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DECOMPOSITION AND FICTITIOUS DOMAINS METHODS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract.

Boundary value problems for elliptic second order equations in three-dimensional domains with piecewise smooth boundaries are considered. Discretization of the problem is performed using a conventional version of the finite element method with piecewise linear basis functions. The main purpose of the paper is the construction of a preconditioning operator for the resulting system of grid equations. The method is based on two approaches: decomposition of the domain into subdomains and using a new version of the method of fictitious domains. The rate of convergence of the corresponding conjugate gradient method is independent of both the grid size and the number of subdomains.

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1. Introduction. In this paper, we consider preconditioning operators for the system of grid equations approximating the following boundary value problem:

$$\left\{\begin{array}{l} -\sum_{i,j=1}^{3} \dfrac{\partial}{\partial x_i} a_{ij}(x) \dfrac{\partial u}{\partial x_j} + a_0(x) u = f(x) \;, \quad x \in \Omega \;, \\ u(x) = 0 \;, \qquad x \in \Gamma_0 \;, \\ \dfrac{\partial u}{\partial N} + \sigma(x) u = 0 \;, \quad x \in \Gamma_1 \;. \end{array}\right. \tag{1.1}$$

Here, $\dfrac{\partial u}{\partial N}$ is the derivative in the conormal direction. We assume that Ω is a polyhedron. Let Ω be a union of n nonoverlapping subdomains Ω_i,

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\Omega}_i \;, \quad \Omega_i \cap \Omega_j = \emptyset \;, \; i \neq j$$

where Ω_i are polyhedrons with diameters on the order of H. Let Ω^h

$$\Omega^h = \bigcup_{i=1}^{n} \Omega_i^h$$

be a regular triangulation of Ω which is characterized by a parameter h.

We use $H^1(\Omega, \Gamma_0)$ to denote a subspace of the Sobolev spaces $H^1(\Omega)$ of real functions

$$H^1(\Omega, \Gamma_0) = \{ v \in H^1(\Omega) \,|\, v(x) = 0 \;, \; x \in \Gamma_0 \}$$

and we introduce the bilinear form

$$a(u, v) = \int_{\Omega} (\sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x) u v) \, dx + \int_{\Gamma_1} \sigma(x) u v \, dx$$

and the linear functional

$$\ell(v) = \int_{\Omega} f(x) v \, dx \;.$$

We assume that the coefficients of the problem (1.1) are such that $a(u, v)$ is a symmetric, coercive and a continuous form in $H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$, i.e.

$$a(u, v) = a(v, u) \;, \qquad \forall\, u, v \in H^1(\Omega, \Gamma_0) \;,$$

$$\mu_0 \|v\|_{H^1(\Omega)}^2 \leq a(v, v) \leq \mu_1 \|v\|_{H^1(\Omega)}^2 \;, \qquad \forall\, v \in H^1(\Omega, \Gamma_0) \;,$$

and that the linear functional $\ell(v)$ is continuous in $H^1(\Omega, \Gamma_0)$, i.e.,

$$|\ell(v)| \leq \mu \|v\|_{H^1(\Omega)} \;.$$

Here μ_0, μ_1, μ are positive constants. A weak formulation of (1.1) is:

Find $u \in H^1(\Omega, \Gamma_0)$ such that

$$a(u, v) = \ell(v) \;, \qquad \forall\, v \in H^1(\Omega, \Gamma_0) \;. \tag{1.2}$$

Denote by W the space of real-valued continuous piecewise linear functions. Using the finite element method we can pass from (1.2) to the linear algebraic system

$$Au = f \tag{1.3}$$

Our purpose is the construction of a preconditioner B for the problem (1.3) such that the following inequalities are valid:

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u) , \qquad \forall\, u \in R^N . \tag{1.4}$$

Here N is the dimension of W, the positive constants c_1, c_2 are independent of h and H, and the multiplication of B^{-1} by the vector can be at a low cost.

