} having diagonal blocks B_{qkl} corresponding to J_{kl}, $l = 1,\ldots,m_k$. Then, for $k = 1,\ldots,\eta$,

$$c'_{kj}(z^0) = \begin{bmatrix} \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{1kl} \\ \vdots \\ \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{\nu kl} \end{bmatrix}\ ,\ \textit{for } j = 1,\ldots,t_k, \tag{36}$$

where the functions c_{kj} are as given in (7) for the factorization (6) of $P(\lambda, z) = \det[\lambda I - A(z)]$ at $z = z^0$.

Proof Let $d \in \mathbb{C}^\nu$ and $\gamma \in \Gamma(z^0, d)$ and define $\hat{A} = A \circ \gamma$. Given $M \in \mathbb{C}^{n\times n}$ denote by M_{kl} that block of M which conforms to the block J_{kl} of J. By [3, Theorem 4] we have

$$\begin{aligned} c'_{kj}(z^0)d &= \sum_{l=1}^{m_k} \mathrm{tr}^{(j)}(S^{-1}\hat{A}'(0)S)_{kl} \\ &= \sum_{l=1}^{m_k} \mathrm{tr}^{(j)}(S^{-1}A'(0)dS)_{kl} \\ &= \sum_{l=1}^{m_k} \mathrm{tr}^{(j)}(S^{-1}(\sum_{q=1}^{\nu} d_q A_q^{(1)})S)_{kl} \\ &= \sum_{q=1}^{\nu} d_q \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{qkl}, \end{aligned}$$

for $j = 1,\ldots,t_k$. Since this holds for all $d \in \mathbb{C}^\nu$, the result follows. □

Thus, given the Jordan form of $A(z^0)$ together with $A'(z^0)$ it is possible to evaluate a lower bound for $\alpha^h(z^0; d)$ by combining Theorems 3 and 5. We do this in the following corollary.

Corollary 6 *Let the assumptions of Theorem 5 hold. If for some $\lambda_k \in \mathcal{A}_0(z^0)$ any one of the conditions*

$$\mathrm{Re} \sum_{q=1}^{\nu} d_q \sum_{l=1}^{m_k} \mathrm{tr}^{\,(2)} B_{qkl} \geq 0,\ \mathrm{Im} \sum_{q=1}^{\nu} d_q \sum_{l=1}^{m_k} \mathrm{tr}^{\,(2)} B_{qkl} = 0,$$

$$\sum_{q=1}^{\nu} d_q \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{qkl} = 0,\ j = 3, \ldots, t_k,$$

is violated, then

$$\alpha^h(z^0; d) = +\infty\ ; \tag{37}$$

otherwise,

$$\alpha^h(z^0; d) \geq \max\left\{ -\frac{\mathrm{Re}\ \sum_{q=1}^{\nu} d_q \sum_{l=1}^{m_k} \mathrm{tr}\ ^{(1)} B_{qkl}}{t_k} \,\middle|\, \lambda_k \in \mathcal{A}_0(z^0) \right\}. \tag{38}$$

Moreover, if the vectors

$$\begin{bmatrix} \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{1kl} \\ \vdots \\ \sum_{l=1}^{m_k} \mathrm{tr}^{(j)} B_{\nu kl} \end{bmatrix},\ \textit{for}\ \lambda_k \in \mathcal{A}_1(z^0, d)\ \textit{and}\ j = 1, \ldots, t_k, \tag{39}$$

are linearly independent, then equality holds in (38).

It should be observed that if any eigenvalue $\lambda_k \in \mathcal{A}_1(z^0)$ is derogatory, then the vectors in (39) cannot be linearly independent. In order to see this note that, for at least one j between 1 and t_k, j exceeds the dimension of all of the blocks J_{kl} making up J_k, and hence the corresponding vector is zero. Thus, if $\mathcal{A}_1(z^0, d)$ contains a derogatory eigenvalue, then the sufficiency condition of Theorem 3 is not satisfied. In this case, one should not expect to obtain equality in (33). Indeed, in the case where $\mathcal{A}_0(z^0)$ contains only semisimple eigenvalues, the resolvent theory for eigenvalue perturbations yields the following result [7], [12, Lemma 3.5].

