On Triangulations of the 3-ball and the Solid Torus

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Abstract

We show that neither the 3-ball nor the solid torus admits a triangulation in which (i) every vertex is on the boundary, and (ii) every tetrahedron has exactly one triangle on the boundary. (Such triangulations are relevant to an unresolved conjecture of Perles.) Our result settles a question posed at the *DIMACS Workshop on Polytopes and Convex Sets.*

1 Introduction

Let M be an n-pseudomanifold with boundary. In the dual graph of M denoted $G(M)$, vertices correspond to the n-cells of M with an edge between two vertices iff the corresponding n-cells share an $(n-1)$-cell.

Micha A. Perles asked the following question [1] : *Let $\mathcal{C}$ be a subset of facets of a simplicial d-polytope P, and $\bar{\mathcal{C}}$ the complement of $\mathcal{C}$. If both $G(\mathcal{C})$ and $G(\bar{\mathcal{C}})$ are connected and if $G(\mathcal{C})$ is $(d-1)$-regular then must $\mathcal{C}$ necessarily be the star of a vertex?* We ask the same question in the more general setting of triangulated spheres (instead of P consider a triangulation of S^{d-1}; call the $(d-1)$-simplices of the triangulation "facets").

Note that the 3-sphere S^3 can be decomposed into either two 3-balls having a common boundary or into two solid tori (solid torus is the product of a 2-ball and a circle) having a common boundary. Hence if the 3-ball or the solid torus has a triangulation

1. that is not the star of a vertex and
2. in which each tetrahedron has one 2-face (triangle) on the boundary (of the ball or the solid torus)

then we could extend that triangulation to a triangulation of S^3 and obtain a 4-dimensional counter-example to the generalization of Perles' question. (The question of whether the 3-ball admits a triangulation having properties (1) and (2), was posed at the *DIMACS Workshop on Polytopes and Convex Sets [2]*, by Jockusch and Prabhu.)

Against this background, we show that neither the 3-ball nor the solid torus admits a triangulation having properties (1) and (2). (It is worth noting that the unshellable triangulation of a tetrahedron that M.E. Rudin describes [3], satisfies property (1) and all but one tetrahedron (of the triangulation) satisfy property (2). In Rudin's triangulation, one tetrahedron has no triangle on the boundary.)

In Section 2 we present two proofs of the result about the 3-ball. In Section 3 we present a proof of an analogous result for the torus (that depends on Proof 1 in Section 2).

2 Triangulation of 3-ball

If X is a manifold with boundary, the relative interior, relative boundary and interior of X will be denoted $relint(X)$, $relbd(X)$ and $int(X)$ respectively. ∂ will denote the boundary operator.

Let Δ be a triangulation of M, a 3-manifold with boundary. A face of a tetrahedron in Δ is called an *exterior face* if it lies in $\partial\Delta$ and an *interior face* otherwise. We say that a tetrahedron in Δ is of *type i* if exactly i of its 2-faces (triangles) are exterior.

Theorem 1 *Excluding triangulations which are the star of a vertex, there is no triangulation of the 3-ball B^3 in which every tetrahedron is of type 1.*

Proof 1 : If the theorem were false we must have a smallest triangulation Δ (a triangulation with fewest tetrahedra) that contradicts the claim. Given such a Δ we show how to obtain a smaller triangulation of B^3 that contradicts the claim.

For a vertex v of Δ let $lk_{\partial\Delta}(v)$ and $lk_{\Delta}(v)$ denote the links of v with respect to $\partial\Delta$ and Δ respectively. $lk_{\partial\Delta}(v)$ is a circle and $lk_{\Delta}(v)$ a 2-ball. We want to show that $\partial(lk_{\Delta}(v)) = lk_{\partial\Delta}(v)$. Let ab be an edge of $\partial(lk_{\Delta}(v))$. Since ab is an edge of exactly one triangle of $lk_{\Delta}(v)$, triangle vab is a face of only one tetrahedron of Δ; *i.e.* vab is an exterior triangle, so $ab \in lk_{\partial\Delta}(v)$. Thus $\partial(lk_{\Delta}(v)) \subset lk_{\partial\Delta}(v)$; since both are topological circles $\partial(lk_{\Delta}(v)) = lk_{\partial\Delta}(v)$.

Let v be a vertex such that $lk_{\Delta}(v)$ is not the star of a vertex. (Such a vertex exists because, for any vertex z if $lk_{\Delta}(z)$ is the star of vertex w, then $lk_{\Delta}w$ cannot be the star of a vertex.) $lk_{\Delta}(v)$ is a triangulation of a 2-ball. If triangle T of $lk_{\Delta}(v)$ has an edge E in $\partial(lk_{\Delta}(v))$ then $v * E$ (* indicates join) is an exterior triangle. Hence T must be interior. Conversely if no edge of T lies in $\partial(lk_{\Delta}(v))$ then T must be exterior since all the other triangles of $v * T$ are interior.

