On a Conjecture of Micha Perles

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Abstract

We prove a conjecture of Micha Perles concerning simple polytopes, for a subclass that properly contains the duals of stacked and crosspolytopes. As a consequence of a special property of this subclass it also follows that, the entire combinatorial structure of a polytope in the subclass can be recovered from its graph, by applying our results recursively.

1 Introduction

Let P be a simple d-polytope and $G(P)$ the graph (1-skeleton) of P. Perles conjectured that every $(d-1)$-regular, induced, connected and non-separating subgraph of $G(P)$ determines a facet of P [2]. In this paper we prove the conjecture for a proper subclass of simple polytopes.

The motivation for our results comes from two subclasses of simplicial polytopes, namely the *stacked polytopes* and the *crosspolytopes*. Polytopes obtained from a simplex by successive addition of pyramids over facets are called *stacked polytopes*. Stacked polytopes form an important subclass of simplicial polytopes in that, only they attain equality in the lower bound theorem [3]. Secondly, if $\{e_1, \ldots, e_d\}$ is a set of linearly independent vectors in R^d then $X = conv\{\pm e_1, \ldots, \pm e_d\}$ is called a *d-crosspolytope*. d-crosspolytopes can be formed by successively building bipyramids $d-1$ times starting with a 1-simplex.

Consider the dual analogues of the operations of stacking and forming bipyramids. It can be shown [3] that, if P is a simplicial polytope and P^* a simple polytope dual to P, then a polytope obtained by forming a pyramid over a facet of P is dual to a polytope obtained by truncating the corresponding vertex of P^*. Also, [4] any bipyramid with basis P is dual to any prism with basis P^*.

Against this background, we show that if P is any polytope for which Perles' conjecture is true, then the conjecture is also true for a polytope obtained by truncating a vertex of P and also for any prism with basis P. Since Perles' conjecture is trivially true for a simplex, one concludes that it is true for any polytope obtained from a simplex by building prisms and truncating vertices finitely many times, in any arbitrary order. The class $\mathcal{C}$ of polytopes so generated, properly contains duals of crosspolytopes and duals of stacked polytopes We also show that $\mathcal{C}$ is a proper subclass of simple polytopes. Moreover the class $\mathcal{C}$ has the interesting property that every face of a polytope in $\mathcal{C}$ also belongs to $\mathcal{C}$. This allows one to recover the entire combinatorial structure of a polytope in $\mathcal{C}$ from its graph, by applying our results recursively.

2 Notation

Please refer to [3] for a discussion on polytopes and for related terminology. Here we merely indicate the convention that we will follow in the sequel.

If P is a polytope then its vertex set will be denoted $V(P)$ and its graph $G(P)$. The vertex set and the edge set of a graph Γ will be denoted $V(\Gamma)$ and $E(\Gamma)$ respectively. Given a d-polytope P, let $\tilde{P} = P + \mathbf{z}$ be a translate of P where $\mathbf{z}$ is a non-zero vector in R^{d+1}. Then the convex hull of P and $\tilde{P}$ is called a *prism with basis* P denoted $Pr(P)$. A hyperplane H in R^d is said to truncate a vertex x of P, if x and $V(P) \setminus \{x\}$ lie in different open half-spaces of H. We denote by $Tr(P)$ the intersection of P with the closed half-space containing $V(P) \setminus \{x\}$. For our purposes, it does not matter which vertex of P is truncated to obtain $Tr(P)$. $Tr(P)$ represents a (not necessarily unique) polytope obtained by truncating some vertex of P.

Define a subclass $\mathcal{C}$ of simple polytopes as follows: *A polytope P belongs to $\mathcal{C}$ iff there is a sequence of polytopes*

$$P_0, P_1, \cdots, P_n = P$$

where P_0 is a k-simplex ($k \geq 1$) and for $1 \leq i \leq n$ either $P_i = Tr(P_{i-1})$ or $P_i = Pr(P_{i-1})$.

¿From the definition of the subclass $\mathcal{C}$ it follows that it contains the duals of stacked and crosspolytopes.

3 Perles' Conjecture for the Subclass $\mathcal{C}$

Theorem 1 *If Perles' conjecture is true for a simple d-polytope P then it is also true for any prism $Pr(P)$ with basis P.*

Proof : Let $Pr(P)$ be the convex hull of P and its translate $\tilde{P} = P + \mathbf{z}$. Then every vertex $v \in V(P)$ is adjacent to the vertex $\tilde{v} = v + \mathbf{z}$ of $\tilde{P}$. If $X \subseteq V(P)$ then $\tilde{X}$ will denote the corresponding subset of $V(\tilde{P})$. If Λ is any induced subgraph of P then $\tilde{\Lambda}$ will denote the subgraph of $G(\tilde{P})$ induced by the corresponding vertices of $\tilde{P}$.

