The Kinematics of Cutting Solid Objects

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Abstract

This paper studies how the cutting of one solid object by another can be described in a formal theory. We present two alternative first-order representations for this domain. The first views an object as gradually changing its shape until it is split, at which time the original object ceases to exist and two (or more) new objects come into existence. The second focusses instead on chunks of material which are part of the overall object. A chunk persists with constant shape until some piece of it is cut away, when the chunk ceases to exist. We prove that the two theories are equivalent under ordinary circumstances, and we show that they are sufficient to support some simple commonsense inferences and algorithms.

1 Introduction

Previous AI studies of reasoning about the physics of solid objects (e.g. [Davis, 88], [Joskowicz, 87], [Faltings, 87]) have, almost without exception, assumed that solid objects are rigid and immutable. The only properties that can change over time are position and its concomitants, such as velocity and energy. A full commonsense theory of solid objects must deal with a range of phenomena that violate this condition, such as bending, breaking, and cutting. This paper deals with the cutting of one solid object, called the *target*, by another, called the *blade*. We show how the geometric aspect of various cutting operations — slicing an object in half, cutting a notch into an object, stabbing a hole through an object, and carving away the surface of an object — can be described in a first-order theory. Our theory characterizes the intermediate states that take place during a cutting process and the geometric relations between the shapes and motions of the blades and targets. It allows great freedom in the combinations of cutting operations that may take place concurrently: a blade may cut many targets at once, a target may be cut by many blades at once, an object may simultaneously be cutting at one end and being cut at another, and so on.

In fact, we present two alternative representations for cutting. The first views an object as gradually changing its shape until it is split, at which time the original object ceases to exist and two (or more) new objects come into existence. The second focusses instead on chunks of material which are part of the overall object. A chunk persists with constant shape until some piece of it is cut away, when the chunk ceases to exist. Under ordinary circumstances (which we will define formally below) the two theories are provably equivalent. Our theories support commonsense inferences about cutting, such as "An object cannot be cut if it is isolated from other objects," and "It is not possible to cut an internal cavity using a blade outside the target." They allow the results of a cutting

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operation to be computed given the initial shape of the targets, the shape of the blades, and the motions of the targets and blades.

A number of limitations of our study should be noted:

- We assume that the cutting operation works by removing and destroying of the material of the target in the path of the blade, rather than by pushing it aside. Thus, the image we are using is more like a saw cutting its way through a board, rather than like a wedge being driven into a crack. Though this is not actually true to life — in most cases, cutting involves some distortion of the material being cut, and in many, such as a knife going into cheese, it involves virtually no destruction of material — still, if the blade is sufficiently thin, it gives a good approximation.

  To go beyond this restriction would require a theory of small distortions; that is, bending. Such a theory will, of course, ultimately be necessary, but appears to be difficult. It is hard to characterize bending and its limits without using the language of partial differential equations, which is unintuitive and does not (at least easily) support qualitative reasoning.

- We deal only with the *kinematics* of cutting, the relations among the positions and shapes of the objects involved, not with its *dynamics*, the forces and velocities required for cutting. In general, dynamic theories are much more difficult than kinematic theories; the formulation of a dynamic theory of rigid objects useful for commonsense inference is still very much an open problem [Davis, 88]. There is also a close relation between this restriction and the previous, since much of the force on a blade comes from the elastic resistance of the target to being bent, and this, of course, can only be characterized in a theory that incorporates the bending of the target.

- We do not consider any restrictions on the shape of the objects involved; we do not require that the blade be sharp. In the absence of a dynamic theory, such restrictions would be superficial and *ad hoc.* Restriction on the materials of the objects are taken as optional. In many commonsense environments, such as cooking, it can be assumed that a hard object, like a knife, is cutting a soft object, like cheese, and this assumption can often simplifies the inference process. However, adding or dropping the assumption makes only a small difference to the structure of the theory, and we will consider both versions.

Section 2 of this paper gives an informal account of the two theories. Section 3 presents a simple algorithm that can be used to calculate the result of cutting given complete knowledge. Section 4 discusses the pros and cons of the two theories. Section 5 discusses some technical issues that arise with certain anomalous cases, and gives a precise definition of the "ordinary circumstances" under which the two theories are equivalent. Section 6 presents the formal theories in first-order language. Section 7 proves that the two theories are equivalent. Section 8 gives an example of commonsense inferences that can be justified by the theories.

2 The Ontology

The major difference between a microworld of immutable objects and one in which cutting occurs is that objects can be created, destroyed, and changed in shape. Our two theories differ in their approach to the conception of these operations and to the identity of objects over time.

As a preliminary step, we observe that, since for our theory to be first-order, existence in the sense of the existential quantifier cannot be time-dependent. Thus, since we wish objects to be denoted by terms and variables, they must be logically be eternal. Whether an object is present "in the flesh" at a given moment is a fluent (a state). Thus if p is a term denoting an object, we suppose that p denotes a ghost at times when the object is not present. Ghosts need not be constrained by physical laws.

In our first theory, we consider a target being cut by a blade as retaining its identity, but changing its shape, up to the moment that it falls into pieces. At the moment when it comes into separate pieces, the original target ceases to exists and each piece becomes a new object. We will call this the "mutable object" theory. (Figure 1)

This conception of object identity does not correspond precisely to the intuitive notion. Intuitively, if a small chip is cut off a large object, the object persists in the large remaining piece, while our theory says that the original object is replaced by two wholly new pieces. Conversely, if a large object is filed down to a small one without ever cutting off a separate piece, our theory will say that the identity of the object remains the same, while intuitively one might say that it has changed. However, our theory is probably as close as one can come to the intuition without drawing arbitrary dividing lines. (How large a chip can be cut off? How much can be filed away?)

There are, then, three kinds of state change in this theory: the shape of an object is cut away; an object comes into existence; and an object ceases to exist. The dynamics of the theory thus consist primarily of specifications of the circumstances and extent of each of these changes. The rules are as follows:

- MO.1 Change of shape: If an object O persists from time $S1$ to $S2$ then its shape in $S2$ is equal to its shape in $S1$ minus the set of all points occupied by some blade between $S1$ and $S2$. (As we will discuss below, boundary points require special care.)
- MO.2 Sufficient condition for destroying and creating objects: If the shape of O is disconnected at time S, then O is a ghost at S and all later times, and each connected component of the shape of O becomes a real object. (We assume that O persists in the same place for the instant S that it is disconnected, though it is a ghost.)
- MO.3 Necessary condition for destruction: If O is real at time $S1$ and a ghost at time $S2 > S1$, then the shape of O is disconnected at some time $S3 \in (S1, S2]$.
- MO.4 Necessary condition for creation: If O is a ghost at time $S1$ and real at time $S2 > S1$, then O came into existence as a connected component of some disconnected object $O2$ at time $S3 \in (S1, S2]$

Additional restrictions on the objects involved can be added as conditions in rule MO.1. For example, the rule that the blade must be harder than the target can be imposed by rewriting MO.1 in the form

- MO.1a If an object O persists from time $S1$ to $S2$ then its shape in $S2$ is equal to its shape in $S1$ minus the set of all points occupied by some blade OB harder than O between $S1$ and $S2$.

Our second theory, called the "immutable chunk" theory, starts with the observation that it should be possible to view all three types of change — creation, destruction, and change of shape — as consequences of a single type of change, namely the destruction of material. If the destruction of the material of an object does not disconnect it, the shape of the object changes; if it does disconnect it, the object changes identity. But locally, at the contact point between the blade and the target, the two look exactly the same, and it should be possible just to characterize the local change, and deduce the global change from that.

Our new theory, in effect, merges reshaping and creation into destruction. We can eliminate creation as a separate process if we take the point of view that the two new pieces *were always* *there;* they were just entrapped inside the larger object. When the larger object is destroyed they are liberated, and free to move separately. Similarly, we can assimilate reshaping into destruction by viewing the new shape as having always been latent in the old shape, and being revealed by the destruction of the old shape.

Thus, we view each object as having latent within it all possible pieces, called *chunks,* that could be cut out. Every reasonable shape (to be defined below) inside the object is a chunk. Cutting has the effect of destroying all the chunks that are cut into. The chunks that are visible at any given moment are those that have been "liberated" by the destruction of all the chunks that contain them. Sometimes, this destruction of chunks will leave one visible chunk; sometimes, it will leave several. In the former case, the object is reshaped; in the latter, it is split. (Figure 2)

During the process of cutting, a continuous infinitude of latent chunks become real for one single instant, and then immediately are cut into and become ghosts. Each such is latent over the interval $(-\infty, T0)$, real for the single instant $T0$, and a ghost for the remaining interval $(T0, \infty)$. There is another infinitude of latent chunks that become ghosts without ever being real, because they are cut into without ever being fully cut out.

The remaining features of the theory are easily fit into this framework. At any given time, there are two primary classes of chunks: *material* chunks, which includes both latent and visible chunks, and *ghosts*. Material chunks are organized in a hierarchy of *sub-chunk* relations; C1 is a sub-chunk of C2 if the region occupied by C1 is a subset of the region occupied by C2. A *visible* or *top-level* chunk is a material chunk that is maximal in the sub-chunk hierarchy; non-maximal chunks are *latent*. Every latent chunk is a sub-chunk of some top-level chunk; every reasonable (still to be defined) subset of the region occupied by a top-level chunk is occupied by some latent chunk. Latent chunks are constrained to move together with the top-level chunk that contains them.

As a target is cut, the current top-level chunk is continually turned into a ghost, thus lowering the top level down to one (or more) of its sub-chunks. The top level thus moves steadily down the sub-chunk hierarchy. The sub-chunks of the new top-level chunk remain latent; chunks in the hierarchy that are not its sub-chunks become ghosts. The target is split when the cutting process splits the live part of the hierarchy into two. (Figure 3)

In this theory there is only one primitive type of change (aside from change of position, which is the same in both theories): the change of a chunk from material to ghost. The dynamics of the theory consist of two rules stating that a chunk becomes a ghost just if it is penetrated by a blade.

IC.1 1. If $C1$ is material at time $S1$ and a ghost at time $S2$, then,
(a) $S1$ precedes $S2$; and
(b) There is a time $S3 \in (S1, S2]$ and a chunk $C2$ such that $C2$ is top-level at $S3$, $C1$ is not a sub-chunk of $C2$, and $C1$ intersects $C2$ at $S3$,

IC.2 If $C1$ and $C2$ are distinct top-level chunks at time S, then they do not intersect at S.

A number of features of these rules are noteworthy:

- IC.1 serves as a frame axiom; that is, a necessary condition for $C1$ to turn from a material chunk to a ghost. IC.2, in its contrapositive, "If $C2$ is top-level and intersects $C1$, then $C1$ is a ghost" serves as a causal axiom; it gives a sufficient condition for $C1$ to be a ghost.
- IC.2 is just the basic rule of solid object kinematics that real objects may not intersect.
- IC.1 incorporates the condition that chunks cannot change from ghosts to material.
- Additional necessary conditions on cutting, such as a rule that the blade must be hard and the target must be soft, can be added as additional consequences in rule IC.1.

It should be noted that, in this model, a blade can completely annihilate a target by pushing through its entire extent. In fact, if we do not require that the blade be harder than the target, two objects can mutually annihilate by pushing into one another, like two soft snowballs being crushed together In order to bring the objects and chunks theories into close correspondence, we must either eliminate this possibility from the chunks theory or add it to the objects theory. The latter turns out to be easier; it suffices to define a "vanishing shape" to be either a disconnected shape or the null shape, and then to replace "disconnected" by "vanishing" in rules MO.3 and MO.5.

