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ADDITIVE SCHWARZ METHODS FOR ELLIPTIC FINITE ELEMENT PROBLEMS IN THREE DIMENSIONS

Maksymilian Dryja* and Olof B. Widlund†

Abstract. Many domain decomposition algorithms and certain multigrid methods can be described and analyzed as additive Schwarz methods. When designing and analyzing domain decomposition methods, we encounter special difficulties in the case of three dimensions and if the coefficients are discontinuous and vary over a large range. In this paper, we first introduce a general framework for Schwarz methods. Three classes of applications are then considered: certain wire basket based iterative substructuring methods, Neumann-Neumann algorithms with low dimensional, global subspaces and a modified form of a multilevel algorithm introduced by Bramble, Pasciak and Xu.

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1. Introduction. In this paper, we discuss additive Schwarz methods for solving systems of linear algebraic equations which result from finite element approximations of second order, elliptic problems in three dimensional, bounded regions. A general framework is presented which is quite useful in the design and analysis of a variety of domain decomposition and some multigrid methods. Three methods are then described and analyzed. They are extensions of methods previously considered in the literature for solving the systems of algebraic equations which correspond to the interfaces between the substructures. We also consider problems with discontinuous coefficients with a great variation in the values.

The first method considered is an iterative substructuring algorithm recently developed by B. Smith [9], [10]. The second is a domain decomposition method developed by R. Glowinski, P. Le Tallec, Y.-H. De Roeck et al., see [1], [3]. Finally, we consider a variant of a multigrid-like method discovered by J. H. Bramble, J. E. Pasciak and J. Xu [2].

The paper is organized as follows. In Section 2 a finite element approximation of second order, elliptic problems is considered. A system of algebraic equations corresponding to the discrete problem is reduced to a system defined on the interfaces of the substructures after eliminating the interior variables associated with the interior nodal points of each substructure. This reduced system with the *Schur complement* matrix can be solved by a variety of iterative methods.

In Section 3, a general abstract framework for additive Schwarz methods is introduced, see also [5], [6]. In Sections 4, 5 and 6 the three algorithms mentioned above are described and analyzed inside this framework. In particular, we discuss problems with discontinuous coefficients and show that the rate of convergence of certain variants of the methods can be made independent of the variation of the coefficients.

The results of Section 4 of this paper have been obtained jointly with Barry Smiths; see further [4].

2. Differential and finite element model problems. To simplify the presentation, we discuss only two model problems, a standard Poisson equation and a special second order problem with discontinuous, piecewise constant coefficients. We call them model problem I and II, respectively. The continuous model problem I is of the form:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \ , \qquad v \in H_0^1(\Omega) \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, ds \ , \quad f(v) = \int_\Omega f v \, dx \tag{2.2}$$

For simplicity, we let Ω be a bounded polyhedral region in three dimensions. A coarse triangulation of Ω is introduced by dividing the region into nonoverlapping simplices Ω_i, $i = 1, \ldots, N$, which are also called substructures. The substructures Ω_i are further divided into elements e_j. We associate parameters H and h with the coarse and fine triangulations and assume that these triangulations are shape regular in the sense common to finite element theory.

Let $V^h(\Omega)$ be the finite element space of continuous, piecewise linear functions defined on the fine triangulation and which vanish on $\partial\Omega$, the boundary of Ω. The discrete model problem I is of the form:

Find $u_h \in V^h$ such that

$$a(u_h, v_h) = f(v_h)\ , \qquad v_h \in V^h \tag{2.3}$$

or alternatively, find the vector x of nodal values such that

$$Kx = b \tag{2.4}$$

Here K is the stiffness matrix, x is the vector of nodal values and b the load vector.

Our goal is to describe and analyze iterative methods for solving (2.2) which can be regarded as additive Schwarz methods.

All our results can be extended to general conforming finite element approximations of any self-adjoint, second order, elliptic problem. This includes the case when there is a great variation in the values of the coefficients. Here we consider only the case when

$$a(u, v) = \int_\Omega \rho(x)\, \nabla u \cdot \nabla v \, dx \tag{2.5}$$

where $\rho(x) > 0$ is a piecewise constant function. Equation (2.1) with the bilinear form (2.5) is called model problem II. We assume that the jumps of $\rho(x)$ occur only at substructure boundaries. Thus, $\rho(x) = \rho_i = \text{constant} > 0$ on the substructure Ω_i. The methods discussed can be generalized to the case when $\rho(x)$ varies continuously in each subregion.

Let K be the stiffness matrix given by the bilinear form (2.2) or (2.5). In the first step of many iterative substructuring methods the unknowns in the interior of the substructures are eliminated. This reduces the system (2.4) to a system of linear algebraic equations associated with the interfaces only. We now describe this procedure.

Let $K^{(i)}$ be the stiffness matrix of the bilinear $a_{\Omega_i}(u_h, v_h)$ which represents the contribution of the substructure Ω_i to the integral $a_\Omega(u_h, v_h) = a(u_h, v_h)$. Let x and y be the vectors of nodal values that correspond to the finite element functions u_h and v_h, respectively. Then the stiffness matrix K of the entire problem can be obtained by using the *method of subassembly*,

$$x^T K y = \sum_i x^{(i)^T} K^{(i)} y^{(i)} \tag{2.6}$$

Here $x^{(i)}$ is the subvector of nodal parameters associated with $\overline{\Omega}_i$, the closure of Ω_i. We represent $K^{(i)}$ as

$$\begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)T} & K_{BB}^{(i)} \end{pmatrix} \tag{2.7}$$

dividing the subvector $x^{(i)}$ into two, $x_I^{(i)}$ and $x_B^{(i)}$, corresponding to the variables which are interior to the substructure and those which are shared with other substructure, i.e. they are associated with the nodal points of $\partial\Omega_i$. Since the interior variables are associated with only one of the substructures, they can be eliminated locally and in parallel. The resulting reduced matrix is a Schur complement and is of the form