Let Γ be a d-regular, induced, connected and non-separating subgraph of $Pr(P)$. We show that Γ must determine a facet of $Pr(P)$.

If Γ is a subgraph of $G(P)$ (resp. $G(\tilde{P})$), since both Γ and $G(P)$ (resp. $G(\tilde{P})$) are d-regular graphs, $\Gamma = G(P)$ (resp. $\Gamma = G(\tilde{P})$). Hence Γ determines a facet of $Pr(P)$. So we may assume that $V(\Gamma) \cap V(P) \neq \emptyset$ and $V(\Gamma) \cap V(\tilde{P}) \neq \emptyset$.

Let Γ_P and $\Gamma_{\tilde{P}}$ be the restrictions of Γ to P and $\tilde{P}$ respectively. Consider any vertex v of Γ_P. v is adjacent to only one vertex in $\tilde{P}$. Also, Γ is d-regular. Hence v has at least $d-1$ neighbors in Γ_P. We consider two cases.

Case 1 : Each vertex in Γ_P has exactly $d-1$ neighbors in Γ_P.

Observe that by symmetry each vertex of $\Gamma_{\tilde{P}}$ is also $(d-1)$-valent in $\Gamma_{\tilde{P}}$ and that the two subgraphs Γ_P and $\Gamma_{\tilde{P}}$ are copies of each other. We also observe that:

1. Γ_P is $(d-1)$-regular.

2. If Γ_P has more than one connected component, pick one and call it C. Then the subgraph Γ_C of Γ induced by $V(C) \cup V(\tilde{C})$ is d-regular and hence not connected to $\Gamma \setminus \Gamma_C$ contrary to the assumption that Γ is connected. Hence Γ_P must be connected.

3. Suppose $x, y \in V(P)$ are separated by Γ_P. Let $C(x)$ and $C(y)$ be the connected components of $G(P) \setminus \Gamma_P$ containing x and y respectively. It is easy to see that $\tilde{C}(x)$ and $\tilde{C}(y)$ are separated by $\Gamma_{\tilde{P}}$ in $G(\tilde{P})$. Then Γ would separate $C(x)$ and $C(y)$, contrary to our assumption. Hence Γ_P cannot separate $G(P)$.

Since Perles' conjecture is true for P, using 1, 2 and 3 we conclude that Γ_P determines a facet F of P. Since $\Gamma_{\tilde{P}}$ is the image of Γ_P it also determines the facet $\tilde{F}$ of $\tilde{P}$. So Γ determines a facet of $Pr(P)$.

Case 2 : At least one vertex in Γ_P has d neighbors in Γ_P.

Let X be the set of all the d-valent vertices in Γ_P, i.e.,

$$X = \{w \mid w \in V(\Gamma_P) \text{ and } w \text{ has } d \text{ neighbors in } \Gamma_P \}$$

Let Y be the set of all vertices in Γ_P that are adjacent to at least one vertex in X, i.e.,

$$Y = \{w \mid w \notin X;\ \exists\ x \in X,\ (w, x) \in E(\Gamma_P)\}$$

Since vertices in Y are $(d-1)$-valent in Γ_P, $\tilde{Y} \subset V(\Gamma_{\tilde{P}})$. In $G(\tilde{P})$, all edges coming out of $\tilde{X}$ terminate in $\tilde{Y}$. In other words, any edge path in $G(Pr(P))$ between a vertex $x \in \tilde{X}$ and a vertex $v \notin \tilde{X}$ must contain a vertex in $\tilde{Y}$. We know that there is a vertex $v \in (G(P) \setminus \Gamma)$ because $G(P)$ being d-regular cannot be a proper subgraph of Γ which is also d-regular (recall that we assumed $\Gamma \cap V(\tilde{P}) \neq \emptyset$.) So $\tilde{Y} \subset V(\Gamma)$ separates v and $\tilde{X}$ contrary to the assumption that Γ does not separate $G(Pr(P))$. Hence Case 2 is impossible.

The above argument, shows that $Pr(P)$ satisfies Perles' conjecture if P does. $\diamond$

Theorem 2 *If Perles' conjecture is true for a simple d-polytope P, then it is also true for the d-polytope $Tr(P)$ obtained by truncating a vertex of P.*

Proof : Assume that vertex $v \in V(P)$ was truncated to obtain $Tr(P)$. Suppose (v, w_1), ... ,(v, w_d) are the d edges incident on v in P. Then $z_1, \ldots, z_d$ are the new vertices in $Tr(P)$ where z_i is the intersection of (v, w_i) and the truncating hyperplane H. Also, the new facet of $Tr(P)$ (namely $H \cap Tr(P)$) is a $(d-1)$-simplex determined by the vertex set $Z = \{z_1, \ldots, z_d\}$.

Let Γ be a $(d-1)$-regular, connected, induced and non-separating subgraph of $G(Tr(P))$.

