Quantum Information Physics II

TR2021-997

Revised March 28, 2022

Davi Geiger and Zvi M. Kedem

Courant Institute of Mathematical Sciences

New York University, New York, New York 10012

Abstract

We study quantum entropy, a measure of randomness over the degrees of freedom of a quantum state and quantified in quantum phase spaces. We show that it is dimensionless, a relativistic scalar, and it is invariant under coordinate and CPT transformations.

We show that the entropy evolution of a coherent state is increasing with time. We augment time reversal with time translation and show that CPT with time translation can transform particles with decreasing entropy evolution for a finite time interval into anti-particles with increasing entropy evolution for the same finite time interval. We revisit transition probabilities of a two state Hamiltonian and show how they relate to entropy oscillations.

We also explore the possibility that entropy oscillations trigger the annihilations and the creations of particles.

CONTENTS

INTRODUCTION

A time arrow emerges in physics only when a probabilistic behavior of ensembles of particles is considered in classical physics. In contrast, quantum physics is presented as time reversible even though a probabilistic behavior is intrinsic even to a single-particle. In [7] we proposed a definition of quantum entropy to measure the randomness of a quantum state, while accounting for all its degrees of freedom (DOFs). That entropy is a sum of two components: the coordinate-entropy and the spin-entropy, each defined in its own quantum phase space. By quantum phase space we mean, the space where the projection of a state in each phase space basis is simultaneously represented. The spin entropy was elaborated in [8]. We analyzed there the possible entropy evolution and conjectured that a law analogous to the classical second law of thermodynamics holds, applicable to all particle physics.

This paper provides more technical depth to further develop the issues studied in [7]. The results are applicable to both the Quantum Mechanics (QM) and the Quantum Field Theory (QFT) settings, but we generally present them in only a more convenient setting. We also further develop the coordinate-entropy for multiple particles. We show that the coordinate-entropy is invariant under changes in continuous 3D coordinate transformations, continuous Lorentz transformations, and discrete CPT transformations. We then analyze the evolution of coherent states. We study time reflection of particles' evolution and the impact of the transformation into anti-particles. We study entropy oscillations for a two-state Hamiltonian and their relation to Fermi's golden rule. Following the results described above, we review a conjectured entropy law that the entropy of a quantum system is an increasing function of time, and end with conclusions.

We also compare our proposed entropy with two different entropy concepts studied in quantum physics, namely the von Neumann's entropy [10] and the Wehrl's entropy [11]. While von Neumann's entropy is a quantification of the randomness of specifying a quantum state, Wehrl's entropy is an attempt to adapt a classical entropy to a quantum state based on the Husimi's quasiprobability distribution and using coherent states as an overcomplete basis representation of a classical phase space. Von Neumann's entropy assigns a zero entropy to any quantum (pure) state and thus does not address the randomness of the observables as we propose. Wehrl's entropy does not satisfy the third Kolmogorov axiom of mutual exclusivity of events, and as a consequence does not satisfy the monotonicity or the complement rules. Also, Wehrl's entropy will not be invariant under the Lorentz group transformations or under pointwise transformations of the position. Thus because of the above shortcomings, Wehrl's entropy will not quantify exactly the randomness of the quantum-phase observables.

QUANTUM ENTROPY IN PHASE SPACE

Given a state $|\psi\rangle_t$ and its density operator $\rho_t = |\psi\rangle_t \langle\psi|_t$, we consider the quantum coordinate phase space to be the space of simultaenous projections of all possible states to the basis $|\mathbf{r}\rangle, |\mathbf{p}\rangle$, i.e., the state $|\psi\rangle_t$ is described in quantum phase space by the pair $(\langle\mathbf{r}|\psi\rangle_t, \langle\mathbf{p}|\psi\rangle_t)$. The coordinate-entropy in quantum phase space was defined in [7] as

$$\mathrm{S} = -\int \rho_{\mathrm{r}}(\mathbf{r},t)\rho_k(\mathbf{k},t)\ \ln\left(\rho_{\mathrm{r}}(\mathbf{r},t)\rho_k(\mathbf{k},t)\right)\ \mathrm{d}^3\mathbf{r}\ \mathrm{d}^3\mathbf{k}\,,$$

where $\mathrm{S}_{\mathrm{r}} = -\int \rho_{\mathrm{r}}(\mathbf{r},t) \ln \rho_{\mathrm{r}}(\mathbf{r},t)\ \mathrm{d}^3\mathbf{r}$, and analogously for S_{k}, $\rho_{\mathrm{r}}(\mathbf{r},t) = \langle\mathbf{r}|\,\rho_t\,|\mathbf{r}\rangle = |\psi(\mathbf{r},t)|^2$ and $\rho_k(\mathbf{k},t) = \langle\mathbf{k}|\,\rho_t\,|\mathbf{k}\rangle = |\tilde{\phi}(\mathbf{k},t)|^2$, with $\psi(\mathbf{r},t)$ and $\tilde{\phi}(\mathbf{k},t)$ representing in QM the wave function and in QFT the coefficients of the Fock states. The momentum is described by the change of variables $\mathbf{p} = \hbar\mathbf{k}$, so that the entropy is dimensionless and invariant under changes of the units of measurements.