$$S^{(i)} = K_{BB}^{(i)} - K_{IB}^{(i)T} K_{II}^{(i)-1} K_{IB}^{(i)} \tag{2.8}$$

From this it follows that the Schur complement corresponding to the global stiffness matrix K is given by S where

$$x_B^T S y_B = \sum_i x_B^{(i)T} S^{(i)} y_B^{(i)} \tag{2.9}$$

If the local problems are solved exactly what remains is to find a sufficiently accurate approximation of solution of the linear system

$$S x_B = g_B \tag{2.10}$$

Note that the elimination of the interior variables of a substructure can be viewed in terms of an orthogonal projection, with respect to the bilinear form, of the solution u_h of (2.3) onto the subspace $H_0^1(\Omega_i) \cap V^h$. It is easy to show that these subspaces are orthogonal, in the sense of $a(u,v)$, to the piecewise discrete harmonic functions given by

$$a(u_h, v_h) = 0 \ , \qquad v_h \in H_0^1(\Omega_i) \cap V^h \ , \qquad i = 1, \ldots, N \tag{2.11}$$

or alternatively by

$$K_{II}^{(i)} x_I^{(i)} + K_{IB}^{(i)} x_B^{(i)} = 0 \tag{2.12}$$

It is convenient to rewrite (2.10) in variational form. Let $s_i(u,v)$ and $s(u,v)$ denote the forms defined by (2.9), i.e.

$$s_i(u,v) = x_B^{(i)} S^{(i)} y_B^{(i)} \quad \text{and} \quad s(u,v) = x_B^T S y_B \tag{2.13}$$

Equation (2.10) can then be rewritten as

$$s(u,v) = (g,v)_{L^2(\Gamma)} \ , \quad v \in V^h(\Gamma) \tag{2.14}$$

Here u is the discrete harmonic part of the solution and $V^h(\Gamma) \subset H^{1/2}(\Gamma)$ the restriction of $V^h(\Omega)$ to Γ.

Problem (2.14) can be solved by different iterative methods of additive Schwarz type.

3. An abstract additive Schwarz method.

We now describe and analyze the convergence of abstract additive Schwarz methods. We note that the theory presented here is a modified version of the theory developed in our previous papers, see [5], [6], [7]. The difference is that we now include the effects of inexact solvers from the very beginning.

Let V be a finite dimensional space with the scalar product $a(u,v)$. We consider the abstract problem

$$a(u,v) = f(v) \ , \qquad v \in V \tag{3.1}$$

Let

$$V = V_0 + V_1 + \cdots + V_N$$

and let the $b_i(u,v)$ be symmetric, positive definite bilinear forms on $V_i \times V_i$. We introduce projections T_i: $V \to V_i$ by

$$b_i(T_i u, v) = a(u,v) \ , \qquad v \in V_i \tag{3.2}$$

and put

$$T = T_0 + T_1 + \cdots + T_N$$

We replace (3.1) by

$$Tu = g \ , \qquad g = \sum_{i=0} g_i \ , \quad g_i = T_i u \ .$$

Theorem 3.1. *Let*

(i) *there exist a constant C_0 such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0} u_i$, $u_i \in V_i$, such that*

$$\sum_{i=0}^{N} b_i(u_i, u_i) \le C_0^2 a(u,u)$$

(ii) *there exist a constant ω such that for $i = 0, \ldots, N$,*

$$a(u,u) \le \omega\, b_i(u,u) \ , \qquad u \in V_i$$

(iii) *there exist constants ϵ_{ij}, for $i,j = 1, \ldots, N$, such that*

$$a(u_i, u_j) \le \epsilon_{ij} a(u_i,u_i)^{1/2} a(u_j,u_j)^{1/2} \ , \quad u_i \in V_i \ , \quad u_j \in V_j$$

Then

$$C_0^{-2} a(u,u) \le a(Tu,u) \le (\rho(\epsilon)+1)\omega\, a(u,u) \ , \qquad u \in V$$

where $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

A proof of this theorem can be found in [8].

4. A wirebasket based method. In this section, we describe and analyze an iterative substructuring method, recently developed by B. Smith, see [9], [10]. For the description and analysis we use the general framework of Section 3. Here $V = V^h(\Gamma)$ and

$$a(u,v) = s(u,v) \tag{4.1}$$

4.1. A method without a vertex space. To describe the method, we need some notations. Let $\mathcal{F}_{ij}$, be the open faces and let $\mathcal{W}_i$ be the wirebasket of the substructure Ω_i. $\overline{\mathcal{F}}_{ij}$ is the closure of the face common to the substructures Ω_i and Ω_j. The wirebasket is the union of the closures of the edges of Ω_i. Let $\mathcal{W}_{ih}$ and $\partial\mathcal{F}_{ijh}$ denote the set of nodal points belonging to $\mathcal{W}_i$ and $\partial\mathcal{F}_{ij}$, the boundary of $\mathcal{F}_{ij}$, respectively. Let

$$\bar{u}_i = \frac{1}{n_i} \sum_{x \in \mathcal{W}_{ih}} u(x) \ , \qquad \bar{u}_{ij} = \frac{1}{n_{ij}} \sum_{x \in \partial\mathcal{F}_{ijh}} u(x) \tag{4.2}$$

where n_i and n_{ij} are the number of nodal points on $\mathcal{W}_i$ and $\partial\mathcal{F}_{ij}$, respectively. We can now introduce the representation

$$V^h(\Gamma) = V_0(\Gamma) + \sum_{ij} V_{ij}(\Gamma) \tag{4.3}$$

Here $V_{ij}(\Gamma)$ is the space of functions $v \in V^h(\Gamma)$ which vanish at all nodal points not on $\mathcal{F}_{ij}$. The space V_0 is defined in a special way. It is a space of continuous, piecewise linear functions defined, on $\partial\Omega_i$, by its values given on the wirebasket $\mathcal{W}_i$ and by constant values at the nodal points of each face $\mathcal{F}_{ij}$. The constant associated with $\mathcal{F}_{ij}$ is the average of the nodal values on $\partial\mathcal{F}_{ij}$. The basis functions of V_0 are of the form

$$\phi_i^{(k)}(x) = \phi_k(x) + \frac{1}{n_{ij}}\theta_{ij}(x) \quad \text{on} \quad \mathcal{F}_{ij} \tag{4.4}$$