Theorem 7 *If $\mathcal{A}_0(z^0)$ contains only semisimple eigenvalues of $A(z^0)$, then for every $d \in \mathbb{C}^\nu$ the ordinary directional derivative $\alpha'(z^0; d)$ exists and satisfies*

$$\alpha'(z^0; d) = \alpha^h(z^0; d) = \max_{\lambda_k \in \mathcal{A}_0(z^0)} \max_{1 \leq l \leq t_k} \mathrm{Re}\ \lambda'_{kl}\ , \tag{40}$$

where λ'_{kl}, $l = 1, \ldots, t_k$ are the eigenvalues of $\sum_{q=1}^{\nu} d_q B_{qk}$.

Therefore, if $\eta = 1$ and λ_1 is semisimple, then the right hand side of (38) is the *average* of the real parts of the eigenvalues of $\sum_{q=1}^{\nu} d_q B_{q1}$, whereas, $\alpha^h(z^0; d)$ is the *maximum* of these quantities.

4 The Spectral Radius

Let us now shift our attention to the study of the function ρ defined in (2). We study the differential properties of ρ in the same manner as we studied these properties for α. That is, we first consider the case when the spectrum near $z^0 \in \mathbb{C}^\nu$ is given as the set of roots of a polynomial of the form (8) and then extend this result to the general case. To this end, consider the directional derivative $\rho^h(z^0;d)$ defined by (3). As was the case for the directional derivative $\alpha^h(x;d)$, the limit infimum in the definition of $\rho^h(z^0;d)$ can be replaced by limit. This is justified in the same way as it was for $\alpha^h(z^0;d)$, that is, by considering the splitting behavior of the eigenvalues under perturbation. Thus, we may write

$$\rho^h(z;d) = \inf_{\gamma\in\Gamma(z^0,d)} \lim_{\epsilon\downarrow 0} \frac{\rho(\gamma(\epsilon)) - \rho(z)}{\epsilon}. \tag{41}$$

where $\Gamma(z^0,d)$ is defined in (4), and $\rho^h(z^0;\cdot)\colon \mathbb{C}^\nu \mapsto \mathbf{R}\cup\{+\infty\}$. Continuing as in Section 2, we begin with the following key result for the case in which (8) holds.

Theorem 8 *Let P be given by (8) near $z^0 \in \mathbb{C}^\nu$ and choose $d \in \mathbb{C}^\nu$. We will consider two cases: $\rho(z^0)=0$ and $\rho(z^0)\neq 0$.*

1. *Assume that $\rho(z^0)=0$ so that $\lambda_0 = 0$. In this case we have*

$$\rho^h(z^0;d) \geq \begin{cases} \frac{1}{t_0}\,|c_1'(z^0)d| & \text{, if } c_j'(z^0)d = 0 \text{ for } j=2,\cdots,t_0, \\ +\infty & \text{, otherwise.} \end{cases} \tag{42}$$

2. *Assume that $\rho(z^0)\neq 0$ and consider the conditions*

$$\mathrm{Re}\ \overline{\lambda}_0 c_2'(z^0)d \geq 0,\ \mathrm{Im}\ \overline{\lambda}_0 c_2'(z^0)d = 0, \tag{43}$$

$$c_j'(z^0)d = 0, \quad j=3,\ldots,t_0. \tag{44}$$

Then

$$\rho^h(z^0;d) \geq \begin{cases} \frac{1}{t_0\rho(z^0)}\left[|c_2'(z^0)d| - \mathrm{Re}\ \overline{\lambda}_0 c_1'(z^0)d\right] & \text{, if (43) and (44) hold,} \\ +\infty & \text{, otherwise.} \end{cases} \tag{45}$$

Here it is understood that the function c_2 is identically zero if $t_0 = 1$.

Moreover, if the rank of $c'(z^0)$ is t_0, where $c\colon \mathbb{C}^\nu \mapsto \mathbb{C}^{t_0}$ is given by

$$c(z) = \begin{bmatrix} c_1(z) \\ \vdots \\ c_t(z) \end{bmatrix},$$

then equality holds in (42) or (45) depending on whether $\rho(z^0)=0$ or $\rho(z^0)\neq 0$ holds, respectively.