As a rule, the most efficient preconditioners for solving boundary value problems in domains with complex geometry can be constructed by simplifying geometry of the original domain. We use two approaches: decomposition of the domain into subdomains [5–7] and a version of the method of fictitious domains; cf. [8–10].

The remainder of the paper is organized as follows. In section 2, we describe abstract results which are useful in constructing the preconditioning operator. Using Lemma 2.1 we split the original space of grid functions into a sum of subspaces and using Lemma 2.2 and Lemma 2.3, on the fictitious domains method, we give equivalent norms in subspaces. In section 3, we construct a decomposition of the original mesh space into a sum of two subspaces. The functions of the first subspace are equal to zero at the boundaries of subdomains. The second subspace corresponds to the values of mesh functions at the boundaries of subdomains extended inside with conservation of the norm. The second subspace has a complex structure, and to simplify it, we decompose it into a sum of a coarse space and local subspaces [1–3,9,11]. In section 4, using the abstract results, we design preconditioning operators for these subspaces. This design uses the fictitious domain (space) method to overcome two obstacles: complex geometry of the substructures and chaotic distribution of the nodes of the triangulation.

2. General Framework. In this section, we outline an abstract method for constructing a preconditioning operator by splitting the original Hilbert space into a sum of subspaces. Here, we completely follow [5–10].

Lemma 2.1. *Let the Hilbert space H with the scalar product (u, v) be split into a sum of subspaces*

$$H = H_1 + H_2 + \cdots + H_m ,$$

let A: $H \to H$ be a linear, self-adjoint, continuous and positive definite operator, and $P_i : H \to H_i$, $i = 1, 2, \ldots, m$, be the orthogonal projections with respect to the scalar product $(u, v)_A$ generated by the operator A

$$(u, v)_A = (Au, v) .$$

Assume that positive constants α and β exist such that for any element $u \in H$ there exists $u_i \in H_i$ such that

$$u_1 + u_2 + \cdots + u_m = u$$

$$\alpha(\,(u_1,u_1)_A + (u_2,u_2)_A + \cdots + (u_m,u_m)_A) \le (u,u)_A$$

and

$$(\,(P_1 + P_2 + \cdots + P_m)u, u)_A \le \beta(u,u)_A$$

for any $u \in H$. *Assume further that there are selfadjoint operators* B_i, $i = 1, 2, \ldots, m$, *such that*

$$B_i : H \to H_i \ ,$$

$$c_1(B_i u, u) \le (Au, u) \le c_2(B_i u, u) \ , \quad \forall\, u \in H_i \ .$$

Then,

$$\alpha c_1(A^{-1}u, u) \le (B^{-1}u, u) \le \beta c_2(A^{-1}u, u) \ , \quad \forall\, u \in H \ ,$$

where $B^{-1} = B_1^+ + B_2^+ + \cdots + B_m^+$, *and* B_i^+ *is a pseudoinverse of* B_i.

Using Lemma 2.1, we will define the preconditioner in the following way. Split the original space of grid functions W into a sum of subspaces, each having a definite structure which makes it possible to determine easily invertible equivalent norms (i.e., operators B_i) in these subspaces. The following lemmas will be used in constructing operators B_i, which give equivalent norms in subspaces, and for which the multiplication of B_i^+ by the vector can be performed efficiently.

Lemma 2.2. *Let* R^m *and* R^n *be real Euclidean spaces, and* Σ *and* S *be symmetric positive definite matrices of order* m *and* n, *respectively. Denote by*

$$\begin{aligned} (\phi,\psi)_\Sigma &= (\Sigma\phi,\psi) \ , \\ (u,v)_S &= (Su,v) \end{aligned}$$

the scalar products generated by these matrices in R^m *and* R^n, *and let* T: $R^m \to R^n$, *be a linear operator such that*

$$\alpha(\phi,\phi)_\Sigma \le (T\phi, T\phi)_S \le \beta(\phi,\phi)_\Sigma$$

for any $\phi \in R^m$. *Here,* α *and* β *are positive constants. Let*

$$C = T\Sigma^{-1}T^* \ ,$$

where T^* *is an operator adjoint to* T *with respect to the Euclidean scalar products in* R^m *and* R^n. *Then,*

$$\alpha(C^+u, u) \le (u,u)_S \le \beta(C^+u, u)$$

for any $u \in E = \operatorname{Im} T$. C^+ *is a pseudo-inverse of* $\mathcal{C}$.