A natural extension of this entropy to an N-particle QM system is

$$\begin{aligned} S = &-\int \mathrm{d}^3\mathbf{r}_1\, \mathrm{d}^3\mathbf{k}_1 \ldots \mathrm{d}^3\mathbf{r}_N\, \mathrm{d}^3\mathbf{k}_N\, \rho_{\mathrm{r}}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)\rho_{\mathrm{k}}(\mathbf{k}_1, \ldots, \mathbf{k}_N, t) \\ &\qquad \times \ln\left(\rho_{\mathrm{r}}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)\rho_{\mathrm{k}}(\mathbf{k}_1, \ldots, \mathbf{k}_N, t)\right) \\ = &-\int \mathrm{d}^3\mathbf{r}_1 \ldots \int \mathrm{d}^3\mathbf{r}_N\, \rho_{\mathrm{r}}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) \ln \rho_{\mathrm{r}}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) \\ &-\int \mathrm{d}^3\mathbf{k}_1 \ldots \int \mathrm{d}^3\mathbf{k}_N\, \rho_{\mathrm{k}}(\mathbf{p}_1, \ldots, \mathbf{k}_N, t) \ln \rho_{\mathrm{k}}(\mathbf{k}_1, \ldots, \mathbf{k}_N, t)\,, \end{aligned}$$

where $\rho_{\mathrm{r}}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = |\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)|^2$ and $\rho_{\mathrm{k}}(\mathbf{k}_1, \ldots, \mathbf{k}_N, t) = |\phi(\mathbf{k}_1, \ldots, \mathbf{k}_N, t)|^2$ are defined in QM via the projection of the state $|\psi_t\rangle^N$ of N particles (the product of N Hilbert spaces) onto the position $\langle\mathbf{r}_1| \ldots \langle\mathbf{r}_N|$ and the momentum $\langle\mathbf{k}_1| \ldots \langle\mathbf{k}_N|$ coordinate systems.

ENTROPY INVARIANT PROPERTIES

Continuous Transformations of the Phase Space

In the QM setting, we investigate a point transformation of coordinates and a translation in phase space of a quantum reference frame [1].

Consider a point transformation of position coordinates $F : \mathbf{r} \mapsto \mathbf{r}'$. It induces the new conjugate momentum operator [3]

$$\hat{\mathbf{p}}' = -\mathrm{i}\hbar \left[\nabla_{\mathrm{r}'} + \frac{1}{2} J^{-1}(\mathbf{r}') \nabla_{\mathrm{r}'} \cdot J(\mathbf{r}') \right]\,, \tag{1}$$

where $J(\mathbf{r}') = J(F^{-1})(\mathbf{r}') = \frac{\partial \mathbf{r}(\mathbf{r}')}{\partial \mathbf{r}'}$ is the Jacobian of F^{-1} at $\mathbf{r}'$.

Theorem 1. *The entropy is invariant under a point transformation of coordinates.*

Proof. Let S be the entropy in phase-space relative to a conjugate Cartesian pair of coordinates $(\mathbf{r}, \mathbf{p})$. Let $\mathbf{p}'$ be the momentum conjugate to $\mathbf{r}'$. As the probabilities in infinitesimal volumes are invariant,

$$|\psi'(\mathbf{r}')|^2\,\mathrm{d}^3\mathbf{r}' = |\psi(\mathbf{r}(\mathbf{r}'))|^2\,\mathrm{d}^3\mathbf{r}(\mathbf{r}') \text{ and } |\tilde{\phi}'(\mathbf{p}')|^2\,\mathrm{d}^3\mathbf{p}' = |\tilde{\phi}(\mathbf{p}(\mathbf{p}'))|^2\,\mathrm{d}^3\mathbf{p}(\mathbf{p}')\,, \quad (2)$$