Proof In this proof we will continue to use the notation of Section 2. Let $\gamma \in \Gamma(z^0,d)$, set $\beta_j^{(1)} = c_j'(z^0)\gamma'(0) = c_j'(z^0)d$ for $j=1,2,\cdots,t_0$ as in (10), and let $\lambda(\epsilon)$ be one of the roots (11) of (9). Since

$$|\lambda(\epsilon)|^2 - |\lambda_0|^2 = 2\mathrm{Re}\ \overline{\lambda}_0(\lambda(\epsilon)-\lambda_0) + |\lambda(\epsilon)-\lambda_0|^2, \tag{46}$$

a necessary condition for

$$\delta\epsilon + o(\epsilon) \geq |\lambda(\epsilon)| - |\lambda_0|, \tag{47}$$

or equivalently,

$$2\delta\, |\lambda_0|\, \epsilon + o(\epsilon) \geq |\lambda(\epsilon)|^2 - |\lambda_0|^2, \tag{48}$$

is that

$$\delta\, |\lambda_0|\, \epsilon + o(\epsilon) \geq \mathrm{Re}\, \overline{\lambda}_0(\lambda(\epsilon) - \lambda_0). \tag{49}$$

It follows from Lemma 1 that if either (16) or (17) with $y_0 = \lambda_0$, or equivalently, (43) or (44), do not hold, then inequality (48) cannot hold for any $\delta \in \mathbf{R}$. Since this is independent of $\gamma \in \Gamma(z^0, d)$, $\rho^h(z^0; d) = +\infty$ if any one of (43) or (44) are violated regardless of the value of $\rho(z^0)$.

Let us now suppose that (43) and (44) hold, i.e. (16) and (17) hold with $y_0 = \lambda_0$. Then for every $\delta > \rho^h(z^0; d)$ there is a $\gamma \in \Gamma(z^0, d)$ such that

$$\lim_{\epsilon \downarrow 0} \frac{\rho(\gamma(\epsilon)) - \rho(z^0)}{\epsilon} < \delta, \tag{50}$$

or equivalently,

$$\rho(\gamma(\epsilon)) - \rho(z^0) < \delta\epsilon \text{ for } \epsilon \in [0, \epsilon_0],$$

for some $\epsilon_0 > 0$. Therefore, inequalities (48) and (49) are satisfied.

As was observed in Section 2, the roots of the equation (9) are necessarily Puiseux series of the form (11). Lemma 1 and inequality (49) imply that either

(i) the exponent p is greater than or equal to 1 corresponding to series of the form

$$\lambda(\epsilon) = \lambda_0 + a\epsilon + o(\epsilon),$$

with $a \in \mathbb{C}$ possibly taking the value zero, or

(ii) the exponent p equals $\frac{1}{2}$ corresponding to m pairs of roots of the form

$$\lambda(\epsilon) = \lambda_0 \pm (r_h\epsilon)^{\frac{1}{2}} + (\tilde{a}_{\pm})\epsilon + o(\epsilon),$$

where, as with a in (i), $\tilde{a}_{\pm}$ takes different values corresponding to different roots.

By summing over *all* the roots, we find that

$$-\beta_1^{(1)} = \sum a + \sum (\tilde{a}_+ + \tilde{a}_-). \tag{51}$$

Moreover, substituting the expressions for $\lambda(\epsilon)$ given in (i) and (ii) into (48) yields

$$2\delta\, |\lambda_0|\, \epsilon \geq (2\mathrm{Re}\, \overline{\lambda}_0 a)\epsilon + o(\epsilon) \tag{52}$$

in case (i) and

$$2\delta\, |\lambda_0|\, \epsilon \geq (|r_h| + 2\mathrm{Re}\, \overline{\lambda}_0 \tilde{a}_{\pm})\epsilon + o(\epsilon) \tag{53}$$

in case (ii). Now summing these inequalities over *all* the roots gives the inequality

$$2\delta t_0\ |\lambda_0|\ \epsilon \geq [2\sum_{h=1}^{m}\ |r_h|\ +2\mathrm{Re}\ \overline{\lambda}_0(\sum a + \sum(\tilde{a}_+ + \tilde{a}_-))]\epsilon + o(\epsilon), \tag{54}$$

where the sums without explicit indexing indicates summing over all the roots, while the factor 2 appearing in the front of $\sum_{h=1}^{m}\ |r_h|$ reflects the fact that there are two roots for each $h = 1, 2, \cdots, m$. Now, as explained in Lemma 1,

$$\sum_{h=1}^{m}\ |r_h| \geq |\sum_{h=1}^{m} r_h| = |\beta_2^{(1)}|\ . \tag{55}$$