Lemma 2.3. *Let* H_0 *and* H *be Hilbert spaces with the scalar products* $(u_0, v_0)_{H_0}$ *and* $(u,v)_H$, *respectively, and let* A *and* B *be selfadjoint positive definite and continuous operators in the spaces* H_0 *and* H:

$$A\colon H_0 \to H_0 \ , \qquad B\colon H \to H \ .$$

Let R be a linear operator such that

$$R\colon H \to H_0 \ ,$$

$$(ARv, Rv)_{H_0} \le c_R (Bv, v)_H$$

for any element $v \in H$. Furthermore, let there exist an operator T such that

$$T\colon H_0 \to H \ , \qquad RTu_0 = u_0 \ ,$$

$$c_T (BTu_0, Tu_0)_H \le (Au_0, u_0)_{H_0}$$

for any element $u_0 \in H_0$. Here, c_R and c_T are positive constants. Then

$$c_T (A^{-1}u_0, u_0) \le (RB^{-1}R^* u_0, u_0)_{H_0} \le c_R (A^{-1}u_0, u_0)$$

for any element $u_0 \in H_0$. Here, R^* is the operator adjoint to R with respect to the scalar products $(u_0, v_0)_{H_0}$ and $(u, v)_H$ such that

$$R^*\colon H_0 \to H \ ,$$

$$(R^* u_0, v)_H = (u_0, R_0 v)_{H_0} \ .$$

3. Decomposition into Subspaces. To design the preconditioning operator B, we will completely follow [9] and decompose the space W into a sum of subspaces

$$W = W_0 + W_1 \ .$$

To this end, divide the nodes of the triangulation Ω^h into two groups: those which lie inside of Ω_i^h and those which lie on boundaries of Ω_i^h. The subspace W_0 corresponds to the first set. Let

$$S^h = \bigcup_{i=1}^{n} \partial\Omega_i^h \ ,$$

$$\begin{aligned} W_0 \ &= \{u^h \in W \,|\, u^h(x) = 0 \ , \ x \in S^h\} \ , \\ W_{0,i} \ &= \{u^h \in W_0 \,|\, u^h(x) = 0 \ , \ x \overline{\in} \Omega_i^h\} \ , \quad i = 1, 2, \ldots, n \ . \end{aligned}$$

It is clear that W_0 is the direct sum of the orthogonal subspaces $W_{0,i}$ with respect to the scalar product in $H^1(\Omega)$:

$$W_0 = W_{0,1} \oplus \cdots \oplus W_{0,n} \ .$$

The subspace W_1 corresponds to the second group of nodes Ω^h and can be defined in the following way. First, define $W_{1/2}$ which is the space of traces of functions from W on S^h:

$$W_{1/2} = \{\phi^h \,|\, \phi^h(x) = u^h|_{S^h} \ , \ u^h \in W\}$$

To define the subspace W_1, we need a norm-preserving extension operator of functions given at S^h into Ω^h. Let us consider the subdomain Ω_i^h. Let (s_1, s_2, n) be a near-boundary coordinate system [7,8] which is defined in a δ-neighborhood of $\partial\Omega_i^h$.