Here $\phi_k(x)$ is the standard nodal basis function for $x_k \in \mathcal{W}_i$ and $\theta_{ij}(x) \in V^h$ is given by

$$\theta_{ij}(x) = \begin{cases} 1 & \text{at the nodal points } x \in \mathcal{F}_{ij} \\ 0 & \text{at the nodal points } x \in \partial\mathcal{F}_{ij} \end{cases}$$

It is also possible to consider the coarse space V_0 as the range of an interpolation operator defined on $\overline{\mathcal{F}}_{ij}$ by

$$\tilde{I}_h u = \sum_{x_k \in \partial\mathcal{F}_{ijh}} u(x_k)\,\phi_k(x) + \bar{u}_{ij}\theta_{ij}(x) \tag{4.5}$$

We now define the quadratic forms corresponding to the different subspaces. Let $b_0(u,v)$: $V_0 \times V_0 \to R$ be of the form

$$b_0(u,v) = (1 + \log\frac{H}{h})\sum_i \sum_{x\in\mathcal{W}_{ih}} h(u(x) - \bar{u}_i)\,(v(x) - \bar{v}_i) \tag{4.6a}$$

and

$$b_0(u,v) = (1 + \log\frac{H}{h})\sum_i \rho_i \sum_{x\in\mathcal{W}_{ih}} h(u(x) - \bar{u}_i)\,(v(x) - \bar{v}_i) \tag{4.6b}$$

for model problems I and II, respectively. To define $b_{ij}(u,v)$: $V_{ij} \times V_{ij} \to R$, let $\Omega_{ij} = \Omega_i \cup \Omega_j \cup \mathcal{F}_{ij}$ and introduce the discrete harmonic extension operator $\mathcal{H}_{ij}$ from $\mathcal{F}_{ij}$ to Ω_{ij} by

$$\begin{aligned} a_{\Omega_{ij}}(\mathcal{H}_{ij}u, v) &= 0\,, \qquad v \in V^h(\Omega_i) \cup V^h(\Omega_j) \\ \mathcal{H}_{ij}u &= u \quad \text{on } \mathcal{F}_{ij}\,,\ \mathcal{H}_{ij}u = 0 \text{ on } \partial\Omega_{ij} \end{aligned} \tag{4.7}$$

Here $V^h(\Omega_i) = V^h(\Omega) \cap H_0^1(\Omega_i)$ and $\mathcal{H}_{ij}u \in V^h(\Omega_{ij})$. We define the bilinear form as

$$b_{ij}(u,v) = a(\mathcal{H}_{ij}u, \mathcal{H}_{ij}v) \tag{4.8}$$

We have now defined all our subspaces and the associated bilinear forms. As in the abstract theory, we define the projections by

$$b_0(T_0 u, v) = s(u, v) , \qquad v \in V_0 \tag{4.9a}$$

and

$$b_{ij}(T_{ij} u, v) = s(u, v) , \qquad v \in V_{ij} \tag{4.9b}$$

and let

$$T = T_0 + \sum_{i,j} T_{ij}$$

Theorem 4.1. *For all $u \in V^h(\Gamma)$, with T_0 and T_{ij} defined by (4.9), we have*

$$\gamma_0 (1 + \log \frac{H}{h})^{-2} \, s(u, u) \le s(Tu, u) \le \gamma_1 \, s(u, u) \tag{4.10}$$

Here γ_0 and γ_1 are constants independent of H and h and the jumps of the coefficients.

To prove this theorem, we use Theorem 3.1 of Section 3; cf. Dryja, Smith and Widlund [4]. For an alternative proof, see Smith [9], [10].

We now briefly discuss how the method can be implemented. Problem (2.14) has been replaced by

$$Tu = g \tag{4.11}$$

where $g = g_0 + \sum g_{ij}$, $g_0 = T_0 u_h$, and $g_{ij} = T_{ij} u_h$.

To solve (4.11) we can use the conjugate gradient method since T is symmetric, positive definite and well conditioned. For simplicity we only consider the first Richardson method. Thus

$$u^{n+1} = u^n - \tau T(u^n - u) , \qquad \tau = 2/(\gamma_0 (1 + \log \frac{H}{h})^{-2} + \gamma_1)$$

Let $r^n = T(u^n - u) = r_0^n + \sum_{ij} r_{ij}^n$, where $r_0^n = T_0(u^n - u)$ and $r_{ij}^n = T_{ij}(u^n - u)$. To find r_0^n and r_{ij}^n, we solve

$$b_0(r_0^n, v) = s(u^n, v) - (g, v) \equiv F(v) , \qquad v \in V_0 \tag{4.12a}$$

and

$$b_{ij}(r_{ij}^n, v) = F(v) , \qquad v \in V_{ij} \tag{4.12b}$$

To compute $s(u^n, v)$ we solve the Dirichlet problems

$$a(\mathcal{H}_k u^n, v) = 0 , \qquad v \in H_0^1(\Omega_k) \cap V^h(\Omega) \tag{4.13}$$

with u^n given on $\partial\Omega_k$. $\mathcal{H}_k u^n$ is the discrete harmonic extension of u^n from $\partial\Omega_k$ to Ω_k. The problem (4.12a) reduces to a system with a sparse matrix and block Gaussian elimination is used to find $r_0^n(x)$ at the nodal points of $x \in \mathcal{W}_i$; cf. Dryja, Smith and Widlund [4]. To find r_{ij}^n, we solve a problem similar to (4.7), after replacing Ω_k with Ω_{ij}.