We still need a definition of a "reasonable" shape for both objects and chunks. Certainly, an object must occupy a connected region, except at the instant when it is split; disconnected pieces do not move in concert. We do not allow infinite objects ; an object must occupy a bounded region. Also there are technical advantages to requiring the shape of a object to be an *open* region, contrary to most previous practice (e.g. [Requicha, 80], [Davis, 88]) which has been to use normal regions. We must prohibit the object from having isolated points or lower-dimensional slits missing; technically, we require the shape to be equal to the interior of its closure. Finally, there is no point in allowing empty objects. These conditions will suffice for our purposes; we will allow a material object or a chunk to occupy any non-empty bounded, connected region that is equal to the interior of its closure. Such a region will be said to be "well-shaped" or "proper."

Since we are taking the shapes of objects to be open regions, we must view the blade as annihilating the points of the target on its boundary, as well as those in its interior.

3 Algorithm

Suppose that blades are hard and targets are soft, and that we have some means of calculating the motion of each object at each time. (Note that it is possible to know the motion of the targets in advance without specifying how many targets there are at each instant. For instance, it may be known that all targets are motionless.) Then the following simulation algorithm, using uniform time steps, will allow us to predict the effect of the cutting operations.

```
Input: A finite set of objects, characterized as blades or targets.
        A specification of the shapes of every object in the starting situation s0.
        The time of some ending situation s1.
        A specification of the motions of every object between s0 and s1.
Output: The characterization of the behavior of the objects from s0 to s1.
Constant: Δ, a small increment of time.

begin
   sb := s0;
   history := the specification of s0;
   loop until sb ≥ s1
        se := sb + Δ;
        For each blade ob, mark that ob is material in se,
            and that its shape in se is the same as in sb.
        For each target ot do begin
            Set rr := the shape of ot in sb;
            For each blade ob, do begin
                Calculate the swath swept out by ob between sb and se,
                    relative to the coordinate system of ot
                    using given motions of ot and ob between se and sb;
                Set rr := rr − the swath computed above.
                end;
            If rr is connected then ot is material in se, and its shape in se is rr
            else mark ot as a ghost in se;
                for each connected component re of the calculated shape of ot in se do
                    create a new object or;
                    The shape of or in se is re and or is material in se.
        end;
        Add the specification of sb to history;
        sb := se;
        end loop
end
```

The algorithm is just a piecewise linear approximation to the true behavior. It is evident that, given the proper well-behavedness conditions, this algorithm converges to the correct answer as the time step Δ goes to zero. Specifically, for Δ sufficiently small, the objects enumerated will correspond one-one to the real objects; their times of creation and destruction will be very near to the real times; and their shape at each time will be very close to their real shape at that time.

The geometrical operations involved — computing the swath swept out by a blade relative to a target, computing the difference between the shape of the target and the swath of the blade, finding the connected components of the difference — are complex but within the scope of existing solid modelling systems [Hoffmann, 90].

4 Comparison of the Two Theories

The main, obvious, advantage of object theory over chunk theory is that over a finite interval of time there are typically uncountably many top-level chunks, while there are only finitely many objects. Thus, a history can be characterized by (1) enumerating the objects that exist; (2) stating when one object is broken off another; and (3) characterizing the changing shape of each object during the time that it is material. By contrast, a representation in terms of chunks must either focus on some specific chunks of interests or it must characterize the continual change of top-level across the infinite space of chunks. The former representation necessarily only partially constrains the history, and some good heuristics will be needed to choose the chunks to focus on. The latter representation is hard to achieve in some way significantly different from bringing objects back in.

The algorithm in section 3 is most naturally presented in the terms of object theory, and, except for the technical difficulties of showing convergence as Δ goes to zero, is easily justified from object theory. It is possible to "translate" this into the language of chunks by changing every record of an object into a record of a top-level chunk; a new top-level chunk is created each time a target changes shape. But this is obviously a clumsy contrivance. Moreover, the justification of the algorithm, and even the definition of convergence, is much more difficult in chunk theory.

Formally, the dynamic structure of chunk theory seems significantly simpler than object theory. As mentioned above, there is essentially only one kind of change in chunk theory, while there are three in object theory. Some types of inference, including the example in section 8, seem to be much easier to justify in terms of chunks than in terms of objects. There is no reason to think that this will be true of commonsense inferences in general, however. Certainly, we will observe in section 7 that proving the correctness of object theory from the axioms of chunk theory is significantly harder than proving chunk theory from object theory.

There are cases where chunk theory seems to be much more suited than object theory. that a sculptor is carving a block of marble on one side while his assistant is carving it on the other side. Then each time the assistant knocks off a piece, the object the sculptor is carving changes its identity. This will generate a large number of quite irrelevant object changes. If the actions of the assistant are not known to the sculptor, then it is not clear how the representation should be structured. Worse yet, the sculptor may not know whether his own actions are creating new objects or merely changing the shape of the existing object; it may depend on connections in places where he cannot see. By contrast, the chunks with which the sculptor is interacting can be characterized in ways that are independent of anything that is happening beyond his view. I do not know how this feature could be used in an implementation.

5 Technical Anomalies

Consider the situations shown in figure 4. In each of these, one small piece on the right is cut off the large block on the left at time t=1/2; then another is cut off at time t=3/4; then another at time t=7/8; . . . What do our theories predict will happen at time t=1 under these peculiar circumstances?

Chunk theory does exactly the right thing. The chunk corresponding to the shape on the left is never penetrated; hence it survives up through time t=1. None of its super-chunks is left unpenetrated; hence it is top-level at time t=1.

Object theory, on the other hand, cannot find this reasonable answer. If a new object, consisting of the material on the left, were to come into existence at time t=1, then by rule MO.4, it must be a piece of some object that ceases to exist at t=1. But there is no object that ceases to exist at t=1; there is one object that ceases to exist at t=1/2, another that ceases to exist at t=3/4 . . . Hence, no new object can come into existence at t=1. What the object theory does allow is that the material on the left should simply disappear at time t=1. The persistence of material that is not carved out by a blade is guaranteed by rule MO.1 while an object persists and by rule MO.2 when an object is split, but there is no rule that guarantees the persistence of material through an infinite sequence of cuttings.

Figure 5, which is similar to the situation in figure 4.a, but taken from the other direction, reveals an even more peculiar anomaly. In figure 5.a, a blade starts to cut into an object in such a way that a piece will be split at t=1/2, and, before that, at t=1/4, at t=1/8 .... The chunk theory accepts this without trouble. The object theory, perhaps reasonably, simply disallows it; the blade provably *cannot* carry out the specified movement. If it did, the original object would persist through time t=0, when it is not split, and then be a ghost at all later times; this is ruled out by MO.3. What is less reasonable of object theory is that it does allow the following astonishing behavior: a blade may be moving along through empty space, and then, suddenly find that it has started to cut a figure like that in figure 5.b, which has appeared literally out of nowhere. Again, rules MO.1 and MO.2 disallow the appearance of material out of nowhere while an object exists, or at the moment when it is cut, but they do not prevent it at times that immediately precede an infinite sequence of cuttings.

Cannot the object theory be reformulated to avoid these anomalies? Certainly it can, but it would seem to require the ability to refer to persisting regions of material; in other words, to chunks. Incorporating the theory of chunks into the theory of objects largely defeats the point of having a separate theory of objects. Instead, we have chosen to posit that situations such as figures 4 and 5 are impossible. Specifically, we assert that in any finite interval and bounded interval there can only be finitely many times at which an object changes status. The formal statement is given in axiom OB.9 below; it is analogous to axiom 9 of [McDermott, 82]. Imposing this condition has the effect of limiting the possible shapes, positions, and motions of objects. These and other consequences of this kind of restrictions will be considered at length in a forthcoming paper [Davis, in prep.]

In a certain sense, this anomaly is not surprising. The appeal of the object theory is that it enables one to reason forward in time in simulation style, tracing the changes in shape until a split occurs, calculating the new shapes, and proceeding. But clearly this technique cannot be applied in cases where one must reason through infinitely many splits.

A second difficulty is illustrated in figure 6. Suppose object O is whole at time t=0. A bullet $OB1$ goes through a hole that splits O at time t=1. But $OB1$ has been preceded by another bullet $OB2$ that went through the same hole at time t=1/2, and by a bullet $OB4$ that went through the same hole at time t=1/4, and by a bullet $OB8$ that went through the same hole at time t=1/8 ...(The case where a single bullet goes through a hole infinitely often in finite time is ruled out by the rule that each object moves continuously, but that does not apply to infinitely many objects.) In chunk theory, this is unobjectionable; in object theory, it is impossible, since there is no moment when the old object disappears and the new objects appear.

Either solution is acceptable in itself, but object theory is here strictly stronger than chunk theory. To establish that the two theories are equivalent, it is necessary to rule out this case. The only reasonable way I have found of stating this restriction is to posit that in any bounded time interval and bounded spatial region, there are only finitely many objects. Physically, this is a sensible restriction. However, I have not found any way of stating this restriction in object theory without using either set theory or higher-order logic; nor have I found any way of stating it in chunk theory without introducing objects as a construct. Therefore, I have not added this restriction as a component of either object or chunk theory at the object level. Rather, I have use it only as a condition in the proof that the two theories are equivalent; specifically, as a condition in the proof of theorem 4, that the axioms of object theory follow from chunk theory (section 7.5).

Finally, there is a third, related anomaly that must be considered. The changing position of an object or chunk in space is a continuous function of time. Moreover, if an object O is split at time S and $O1$ is a piece of O, then the position of $O1$ at S must be the limit of its position from previous times. Hence the position of O approaches a limit as time approaches S from previous times. But what happens if O disappears at S? Must the position of O converge to a limit at S? Or can the position of O fail to converge at S; i.e. either wander off to infinity or moves around wildly within a bounded space (figure 7)? Within object theory, it makes better sense to suppose that the position of O converges at S; this is both easier to state, and more consistent with the principle outlawing infinite loops in finite time. However, this restriction is not at all a natural one within chunk theory.[1] The problem is that no single chunk is behaving anomalously; each chunk turns into a ghost strictly before S, and therefore moves continuously throughout its lifetime. In this case we have decided not to enforce the restriction. Rather, to maintain the equivalence of the two theories, we have modified the rule for objects to read that the position of object O is continuous at S unless O vanishes at S.

6 Formal theories

In this section, we give a first-order representation and axiomatization of object and chunk theory. Our theory is an extension of the kinematics of extended solid object presented in [Davis, 90]. As in [Davis, 90], we use a sorted logic; we use lower-case symbols for non-logical symbols and italicized capital symbols for variables; we take free variables to be universally quantified; and we will indicate the sort of a variable by its first letter. Space is taken to be R^2 or R^3. Time is taken to be R^1. The theory will work equally well with the branching time of [McDermott, 82], in which each individual chronicle has the topology of the real line. Our temporal notation uses fluents as first-order entities. We use the temporal primitives "true_in(S, F)", a predicate meaning that Boolean fluent (state) F holds in situation S; "value_in(S, F)", a function giving the value of fluent F in situation S; and $S1 < S2$, a predicate meaning that situation $S1$ precedes $S2$. We adopt the following notational convention: If $\beta(\tau_1 \ldots \tau_k)$ is a term denoting a fluent, then the situation may be added as a final argument.

$$\beta(\tau_1 \ldots \tau_k, S) = \text{value\_in}(S, \beta(\tau_1 \ldots \tau_k))$$

For example, "place(O)" is a fluent giving the region occupied by an object O in each situation. We may write either "value_in(s1,place(o1))" or "place(o1,s1)" to mean the region occupied by object o1 in situation s1.