Here s_1, s_2 define a point P at $\partial\Omega_i^h$ and n is the distance between the given point and $\partial\Omega_i^h$. Set

$$t\colon H^{1/2}(\partial\Omega_i^h) \to H^1(\Omega_i^h) \ ,$$

$$t\phi = u \ ,$$

$$u(s_1,s_2,n) = (1-\frac{n}{\delta})\,\frac{1}{\text{meas}\,K(P,n)}\int_{K(P,n)} \phi(x)\,dx \ , \qquad (3.1)$$

$$P \in \partial\Omega_i^h \ , \qquad 0 \le n \le \delta \ ,$$

where

$$K(P,n) = \{(s_1',s_2',0) \in \partial\Omega_i^h \,|\, |s_i' - s_i| \le n \ , \ i = 1,2\} \ .$$

The function u is extended by zero in the rest of Ω_i. Using Hardy's inequality [4], it is not difficult to see that t is a norm-preserving operator, i.e.

$$\|u\|_{H^1(\Omega_i)} = \|t\phi\|_{H^1(\Omega_i)} \le c_1\|\phi\|_{H^{1/2}(\partial\Omega_i)} \ , \ \forall\,\phi \in \ \ ,$$

where c_1 is a positive constant. Using the auxiliary mesh, which is topologically equivalent to a uniform mesh, we can define the finite-element analogy t_h of the operator t from (3.1), such that

$$\|t_h\phi^h\|_{H^1(\Omega_i)} \le c_2\|\phi^h\|_{H^{1/2}(\partial\Omega_i)} \ , \quad \forall\,\phi^h \qquad (3.2)$$

The cost of the multiplication of t and t^* by a vector is proportional to the number of mesh domain nodes. Here c_2 independent of h, H and t^* is an adjoint operator to t with respect to the Euclidean scalar product (see [8,9] for more details). Set

$$W_1 = t_h W_{1/2} \ .$$

It is obvious that $W = W_0 + W_1$ and according to (3.2) this decomposition of W satisfies Lemma 2.1 with α independent of h and H.

According to Lemma 2.2, to define the preconditioning operator B_1 for the subspace W_1 we need to define the preconditioning operator $B_{1/2}$ which generates the norm in the space $W_{1/2}$. The set S^h has a complex structure. Therefore we will split $W_{1/2}$ into subspaces associated with the following substructures. Let S^h be a union of k nonoverlapping triangles T_i:

$$S^h = \bigcup_{i=1}^{k} \overline{T}_i \ , \quad T_i \cap T_j = \emptyset \ , \ i \ne j \ . \qquad (3.3)$$

We assume that k and n are of the same order of magnitude. Split the space $W_{1/2}$ into a sum of subspaces

$$W_{1/2} = V_0 + V_1 + \cdots + V_k \ , \qquad (3.4)$$

where V_0 is the coarse space which consists of continuous functions linear on the triangles T_i, $i = 1, 2, \ldots, k$, and V_i, $i = 1, \ldots, k$ correspond to T_i and are defined below. Denote by

$$\begin{aligned} W_{1/2,i} &= \{\phi_i^h \in H^{1/2}(T_i) \,|\, \phi_i^h(x) = \phi^h(x) \ , \ x \in T_i \ , \ \phi^h \in W_{1/2}\} \ , \\ \overset{o}{W}_{1/2,i} &= \{\phi_i^h \in W_{1/2,i} \,|\, \phi_i^h(x) = 0 \ , \ x \in \partial T_i\} \ . \end{aligned}$$

For any triangle T_i, we define the explicit extension operator τ_i:

$$\tau_i \colon W_{1/2,i} \to W_{1/2} \ .$$

Denote by T_i' the union of adjacent triangles:

$$T_i' = \bigcup_{\overline{T}_j \cap \overline{T}_i \neq \emptyset} \overline{T}_j \ .$$

Let ϕ^h be a function given on T_i. We will define the extension of the function ϕ^h using two tricks: *reflection* and *semi-reflection*. First let us define reflection $\tilde{\phi}^h$ of the function ϕ^h. Let T_j be a triangle from T_i', which has a common side with T_i (see Fig. 3.1a).