4.2. The method with vertex spaces. In this subsection, we discuss a variant of a method discussed in Chapter 4 of Smith [9]. The estimate in Theorem 4.1 contains two log factors. We can remove one of them for model problem I by adding *vertex spaces* to the representation (4.3) of $V^h(\Gamma)$; we call this a wirebasket based method with vertex spaces. For each substructure vertex x_k, we define a space $V_k(\Gamma)$ as follows. Let $\widetilde{\mathcal{F}}_k$ be the union of parts of the faces which have the vertex x_k in common. We assume that $\text{dist}(x_k, \partial\widetilde{\mathcal{F}}_k)$ is on the order of H. The space $V_k(\Gamma)$ is the subspace of functions belonging of $V^h(\Gamma)$ which vanish at the nodal points not in $\widetilde{\mathcal{F}}_k$. Clearly

$$V^h(\Gamma) = V_0 + \sum_k V_k + \sum_{i,j} V_{ij}$$

Let $\widetilde{\mathcal{H}}_k u$ be the extension of u from $\widetilde{\mathcal{F}}_k$ to $\tilde{\Omega}_k$ here $\tilde{\Omega}_k$ is a neighborhood of x_k which which contains $\widetilde{\mathcal{F}}_k$. Let

$$b_k(u, v) = a(\widetilde{\mathcal{H}}_k u, \widetilde{\mathcal{H}}_k v) \ , \qquad u, v \in V_k(\Gamma) \tag{4.14}$$

where $a(u, v)$ is given by (2.2). Let T_k be given by

$$b_k(T_k u, v) = s(u, v) \ , \qquad v \in V_k(\Gamma) \tag{4.15}$$

and let

$$T = T_0 + \sum_k T_k + \sum_{ij} T_{ij} \tag{4.16}$$

Theorem 4.2. *For all $u \in V^h(\Gamma)$, with T_0, T_{ij} and T_k defined by (4.9) and (4.15) and for model problem I,*

$$\gamma_0 (1 + \log \frac{H}{h})^{-1} s(u, u) \le s(Tu, u) \le \gamma_1 s(u, u)$$

where γ_0 and γ_1 are constants independent of H, h.

A proof of this theorem is based on Theorem 3.1 of Section 3; cf. Dryja, Smith and Widlund [4].

5. The Neumann-Neumann method. In this section, we give a description of a method, introduced by R. Glowinski, P. Le Tallec, Y.-H. De Roeck et al., see [1], [3], as an additive Schwarz method. We then extend the method to the case of a large number of substructures introducing a coarse space V_0, which is similar but not identical to that of Section 4. We also modify this method to handle problems with discontinuous coefficients.

5.1. The method without a coarse space. We first consider model problem I, i.e., when $s(u, v)$ corresponds to the bilinear form $a(u, v)$ given by (2.2). Let $V_i(\Gamma)$ be the subspace of functions $v \in V^h(\Gamma)$ which vanish at all nodal points on $\Gamma \setminus \partial\Omega_i$. It is easy to verify that

$$V^h(\Gamma) = V_1(\Gamma) + \cdots + V_N(\Gamma) \tag{5.1}$$

Let $s_i(u,v)$ be the bilinear form on $V_i \times V_i$ defined in Section 2,

$$s_i(u,v) = x_B^{(i)T} S^{(i)} y_B^{(i)} \tag{5.2}$$

For the method considered, using the notations from Section 3, $V = V^h(\Gamma)$, $V_i = V_i(\Gamma)$, $i = 1,\ldots,N$, and $a(u,v) = s(u,v)$. There is no coarse space V_0. To define $b_i(u,v)$: $V_i(\Gamma) \times V_i(\Gamma) \to R$, we introduce a counting function $\alpha_i \in V^h(\Gamma)$, associated with $\partial\Omega_i$.

$$\alpha_i(x) = \begin{cases} 2 & x \in \mathcal{F}_{ijh} \\ n_i(x) & x \in \mathcal{W}_{i,h} \\ 0 & \text{all other nodal points on } \Gamma \ . \end{cases}$$

Here $n_i(x)$ is the number of substructures which have the nodal point $x \in \mathcal{W}_i$ in common. Let

$$b_i(u,v) = s_i(I_h(\alpha_i u)\ ,\ I_h(\alpha_i v))\ , \quad u,v \in V^h(\Gamma) \tag{5.3}$$

Here I_h is the linear interpolation operator on the fine triangulation. We note that $b_i(u,v)$ is symmetric and positive definite on $V_i \times V_i$.

We introduce a projection T_i: $V^h \to V_i$ by

$$b_i(T_i u, v) = s(u,v)\ , \qquad v \in V_i \tag{5.4}$$

and put

$$T = T_1 + \cdots + T_N\ .$$

Theorem 5.1. *For all $u \in V^h(\Gamma)$, with T_i defined by* (5.4) *and for model problem I*

$$\gamma_0 s(u,u) \le s(Tu,u) \le \gamma_1 \frac{(1+\log H/h)^2}{H^2} s(u,u) \tag{5.5}$$

where γ_0 and γ_1 are constants independent of H and h.

This result is proved in [3] using other tools. We believe that our proof, which is based on Theorem 3.1 from Section 3, is simpler; cf. [8]. We note that an estimate such as (5.5) but with four log factors was given already in [4]. For the case when there is a red-black ordering of the substructures, we also showed that three log factors are enough and we also derived similar estimates for algorithms with coarse spaces.