We also introduce, for convenience, the predicate, "just_before(S, F)" meaning that fluent F was true in some open interval ending in S.

$$\text{just\_before}(S, F) \Leftrightarrow \exists_{S1} \forall_{S2 \in (S1, S)} \text{true\_in}(S2, F).$$

We use three fluents to describe the spatial behavior of an object over time. The region occupied by an object O over time is denoted by the fluent "place(O)". The shape of O, as described in a coordinate system attached to O, is denoted by the fluent shape(O). The mapping from the object-centered coordinate system to the external coordinate system is denoted "placement(O)".[2] Thus, in each situation S, place(O, S) is the spatial region occupied by O; shape(O, S) is the shape of O, a spatial region that changes with the cutting of O by blades but not with the motions of O through space; and placement(O, S) is a rigid mapping — a composition of a translation and a rotation — that maps the shape to the place. We have, then, the rule

$$\text{place}(O, S) = \text{image}(\text{placement}(O, S), \text{shape}(O, S))$$

Chunks are ontologically very similar to objects; like objects, they have a place, a shape, and a placement and may be either material or ghosts. The major differences are that chunks are related by

[1] The rule can be stated within chunk theory by combining a rule that prohibits infinitely many splits in finite time with a convention that that the position of a chunk remains equal to that of its sub-chunks until it has two top-level sub-chunks. But this is decidedly clumsy.

[2] In previous work [Davis, 88] [Davis, 90] I have called this "position(O)", but I think that "placement" is more suggestive.

a sub-chunk relation and that the shape of a chunk is time-invariant. We use the time-independent function "cshape(C)" mapping chunk C directly into its shape.

We can specify the spatial relation between a sub-chunk $C1$ and a super-chunk $C2$ by positing that the shape of $C1$ is a subset of $C2$ and that they have the same placement, as long as $C2$ is material. Given these conditions, it follows that $C1$ occupies a subset of $C2$, as long as $C2$ is material.

Table 1 shows the sorts we use, together with the key letter. Table 2 defines the non-logical symbols. Table 3 shows the axioms of object theory. Table 4 shows the axioms of chunk theory. We omit temporal and spatial axioms. They will be mentioned, as needed, in the inferences in section 7 and 8.

In tables 3 and 4, we distinguish between definitions of complex primitives in terms of fundamentals, and physical axioms constraining behavior. The distinction is needed in section 7 when we come to prove object theory from chunk theory and vice versa. The definitions cannot be proven; they must be taken as given linguistic conventions. The physical axioms are proven.

Sort	Letter
Point	X
Spatial regions (set of points)	R
Rigid mappings	M
Temporal situations	S
Fluents	F
Objects	O
Chunks	C
Either object or chunk	Q

Table 1: Logical Sorts

Temporal:

true_in(S, F) — Predicate. Boolean fluent F holds in situation S.
value_in(S, F) — Function. Value of fluent F in situation S.
$S1 < S2$ — Predicate. Situation $S1$ precedes $S2$.
just_before(S, F) — Predicate. Boolean fluent F holds in an open interval ending in S.

Spatial: (This includes the spatial primitives we use in the axioms. In the inferences of section 4, we will introduce additional, special purpose, spatial primitives.)

$X \in R$ — Predicate. Point X is in region R.
$R1 \subset R2$ — Predicate. Region $R1$ is a proper subset of $R2$.
$R1 - R2$ — Function. The set difference of $R1$ and $R2$.
intersect($R1, R2$) — Predicate. Region $R1$ intersects $R2$.
$\emptyset$ — Constant. The empty region.
good_shape(R) — Predicate. Region R is non-empty, bounded, connected, and equal to the interior of its closure. (see section 2).
closure(R) — Function. The closure of region R.
interior(R) — Function. The interior of region R.
image(M, R) — Function. The image of region R under mapping M.
continuous(F, S) — Predicate. F is a continuous function of time at situation S. F is a fluent whose value in each situation is a rigid mapping.
connected_component($R1, R2$) — Region $R1$ is a connected component of $R2$.

Physical: Primitive Symbols

material(Q) — Function. The fluent of object or chunk Q being material.
placement(Q) — Function. The fluent of the mapping from the shape of Q to the place of Q.
shape(O) — Function. The point set occupied by O in a standard orientation.
cshape(C) — Function. The time-invariant shape of chunk C.

Physical: Defined Symbols

ghost(Q) — Function. The fluent of Q being a ghost.
place(Q) — Function. The fluent of the region occupied by Q in situation S.
blade_swath($S1, S2, O$) — Function. The swath cut by blades between situations $S1$ and $S2$, relative to the coordinate system attached to object O.
destroyed(S, O) — Function. Object O is destroyed at time S.
top_level(C) — Function. The fluent of chunk C being top-level (visible).
sub_chunk($C1, C2$) — Predicate. Object $C1$ is (non-strictly) a sub-chunk of $C2$.

Table 2: Non-logical primitives

Definitions of Object Theory

OD.1 $\text{ghost}(O,S) \Leftrightarrow \neg\text{material}(O,S)$.
(Definition of ghost: An object or chunk is a ghost iff it is not material.)

OD.2 $\text{place}(O,S) = \text{image}(\text{placement}(O,S), \text{shape}(O,S))$.
(Definition of place: The region occupied by O in S is the image of its shape under its placement.)

OD.3 $X \in \text{blade\_swath}(S1,S2,O) \Leftrightarrow$
$\exists_{S3,OB}\ S1 \leq S3 \leq S2 \wedge OB \neq O \wedge \text{image}(\text{placement}(O,S3),X) \in \text{place}(OB,S3)$.
(Definition of blade-swath: The blade-swath between $S1$ and $S2$, relative to O, is the region swept out by all blades between $S1$ and $S2$, as measured from a coordinate system attached to O.)

OD.4 $\text{destroyed}(S,O) \Leftrightarrow [\text{just\_before}(S,\text{material}(O)) \wedge \neg\text{good\_shape}(\text{shape}(O,S))]$
(An object is destroyed at S if it existed up to S, but became disconnected or null at S.)

Axioms of Object Theory

OB.1 $[\text{material}(O1,S) \wedge \text{material}(O2,S) \wedge O1 \neq O2] \Rightarrow$
$\neg\text{intersect}(\text{place}(O1,S),\ \text{place}(O2,S))$.
(Two material objects do not overlap.)

OB.2 $[S1 < S2 < S3 \wedge \text{material}(O,S1) \wedge \text{material}(O,S3)\ ] \Rightarrow$
$\text{material}(O,S2)$.
(Objects do not change from material to ghost to material.)

OB.3 $\text{material}(O,S) \Rightarrow \text{good\_shape}(\text{shape}(O,S))$.
(Material objects have good shapes.)

OB.4 $\forall_{S,O}\ \text{shape}(O,S) \neq \emptyset \Rightarrow \text{continuous}(\text{placement}(O),S)$.
(The placement of object O is continuous in any situation S where the shape of O is non-null. See section 5.)

OB.5 $[S1 < S2 \wedge \text{material}(O,S1) \wedge \text{just\_before}(S2,\text{material}(O))] \Rightarrow$
$\text{shape}(O,S2) = \text{interior}(\text{shape}(O,S1) - \text{blade\_swath}(S1,S2,O))$
(The material removed from O between $S1$ and $S2$ is the blade-swath between $S1$ and $S2$ relative to O, plus boundary points. This is rule MO.1 of section 2.)

OB.6 $[\text{destroyed}(S,O) \wedge \text{connected\_component}(R,\text{shape}(O,S))] \Rightarrow$
$\exists_{OR}\ \text{shape}(OR,S) = R \wedge \text{placement}(O2,S) = \text{placement}(O1,S) \wedge$
$\text{just\_before}(S,\text{ghost}(OR)) \wedge \text{material}(OR,S)$.
(Rule MO.2: If O becomes disconnected or null at S, then each of its connected components become material.)

OB.7 $[\text{material}(O,S1) \wedge \text{ghost}(O,S2) \wedge S1 < S2] \Rightarrow$
$\exists_{S3 \in (S1,S2]}\ \text{destroyed}(S3,O)$
(Rule MO.3: An object turns from material to ghost only if it is destroyed in the sense of OD.4.)

OB.8 $[\text{ghost}(O,S1) \wedge \text{material}(O,S2) \wedge S1 < S2] \Rightarrow$
$\exists_{S3,O3}\ \text{destroyed}(S3,O3) \wedge S1 < S3 \leq S2 \wedge$
$\text{connected\_component}(\text{place}(O,S3),\ \text{place}(O3,S3))$.
(Rule MO.4: An object can come into existence between $S1$ and $S2$ only if it is a connected component of some object $O3$ that is destroyed at some $S3 \in (S1,S2]$.)

OB.9 $\forall_{S,R}$ good_shape(R) $\Rightarrow$
$[\exists_{S1<S} \forall_{SA\in(S1,S),O}$ intersect(place(O, SA), R) $\Rightarrow$ [material($O, S1$) $\Leftrightarrow$ material(O, SA)] $]\wedge$
$[\exists_{S2>S} \forall_{SB\in(S,S2),O}$ intersect(place(O, SB), R) $\Rightarrow$ [material(O, SB) $\Leftrightarrow$ material($O, S2$)]]
(Given any well-shaped region R and situation S, there are a time intervals before and after S in which no object that intersects R changes status. See section 5.)

Table 4: The "mutable objects" theory.

Definitions in Chunk Theory

CD.1 ghost(C, S) $\Leftrightarrow$ $\neg$material(C, S).
(Definition of ghost: An object or chunk is a ghost iff it is not material.)

CD.2 place(C, S) = image(placement(C, S), cshape(C))
(Definition of place: The region occupied by C in S is the image of its shape under its placement.)

CD.3 sub_chunk($C1, C2$) $\Leftrightarrow$
$\exists_S$ material($C2$,S) $\wedge$ place($C1, S$) $\subseteq$ place($C2, S$).
(Definition of sub-chunk: $C1$ is a sub-chunk of $C2$ iff $C1$ occupies a subset of $C2$ in some situation where $C2$ is material.)

CD.4 top_level(C, S) $\Leftrightarrow$
[material(C, S) $\wedge$ $\forall_{C1}$ [material($C1, S$) $\wedge$ sub_chunk($C, C1$)] $\Rightarrow$ $C1 = C$].
(A top-level chunk is a maximal material chunk relative to the sub-chunk relation.)

Axioms of Chunk Theory

CH.1 good_shape(cshape(C)).
(Chunks have a good shape.)

CH.2 [good_shape($R1$) $\wedge$ $R1 \subseteq$cshape($C2$)] $\Rightarrow$
$\exists^1_{C1}$ $R1$=cshape($C1$) $\wedge$ sub_chunk($C1, C2$).
(Every reasonably-shaped subregion of a chunk is a chunk.)

CH.3 continuous(placement(C), S).
(The placement of chunk C is continuous in every situation.)

CH.4 [sub_chunk($C1, C2$) $\wedge$ material($C2, S$)] $\Rightarrow$ material($C1, S$).
(A sub-chunk of a material chunk is itself material.)

CH.5 [sub_chunk($C1, C2$) $\wedge$ material($C2, S$)] $\Rightarrow$
placement($C1, S$) = placement($C2, S$).
(A sub-chunk of a material chunk has the same placement.)

CH.6 material(C, S) $\Rightarrow$ $\exists_{C1}$ top_level($C1, S$) $\wedge$ sub_chunk($C, C1$).
(Every material chunk is a sub-chunk of a top-level chunk (possibly itself).)

CH.7. [material($C1, S1$) $\wedge$ ghost($C1, S2$)] $\Rightarrow$
$[S1 < S2 \wedge$
$\exists_{S3,C2}$ $S1 < S3 \leq S2$ $\wedge$ $\neg$sub_chunk($C1, C2$) $\wedge$ top_level($C2, S3$) $\wedge$
intersect(place($C1, S3$), place($C2, S3$))].
(A material chunk $C1$ can only turn into a ghost if its interior is penetrated by a visible chunk.)