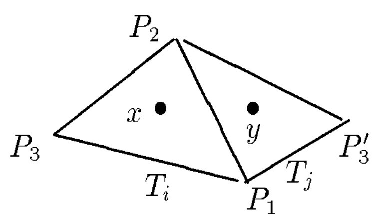

Figure 3.1

Let u, v, w, $u + v + w = 1$ be the barycentric coordinates on the triangle T_i. Then any point x from T_i can be represented in the following form

$$x = u \cdot P_1 + v \cdot P_2 + w \cdot P_3$$

where P_1, P_2, P_3 are the vertices of the triangle T_i. Set

$$\tilde{\phi}(y) \equiv \tilde{\phi}(u \cdot P_1 + v \cdot P_2 + w \cdot P_3') = \phi(c) \equiv \phi(u \cdot P_1 + v \cdot P_2 + w \cdot P_3)$$

Now let us define semi-reflection $\tilde{\tilde{\phi}}$ of the function ϕ given on T_i for the case when a triangle T_j has a vertex common with the triangle T_i (see Fig. 3.2).

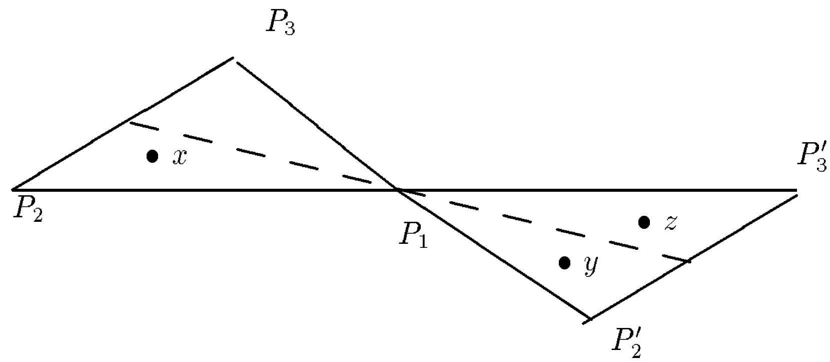

Fig. 3.2

Set

$$\begin{aligned}\tilde\phi(z) &\equiv \tilde\phi(u\cdot P_1 + v\cdot P_3' + w\cdot 0.5\cdot(P_2'+P_3'))\\ &= \tilde\phi(y) \equiv \tilde\phi(u\cdot P_1 + v\cdot P_2' + w\cdot 0.5\cdot(P_2'+P_3'))\\ &= \phi(x) \equiv \phi(u\cdot P_1 + v\cdot P_2 + w\cdot 0.5\cdot(P_2+P_3))\ .\end{aligned}$$

It is clear that we can define the "semi-reflection" for the case when a triangle T_j has a side common with the triangle T_i too. Combining both approaches, we can define an extension $\tilde\phi^h$ of the function ϕ^h onto T_i. Using the piecewise linear cutoff function ξ_i defined by

$$\xi_i(x) = \begin{cases} 1\ , & x\in T_i\\ 0\ , & x\notin T_i'\ ,\end{cases}$$

we can define the extension operator τ_i by

$$\tau_i\phi^h = \xi_i\tilde\phi^h\ .$$

Set

$$V_i = \tau_i W_{1/2,i} + \overset{\circ}{W}_{1/2,i}\ , \quad i = 1,2,\ldots,k\ . \tag{3.5}$$

Using Lemma 4.1 from [9] it is easy to see that the decomposition (3.4), (3.5) satisfies Lemma 2.1 with α independent of h and H.

4. The Fictitious Space Method. In this section we will design a preconditioning operators for the subspaces $W_{0,i}$, $i = 1,2,\ldots,n$, and V_i, $i = 0,1,\ldots,k$ constructed in Section 3 using Lemmas 2.2 and 2.3.

There are two obstacles to using the finite-element approximation of the Laplace operator as a preconditioning operator for $W_{0,i}$. The first is related to the geometry of Ω_i while the second relates to the chaotic distribution of nodes of the triangulation Ω_i^h.