5.2. A method with a coarse space. We now describe the Neumann-Neumann method with a coarse space V_0. The factor $1/H^2$ can now be removed from the estimate (5.5). A function $v \in V_0$ is a continuous, piecewise linear function defined by the nodal values on the wirebasket $\mathcal{W}_i$ and it is constant on the faces $\mathcal{F}_{ij}$ with different constants on different faces. This space is different from the coarse space used in Section 4. The basis functions of V_0 are of the following form. For a nodal point $x_k \in \mathcal{W}_{i,h}$, we use a standard nodal basis function $\phi_k(x)$ and for a face $\mathcal{F}_{ij}$ the function $\theta_{ij}(x)$ defined in Section 4. V_0 can be regarded as the range of an interpolation operator defined on $\partial\Omega_i$ by

$$\widehat{I}_h u = \sum_{x_k \in \mathcal{W}_{ih}} u(x_k)\,\phi_k(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} u(\bar{x}_{ij})\,\theta_{ij}(x) \tag{5.6}$$

on $\partial\Omega_i$. Here $\bar{x}_{ij}$ is a fixed nodal point on $\mathcal{F}_{ij}$, and the summation in the second term is taken over the faces $\mathcal{F}_{ij}$. The spaces $V_i(\Gamma)$, $i = 1, \ldots, N$, are defined by

$$V_i = \{v \in V^h(\Gamma)\colon\, v(x) = 0\ ,\ x \in \Gamma \setminus \partial\Omega_i\ ,\ \bar{v}_i^\alpha = 0\} \tag{5.7}$$

Here $\bar{u}_i^\alpha$ is a weighted discrete average of u on $\mathcal{W}_{ih}$, i.e.

$$\bar{u}_i^\alpha = \frac{1}{n_i} \sum_{x \in \mathcal{W}_{ih}} \alpha_i(x)\,u(x)$$

It is easy to verify that

$$V^h(\Gamma) = V_0 + V_1 + \cdots + V_N \tag{5.8}$$

We now introduce bilinear forms $b_i(u,v)$: $V_i \times V_i \to R$. For $i = 1, \ldots, N$, they are defined as in (5.4). For $i = 0$, we use

$$\begin{aligned} b_0(u,v) \; &= (1+\log\tfrac{H}{h})^{-1} \textstyle\sum_i \{h \sum_{x\in\mathcal{W}_{ih}} (u(x)-\bar{u}_i)\,(v(x)-\bar{v}_i) \\ &\quad + \textstyle\sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} H(u(\bar{x}_{ij})-\bar{u}_i)\,(v(\bar{x}_{ij})-\bar{v}_i)\} \end{aligned} \tag{5.9}$$

Here $\bar{u}_i$ is the discrete average of u on $\mathcal{W}_{ih}$ introduced in (4.2). Let

$$T = T_0 + T_1 + \cdots + T_N$$

where

$$b_i(T_i u, v) = s(u,v)\ , \qquad v \in V_i\ , \qquad i = 0, \ldots, N \tag{5.10}$$

Theorem 5.2. *For all $u \in V^h(\Gamma)$, with T_i defined by* (5.10) *and for model problem I*

$$\gamma_0 s(u,u) \le s(Tu,u) \le \gamma_1 (1+\log\frac{H}{h})^2 s(u,u) \tag{5.11}$$

where γ_0 and γ_1 are constants independent of H and h.

A proof of this theorem is given in [8].

5.3. A method with a coarse space for model problem II. In this subsection, we describe a Neumann-Neumann method for model problem II. The bilinear form is now given by

$$a(u,v) = \int_\Omega \rho(x)\,\nabla u \cdot \nabla v\,dx \tag{5.12}$$

where $\rho(x) = \rho_i = \text{constant} > 0$ on substructure Ω_i. Let V_i, $i = 1, \dots, N$, associated with $\partial\Omega_i$, be a space of functions $v \in V^h(\Gamma)$ which vanish on the wirebasket $\mathcal{W}_i$ of Ω_i and at all nodal points on Γ which are not on $\partial\Omega_i$. The space V_0 is the same as in Section 4 i.e. it is the range of the interpolation operator $\widetilde{I}_h$; cf. (4.5). We have

$$V^h(\Gamma) = V_0 + V_1 + \cdots + V_N \tag{5.13}$$

To define $b_i(u, v)$: $V_i \times V_i \to R$, $i = 1, \dots, N$, we introduce a piecewise constant function $\bar{\rho}_i$, defined on $\partial\Omega_i$, which is equal to $(\rho_i + \rho_j)^{1/2}$ on the face $\mathcal{F}_{ij}$ common to Ω_i and Ω_j. Let

$$b_i(u, v) = s_i(I_h(\bar{\rho}_i u), I_h(\bar{\rho}_i u)) \ , \quad u, v \in V_i \tag{5.14}$$

where $s_i(u, v)$ corresponds to the form $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$; cf. (2.2) and (2.13). For $i = 0$

$$b_0(u, v) = \sum_i \rho_i \sum_{x \in \mathcal{W}_{ih}} h(u(x) - \bar{u}_i)(v(x) - \bar{v}_i) \tag{5.15}$$

where $\bar{u}_i$ is defined in (4.2). Note that for $u \in V_i$, $I_h(\bar{\rho}_i u)$ is defined uniquely on the wirebasket $\mathcal{W}_i$ since $u = 0$ there. Let T_i: $V^h(\Gamma) \to V_i$ be a projection defined by

$$b_i(T_i u, v) = s(u, v) \ , \qquad v \in V_i \ , \quad i = 0, 1, \dots, N \tag{5.16}$$

and let

$$T = T_0 + T_1 + \cdots + T_N$$

Theorem 5.3. *For all $u \in V^h(\Gamma)$, with the T_i defined by* (5.16), *and for model problem II*

$$\gamma_0 (1 + \log \frac{H}{h})^{-2} s(u, u) \le s(Tu, u) \le \gamma_1 (1 + \log \frac{H}{h})^2 s(u, u) \ . \tag{5.17}$$

Here γ_0 and γ_1 are constants independent of H and h and the jumps of $\rho(x)$.