CH.8 $[\text{top\_level}(C1,S) \wedge \text{top\_level}(C2,S) \wedge C1 \neq C2] \Rightarrow$
$\neg\text{intersect}(\text{place}(C1,S),\ \text{place}(C2,S))$.
(Two visible chunks cannot intersect.)

Table 5: Chunk Theory

7 Equivalence of the theories

In this section we prove that the object and chunk theories presented above are equivalent. Section 7.1 presents the definition of each concept in terms of the other. An object is defined, essentially, as a fluent from time to the corresponding top-level chunk. A chunk is defined as a specification of a well-shaped region (the shape of the chunk) and a fluent from time to the corresponding object. The remainder of this section proves the equivalence of the two theories in terms of four theorems. Theorem 1, in section 7.2, shows that every material object has a corresponding material chunk for every well-shaped subset of its shape. Theorem 2, in section 7.3, shows that the axioms of chunk theory can be proven from object theory. Theorem 3, in section 7.4, shows that each top-level chunk has a corresponding object. Theorem 4, in section 7.5, shows that the axioms of object theory can be proven from the axioms of chunk theory. Thus, theorems 1 and 2 use the axioms of object theory and show that chunks with the desired properties can be defined in terms of objects. Theorems 3 and 4 use the axioms of chunk theory and show that objects with the desired properties can be defined in terms of chunks.

The definitions in section 7.1 are essentially second-order or set-theoretic in flavor; they define each type as a function over the other type. Also theorems 1-4 are meta-level rather than object-level results; they are proofs about chunk and object theory, rather than proofs in those theories. Hence though the definitions and proofs are entirely rigorous (indeed, perhaps excessively detailed) we have not written them out in an entirely formal language.

The proofs of theorems 1-4 are long and occasionally tricky but not deep. The reader who has some knowledge of elementary real analysis or topology will no difficulty with them, except impatience.

7.1 Defining one theory in terms of the other

Axioms CtoO.1-7 in table 5 define objects in terms of chunks; an object is essentially a fluent from time to the corresponding top-level chunk. Axioms OtoC.1-7 in table 6 define chunks in terms of objects; a chunk is a fluent from time to the corresponding objects, together with a specification of the shape of the chunk.

CtoO.1 A *proto-object* O is a pair $< I, F >$ of a non-empty interval I and a fluent F from I to chunks satisfying the following:

a. For all $S \in I$, value_in(S, F) is a top-level chunk.

b. Let $S1, S2 \in I$, $S1 < S2$, $C1$=value_in$(S1, F)$, $C2$=value_in$(S2, F)$. Then

   i. $C2$ is a sub-chunk of $C1$.

   ii. There is no other chunk $C2A \neq C2$ that is also both top-level in $S2$ and a sub-chunk of $C1$

CtoO.2 An *object* is a proto-object with a maximal time interval. That is, if $< I1, F1 >$ is an object, $< I2, F2 >$ is a proto-object, $I1 \subseteq I2$, and, for all $S \in I1$, value_in$(S, F1)$ = value_in$(S, F2)$, then $I1 = I2$.

If object $O =< I, F >$, we write I=time_of(O), F=chunk_of(O).

CtoO.3 An object $O =< I, F >$ is *material* in situation S if $S \in I$.
material$(O, S) \Leftrightarrow S \in$time_of$(O)$

CtoO.4 The *placement* of an object is equal at each time to the placement of the associated chunk.
$S \in$time_of$(O) \Rightarrow$ placement(O, S) = placement(chunk_of(O, S),S).

CtoO.5 The *shape* of an object is shape of the associated chunk.
$S \in$time_of$(O) \Rightarrow$ shape(O, S) = cshape(chunk_of(O, S))

CtoO.6 If situation S is before I, then the shape of O in S is the interior of the closure of the union over all $SA \in I$ of the shape of O in SA. If S is after I, then the shape of O in S is the interior of the closure of the intersection over all $SA \in I$ of the shape of O in SA.

CtoO.7 If I consists of a single situation S, then the placement of O throughout all time is equal to its value in S. Otherwise, if situation S is before I, then the placement of O in S is equal to the limit of the placement of O in SA as SA approaches the beginning of I from the right. If situation S is after I, and the shape of O in S is non-null, then the placement of O in S is equal to the limit of the placement of O in SA as SA approaches the end of I from the left. (We shall show below that the limit of the placement of O at the beginning of I always exists, and that the limit of the placement of O at the end of I exists as long as the shape of O is non-null at the end of I. See the proof of OB.4 in section 7.5. As we have discussed in section 5, the placement of O may not be continuous if the shape of O is null.)

Table 5: Defining objects in terms of chunks.

OtoC.1 Object $O2$ is *broken off* object $O2$ if there is a situation S in which $O1$ is destroyed, $O2$ is material in the scene S, and the place of $O2$ in S is a subset of the place of $O1$.

$$\text{broken\_off}(O2, O1) \Leftrightarrow$$
$$\exists_S \text{ destroyed}(O1, S) \wedge \text{material}(O2, S) \wedge \text{place}(O2, S) \subset \text{place}(O1, S).$$

OtoC.2 The relation "piece_of($O1, O2$)" is the transitive closure of "broken_off($O1, O2$)". (An object is also considered a piece of itself.) Note: this is a second-order definition. We will use proofs by induction from broken_off to piece_of of the following form: For any property Φ, given that $\Phi(O, O)$ and that $\Phi(O1, O2) \wedge \text{broken\_off}(O2, O3) \Rightarrow \Phi(O1, O3)$, conclude that $\text{piece\_of}(O1, O2) \Rightarrow \Phi(O1, O2)$.

If $O1$ is a piece of $O2$, we say that $O2$ is a source of $O1$.

OtoC.3 A *proto-chunk* C is a triple $< I, R, F >$ of an non-empty interval I, a well-shaped region R, and a fluent F from time to objects satisfying the following:

a. For all $S \in I$, value_in(S, F) is a material object.
b. For all $S \in I$, R is a subset of shape(value_in(S, F)).
c. If $S1, S2 \in I$, $S1 < S2$ then value_in($S2, F$) is a piece of value_in($S1, F$).

OtoC.4 A *chunk* is a proto-chunk with a maximal interval. If $< I1, R, F1 >$ is a chunk and $< I2, R, F2 >$ is a proto-chunk such that $I1 \subset I2$ and, for all $S \in I1$, value_in($S, F1$)=value_in($S, F2$) then $I2 = I1$.

If chunk $C =< I, R, F >$ we write I=time_of(C), R=cshape(C), F=object_of(C).

OtoC.5 C is *material* in S if S is in time_of(C).
$\text{material}(C, S) \Leftrightarrow S \in \text{time\_of}(C))$

OtoC.6 The *placement* of chunk C in S is the placement of the corresponding object.
$S \in \text{time\_of}(C)) \Rightarrow \text{placement}(C, S) = \text{placement}(\text{object\_of}(C, S), S)$

OtoC.7 The placement of chunk C in S at times after I is the limit of the placement up to the end of I.

Table 6: Defining chunks in terms of objects.

7.2 Existence and Uniqueness of Chunks

Given: The definitions and axioms of object theory (OD.1-4, OB.1-9) and the definition of chunks in terms of objects (OtoC.1-7).

To prove: There is a unique chunk corresponding to each well-shaped region of each object in each situation.

Lemma 1.1: If $S2 > S1$, then the shape of O in $S2$ is a subset of the shape of O in $S1$.

Proof: It is immediate from OD.3 that, for fixed $S1$ and O, the function of $S2$, blade_swath($S1, S2, O$) is an increasing function of time. (The right-hand side of the definition just becomes more inclusive over time.) Therefore, by OB.5, the shape of O is a decreasing function of time.□

Lemma 1.2: For each object O there is at most one situation S in which S is destroyed. O is a ghost in S. Any time in which O is material precedes S.

Proof: That O is a ghost follows directly from OD.4 and OB.3. That O is not destroyed twice and that S follows the times when O is material follow from OB.2. □

By virtue of lemma 1.2, we may speak of *the* situation when O is destroyed. This situation may not exist, but if it does then it is unique.

Lemma 1.3: For any region R and object $O1$, there is at most one object $O2$ such that $O2$ is broken off $O1$ and R is a subset of shape($O2$) in the situation where $O1$ is destroyed.

Proof: Let $O2A$ and $O2B$ be two such objects broken off $O1$ containing R. In the situation S where $O1$ is destroyed, $O2A$ and $O2B$ are both material and both contain R. Hence, by OB.1, $O2A = O2B$.□

Lemma 1.4: If $O1$ is broken off $O2$, and $O1$ is material in S, then S is equal to or later than the situation SD where $O2$ is destroyed.

Proof: By OtoC.1, $O1$ is material in SD. Suppose that $S < SD$ and $O1$ is material in S. By OB.2, $O1$ is material throughout the interval $[S, SD]$. By OD.4, $O2$ is material over some interval ending in SD. Let SA be an interval such that both $O1$ and $O2$ are material throughout (SA, SD). By OtoC.1, place($O1, SD$) is a subset of place($O2, SD$). Then, from the facts that that placement($O1$) and placement($O2$) are continuous throughout (SA, SD) (OB.4) and that shape($O1$) and shape($O2$) are decreasing functions of time (lemma 1.1), using OD.2, it follows that place($O1, SB$) must overlap place($O2, SB$) for some $SB \in (SA, SD)$. But this is contrary to axiom OB.1.□

Lemma 1.5: If $O2$ is broken off $O1$, then $O2$ starts to be material just when $O1$ is destroyed.

Proof: Immediate from OtoC.1 and Lemma 1.4.□

Lemma 1.6: If $O2$ is broken off $O1A$ and $O2$ is broken off $O1B$ then $O1A = O1B$.

Proof: By lemma 1.5, there is some situation SD in which both $O1A$ and $O1B$ are destroyed and $O2$ begins to be material. By OtoC.1, place($O2, SD$) is a subset of both place($O1A, SD$) and place($O1B, SD$). Thus place($O1A, SD$) and place($O1B, SD$) overlap. As in the proof of lemma 1.4, it follows from OB.4, OD.2, and lemma 1.1, that place($O1A$) and place($O1B$) must overlap for some interval before SD. So, by OB.1, $O1A = O1B$.□

Lemma 1.7: If $O2$ is broken off $O1$, $O1$ is material in $S1$, and $O2$ is material in $S2$, then shape($O2, S2$) is a subset of shape($O1, S1$)).

Proof: Immediate from Lemma 1.1, lemma 1.4, and OtoC.1.□

Lemma 1.8: If $O2$ is a piece of $O1$, $O1 \neq O2$, $O1$ is material in $S1$, and $O2$ is material in $S2$ then $S1 < S2$ and shape($O2, S2$) is a subset of shape($O1, S1$)).

Proof: That $S1 < S2$ follows by induction from lemma 1.4. That the shape of $O2$ is a subset of the shape of $O1$ follows by induction from lemma 1.7.□

Lemma 1.9: For any object $O2$ and scene S there is at most one object $O1$ that is a source of $O2$ and is material in S.

Proof: By induction, using lemma 1.4 and lemma 1.6.□

Lemma 1.10: If $O1$ is material in $S1$, $O2$ is material in $S2$, and $O2$ is a piece of $O1$, then for any $S3 \in (S1, S2)$ there is an object $O3$ such that $O3$ is a piece of $O1$ and $O3$ is material in $S3$.

Proof: By induction.

Base step: If $O1 = O2$, then the result holds for $O3 = O1$ by OB.2.