Let the domain Ω_i be embedded in the cube Π_i. We extend the triangulation Ω_i^h to the entire cube Π_i^h in such a way that Π_i^h is a quasi-uniform. We will denote by $W(\Pi_i^h)$ the space of continuous functions which are linear on each simplex of the triangulation Π_i^h, and by $\Delta_{\Pi_i^h}$ the approximation of a symmetric elliptic operator in $W(\Pi_i^h)$. According to Lemma 2.3, we need a continuous operator

$$R_{\Pi_i}\colon W(\Pi_i^h) \to W_{0,i} \ .$$

Let

$$R_{\Pi_i} = I_{\Omega_i} - t_h \cdot I_{\partial\Omega_i} \ ,$$

where I_{Ω_i}, $I_{\partial\Omega_i}$ are the trace operators from Π_i^h onto Ω_i^h, $\partial\Omega_i^h$, respectively. The operator t_h is the extension operator which was described in section 3. From the continuity of I_{Ω_i}, $I_{\partial\Omega_i}$ and t_h, we obtain the continuity of R_{Π_i}. We can define the operator

$$T_{\Pi_i}\colon W_{0,i} \to W(\Pi_i^h) \ ,$$

as an extension by zero. It is easy to see that the hypotheses of Lemma 2.3 are valid. Hence there exist positive constants c_1 and c_2, independent of h and H, such that

$$c_1\|u^h\|^2_{h^1(\Omega_i)} \le (\,(R_{\Pi_i}(-\Delta_{\Pi_i^h})^{-1}R^*_{\Pi_i})^{-1}u, u) \le c_2\|u^h\|^2_{H^1(\Omega_i)}$$

for any $u^h \in W_{0,i}$.

Further, along with the mesh Π_i^h in the cube Π_i we will also consider an auxiliary uniform mesh Q_i^h with a mesh step h_0: $c_3 h \le h_0 \le h_{\min}/\sqrt{3}$. Here $h_{\min}$ is the length of the minimal edge of simplices of the triangulation Π_i^h, and constant c_3 is independent of h and H. Denote the nodes of the mesh Q_i^h by $Z_{j,k,\ell} = (X_j, Y_k, Z_\ell)$, $j,k,\ell = 0,1,\ldots,M_i$ and denote the cells of the mesh Q^h by

$$Q_{\ell,j,k} = \{(x,y,z)\,|\,X_j \le x < X_{j+1} \ , \ Y_k \le y < Y_{k+1} \ , \ Z_\ell \le z \le Z_{\ell+1}\} \ , \\ \ell, j, k = 0,1,\ldots,M_i - 1 \ .$$

On the mesh Q_i^h, we will consider the space $W(Q_i^h)$ of mesh functions $U(Z_{j,k,\ell})$. Let us define the operator R_{Q_i}:

$$R_{Q_i}\colon W(Q_i^h) \to W(\Pi_i^h)$$

which introduce a correspondence of each function $U(Z_{j,k,\ell}) \le W(Q_i^h)$ to a function $u^h \in W(\Pi_i^h)$ in the following way. Let z_m be a node of the triangulation Π_i^h and let $z_m \in Q_{j,k,\ell}$. Let

$$u^h(z_m) = U(Z_{j,k,\ell}) \ .$$

Note that by the assumptions on h_0 at most one node z_m of the triangulation Π_i^h can belong to the cell $Q_{j,k,\ell}$. The operator T_{Q_i},

$$T_{Q_i}\colon W(\Pi_i^h) \to W(Q_i^h)$$

is now defined as follows. If the cell $Q_{j,k,\ell}$ contains a node z_m of the triangulation Π_i^h, we set

$$U(Z_{j,k,\ell}) = u^h(z_m) \ .$$

At the other nodes of the mesh Q_i^h the function $U(z_{j,k,\ell})$ can be defined in a relatively arbitrary way, for instance, as follows. Let the node $z_{j,k,\ell}$ belong to the simplex τ_m of the triangulation Π_i^h with the vertices $z_{m_1}, \ldots, z_{m_4}$. Set

$$U(z_{j,k,\ell}) = \frac{1}{4}(u^h(z_{m_1}) + \cdots + u^h(z_{m_4})\ ,) \ .$$

It can be easily shown that the operators R_{Q_i}, T_{Q_i} defined in this way satisfy the hypotheses of Lemma 2.3 and that the corresponding constants do not depend on h and H. Thus, there exist positive constants c_4, c_5, independent of h and H, such that