A proof of this theorem is given in [8].

6. A multilevel method. In this section we discuss a modification of the multilevel method suggested in J. Bramble et al. [2] for the problem (2.14) in terms of the additive Schwarz method described in Section 3. We first consider model problem I and then model problem II. We note that we have previously described this multilevel method as an additive Schwarz method for the original problem (2.3); cf. Dryja and Widlund [7].

6.1. The method for model problem I. We discuss problem (2.3) with the bilinear form given by (2.2) defined on a triangulation of Ω obtained by successive refinements. We consider $(\ell + 1)$ levels of triangulations of Ω, associated with parameters h_k, which satisfy $V^{h_{k-1}}(\Omega) \subset V^{h_k}(\Omega)$. Here the $V^{h_k}(\Omega)$ are standard finite element spaces with $V^{h_0}(\Omega)$ and $V^{h_\ell}(\Omega)$ the coarsest and finest spaces, respectively. $V^{h_k}(\Gamma)$ is the restriction of $V^{h_k}(\Omega)$ to Γ. The problem considered, (2.3), is defined in $V^{h_\ell}(\Gamma)$. As a representation of $V^h = V^{h_\ell}$, we take

$$V^h = V_0 + V_1 + \cdots + V_\ell \ , \qquad V_k = V^{h_k} \tag{6.1}$$

We introduce, for $k \geq 1$, $b_k(u,v)$: $V_k \times V_k \to R$ by

$$b_k(u,v) = \sum_{x \in N_k} h_k u(x) v(x) \tag{6.2}$$

Here N_k is the set of nodal points of the level k triangulation on Γ. We introduce T_k: $V^h \to V_k$ by

$$s(T_0 u, v) = s(u,v) \qquad v \in V_0 \ , \tag{6.3a}$$

$$b_k(T_k u, v) = s(u,v) \ , \qquad v \in V_k \ , \qquad k = 1, \ldots, N \tag{6.3b}$$

and put

$$T = T_0 + T_1 + \cdots + T_\ell$$

Theorem 6.1. *Let h_k/h_{k+1} be uniformly bounded. Then for all $u \in V^h(\Gamma)$, with T_i defined by* (6.3), *and for model problem I*

$$\gamma_0(\ell+1)^{-1} s(u,u) \leq s(Tu,u) \leq \gamma_1(\ell+1) s(u,u) \tag{6.4}$$

If Ω is convex then

$$\gamma_2 s(u,u) \leq s(Tu,u) \leq \gamma_3(\ell+1) s(u,u) \tag{6.5}$$

Here γ_i, $i = 0, \ldots, 3$, are constants independent of $h_0, \ldots, h_\ell$.

Proof: The proof uses Theorem 3.1 of Section 3.

Assumption (i): Let $\mathcal{H}u$ be the discrete harmonic extension in the sense of $a(u,v)$ of u from $\partial\Omega_i$ to Ω_i, for all i. We use the partitioning

$$u = \sum_{k=0}^{\ell} u_k \ , \quad u_0|_\Gamma = Q_0 \mathcal{H} u|_\Gamma \in V_0(\Gamma) \ , \quad u_k|_\Gamma = (Q_k - Q_{k-1})\mathcal{H}u|_\Gamma \in V_k(\Gamma) \ , \ k = 1, \ldots, \ell$$

Here Q_k is the L_2-projection from $V^h(\Omega) \to V_k(\Omega)$, i.e.,

$$(Q_k u, v)_{L^2(\Omega)} = (u,v)_{L^2(\Omega)} \ , \quad v \in V_k \ , \ k = 0, \ldots, \ell$$

It is well known that

$$|Q_i u|_{H^1(\Omega)} \leq C |u|_{H^1(\Omega)} \tag{6.6}$$

and

$$\|u - Q_i u\|_{L^2(\Omega)} \leq h_i |u|_{H^1(\Omega)} \tag{6.7}$$

Using (6.6), we have

$$s(u_0, u_0) \leq C |Q_0 \mathcal{H} u|^2_{H^1(\Omega)} \leq C |\mathcal{H} u|^2_{H^1(\Omega)} \leq C s(u,u)$$

For $k = 1, \ldots, \ell$, we obtain, by using an elementary inequality and (6.7),

$$\begin{aligned} b_k(u_k, u_k) &\leq C h_k^{-2} \|u_k\|^2_{L^2(\Gamma)} \leq C h_k^{-2} \|Q_k \mathcal{H}u - Q_{k-1}\mathcal{H}u\|^2_{L^2(\Omega)} \\ &\leq C h_k^{-2} \{ \|\mathcal{H}u - Q_k \mathcal{H}u\|^2_{L^2(\Omega)} + \|\mathcal{H}u - Q_{k-1}\mathcal{H}u\|^2_{L^2(\Omega)} \} \\ &\leq C |\mathcal{H}u|^2_{H^1(\Omega)} \leq C s(u,u) \end{aligned}$$

Thus

$$s(u_0, u_0) + \sum_{i=1}^{N} b_i(u_i, u_i) \leq C(\ell + 1)\, s(u,u)$$

which proves Assumption (i).