Combining this inequality with (51) and (54) gives

$$\delta\ |\lambda_0| \geq \frac{1}{t_0}(|\beta_2^{(1)}|\ -\mathrm{Re}\ \overline{\lambda}_0\beta_1^{(1)}) + \frac{o(\epsilon)}{\epsilon}. \tag{56}$$

Letting $\epsilon \downarrow 0$, we obtain the inequality

$$\delta\ |\lambda_0| \geq \frac{1}{t_0}(|\beta_2^{(1)}|\ -\mathrm{Re}\ \overline{\lambda}_0\beta_1^{(1)}). \tag{57}$$

We now consider the two cases $\rho(z^0) = 0$ and $\rho(z^0) \neq 0$ separately. If $\rho(z^0) = 0$, then inequality (57) implies that $\beta_2^{(1)} = 0$. Thus, $\rho^h(z^0; d) = +\infty$ unless $\beta_j^{(1)} = 0$ for $j = 2, \cdots, t_0$. Furthermore, observe that

$$|\sum \lambda(\epsilon)| \leq \sum\ |\lambda(\epsilon)| < t_0\delta\epsilon\ ,$$

for all $\epsilon \in [0, \epsilon_0]$ where the sum is taken over all branches $\lambda(\epsilon)$. Letting $\delta \downarrow \rho^h(z^0; d)$ yields (42). On the other hand, if $\rho(z^0) \neq 0$, then (45) follows immediately from (57) by letting $\delta \downarrow \rho^h(z^0; d)$.

Next, suppose that the rank of $c'(z^0)$ is t_0 and that (43) and (44) hold (otherwise equality in either (42) or (45) is trivially satisfied). We will only consider the case $\rho(z^0) \neq 0$ since the case $\rho(z^0) = 0$ follows in a similar manner. Moreover, in the case $t_0 = 1$, we take $\beta_2^{(1)} = 0$ as usual. Consider the coefficients of the powers of $(\lambda - \lambda_0)$ in the polynomial

$$\begin{aligned}&((\lambda - \lambda_0) + \sigma\epsilon)^{t_0 - 2}[(\lambda - \lambda_0) + \sqrt{-\beta_2^{(1)}}\sqrt{\epsilon} + \tau\epsilon][(\lambda - \lambda_0) - \sqrt{-\beta_2^{(1)}}\sqrt{\epsilon} + \tau\epsilon]\\ &= (\lambda - \lambda_0)^{t_0} + \beta_1^{(1)}\zeta(\lambda - \lambda_0)^{t_0 - 1} + (\beta_2^{(1)}\zeta + O(\zeta^2))(\lambda - \lambda_0)^{t_0 - 2} + \dots\ ,\end{aligned} \tag{58}$$

where σ and τ are defined by the expressions

$$\sigma = \frac{-1}{t_0}[|\beta_2^{(1)}|\ -\mathrm{Re}\ \overline{\lambda}_0\beta_1^{(1)}]\,\frac{\lambda_0}{|\lambda_0|^2}$$

and

$$\tau = \frac{1}{2}[\beta_1^{(1)} - (t_0 - 2)\sigma].$$

These coefficients are chosen so that not only are (16) and (17) satisfied with $y_0 = \overline{\lambda}_0$, but also the expansion of each root satisfies

$$|\lambda(\epsilon)|^2 = |\lambda_0|^2 + \frac{1}{t_0\ |\lambda_0|}[|\beta_2^{(1)}| - \mathrm{Re}\ \overline{\lambda}_0 \beta_1^{(1)}]\epsilon + o(\epsilon).$$

The proof now follows the argument given in the proof of Theorem 2, that is, one uses the rank condition and the implicit function theorem to establish the existence of a curve $\gamma \in \Gamma(z^0, d)$ such that (9) has the same coefficients as (58). With this choice of γ equality holds in (45). □

The main theorem of this section now follows easily from Theorem 8. It is derived from Theorem 8 in the same way that Theorem 3 was derived from Theorem 2 and so its proof is omitted.