$$c_4\|u^h\|^2_{H^1(\Omega_i)} \leq (\,(R_{\Pi_i}R_{Q_i}(-\Delta_{Q_i})^{-1}R^*_{Q_i}R^*_{\Pi_i})^{-1}u, u) \leq c_5\|u^h\|^2_{H^1(\Omega_i)}$$

for any $u^h \in W_{0,i}$. Here Δ_{Q_i} is the mesh approximation of a symmetric elliptic operator in $W(Q_i^h)$. The preconditioning operator $B^+_{0,i}$ for $W_{0,i}$ can now be defined as an extension by zero of the operator $R_{\Pi_i}R_{Q_i}(-\Delta_{Q_i})^{-1}R^*_{Q_i}R^*_{\Pi_i}$. Set

$$B^+_0 = B^+_{0,1} + \cdots + B^+_{0,n} \ .$$

We now define the preconditioning operator B^+_1 for the subspace W_1. According to Lemma 2.2, we can define

$$B^+_1 = t_h\Sigma^{-1}t^*_h$$

where the operator Σ generates an equivalent norm in the trace space $W_{1/2}$ and the extension operator t_h was described in section 3. To define the operator Σ we will use the decomposition (3.3). Let us first consider the coarse subspace V_0. Let W_0 be a mesh space which corresponds to the values of the functions $\phi^h \in V_0$ at the nodes of the triangles T_i from (3.3). Define, in the space W_0, the symmetric operator S_0:

$$(S_0\phi, \phi) = \sum_{i=1}^{k}(\sum_{j=1}^{3}(H^3(\phi_i^{(j)})^2 + H(\phi_i^{(j)} - \phi_i^{(j-1)})^2)\ ,$$

where $\phi_i^{(1)}$, $\phi_i^{(2)}$, $\phi_i^{(3)} = \phi_i^{(0)}$ are the values of the function $\phi^h \in V_0$ at the nodes of the triangle T_i. The dimension of the operator S_0 is equal to the number of the nodes which belong to $S^h \setminus \Gamma_0$. Let τ_0 be a piecewise linear interpolation operator:

$$\tau_0\colon W_0 \to W_{1/2} \ .$$

According to Lemma 2.2, we can set

$$\Sigma^+_0 = \tau_0 S_0^{-1}\tau_0^* \ .$$

Let us now consider the subspaces V_i, $i = 1, \ldots, k$. From Lemmas 2.1, 2.2 and the definition of V_i, we can define

$$\Sigma^+_i = \tau_i S^{-1}_{i,N}\tau^*_i + S^+_{i,\mathcal{D}}\ ,$$

where $S_{i,N}$, $S_{i,\mathcal{D}}$ generate norms in $W_{1/2,i}$ and $\overset{\circ}{W}_{1/2,i}$, respectively. To define $S_{i,N}$ and $S^+_{i,\mathcal{D}}$ we will again use Lemma 2.3. Let T_i be embedded into the square Π_i and the triangulation of T_i be complemented to triangulation Π_i^h of the larger region. In the square Π_i, we will also consider the auxiliary square mesh Q_i^h with the step h_0: $c_6 h \le h_0 \le h_{\min}/\sqrt{2}$. Let $W(Q_i^h)$ be a space of mesh functions which is defined on Q_i^h and let Σ_{Q_i} be an operator which generates an equivalent norm in the trace space $W(Q_i^h)$. According to Lemma 2.3, we can define

$$\begin{aligned} S_{i,N} &= R_{N,\Pi_i} R_{Q_i} \Sigma_{Q_i}^{-1} R_{Q_i}^* R_{N,\Pi_i}^* \ , \\ \widetilde{S}_{i,\mathcal{D}} &= R_{\mathcal{D},\Pi_i} R_{Q_i} \Sigma_{Q_i}^{-1} R_{Q_i}^* R_{\mathcal{D},\Pi_i} \ , \end{aligned}$$