For a convex region Ω, we can instead use $u_0 = P_0 \mathcal{H}u$ and $u_k = (P_k - P_{k-1})\mathcal{H}u$, where P_k is the H^1-projection and use the fact that the resulting u_k are H^1-orthogonal; cf. Dryja and Widlund [5]. For more details see below.

Assumption (ii) follows from the inverse inequality.

Assumption (iii) is proved by using standard arguments.

6.2. A new method for model problem II. We now discuss a new method for model problem II, i.e. when the bilinear form is given by

$$a(u,v) = \int_\Omega \rho(x)\, \nabla u \cdot \nabla v \, dx \ ,$$

where $\rho(x) = \rho_i = \text{constant} > 0$ on Ω_i. The decomposition of $V^h(\Gamma)$ is now of the form

$$V^h(\Gamma) = V_0(\Gamma) + V_1(\Gamma) + \cdots + V_\ell(\Gamma) \tag{6.8}$$

Here V_0 is the space from Section 4, i.e. the range of the interpolation operator $\widetilde{I}_h$; cf. (4.5).

We let V_k, $k = 1, \ldots, \ell$, be the space of functions in $V^{h_k}(\Gamma)$ which vanish at the nodal points of the wirebasket of $\partial\Omega_i$. It is easy to verify that the representation (6.8) of V^h holds.

We introduce $b_k(u,v)$: $V_k(\Gamma) \times V_k(\Gamma) \to R$, by

$$b_0(u,v) = (1 + \log \frac{H}{h}) \sum_i \sum_{x \in \mathcal{W}_{ih}} h \rho_i (u(x) - \bar{u}_i)(v(x) - \bar{v}_i) \ , \tag{6.10}$$

cf. (4.6), and for $k = 1, \ldots, \ell$,

$$b_k(u,v) = \sum_{x \in N_k} h_k \tilde{\rho}(x)\, u(x)\, v(x) \ , \quad u, v \in V_k(\Gamma) \tag{6.11}$$

Here $\tilde{\rho}(x)$ is piecewise constant and equal to $(\rho_i + \rho_j)/2$ on the face $\mathcal{F}_{ij}$ which is common to the substructures Ω_i and Ω_j. Note that for $u \in V_k$, $\tilde{\rho}(x)u(x) = 0$ on $\mathcal{W}_i$ since $u(x) = 0$ there.

We now introduce T_k: $V^h(\Gamma) \to V_k$, by

$$b_k(T_k u, v) = s(u,v) \ , \qquad v \in V_k \ , \ k = 0, \ldots, \ell \tag{6.12}$$

and put

$$T = T_0 + T_1 + \cdots + T_\ell \tag{6.13}$$

Theorem 6.2. *Let h_k/h_{k+1} be uniformly bounded. Then for all $u \in V^h(\Gamma)$, with T_i defined by* (6.12), *and for model problem II*

$$\gamma_0(\ell+1)^{-2}s(u,u) \leq s(Tu,u) \leq \gamma_1(\ell+1)\, s(u,u) \tag{6.14}$$

where γ_0 and γ_1 are constants independent of h_0, $h_1, \ldots, h_\ell$ and the jumps of $\rho(x)$.

Proof: We again use Theorem 3.1.

Assumption (i): Let $u_0 = \widetilde{I}_h u$ where $\widetilde{I}_h u$ is defined in (4.5). It can be established that

$$b_0(u_0, u_0) \leq C(\ell+1)^2 s(u,u) \tag{6.15}$$

see Dryja, Smith and Widlund [4] and Smith [9]. We now define u_k. Let $w = u - u_0$ and note that it vanishes on the wirebasket $\mathcal{W}_i$. The function w, given on $\mathcal{F}_{ij}$, the face common to Ω_i and Ω_j, is extended to Ω_i as a discrete harmonic function ω_{ij}, i.e.,

$$(\nabla w_{ij}, \nabla v)_{L^2(\Omega_i)} = 0\ , \qquad v \in V^h(\Omega_i) \tag{6.16}$$

$$w_{ij} = w \quad \text{on} \quad \mathcal{F}_{ij} \quad \text{and} \quad w_{ij} = 0 \quad \text{on} \quad \partial\Omega_i \setminus \mathcal{F}_{ij}$$

Let $\hat{V}^{h_k}(\Omega_i), k = 0, \cdots, \ell$, be the subspace of $V^{h_k}(\Omega_i)$, the restriction of V^{h_k} to Ω_i, of functions which vanish on $\partial\Omega_i \setminus \mathcal{F}_{ij}$. We note that $w_{ij} \in \hat{V}^{h_\ell}(\Omega_i)$. This function is represented as

$$w_{ij} = P_{ij,1}w_{ij} + (P_{ij,2} - P_{ij,1})w_{ij} + \cdots + (P_{ij,\ell} - P_{ij,\ell-1})w_{ij} \tag{6.17}$$

where $P_{ij,k}$ is the H^1-projection: $\hat{V}^h(\Omega_i) \to \hat{V}^{h_k}(\Omega_i)$, i.e.

$$(\nabla P_{ij,k}v, \nabla\phi)_{L^2(\Omega_i)} = (\nabla v, \nabla \phi)_{L^2(\Omega_i)}\ , \quad \phi \in \hat{V}^{h_k}(\Omega_i)$$

It is known that, since Ω_i is convex,

$$\|u - P_{ij,k}u\|_{L^2(\Omega_i)} \leq M h_k |u|_{H^1(\Omega_i)} \tag{6.18}$$

On $\mathcal{F}_{ij}$ we define u_k as the traces of the functions defined in (6.17)

$$u_1 = P_{ij,1}w_{ij}\ , \quad u_k = (P_{ij,k} - P_{ij,k-1})w_{ij}\ , \qquad k = 2, \ldots, \ell$$