Inductive step: Suppose that OZ is broken off $O1$, $O2$ is a piece of OZ, $O1$ is material in $S1$, $O2$ is material in $S2$, OZ is material in SZ, and for every $SQ \in (SZ, S2)$ there is an object OQ such that OQ is a piece of $O1$ and OQ is material in SQ. Let $S3$ be any scene in $(S1, S2)$. We wish to find an $O3$ which is a piece of $O1$, a source of $O2$, and material in $S3$. Then there are three cases to be considered:

i. $S3 \in (SZ, S2)$. $O3$ exists by the induction hypothesis.

ii. $S3 \leq SZ$ and $S3$ is later than the situation where $O1$ is destroyed and OZ created. Choose $O3 = OZ$.

iii. $S3$ is between $S1$ and the situation where $O1$ is destroyed. Choose $O3 = O1$.□

Lemma 1.11: If $O2$ is material in $S2$ and $S1 < S2$, then there is an object $O1$ that is a source of $O2$ and is material in $S1$.

Proof: By contradiction. Suppose there is no source of $O2$ in $S1$. By lemma 1.10, the situations before $S2$ when there exists a source of $O2$ form a connected interval. Therefore, there must be a time SQ such that in any $S < SQ$ there does not exist a material source of $O2$, while in any $S \in (SQ, S2]$, there does exist such an O. Let SZ be any such situation in $(SD, S2]$, and let OZ be the source of $O2$ that is material in SZ. Since OZ is a ghost in $S1$ and material in SZ, by OB.8 there is a situation $SZ1 < SZ$ and an object $OZ1$ such that OZ is material in $SZ1$, $OZ1$ is destroyed in $SZ1$, and OZ is broken off $OZ1$. Clearly $SZ1 > SQ$. Thus in any neighborhood of SQ, these is a situation in which one object is destroyed and another created. Moreover, since all these objects are ancestors of $O2$, it is easily shown that they all remain within a bounded spatial region. But this contradicts axiom OB.9.□

Lemma 1.12: Let $O1$ be an object that is material in $S1$, and let $S2 > S1$. For any region R, there is at most one object $O2$ that is a piece of $O1$, is material in $S2$, and contains R in its shape.

Proof: By induction from Lemma 1.3.□

Lemma 1.13: Let $S2 < S3$, let $O2$ be a piece of $O1$ that is material in $S2$, and let $O3$ be a piece of $O1$ that is material in $S3$. If shape($O2$) and shape($O3$) intersect, then $O3$ is a piece of $O2$.

Proof: By lemma 1.11, there is an object OA that is a source of $O3$ and is material in $S2$. By lemma 1.8, shape($OA, S2$) is a superset of shape($O3, S2$); it therefore overlaps shape($O2, S2$). Then, by lemma 1.12 $OA = O2$.□

Lemma 1.14: If $O1$ is a piece of $O2$ and $O2$ is a piece of $O3$ then $O1$ is a piece of $O3$.

Proof: Immediate from OtoC.2. (Transitivity is a defining property of any transitive closure.)□

Theorem 1: Let S be a situation; let O be an object that is material in S; and let R be a well-shaped sub-region of the shape of O in S. Then there exists a unique chunk C such that object_of(C, S)=O and cshape(C)=R.

C may be constructed as follows: Let $I1$ be the set of all times $S1 > S$ in which there is a material piece $O1$ of O containing R in shape($O1, S1$). By lemmas 1.1 and 1.8, $I1$ is an interval. Let $I = (-\infty, S] \cup I1$. Let F be a fluent from I to objects defined as follows: for $S0 \leq S$, value_in($S0, F$) is the material source of O in $S0$; for $S0 \in I1$ value_in($S0, F$) is the material piece of O that contains R. Then $C =< I, R, F >$.

Proof: We observe:

- By lemma 1.11, F exists and is a material object for $S0 < S$; by lemma 1.9, F is uniquely defined. Clearly, from conditions OtoC.3.a and OtoC.3.c, any chunk equal to O on S will have to be equal to F for those times.
- By construction, F exists and is a material object for $S1 \in I1$; by lemma 1.11, F is uniquely defined. Clearly, from condition OtoC.3, any chunk equal to O on S will be equal to F for $S1 \in I1$. Also, by construction, for $S1 \notin I$, there is no $O1$ that is a piece of O, material in $S1$ and contains R. Hence no chunk equal to O on S can extend past $I1$, so I is maximal, satisfying OtoC.4.

- From lemmas 1.13 and 1.14, it is easily shown that if $S1, S2 \in I$, $S1 < S2$, then value_in($S2, F$) is a piece of value_in($S1, F$).
- It is immediate from OtoC.4 that cshape(C)=R.

This completes the proof of Theorem 1.□

Note that the proof of theorem 1 above relies on the existence of a fluent of the described type. This can be justified, either by a higher-order logic defining a fluent as a function, or by a set-theoretic definition defining a fluent as a set of ordered pairs, or by a first-order axiom schema stating the existence of a fluent corresponding to any uniquely defined property.

7.3 Properties of Chunks

Givens: The definitions and axioms of object theory (OD.1-4, OB.1-9), the definitions of chunks in terms of objects (OtoC.1-7) and the definitions of the complex primitives of chunk theory in terms of the fundamental primitives (CD.1-4).

To prove: The axioms of chunk theory (CH.1-8).

Theorem 2: The chunks defined as above in terms of objects satisfy the axioms of chunks theory.

We check each axiom in turn. First, we categorize the sub-chunk relation in terms of objects.

Lemma 2.1: If chunks $C1$ and $C2$ are material in S and place($C1, S$) overlaps place($C2, S$), then object_of($C1, S$) = object_of($C2, S$).

Proof: Immediate from OtoC.4 and 5, CD.2, OD.2, and OB.1.□

Lemma 2.2: Chunk $C1$ is a sub-chunk of $C2$ just if cshape($C1$) $\subset$ cshape($C2$) and in some situation S, object_of($C1, S$) = object_of($C2, S$).

Proof: Immediate from CD.3, OtoC.3, and lemma 2.1.□

Lemma 2.3 If $C1$ is a sub-chunk of $C2$ and $C2$ is material in S, then $C1$ is material in S and object_of($C1, S$) = object_of($C2, S$).

Proof: Follows directly from lemma 2.2, together with the uniqueness of the construction in Theorem 1. Note that if $O2$ is a piece of $O1$ whose shape in S contains RA, and $RB \subset RA$, then $O2$ also contains RB. Hence, by the construction of theorem 1, chunks corresponding to sub-regions persist for a longer interval.□

CH.1: A chunk has a good shape.

Proof: Immediate from OtoC.3.□

CH.2: If $C1$ is a chunk, R is a good shape, and $R \subset$cshape($C1$) then there is a sub-chunk CR of $C1$ such that R =cshape($C2$).

Proof: Let S be an a situation where $C1$ is material. Using theorem 1, choose $C2$ to be the chunk such that object_of($C2, S$) = object_of($C1, S$) and R=cshape($C2$). By lemma 2.2 $C2$ is a sub-chunk of $C1$.

CH.3: The placement of chunk C is continuous.

Proof: We wish to show that placement(C) is continuous in every situation S. By the construction of theorem 1 and lemmas 1.2 and 1.5, there are three cases to be considered:

i. S is in the interior of the lifetime of O=object_of(C, S). Then by CtoO.6, placement(C) is equal to placement(O, S) in a neighborhood of S. The continuity of placement(C) in S follows directly from the continuity of placement(O) (OB.4)

ii. O=object_of(C, S) comes into existence at S, being broken off $O1$. By the construction of theorem 1, $O1$ is the object of C in an open interval preceding S. By OB.6, placement(O, S)=placement($O1, S$). Since placement(C) is equal to placement($O1$) up to S, and equal to placement(O) starting in S and for a finite interval afterward, and both placement(O) and placement($O1$) are continuous (OB.4), it follows that placement(C) is continuous at S.

iii. S is after the lifetime of C. Then by OtoC.7, placement(C) is continuous in S.□

CH.4: If $C1$ is material in S and $C2$ is a sub-chunk of $C1$ then $C2$ is material in S.

Proof: Part of lemma 2.3.□

CH.5: If $C1$ is material in S and $C2$ is a sub-chunk of $C1$ then $C1$ and $C2$ have the same placement in S.

Proof: Immediate from lemma 2.3 and OtoC.5.□

CH.6: If C is material in S then C is a sub-chunk of some chunk CT that is top-level in S.

Proof: Using theorem 1, choose CT to be the chunk such that object_of(CT, S) = object_of(C, S) and cshape(CT) = shape(object_of(C, S),S). By lemma 2.2 and OtoC.3, CT has no super-chunks; by lemma 2.2, C is a sub-chunk of CT.□

Corollary 2.4: If object O is material in situation S, then there is a unique chunk C such that top_level(C, S) and O=object_of(C, S).

Proof: Immediate from the proof of CH.6.□

Lemma 2.5: If chunk C is material in $S1$ and a ghost in $S2$ then $S1 < S2$. Equivalently, time_of(C) is unbounded on the left.

Proof: Immediate from Theorem 1.

CH.7: If chunk C turns from material in $S1$ to a ghost in $S2$, then, at some time $S3 \in (S1, S2]$, place(C, S) is intersected by some top-level chunk CB which is not a super-chunk of C.

Proof: Let $O1$=object_of($C, S1$). Let $S3$ be the upper bound of time_of(C). Then by the construction in theorem 1, for all $SA < S3$ there is an object OA that is material in SA, that is a piece of $O1$, and whose shape in SA contains cshape(C); for all $SA > S3$, there is no such object OA. By the axiom of finite variation OB.9 and lemma 1.15, there exists a situation $SZ < S3$ such that there is a single object OZ that is the piece of $O1$ containing cshape(C) throughout the interval ($SZ, S3$). There are now two cases to consider: (A) OZ is material for some $SQ > S3$; (B) OZ is a ghost for all $SQ > S3$.

Case A: OZ is material for some $SQ > S3$. Since C is not material in SQ, it follows from the construction of theorem 1 that shape(OZ, SQ) is not a superset of cshape(C). Hence, by axiom OB.5, the blade-swath cutting into OZ between $S3$ and SQ overlaps cshape(C). (Note that both the blade-swath and and cshape(C) are open regions.) By definition of blade-swath OD.3, this means that there is some object OB and some situation SB such that the place of OB in SB, translated into OZ's frame of reference, overlaps cshape(C). Choosing CB to be the top-level chunk corresponding to OB in S (CH.6) and translating back into the standard frame of reference, we find that the places of CB and C overlap in S.

Case B: OZ is not material in any $SQ > S3$. In this case we can apply axiom OB.7 to deduce that OZ is destroyed in $S3$. If cshape(C) were in the shape of OZ in $S3$, then by OB.6 there would be a new object OR that comes into existence in $S3$ and whose shape contains cshape(C). However, we know that there is no such object, so shape(OZ) does not contain cshape(C) in OZ. The rest of the argument proceeds as in case A. (In fact, following that argument, it is easily shown that case B is impossible.)□

CH.8: Two top-level chunks do not occupy overlapping positions in a single situation.

Proof: Immediate from lemma 2.1.□

This completes the proof of Theorem 2.□

7.4 Existence and Uniqueness of Objects:

Given: The axioms and definitions of chunk theory (CD.1-4, CH.1-8) and the definition of objects in terms of chunks (CtoO.1-7)

To prove: There exists a unique object for every chunk C and situation S.

Lemma 3.1: If C is material in $S2$ and $S1 < S2$ then C is material in $S1$.

Proof: Just the first part of CH.7. We separate this, since we will often want this result without the second, more complicated part of CH.7.□

Lemma 3.2: If $C2$ is material in $S2$ and $S1 < S2$ then there is a chunk $C1$ that is top-level in $S1$ and that is a super-chunk of $C2$.

Proof: Immediate from lemma 3.1 and CH.6.□.