where

$$R_{N,\Pi_i} = P_i (I_i O_i) P_i^* \ .$$

Here I_i is the identity matrix which corresponds to nodes belonging to $\overline{T}_i$, O_i the nullmatrix which corresponds to nodes belonging to $\Pi_i \setminus \overline{T}_i$, and P_i a permutation matrix. The operators $R_{\mathcal{D},\Pi_i}$, R_{Q_i} are defined exactly as were those used when constructing $B_{0,i}$ in the case of three dimensions. The operator $S_{i,\mathcal{D}}$ is the extension of the operator $\widetilde{S}_{i,\mathcal{D}}$ by zero. We can thus define

$$\begin{aligned} \Sigma^{-1} &= \Sigma_0^+ + \Sigma_1^+ + \cdots + \Sigma_k^+ \ , \\ B_1^+ &= t_h \Sigma^{-1} t_h^* \ , \\ B^{-1} &= B_0^+ + B_1^+ \end{aligned} \tag{4.1}$$

The following theorem is valid.

Theorem. *There exist positive constants c_7, c_8, independent of h and H, such that*

$$c_7 (Bu, u) \in (Au, u) \in c_8 (Bu, u) \ .$$

Here A is defined by (1.3) *and B is defined by* (4.1).

REFERENCES

1. J. H. Bramble, J. E. Pasciak and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring*, I. Math. Comp. 47 (1988), 103–134.

2. M. E. Dmitrienko and L. A. Oganesian, *Version of the Schwarz method for adjacent grid domains, in Computations with Sparse Matrices*, Novosibirsk, 1981, pp. 33–44 (in Russian).

3. M. Dryja and O. Widlund, *Some domain decomposition algorithms for elliptic problems, Iterative Methods for Large Linear Systems*, L. Hayes and D. Kincaid, eds., Academic Press, Orlando, Florida, 1989.

4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, 1959.

5. P. L. Lions, *On the Schwarz alternating method*, I, First International Symposium on Domain Decomposition Methods for Partial Differential Equations, P. Glowinski, G. H. Golub, G. Meurant and J. Peraiux, eds., SIAM, Philadelphia, 1988.

6. A. M. Matsokin and S. V. Nepomnyaschikh, *Schwarz alternating method in subspaces*, Soviet Mathematics, 29 (10) (1985), pp. 78–84.

7. A. M. Matsokin and S. V. Nepomnyaschikh, *Norms in the space of traces of mesh functions*, Sov. J. Numer. Anal. Math. Modeling, Vol. 3, No. 3 (1988), pp. 199–216.

8. S. V. Nepomnyaschikh, *Method of splitting into subspaces for solving elliptic boundary value problems in complex form domains*, Sov. J. Numer. Anal. Math. Modeling, Vol. 6, No. 2 (1991), pp. 1–20.

9. S. V. Nepomnyaschikh, *Application of domain decomposition to elliptic problems with discontinuous coefficients*, Preprint No. 891, Comp. Cent. Sib. Branch, USSR Acad. Sci., Novosibirsk,1990 (in Russian), to appear in Proc. 4th Intl. Symp. on Domain Decomposition, held in Moscow, USSR, May 1990, R.Glowinski, Yu. Kuznetsov, G. Méurant, J. Périaux and O. Widlund, eds., SIAM, 1991.

10. S. V. Nepomnyaschikh, *Mesh theorems of traces, normalizations of function traces and their inversion*, preprint, No. 930, Comp. Cent. Sib. Branch, USSR Acad. Sci. Novosibirsk, 1991 (in Russian). See also: Sov. J. Numer. Anal. Math. Modeling, Vol. 6, No. 3 (1991), pp. 1–25.

11. B. Smith, *Domain Decomposition Algorithm for the Partial Differential Equations of Linear Elasticity*, Ph.D. Thesis, Courant Inst., New York Univ., New York, NY 10012, 1990.