It is easy to see that $P_{ij,k} - P_{ij,k-1}$ is an orthogonal projection from $V^h(\Omega_i)$ into $\widetilde{V}^{h_k}(\Omega_i) = \text{Range}(P_{ij,k} - P_{ij,k-1}) \subset V^{h_k}(\Omega_i)$. We now show that

$$\sum_{k=1}^{\ell} b_k(u_k, u_k) \leq C(\ell+1)^2 s(u,u) \tag{6.19}$$

for the u_k defined above. We obtain, by using the definition of $\tilde{\rho}(x)$,

$$b_k(u_k, u_k) = \sum_{x \in N_k} h_k \tilde{\rho}(x)\, u_k^2(x) = \frac{1}{2} \sum_i h_k \sum_j \sum_{x \in N_{ij,k}} \rho_i u_k^2(x) \tag{6.20}$$

where $N_{ij,k}$ is the set of nodal points of the k-level triangulation of $\mathcal{F}_{ij}$ and the summation with respect to j is over the different faces of $\partial\Omega_i$. For one face, we obtain

$$\begin{aligned} h_k \textstyle\sum_{x \in N_{ij,k}} \rho_i u_k^2(x) &\le C h_k^{-2} \rho_i \|u_k\|_{L^2(\Omega_i)}^2 = C h_k^{-2} \rho_i \|(P_{ij,k} - P_{ij,k-1}) u_k\|_{L^2(\Omega_i)}^2 \\ &\le C h_k^{-2} \rho_i \{\|P_{ij,k} u_k - u_k\|_{L^2(\Omega_i)}^2 + \|P_{ij,k-1} u_k - u_k\|_{L^2(\Omega_i)}^2\} \\ &\le C \rho_i |u_k|_{H^1(\Omega_i)}^2 \end{aligned} \tag{6.21}$$

Adding these inequalities with respect to k, we obtain

$$\begin{aligned} &\rho_i \textstyle\sum_{k=1}^{\ell} h_k \sum_{x \in N_{ij,k}} u_k^2(x) \le C \rho_i \sum_{k=1}^{\ell} (\nabla u_k, \nabla u_k)_{L^2(\Omega_i)} \\ &\quad = C \rho_i \{(\nabla P_{ij,1} \omega_{ij}, \nabla \omega_{ij})_{L^2(\Omega_i)} + \textstyle\sum_{k=2}^{\ell} (\nabla (P_{ij,k} - P_{ij,k-1}) \omega_{ij}, \nabla \omega_{ij})_{L^2(\Omega_i)}\} \\ &\quad = C \rho_i (\nabla P_{ij,\ell} \omega_{ij}, \nabla \omega_{ij})_{L^2(\Omega_i)} = C \rho_i \|\nabla \omega_{ij}\|_{L^2(\Omega_i)}^2 \\ &\quad \le C \rho_i |\omega|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 \le C (\ell + 1)^2 s_i(u, u) \end{aligned}$$

For the last inequality, see for example Dryja and Widlund [6]. Combining this inequality with (6.20), we obtain

$$\sum_{k=1}^{\ell} b_k(u_k, u_k) \le C (\ell + 1)^2\, s(u, u)\ .$$

Together with (6.15), this provides the bound for Assumption (i).

Assumption (ii): It is known that for $u \in V_0$

$$a(u, u) \le C b_0(u, u)$$

cf. [4]. For $k = 1, \ldots, \ell$,

$$a(u, u) \le C b_k(u, u)\ , \qquad u \in V_k$$

follows from an inverse inequality.

Assumption (iii) is proved by using standard arguments.

REFERENCES

[1] J.-F. Bourgat, R. Glowinski, P. Le Tallec and M. Vidrascu, Variational Formulation and Algorithm for Trace Operator, in Domain Decomposition Calculations in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux and O. Widlund, eds., Philadelphia, PA, 1989, SIAM.

[2] J. H. Bramble, J. E. Pasciak and J. Xu, Parallel Multilevel Preconditioners, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20–22, 1989, T. Chan, R. Glowinski, J. Périaux and O. Widlund, eds., SIAM, Philadelphia, 1990.

[3] Y.-H. De Roeck and P. Le Tallec, Analysis and Test of a Local Domain Decomposition Preconditioner in Fourth International. Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Moscow, USSR, May 21–25, 1990, R. Glowinski, Yu. A. Kuznetsov, G. Meurant, J. Périaux and O. Widlund, eds., SIAM, Philadelphia, PA, 1991.

[4] M. Dryja, B. F. Smith and O. B. Widlund, Schwarz Analysis of Iterative Substructuring Methods for Problems in Three Dimensions, to appear as a Tech. Rep., Dept. of Computer Science, Courant Institute, 1991.

[5] M. Dryja and O. B. Widlund, An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions, Tech. Rep. 339, Dept. of Computer Science, Courant Institute, 1987.

[6] ———-, Towards a Unified Theory of Domain Decomposition Algorithms for Elliptic Problems, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20–22, 1989, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1990.

[7] ———-, Multilevel Additive Methods for Elliptic Finite Element Problems, in Parallel Algorithms for Partial Differential Equations, Proc. of the 6th GAMM-Seminar, Kiel, January 19–21, 1990, W. Hackbusch, ed., 1991, Vieweg & Sons, Braunchweig.

[8] ———-, The Neumann-Neumann Method as an Additive Schwarz Method for Finite Element Elliptic Problems, to appear as a Tech. Rep., Dept. of Computer Science, Courant Institute, 1991.

[9] B. F. Smith, Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity, Ph.D. Thesis, Tech. Rep. 517, Dept. of Computer Science, Courant Institute, 1990.

[10] ———–, A Domain Decomposition Algorithm for Elliptic Problems in Three Dimensions, Tech. Rep. 519, Dept. of Computer Science, Courant Institute, 1990, to appear in Numerische Mathematik.