Lemma 3.3: If CB is a sub-chunk of CA, and CA is material in S then cshape(CB) $\subset$ cshape(CA) and place(CA, S) $\subset$ place(CB, S).

Proof: By CD.5 there is a situation $S1$ in which CA is material and the place of CB is a subset of CA. From CH.5, CA and CB have the same placement as long as CA is material. Applying CD.2 twice, we can determine that cshape(CB) is a subset of cshape(CA) and that place(CB, S) $\subset$ place(CA, S).□

Lemma 3.4: If $C1$ is a sub-chunk of $C2$ and $C2$ is a sub-chunk of $C3$, then $C1$ is a sub-chunk of $C3$.

Proof: By lemma 3.1, there is a situation $S0$ in which $C1$, $C2$, and $C3$ are all material. By CH.5, placement($C1, S0$) = placement($C2, S0$) = placement($C3, S0$). By lemma 3.3, cshape($C1$) is a subset of cshape($C2$) which is a subset of cshape($C3$). From CD.2, it follows that place($C1, S0$) $\subset$ place($C3, S0$), so by CD.3 $C1$ is a sub-chunk of $C3$.□

Lemma 3.5: If $C1$ is top-level in $S1$, $C2$ is material in $S2 \geq S1$, and CS is a sub-chunk of both $C1$ and $C2$, then $C2$ is a sub-chunk of $C1$.

Proof: By lemma 3.2, $C2$ is a sub-chunk of some CQ that is top-level in $S1$. By lemma 3.3, place($CS, S1$) $\subset$ place($C2, S1$) $\subset$ place($CQ, S1$) and place($CS, S1$) $\subset$ place($C1, S1$). Hence place($CQ, S1$) and place($C1, S1$) intersect (their intersection includes place($CS, S1$)). By CH.8, therefore $C1 = CQ$, but then $C2$ is a sub-chunk of $C1$.□

Corollary 3.6: If $C2$ is a sub-chunk of $C1A$ and $C1B$, and both $C1A$ and $C1B$ are top-level in $S1$, then $C1A = C1B$.

Proof: By lemma 3.3, place($C2, S$) is a subset of both place($C1A, S$) and place($C1B, S$). Since these two top-level chunks intersect, they must be equal by CH.8.□

Based on lemma 3.2 and corollary 3.6, we may make the following definition: If C is material in

$S2$ and $S < S2$, then the *ancestor* of C in S is the unique super-chunk of C that is top-level in S.

Lemma 3.7: If $S1 < S2 < S3$ are three situations, $C3$ is material in $S3$, $C1$ is the ancestor of $C3$ in $S1$ and $C2$ is the ancestor of $C3$ in $S2$, then $C1$ is the ancestor of $C2$ in $S2$. (That is, the "ancestor" relation is a forward-branching tree.)

Proof: Immediate from lemma 3.5.□

Lemma 3.8: If $S1 < S2$, $C1$ is the ancestor of $C2$ in $S1$, and $C2$ is the ancestor of $C3$ in $S2$, then $C1$ is the ancestor of $C3$ in $S1$.

Proof: Immediate from lemma 3.4.

Definition CD.5: Chunk C is *split* from $S1$ to $S2$ if (i) C is top-level in $S1$ and (ii) C has two sub-chunks $C2A \neq C2B$ that are both top-level in $S2$.

CD.5 $\text{split}(C, S1, S2) \Leftrightarrow$
$[\text{material}(C, S1)) \wedge$
$\exists_{C2A,C2B}\ C2A \neq C2B \wedge \text{sub\_chunk}(C2A, C1) \wedge \text{sub\_chunk}(C2B, C1) \wedge$
$\text{top\_level}(C2A, S2) \wedge \text{top\_level}(C2B, S2)$.

Theorem 3: (Existence and Uniqueness of Objects:) Let C be top-level in S. There exists a unique object O such that C=chunk_of(O, S).

O may be constructed as follows:

Define the following time intervals: $I1$ is the set of all times $S1$ before S such that the ancestor of C in $S1$ is not split at any time SQ before or including S. $I2$ is the set of all times $S2$ after S such that C is not split in any time SQ between S and $S2$.

$$I1 = \{\ S1 \mid S1 \leq S \wedge \neg\exists_{SQ}\ S1 < SQ \leq S \wedge \text{split}(\text{ancestor}(C, S1), S1, SQ)\ \}$$
$$I2 = \{\ S2 \mid S \leq S2 \wedge \neg\exists_{SQ}\ S < SQ \leq S2 \wedge \text{split}(C, S, SQ)\ \}$$

Let $I = I1 \cup I2$. Let F be the fluent from I to top-level chunks defined as equal to ancestor(C, S) for $S \in I1$, and as the unique top-level sub-chunk of C in $I2$. Now $O =< I, F >$ is the desired object.

Proof: From lemma 3.8, it follows directly that, if $S1A \in I1$ and $S1B \in (S1A, S)$ then $S1B \in I1$; and that if $S2B \in I2$ and $S2A \in (S, S2B)$ then $S2A \in I3$. Moreover S is in both $I1$ and in $I2$. Hence $I1 \cup I2$ is an interval.

Property CtoO.1.a, that F is always a top-level chunk, holds on O by construction. Property CtoO.1.b states the following: if $S1$ and $S2$ are situation in I, $S1 < S2$, $C1$=value_in$(S1, F)$, and $C2$=value_in$(S2, F)$, then

i. $C1$ is an ancestor of $C2$;

ii. $C1$ is not split in $S2$; that is, $C2$ is the only top-level descendant of $C1$ in $S2$. is top-level in $S2$.

We must check these two conditions in three cases:

A. $S1$ and $S2$ are both in $I1$. Then $C1$ is an ancestor of $C2$ by lemma 3.7, since both are ancestors of C. $C1$ is not split in $S2$ directly by definition of $I1$.

B. $S1$ and $S2$ are both in $I2$.

(i) Suppose that $C1$ were not the ancestor of $C2$. Let $C1A$ be the ancestor of $C2$ in $S1$. Then by lemma 3.7, $C1A$ is a descendant of C, which means that C has been split. But this is impossible by definition of $I2$.

(ii) Suppose that $C1$ has another top-level descendant $C2A$ in $S2$. By lemma 3.8, $C2A$ is a descendant of C. But this is impossible, by definition of $I2$

C. $S1$ is in $I1$ and $S2$ is in $I2$.

(i) Since $C1$ is an ancestor of C and C is an ancestor of $C2$, by lemma 3.4 $C1$ is an ancestor of $C2$.

(ii) Suppose that $C1$ has another descendant $C2A$ in $S2$. Let CA be the ancestor of $C2A$ in S. By lemma 3.7, CA is a descendant of $C1$. If $CA \neq C$, then $C1$ is split in S, which contradicts the definition of $I1$. If $CA = C$, then C is split in $S2$, which contradicts the definition of $I2$.

Thus, O satisfies all the properties of a proto-object.

It is clear that properties CtoO.1.a and CtoO.1.b.i uniquely define the value of F over the interval I; hence any proto-object which is equal to C at S must agree with F over I. Also, property CtoO.b.ii makes it impossible for any proto-object that is equal to C at S to extend earlier than $I1$ or later than $I2$. Hence O is maximal and unique.□

Like the proof of theorem 1, this proof relies on the existence of a fluent of the prescribed type.

7.5 Properties of Objects

Given: The definitions and axioms of chunks (CD.1-4, CH.1-8); the definition of objects in terms of chunks (CtoO.1-7); the definitions of complex primitives of object theory in terms of fundamental primitives (OD.1-4); and the following finiteness assumption:

FIN. Let I be a bounded interval and let R be a bounded region. There are only finitely many objects O such that for some $S \in I$, place(O, S) intersects R.

To prove: the axioms of object theory OB.1-9.

Theorem 4: Objects, as defined above in terms of chunks, satisfy the axioms of object theory.

The proof proceeds just by proving OB.1-9 in order.

Lemma 4.1: If O is material in S, then place(O, S) = place(chunk_of(O, S),S).

Proof: Immediate from OD.2, CD.2, CtoO.4, and CtoO.5.

OB.1: If $O1 \neq O2$, and $O1$ and $O2$ are both material in S, then place($O1, S$) and place($O2, S$) do not intersect.

Proof of OB.1: By theorem 3, chunk_of($O1, S$) is not equal to chunk_of($O2, S$). By axiom CH.8, place(chunk_of($O1, S$),S) and place(chunk_of($O2, S$),S) do not overlap. By lemma 4.1, place($O1, S$) and place($O2, S$) do not overlap.□

OB.2: If $S1 < S2 < S3$, O is material in $S1$, and O is material in $S3$, then O is material in $S2$.

Proof of OB.2: Immediate from the definitions CtoO.1-3. using the fact that the interval time_of(O) cannot be missing any interior points. □

OB.3: If O is material in S then shape(O, S) is a good shape.

Proof of OB.3: Immediate from the fact (CH.1) that chunk_of(O, S) has a good shape, together with lemma 4.1.

Definition: A region is *almost well-shaped* if it is non-null bounded, open, and equal to the interior of its closure. (This is weaker than the definition of a well-shaped region in that it is not required to be connected.)

Lemma 4.2: If R is almost well-shaped then each of its connected components is well-shaped.

Proof: Immediate from the definitions.□

Lemma 4.3: If R is non-null and bounded, then the interior of the closure of R is almost well-shaped.

Proof: By an easy geometric argument, omitted here.□

Lemma 4.4: If S is after time_of(O), then the shape of O in S is either null or almost well-shaped.

Proof: Immediate from CtoO.6 and Lemma 4.3.□

Lemma 4.5: Let $C1$=chunk_of$(O, S1)$, $C2$=chunk_of$(O, S2)$. Let R be a well-shaped region that is a subset of both cshape$(C1)$ and cshape$(C2)$. Then the sub-chunk of $C1$ that has shape R is the same as the sub-chunk of $C2$ with shape R.

Proof: Assume without loss of generality that $S1 < S2$. By CtoO.3.a, $C2$ is a sub-chunk of $C1$. By CH.2, there exists a unique sub-chunk CR of $C2$ that has shape R. By Lemma 3.4, CR is a sub-chunk of $C1$. By CH.2, CR is the only sub-chunk of $C1$ with shape R.

OB.4: If shape(O, S) is non-null, then placement(O) is continuous in S.

Proof: We must consider five cases:

A. time_of(O) consists of a single situation. Then continuity for all time is immediate from CtoO.7.

B. S is in the interior of time_of(O). Let $S1$ and $S2$ be two situations in time_of(O) such that $S1 < S < S2$. Let $C2$=chunk_of$(O, S2)$. Let SA be an arbitrary situation in $[S1, S2]$ and let CA=chunk_of(O, SA), By CtoO.3, CA is the ancestor of $C2$ in SA. In SA the placement of O is equal to the placement of CA (CtoO.4) which is equal to the placement of $C2$ (CH.5). Thus, the placement of O is equal to the placement of $C2$ throughout $[S1, S2]$. Since the placement of $C2$ is continuous at S, (CH.3), so is the placement of O.

C. S is the lower limit of time_of(O). Let $S2$ be a time in time_of(O) that is greater than S, and let $C2$=chunk_of$(O, S2)$. By the same argument in (B), the placement of O is equal to the placement of $C2$ throughout $(S, S2)$. Hence, by the continuity of placement$(C2)$, the limit of placement(O) from the right exists at S. If $S \in$time_of(O), then this limit is equal to the placement in S by the continuity of placement$(C2)$; if $S \notin$time_of(O), then the limit is equal to the placement in S by the definition in CtoO.7. It is immediate from CtoO.7 that the limit of placement(O) from the left at S is equal to placement(O, S).