Theorem 9 *Let P have the representation (6) where each μ_k is given by (7), choose $d \in \mathbb{C}^\nu$, and define $\mathcal{R}_0(z^0) = \{\lambda_k \in \Sigma(z^0) : |\lambda_k| = \rho(z^0)\}$. We consider the two cases $\rho(z^0) = 0$ and $\rho(z^0) \neq 0$ separately.*

1. *Suppose $\rho(z^0) = 0$ so that $\mathcal{R}_0(z^0) = \Sigma(z^0) = \{\lambda_1\}$ and $t_1 = n$. If any one of the conditions*

$$c'_{1j}(z^0)d = 0,\ j = 2, \ldots, n,$$

is violated, then

$$\rho^h(z^0; d) = +\infty\ ; \tag{59}$$

otherwise,

$$\rho^h(z^0; d) \geq \frac{1}{n}\ |c'_{11}(z^0)d|\ . \tag{60}$$

2. *Suppose $\rho(z^0) \neq 0$. If for some $\lambda_0 \in \mathcal{R}_0(z^0)$ any one of the conditions*

$$\mathrm{Re}\ \overline{\lambda}_0^{2} c'_{k2}(z^0)d \geq 0,\ \mathrm{Im}\ \overline{\lambda}_0^{2} c'_{k2}(z^0)d = 0, \tag{61}$$

$$c'_{kj}(z^0)d = 0,\ j = 3, \ldots, t_k, \tag{62}$$

is violated, then

$$\rho^h(z^0; d) = +\infty\ ; \tag{63}$$

otherwise,

$$\rho^h(z^0; d) \geq \max\{\Psi(\lambda_k) : \lambda_k \in \mathcal{R}_0(z^0)\}, \tag{64}$$

where

$$\Psi(\lambda_k) = \frac{1}{t_k \rho(z^0)}[|c'_{k2}(z^0)d| - \mathrm{Re}\ \overline{\lambda}_0 c'_{k1}(z^0)d]$$

with c_{k2} understood to be the zero map if $t_k = 1$.

Moreover, if the vectors $\{c'_{kj}(z^0) : \lambda_k \in \mathcal{R}_1(z^0, d),\ j = 1, \ldots, t_k\}$ *are linearly independent, where* $\mathcal{R}_1(z^0, d)$ *is the set of* $\lambda_k \in \mathcal{R}_0(z^0)$ *such that* $\Psi(\lambda_k) = \max\{\Psi(\xi) : \xi \in \mathcal{R}_0(z^0)\}$ *when* $\rho(z^0) \neq 0$ *and* $\mathcal{R}_1(z^0, d) = \Sigma(z^0)$ *otherwise, then equality holds in (60) or (64) depending on whether* $\rho(z^0) = 0$ *or* $\rho(z^0) \neq 0$ *holds, respectively.*

For the case in which P is given by (35) one can apply Theorem 9 in conjunction with Theorem 5 to directly obtain a result in terms of matrices. But, we again caution that if any of the eigenvalues $\lambda_k \in \mathcal{R}_1(z^0, d)$ is derogatory, then the linear independence condition used to establish equality in (60) and (64) is not satisfied. Thus, in this case, one should not expect equality to hold. Indeed, when $\mathcal{R}_0(z^0)$ contains only semisimple eigenvalues, we obtain the following result as an immediate consequence of [12, Lemma 3.4].

Theorem 10 *Suppose* P *is given by (35). If* $\mathcal{R}_0(z^0)$ *contains only semisimple eigenvalues of* $A(z^0)$*, then for every* $d \in \mathbb{C}^\nu$ *the ordinary directional derivative* $\rho'(z_0; d)$ *exists and satisfies*

$$\rho'(z^0; d) = \rho^h(z_0; d) = \begin{cases} \max_{k \in \mathcal{R}_0(z^0)} \max_{1 \leq l \leq n_k} |\lambda'_{kl}| & \text{if } \rho(z_0) = 0, \\ \frac{1}{\rho(z_0)} \max_{k \in \mathcal{R}_0(z^0)} \max_{1 \leq l \leq n_k} \operatorname{Re} \overline{\lambda}_k \lambda'_{kl} & \text{if } \rho(z_0) \neq 0. \end{cases} \tag{65}$$

where $\lambda'_{kl},\ l = 1, \ldots, t_k$ *are the eigenvalues of* $\sum_{q=1}^{\nu} d_q B_{qk}$ *and the matrices* B_{qk} *are defined in Theorem 5.*

Therefore, if, for example, $\rho(z^0) \neq 0$, $\mathcal{R}_0(z^0)$ contains only the single element λ_1 and λ_1 is semisimple, then the right hand side of (64) is the average of the values $\frac{1}{|\lambda_1|} \operatorname{Re} \overline{\lambda}_1 \lambda'_{1l},\ l = 1, \ldots, t_k$, whereas $\rho^h(z^0; d)$ is the maximum.

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