If $Z \cap V(\Gamma) = \emptyset$, there is nothing to prove. Also, if $Z \subseteq V(\Gamma)$ then since Z induces a $(d-1)$-regular subgraph $Z = V(\Gamma)$ and hence Γ determines a facet of

$Tr(P)$. So the only case left to consider is where Γ contains a proper nonempty subset of Z. Since Γ is $(d-1)$-regular, if it contains a vertex of Z, it must contain at least $d-2$ of its neighbors in Z. Therefore at most one vertex of Z can be left out and without loss of generality we assume $z_d \notin V(\Gamma)$. Hence, $w_i \in V(\Gamma)$ for $1 \leq i \leq d-1$.

Consider the subgraph Γ' of $G(P)$ induced by the vertex set $(V(\Gamma) \setminus Z) \cup \{v\}$ Γ' is a $(d-1)$-regular, connected, induced subgraph. z_d has only one neighbour in $V(Tr(P)) \setminus V(\Gamma)$. Therefore $V(\Gamma) \cup \{z_d\}$ does not separate $G(Tr(P))$ which means Γ' does not separate $G(P)$. So Γ' determines a facet F of P and $w_1, \ldots, w_{d-1}$ are the neighbors of v in F. Let H be the supporting hyperplane for F in P. $H \cap Tr(P)$ is a facet of $Tr(P)$ and the graph of this facet is Γ; that completes the proof. $\diamond$

As an immediate consequence of theorems 1 and 2 we have,

Corollary 1 *Perles' conjecture is true for every polytope in the subclass $\mathcal{C}$.*

The subclass $\mathcal{C}$ has the property that any face of a polytope in $\mathcal{C}$ also belongs to $\mathcal{C}$. We prove this property in the following lemma.

Lemma 1 *If $Q \in \mathcal{C}$ and F is a facet of Q then $F \in \mathcal{C}$.*

Proof : Let $Q \in \mathcal{C}$ be a polytope for which the lemma is true. Let X be a facet of $Pr(Q)$. If $X = Q$ or $X = \tilde{Q}$ then by assumption $X \in \mathcal{C}$. If however $X = Pr(F)$ where F is a facet of Q, then since the lemma is true for Q, $F \in \mathcal{C}$ and hence $Pr(F) = X \in \mathcal{C}$. So, if the lemma is true for a $Q \in \mathcal{C}$ then it is also true for $Pr(Q)$.

Now we consider $Tr(Q)$. Let $v \in V(Q)$ be truncated to obtain $Tr(Q)$ and let H be the truncating hyperplane. Assume that H^+ contains $Tr(Q)$ (H^+ denotes one of the closed half-spaces of H). Let Y be a facet of $Tr(Q)$.

If $Y = H \cap Tr(Q)$ then Y is a simplex and hence $Y \in \mathcal{C}$. So assume $Y = F \cap H^+$ where F is a facet of Q. Since the lemma is true for Q, $F \in \mathcal{C}$. Hence $Tr(F) = Y \in \mathcal{C}$. The only case left to consider is when $Y = F$ where F is a facet of Q. Once again $Y \in \mathcal{C}$.

Therefore if the lemma is true for a $Q \in \mathcal{C}$ it is also true for $Tr(Q)$; that completes the proof.$\diamond$

As an immediate consequence of this lemma we obtain

Corollary 2 *The entire combinatorial structure of a polytope $P \in \mathcal{C}$ can be determined from $G(P)$ by repeated application of theorems 1 and 2 and lemma 1.*

The following lemma shows that $\mathcal{C}$ is properly contained in the class of simple polytopes.

Lemma 2 *$\mathcal{C}$ is a proper subclass of simple polytopes.*

Proof : We show that $\mathcal{C}$ does not contain a simple 4-polytope with 9 vertices. Suppose it did. Then, to construct the polytope we can either start with a simple 3-polytope or a 4-simplex. Suppose we started with a 4-simplex which has 5 vertices. In this case we may only truncate vertices. But each truncation (when d=4) increases the vertex count by 3; so we get 4-polytopes with 5, 8, 11, $\cdots$ vertices but not with 9 vertices. On the contrary suppose we started with a 3-polytope. Since constructing a prism doubles the vertex count we can only construct a prism over a 3-polytope with 4 vertices. The same argument as before shows that again we cannot obtain a 4-polytope with 9 vertices.

Now consider $C(6,4)$ - the cyclic 4-polytope with 6 vertices has 9 facets. (refer to [3] for details). It is a simplicial polytope. Its dual which is simple has 9 vertices and is hence not in $\mathcal{C}$. $\diamond$

Also, it is easy to show that the dual-stacked and the dual-crosspolytopes form proper subclasses of $\mathcal{C}$.

4 Remarks

Perles' conjecture is true for any simple 3-polytope[5]. So we could as well start with any simple 3-polytope and build prisms and truncate vertices finitely many times. The foregoing results would still be valid without any modification for a polytope so obtained.

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