A straightforward argument shows that $lk_{\Delta}(v)$ must contain two 2-balls C_1 and C_2 with disjoint relative interiors, having the property that $relint(C_i) \subset int(B^3)$ and $relbd(C_i) \subset \partial(B^3)$. Hence each C_i divides Δ into two parts; C_1 and C_2 cut Δ into three pieces : Δ_1, Δ_2 and Δ_3. Say the arrangement is $\Delta_1\ C_1\ \Delta_2\ C_2\ \Delta_3$; v lies in Δ_2.

Call an interior triangle a 'cutting triangle' if all of its edges are exterior and an 'almost cutting triangle' if two of its edges are exterior. Pick two almost cutting triangles A_1 and A_2 from C_1

and C_2 respectively. Let a, b and c be the vertices of A_1 and ab and bc the exterior edges. The portion of $lk_{\partial\Delta}(b)$ contained in Δ_1 is an arc, say $a \leftrightarrow x_1 \leftrightarrow \ldots \leftrightarrow x_n \leftrightarrow c$. Let $wabc$ be the tetrahedron in Δ_1 containing triangle abc. ac is an interior edge, hence wac cannot be an exterior triangle; so either wab or wbc is, which means w must be either x_1 or x_n. If $w = x_1$ then bx_1c is an almost cutting triangle and we repeat the argument on the arc $x_1 \leftrightarrow \ldots \leftrightarrow x_n \leftrightarrow c$; else bx_na must be an almost cutting triangle and we consider the arc $a \leftrightarrow x_1 \leftrightarrow \ldots \leftrightarrow x_n$. Repeating this process we eventually reach a cutting triangle T_1 in Δ_1. Similarly starting with A_2 we find a cutting triangle T_2 in Δ_3.

The two cutting triangles T_1 and T_2 cut Δ into three pieces, say Δ_1', Δ_2' and Δ_3'; say the arrangement is Δ_1' T_1 Δ_2' T_2 Δ_3'. Removing Δ_2' and pasting Δ_1' and Δ_3' by identifying T_1 and T_2, we obtain a smaller triangulation of B^3 which is not the star of a vertex and in which every tetrahedron is of type 1. Thus Δ cannot exist. $\Box$

Proof 2 : We use the same notation as in Proof 1. Assume Δ is a triangulation that contradicts the claim. Observe that Δ cannot have any interior vertices. Let $n = f_0(\Delta)$. $\partial\Delta$ is a triangulation of 2-sphere and hence satisfies Euler's relation $f_0(\partial\Delta) - f_1(\partial\Delta) + f_2(\partial\Delta) = 2$. Also, each edge in $\partial\Delta$ is contained in two triangles. Hence $f_2(\partial\Delta) = 2n - 4$.

For a vertex v of Δ, let $p(v)$ be the number of triangles of $lk_\Delta(v)$ contained in $\partial\Delta$. Then $\sum_v p(v) = f_2(\partial\Delta) = f_3(\Delta) = 2n - 4$. On the other hand we show that that $\sum_v p(v) \geq 2n$, to obtain a contradiction.

For a vertex v of Δ, $lk_\Delta(v)$ is a triangulated polygon. A triangle T of $lk_\Delta(v)$ lies in $\partial\Delta$ iff exactly one edge of T lies in $\partial(lk_\Delta(v))$ (see para. 3, Proof 1). If we think of $\partial(lk_\Delta(v))$ as bounding a cell C, then $lk_\Delta(v)$, together with cell C, forms a cell-decomposition of 2-sphere which satisfies Euler's relation (above). Hence, a simple calculation shows that if $lk_\Delta(v)$ has k vertices in its relative interior, then $p(v) = 2k - 2$.

Case 1: Assume that $lk_\Delta(v)$ has exactly one vertex w in its relative interior. In this case $p(v) = 0$ and we call w the *interior neighbor* of v.

Case 2: Assume v is the interior neighbor of at least one vertex (see Case 1). Let $\{v_1, \ldots, v_q\}$

be the set of vertices for which v is the interior neighbor. We show that $lk_\Delta(v)$ must have at least $q+2$ vertices in its relative interior and hence $p(v) \geq 2(q+1)$; i.e., we show $p(v)+p(v_1)+p(v_2)+\cdots+p(v_q) \geq 2(q+1)$.