D. S is the upper limit of time_of(O). Since shape(O, S) is non-null, let R be a connected component of shape(O, S). By lemmas 4.2 and 4.4, R is well-shaped. By CtoO.6, R is a subset of shape(O, SA) for every $SA \in$time_of(O). By lemma 4.5, there is a single chunk CR that has shape R and that is a sub-chunk of chunk_of(O, SA) for every $SA \in I$. By the same argument as in (A), placement(O) is equal to placement(CR) throughout time_of(O). Since placement(CR) approaches a limit at end of time_of(O), so does placement(O). By the construction of CtoO.7, the value at S and after S is equal to the limit at S from the left.

E. S is either strictly before the beginning of time_of(O), or strictly after its end. Then by CtoO.7, placement(O) is constant around S, so it is continuous. (Note that, by CtoO.6, shape(O) is non-null after time_of(O) just if it is non-null at the end of time_of(O).□

Lemma 4.6: If C is material in all situations $S1 < S$ then C is material in S. (The lifetime of a chunk is closed on the right.)

Proof: By CH.8, since C is material before S, there is no top-level chunk CB other than its own super chunk such that place(CB) intersects place(C) in $S1$ before S. Since the shape of a chunk is an open set (CH.1) and since chunks move continuously (CH.3), it follows by a geometric argument that the places of CB and C do not overlap in S, either. Hence, by CH.7, C is still material in S.

Lemma 4.7: If $S1 < S2$ and O is material in both $S1$ and $S2$, then shape($O, S1$) $\subseteq$ shape($O, S2$).

Proof: Let $C1$=chunk_of($O, S1$), $C2$=chunk_of($O, S2$). By CtoO.1, $C2$ is a sub-chunk of $C1$. By lemmas 3.3 and 4.1, shape($O, S2$)=cshape($C2$) $\subset$ cshape($C1$)=shape($O, S1$).□

Lemma 4.8: Let $C1$ be material in $S1$. Let R be a well-shaped region whose closure is a subset of cshape($C1$). Let CR be the sub-chunk of $C1$ whose shape is R. Then there is an $S2 > S1$ such that CR is material in $S2$.

Proof: Geometrically, there must be a finite distance from R to the boundary of cshape($C1$). Since $C1$ is material in $S1$, there is no top-level chunk CB other than the super-chunk of $C1$ that intersect place($C1$). Thus, for any object OB, if chunk_of($OB, S1$) $\neq C1$ and OB is material in $S1$, then place($OB, S1$) is a finite distance from place($CR, S1$). Since OB and CR move continuously (OB.4,CH.3), and shape(OB) is a decreasing function of time (lemma 4.7) it follows that there is some interval $(S1, SB]$ after $S1$ during which OB does not overlap R. By the finiteness principle FIN, in any finite interval there can only be finitely many such external blades OB to consider. Hence, if we choose $S2$ to be the minimum of such an SB over all the external blades OB, then no blade ovelaps place(CR) between $S1$ and $S2$. By CH.7, CR is still material in $S2$.

OB.5: The material removed from O between $S1$ and $S2$ is the blade-swath between $S1$ and $S2$ relative to O plus boundary points.

Proof: We must consider two cases: (I) O is material in $S2$; (II) O is material just before $S2$.

Case I (O is material in $S2$.): Let $C1$=chunk_of($O, S1$), $C2$=chunk_of($O, S2$). By definition OD.3, a point X is in the blade-swath of O from $S1$ to $S2$, just if there is a situation $S3 \in (S1, S2]$ and an object $OB \neq O$ such that the image of X under the placement of O in $S3$ is inside the place of OB in $S3$. By theorem 3, the objects in a situation correspond one-to-one with the top-level chunks, and by CtoO.3 and CtoO.4, the shapes and placements of an object are the same as those of the associated chunk. It follows that X is in the blade-swath just if there is a situation $S3 \in (S1, S2]$ a chunk $C3$=chunk_of($O, S3$), and a top-level chunk $CB \neq C3$ such that the image of XB under the mapping placement($C3, S3$) is in place($CB, S3$).

We break the remainder of case I into two parts:

A. The blade-swath is disjoint from the shape of $C2$. Since cshape($C2$) is open, it follows that the closure of the blade-swath is disjoint from the cshape($C2$).

B. The interior of cshape($C1$) minus the blade-swath is a subset of cshape($C2$).

Part A: Let X be in the blade-swath. Let $S3$,OB, $C3$, and CB be as above. Then the image of X under the mapping placement($C3, S3$) is in place($CB, S3$). It follows, from CD.2 and CH.8, that X is not in cshape($C3, S3$). Since $C3$ is the chunk of O in $S3$, and $C2$ is the chunk of O in $S2$ which is later, by CtoO.1.b.i, $C2$ must be a sub-chunk of $C3$. Hence by CD.4 XB is not in $C2$.

Part B: Let region R be the interior of cshape($C1$) minus the blade-swath. By definition of the blade-swath, the image of R under placement($O, S3$) is disjoint from the place of OB in $S3$ for any $OB \neq O$ and $S3 \in (S1, S2]$. By CH.2 there is a sub-chunk CR of $C1$ whose shape is R. By CH.4, CR is material in $S1$. By definition of the object O, O has only one top-level descendant throughout time_of(O); hence CR is a sub-chunk of chunk_of(O, S) in any situation S where both O and CR are material. Therefore, the placement of CR is equal to the placement of O as long as CR and O are material.

Suppose that R is not a subset of cshape($C2$). Since R is open, and cshape($C2$) is equal to the interior of its closure, it is easily shown that $R-$cshape($C2$) has non-empty interior. Let RI be a well-shaped region whose closure is a subset of the interior of $R-$cshape($C2$). Since RI is a subset of cshape($C1$), there is a sub-chunk CI of $C1$ with shape RI (CH.2). By CtoO.b.ii, O has only one top-level descendant throughout time_of(O); hence CR is a sub-chunk of chunk_of(O, S) in any situation S where both O and CR are material (lemma 3.7). However, CI is not a sub-chunk of $C2$ (lemma 3.1), so CI is not material in $S2$. By lemma 4.5, there is last situation $S3 \in [S1, S2)$ in which CI is material. Since CI is a sub-chunk of $C3$=chunk_of($O, S3$), the placement of CI is equal to that of $C3$. But then CI satisfies the conditions of lemma 4.8, so CI is material for some interval after $S3$. This completes the contradiction, so R must be a subset of cshape($C2$).

Case II (O is material just before $S2$ but not in $S2$): From the definition of the blade-swath, plus the facts that the placements of all the blades are continuous functions of time (OB.4) and that the shape of the blades are decreasing functions of time (lemma 4.7), it is easily shown that the closure of the blade-swath from $S1$ to $S2$ is equal to the closure of the union over $SA \in [S1, S2)$ of the blade-swath from $S1$ to SA. Now

1. shape($O, S2$) = 2. interior(closure($\bigcap_{SA \in [S1,S2)}$ shape(O, SA)))) =
3. interior(closure($\bigcap_{SA \in [S1,S2)}$ interior(shape($O, S1$)) − blade_swath($O, S1, SA$)))) =
4. interior(closure(shape($O, S1$) − closure($\bigcup_{SA \in [S1,S2)}$ blade_swath($O, S1, SA$)) =
5. interior(closure(shape($O, S1$) − closure(blade_swath($O, S1, S2$)))) = 6. interior(shape($O, S1$) − blade_swath($O, S1, S2$))

which is the desired result.

In the derivation above, 1-2 is justified by CtoO.6. 2-3 is justified by case I of this proof. 3-4 is justified by a geometric argument, using the fact that shape($O, S1$) is an open set. 4-5 is justified by the above remark that the blade-swath from $S1$ to $S2$ has the same closure as the union of the blade-swath from previous times. 5-6 follows geometrically from the facts that shape(O, S) is a good shape and that the blade-swath is an open regions.□

OB.6: If O is destroyed in S, then each connected component of the shape of O in S becomes a new object.

Proof: By OB.1, O is not material in S. By lemma 4.4, O is almost well-shaped in S. Let R be a connected component of the shape of O in S. By lemma 4.2, R is a well-shaped region. By CtoO.6, R is a subset of shape(O, SA) in every situation $SA < S$. By lemma 4.5, there is a chunk CR such that CR is a sub-chunk of chunk_of(O, SA) for all SA in the lifetime of O and such that cshape(CR)=R. By lemma 4.6, CR is material in S.

Suppose CT is a proper super-chunk of CR. Then cshape(CT) is a proper superset of cshape(CR). Since R is a connected component of the shape of O in S, and since cshape(CT) is connected, it follows that cshape(CT) is not a subset of the shape of O in S. By CtoO.6, there is some $SA < S$ such that cshape(CT) is not a subset of shape(O, SA). Therefore, CT is not material in SA, so CT is not material in S. Since no super-chunk of CR is material in S, CR is top-level in S. By the existence theorem 3, there is an object OR such that chunk_of(OR, S)=CR. Then shape(O, S)=R.□

Lemma 4.9: Let $C2A$ and $C2B$ be two distinct top-level chunks in $S2$ that have a common ancestor prior to $S1$. Then there exists a situation $S1$ such that $C2A$ and $C2B$ have distinct ancestors in $S1$ and in all later situations, but have the same ancestors prior to $S1$.

Proof: By lemma 3.7, if $C2A$ and $C2B$ have the same ancestor in $S0$, then they have the same ancestor in all earlier situations. Let $S1$ be the least upper bound of the times in which $C2A$ and $C2B$ have the same ancestor. Then $C2A$ and $C2B$ have the same ancestor in all situations before $S1$ and different ancestors in all situations in all situations after $S1$. What remains to prove is that they have different ancestors in $S1$ itself.

Suppose not. Let $C1$ be the common ancestor of $C2A$ and $C2B$ in $S1$. Let RA be the union over $SZ > S1$ of cshape(ancestor($C2A, SZ$)), and let RB be the union over $SZ > S1$ of cshape(ancestor($C2B, SZ$))) It is easily shown that RA and RB are disjoint open subsets of cshape($C1$). Since cshape($C1$) is a connected set, there is a point X that is in the boundary of RA and in the interior of cshape($C1$). Since cshape($C1$) is an open region, there is an open region RX containing X such that the closure of RX is a subset of cshape($C1$). Let CX be the sub-chunk of $C1$ with shape RX. Thus CX is a finite distance from the boundary of $C1$. By lemma 4.8, there is a situation $SX > S1$ such that CX is still material in SX. But then between $S1$ and SX, the ancestor of $C1A$ is overlapped by CX, which is material but not a sub-chunk. This is impossible by CH.8, thus completing the contradiction.

OB.7: An object O can change from material in $S1$ to a ghost in $S2$ if there is a situation $S3$ between $S1$ and $S2$, such that O is material just before $S3$, and O is an improper shape in $S3$.

Proof: Let $S3$ be the upper bound of time_of(O). Then, by definition, O is material just before $S3$. We show first that O is not material in $S3$ and then that O is not a proper shape in $S3$.

Proof that O is not material in $S3$: By contradiction. Suppose that O is material in $S3$. Let $C3$=chunk_of($O, S3$). Since $S3$ is the upper bound of time_of(O), it follows that O is a ghost in every situation $S4 > S3$. By lemma 4.6, this means that, for every $S4 > S3$, there is a situation $S5 \in (S3, S4]$ in which there is not a unique top-level sub-chunk of $C3$.