All triangles of $\partial\Delta$ that contain v_i lie in $lk_\Delta(v)$. Hence $v_1,\ldots,v_q$ lie in $relint(lk_\Delta(v))$. In a triangulated 2-ball B, we call a triangle with one edge in ∂B a *boundary triangle.* Observe that none of the boundary triangles of $lk_\Delta(v)$ can be incident on any of the v_i's. For each boundary triangle of $lk_\Delta(v)$, having all three vertices on $\partial(lk_\Delta(v))$, contract the edge in $\partial(lk_\Delta(v))$ to obtain a reduced triangulation. None of the contractions can destroy a triangle that contains v_i. The result of all the contractions is a triangulated 2-ball M. $v_1,\ldots,v_q$ still lie in the relative interior of M. A boundary triangle of M cannot be incident on any of the v_i's. If M has fewer than two (it must have at least one) interior vertices different from $v_1,\ldots,v_q$, then all the boundary triangles of M are incident on a vertex, i.e., M is the star of a vertex, which is a contradiction.

Case 3: Assume v falls neither into Case 1 nor into Case 2. Then $lk_\Delta(v)$ has $k \geq 2$ interior vertices. So $p(v) \geq 2$.

Combining the three cases, we see that $\sum_v p(v) \geq 2n$. □

3 Triangulation of Solid Torus

In this section we prove an analogue of Theorem 1 for the solid torus. Both the main proof and the following lemma depend on Proof 1 above.

Lemma 1 *There is no triangulation of B^3 in which two tetrahedra that share a vertex v are of type 2, and the remaining tetrahedra are of type 1.*

Proof: (We borrow notation from Proof 1 above.) Assume Δ is a triangulation that contradicts the claim. One can easily show that $lk_\Delta(v)$ contains a 2-ball C with $relint(C) \subset int(B^3)$ and $relbd(C) \subset \partial(B^3)$. C cuts Δ into two pieces, say Δ_1 and Δ_2. $lk_\Delta(v)$ is contained in one of the pieces, say in Δ_2. C must have an almost cutting triangle and arguing as in Proof 1, we find a cutting triangle T in Δ_1. T cuts Δ into two pieces, one of which contains $lk_\Delta(v)$. Pasting two copies of the other piece along triangle T, we obtain a triangulation of B^3 that contradicts Theorem 1. $\Box$

Theorem 2 *There is no triangulation of the solid torus T in which every tetrahedron is of type 1.*

Proof : If possible let Δ be a triangulation of T that contradicts the claim. Let $lk_\Delta(v)$ and $lk_{\partial\Delta}(v)$ denote the links of a vertex v with respect to T and $\partial(T)$ respectively. $lk_\Delta(v)$ is a 2-ball and $lk_{\partial\Delta}(v)$ a circle.

Arguing as in Proof 1 of Theorem 1 one can show:

1. $\partial(lk_\Delta(v)) = lk_{\partial\Delta}(v)$ and
2. $lk_\Delta(v)$ contains a 2-ball C with $relint(C) \subset int(T)$ and $relbd(C) \subset \partial(T)$.

Observe that since $relbd(C)$ (a circle) is homotopic to a point within T, C either cuts Δ into a 3-ball Δ_1 and its complement (Fig. 1), or it cuts T into a cylinder (Fig. 2).

We look at an almost cutting triangle abc of C with exterior edges ab and bc. C divides $lk_{\partial\Delta}(b)$ into two arcs each of which yields a cutting triangle. Call those cutting triangles T_1 and T_2. T_1 and T_2 must be distinct and they share the vertex b.

If either T_1 or T_2 cuts T as in Fig. 1, we obtain a contradiction to Theorem 1. So assume both T_1 and T_2 cut T into a cylinder (as in Fig. 2). Then T_1 and T_2 cut Δ into a 3-ball Δ_1' and its complement. In Δ_1', if T_1 and T_2 are faces of the same tetrahedron then we can remove that tetrahedron, leaving a triangulation of a 3-ball with one tetrahedron of type 2 and the rest of type 1; this contradicts Theorem 1. On the other hand, if T_1 and T_2 belong to different tetrahedra in Δ_1', Lemma 1 is contradicted. Thus Δ cannot exist. $\Box$

4 Remarks

It is not known if a d-ball ($d > 3$) admits a triangulation with no interior vertices, in which all the d-simplices have exactly one $(d-1)$-dimensional face on the boundary of the ball. It is also not known if any 3-manifold with boundary (other than the 3-ball and the solid torus) can be triangulated such that every tetrahedron is of type 1.

5 Acknowledgment

We thank Richard Stanley for finding an error in an earlier version of Proof 1 (of Theorem 1).

References

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