By lemma 4.8, however, there are sub-chunks of $C3$ that persist for intervals after $S3$. Therefore, they must not be unique in any such interval. That is, for every $S4 > S3$ there exists a situation $S5 \in (S3, S4]$ in which there are two top-level chunks $C5A$ and $C5B$ that are descendants of $C3$. But then, by lemma 4.8, there is a situation $S6$ such that $S6 \leq S5$ such that $C5A$ and $C5B$ have distinct ancestors in $S6$ and after $S6$ and common ancestors in all situations prior to $S6$. Since $C5A$ and $C5B$ have the common ancestor $C3$ in $S3$, $S6$ must be after $S3$. Thus, we can choose an $S7 > S3$ such that $C3$ has a unique descendant in $S7$ and is split in $S6$. Summarizing, for any $S4 > S3$, it is possible to find situations $S7$ and $S6$ and a chunk $C7$ such that $C7$ is material in $S7$ and split in $S6$. But each of these splitting corresponds to a new object. Thus, there are infinitely many objects corresponding to sub-chunks of $C3$ in any finite interval following $S3$. Since these objects must all lie within a bounded neighborhood of place($O, S3$), this contradicts the finiteness principle (FIN).

This completes the argument that O is not material in $S3$. We must now show that O is not a proper shape in $S3$. Let R=shape($O, S3$). By CtoO.6, R is a subset of shape(O, SA) for all $SA \in [S1, S3)$. Suppose that R were a proper shape. By lemma 4.5, there is a chunk CR of shape R that is a sub-chunk of chunk(O, SA) for all $SA \in [S1, S3)$. But then by lemma 4.6, CR is material in $S3$. Since R is the interior of the intersection of shape(O, SA) over $SA < S3$, it is clear that there is no room for either a second disjoint chunk or a super-chunk of CR to persist until $S3$. Therefore, we could extend O so that chunk_of($O, S3$)=CR without violating CtoO.1.a or CtoO.1.b. But this contradicts the maximality of time_of(O).□

Lemma 4.10: Let C be material in S. There exists a situation $S1 < S$ such that the ancestor of C in $S1$ is not split in any $S2 \in (S1, S)$.

Proof: Suppose not. Then, for any $S1 < S$, there exists an $S2$ such that $S1 < S2 < S$ and the ancestor of C in $S1$ is split in $S2$. By the same token, the ancestor of C in $S2$ is split in some $S3 \in (S2, S)$, the ancestor of C in $S3$ is split in $S4$ ... Each of these ancestors must correspond to a different object. Since all are super-chunks of C, they must all lie within a bounded region within the interval $(S1, S)$. But this contradicts the finiteness principle FIN.

OB.8: If $O1$ turns from being a ghost in $S1$ to being material in $S2$, then there a situation $S3 \in (S1, S2]$ and an object $O3$ such that $O3$ is destroyed in $S3$, $O1$ is material in $S3$, and $O1$ occupies a connected component of $O3$ in $S3$.

Proof: Let $S3$ be the lower bound of time_of($O1$). Let $C3$ be the ancestor in $S3$ of the chunks of O. (By lemma 3.7, they all have the same ancestor.) By lemma 4.10, there is a situation $S4$ such that the ancestor of $C3$ in $S4$ is not split in any $SA \in (S4, S3)$. Let $C4$=ancestor($C3, S4$) and let $O3$ be the object such that chunk_of($O3, S4$) $= C4$ (theorem 3).

We claim that $C4$ is split in $S3$. Suppose not. Then define an object OZ such that time_of(OZ) is the union of time_of($O3$) with time_of(O), and such that chunk_of(OZ, S)=chunk_of($O3, S$) for $S \in$time_of($O3$) and chunk_of(OZ, S)=chunk_of(O, S) for $S \in$time_of(O). Then properties CtoO.1.a and CtoO.1.b hold on O, by argument similar to those used to show that the construction of theorem 3 satisfies these properties. But then this contradicts the maximality of O. This completes the contradiction, allowing us to conclude that $C4$ is split in $S3$. It is then immediate that $O3$ does not have a proper shape, and that cshape($C3$) is a connected component of shape($O, S3$).□

OB.9: Before and after each situation, there is a time interval in which no object intersecting a finite region changes status.

Proof: Immediate from the finiteness principle FIN together with the fact that an object can change status only twice (OB.2).□

This complete the proof of Theorem 4.□

8 Sample Inference

In this section, we sketch how the theory of chunks can be used to prove that it is not possible to carve out an internal cavity within an object which initially has no internal cavity.

To represent and prove this statement, we introduce a number of new primitives. The predicate "filled_in(R)" means that region R has no internal cavities. It can be defined topologically as follows: A bounded region R is filled in iff the complement of R has only one connected component. The function "filling_in(R)" maps a region R to the smallest filled-in region containing R. The region filling_in(R) can be defined either as the intersection of all the filled-regions that contain R, or as the union of R with all its internal cavities. The function "boundary(R)" maps a region R into its boundary.

To prove our target theorem, we need a number of geometric facts. Some of the deeper of these facts we state here as lemmas without proof (the proofs are not difficult). More basic results, such as the fact that the composition of continuous functions is continuous, will be used without explicit discussion.

Lemma 5.1: (Properties of filling in) Let RF be the filling in of R. Then

a. RF is filled in.

b. RF contains R.

c. If R is a good shape then RF is also a good shape

d. The boundary of RF is a subset of the boundary of R.

e. If $R2$ is filled in and contains R, then $R2$ contains RF.

RF=filling_in(R) $\Rightarrow$
[filled_in(RF) $\wedge$ $R \subset RF$ $\wedge$ [good_shape(R) $\Rightarrow$ good_shape(RF)] $\wedge$
boundary(RF) $\subset$ boundary(R) $\wedge$
[[filled_in($R2$) $\wedge$ $R \subset R2$] $\Rightarrow$ $RF \subseteq R2$]]

Lemma 5.2: Let $R1$ and $R2$ be well-shaped regions. If $R2$ intersects $R1$ then either $R2$ is a subset of $R1$, or $R2$ intersects the boundary of $R1$.

[good_shape($R1$) $\wedge$ good_shape($R2$) $\wedge$ intersect($R2, R1$)] $\Rightarrow$
[$R2 \subset R1$ $\vee$ intersect($R2$, boundary($R1$))]

Lemma 5.3: If $R1$ is an open region and $R1$ intersects the boundary of $R2$ then the boundary of $R2 - R1$ is not a superset of the boundary of $R2$.

$\neg$[good_shape($R1$) $\wedge$ intersects($R1$,boundary($R2$)) $\wedge$
boundary($R2$) $\subset$ boundary($R2 - R1$)]

Lemma 5.4: If $R1$ and $R2$ are regions of good shape that intersect then their intersection contains another region of good shape.

[good_shape($R1$) $\wedge$ good_shape($R2$) $\wedge$ intersect($R1, R2$)] $\Rightarrow$
$\exists_{RO}$ good_shape(RO) $\wedge$ $RO \subset R1$ $\wedge$ $RO \subset R2$

Lemma 5.5: Let $F1$ and $F2$ be a continuous fluents from time to placements, let $R1$ and $R2$ be regions, and let SA and SB be situations with $SA < SB$. Let us write $F(R, S)$ as an abbrevation for image(value_in(F, S),R), the image of region R under F at S. If $F1(SA, R1)$ is not a subset of $F2(SA, R2)$ and $F1(SB, R1)$ intersects $F2(SB, R2)$, then there is an $SC \in [SA, SB]$ such that $F1(SC, R1)$ intersects the boundary of $F2(SC, R2)$

[$SA < SB$ $\wedge$ $\forall_S$ continuous($F1, S$) $\wedge$ continuous($F2, S$) $\wedge$
$F1(SA, R1) \not\subset F2(SA, R2)$ $\wedge$ intersect($F1(SB, R1)$, $F2(SB, R2)$)] $\Rightarrow$
$\exists_{SC \in [SA,SB]}$ intersect($F1(SC, R1)$, boundary($F2(SC, R2)$))

Lemma 5.6: If $R1$ and $R2$ are good shapes and $R1$ intersects the boundary of $R2$, then $R1$ intersects $R2$.

[good_shape($R1$) $\wedge$ good_shape($R2$) $\wedge$ intersect($R1$, boundary($R2$))] $\Rightarrow$
intersect($R1, R2$).

Lemma 5.7: If place($C1, SA$) and place($C2, SA$) intersect, then $C1$ and $C2$ can never afterward be sub-chunks of two different top-level chunks.

$$[\text{intersect}(\text{place}(C1, SA), \text{place}(C2, SA)) \wedge SA \leq SB$$
$$\text{top\_level}(CT1, SB) \wedge \text{top\_level}(CT2, SB) \wedge$$
$$\text{sub\_chunk}(C1, CT1) \wedge \text{sub\_chunk}(C2, CT2)] \Rightarrow$$
$$CT1 = CT2$$

Proof: Assume that the antecedent is satisfied. By CH.4 and lemma 3.1, $C1$ and $C2$ are material in SA. By CD.2, cshape($C1$) overlaps cshape($C2$). By CH.2 and lemma 5.4, there is a region RR of good shape in the intersection of cshape($C1$) with cshape($C2$). By CH.2 there is a sub-chunk CR of $C1$ with shape RR; by CD.3 CR is also a sub-chunk of $C2$. Hence, using lemma 3.4, CR is a sub-chunk of both $CT1$ and $CT2$. By lemma 3.3, place($CT1, S$) and place($CT2, S$) must overlap, so by CH.8, $CT1 = CT2$.□

We can now prove our target result:

Theorem 5: It is not possible to cut out an internal cavity if all blades are originally outside the target. Formally, if $C2$ is top-level in $S2 > S1$, $C1$ is the ancestor of $C2$ in $S1$, and cshape($C1$) is filled in, then $C2$ is filled in.

$$[\ S1 < S2 \wedge \text{top\_level}(C2, S2) \wedge C1 = \text{ancestor}(C2, S1) \wedge \text{filled\_in}(\text{cshape}(C1))] \Rightarrow$$
$$\text{filled\_in}(\text{cshape}(C2))$$

Proof: By contradiction. Suppose that, contrary to the result to be proven, $S1 < S2$, $C2$ is top-level in $S2$, $C1$ is the ancestor of $C2$ in $S1$, $C1$ is filled in, but $C2$ is not filled in. Let RF=filling_in(cshape($C2$)). By lemma 5.1.a,b, RF is filled in and contains cshape($C2$). By CH.1 cshape($C2$) is a good shape, hence from lemma 5.1.c, RF is a good shape. By lemma 3.3, cshape($C2$) is a subset of cshape($C1$); hence, by lemma 5.1.e, RF is a subset of cshape($C1$). By CH.2 and CD.3, there is a chunk CF whose shape is RF and which is a sub-chunk of $C1$ and a proper super-chunk of $C2$.

Since CF is a sub-chunk of $C1$, and $C1$ is top-level in $S1$, by CH.4, CF is material in $S1$. Since CF is a proper super-chunk of $C2$, and $C2$ is top-level in $S2$, by CD.4, CF is a ghost in $S2$. Hence, CH.7 applies: there must be a top-level chunk CB that is not a super-chunk of CF and a situation $S3 \in (S1, S2]$ such that CB intersects CF in $S3$.

By lemma 3.1, both CF and CB are material at $S1$. Hence place($CB, S1$) is disjoint from place($CF, S1$). (If CF and CB are in the same top-level objects at $S1$ this follows from Lemma 5.8; if they are in different top-level objects, it follows from CH.8). On the other hand, place($CB, S3$) does intersect place($CF, S3$). Hence, by lemma 5.5 there is a time $S4 \in [S1, S3]$ at which place($CB, S4$) intersects the boundary of place($CF, S4$). By lemma 5.1.d the boundary of cshape(CF) is a subset of the boundary of cshape($C2$), so place($CB, S4$) intersects the boundary of place($C2, S4$). Then, by lemma 5.6, place($CB, S4$) intersects place($C2, S4$). But then by CH.8, $C2$ must a ghost at $S4$, and therefore at $S2 > S4$ as well.□

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