where $\mathbf{r}(\mathbf{r}') \equiv F^{-1}(\mathbf{r}')$ and $\mathbf{p}(\mathbf{p}') \equiv G^{-1}(\mathbf{p}')$ with $G : \mathbf{p} \mapsto \mathbf{p}'$ specified by (1). Thus, by Born's rule the probability density functions are $|\psi'(\mathbf{r}')|^2$ and $|\tilde{\phi}'(\mathbf{p}')|^2$. The Jacobian satisfies $\det J(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}' = \mathrm{d}^3\mathbf{r}(\mathbf{r}')$, and applying this to (2) we get $|\psi'(\mathbf{r}')|^2\,\mathrm{d}^3\mathbf{r}' = |\psi(F^{-1}(\mathbf{r}'))|^2 \det J(\mathbf{r}')\ \mathrm{d}^3\mathbf{r}'$, i.e., $|\psi'(\mathbf{r}')|^2 = |\psi(F^{-1}(\mathbf{r}'))|^2 \det J(\mathbf{r}')$. Similarly, we define $g(\mathbf{p}') = \det J(G^{-1})(\mathbf{p}')$ and so $g(\mathbf{p}')\,\mathrm{d}^3\mathbf{p}' = \mathrm{d}^3\mathbf{p}$, and to satisfy the infinitesimal probability invariant in momentum space $|\tilde{\phi}(\mathbf{p})|^2\ \mathrm{d}^3\mathbf{p} = |\tilde{\phi}'(\mathbf{p}')|^2\,\mathrm{d}^3\mathbf{p}'$ at $\mathbf{p}' = G(\mathbf{p})$ we obtain $|\tilde{\phi}'(\mathbf{p}')|^2 = |\tilde{\phi}(G^{-1}(\mathbf{p}')|^2 g(\mathbf{p}')$.

As noted in [3], there is an arbitrariness in the choice of G that allows a new transformation G' to be specified by (1) with $\det J(G'^{-1})(\mathbf{p}') = \frac{g(\mathbf{p}')}{f(\mathbf{p}')}$, i.e., the arbitrariness of G' is equivalent to the choice of a function $f(\mathbf{p}')$ to define the determinant of its (inverse) Jacobian. Then, associated with such a G' we must also define a new density function $|\tilde{\phi}'(\mathbf{p}')|^2$ scaled by $\frac{1}{f(\mathbf{p}')}$, producing an equally valid conjugate solution. Thus,

$$\begin{aligned}
\mathrm{S_r} + \mathrm{S_p} &= -\int \mathrm{d}^3\mathbf{r}\ \mathrm{d}^3\mathbf{p}\,\rho_{\mathrm{r}}(\mathbf{r},t)\rho_p(\mathbf{p},t)\ln\left(\rho_{\mathrm{r}}(\mathbf{r},t)\rho_p(\mathbf{p},t)\right) - 3\ln\hbar \\
&= \mathrm{S_{r'}} + \mathrm{S_{p'}} - \langle \ln\det J^{-1}(\mathbf{r}')\rangle_{\rho'_{r'}} + \langle \ln g(\mathbf{p}')\rangle_{\rho'_{p'}} \\
&= \mathrm{S_{r'}} + \mathrm{S_{p'}}\,,
\end{aligned}$$

and given the arbitrariness of G, we chose $g(\mathbf{p}')$ to satisfy $\langle \ln g(\mathbf{p}')\rangle_{\rho'_{p'}} = \langle \ln\det J^{-1}(\mathbf{r}')\rangle_{\rho'_{r'}}$. □

We next investigate translation transformations. When a quantum reference frame is translated by x_0 along x, the state $|\psi_t\rangle$ in the position representation becomes $\psi(x - x_0, t) = \langle x - x_0|\psi_t\rangle = \langle x|\,\hat{T}_P(-x_0)\,|\psi_t\rangle$, where $\hat{T}_P(-x_0) = \mathrm{e}^{\mathrm{i}x_0\hat{P}}$, and $\hat{P}$ is the momentum operator conjugate to $\hat{X}$. When the reference frame is translated by p_0 along p, the state $|\psi_t\rangle$ in the momentum representation becomes $\tilde{\phi}(p-p_0,t) = \langle p - p_0|\psi_t\rangle = \langle p|\,\hat{T}_X(-p_0)\,|\psi_t\rangle$, where $\hat{T}_X(-p_0) = \mathrm{e}^{\mathrm{i}p_0\,\hat{X}}$, and $\hat{X}$ is the position operator conjugate to $\hat{P}$.

Theorem 2 (Frames of reference)**.** *The entropy of a state is invariant under a change of a quantum reference frame by translations along x and along p.*

Proof. Let $|\psi_t\rangle$ be a state and S its entropy. We start by showing that $\mathrm{S}_x = -\int_{-\infty}^{\infty}\mathrm{d}x\,|\psi(x,t)|^2\ln|\psi(x,t)|^2$ is invariant under two types of translations:

(i) translations along x by any x_0

$$\mathrm{S}_{x+x_0} = -\int_{-\infty}^{\infty}\mathrm{d}x\,|\psi(x+x_0,t)|^2\ln|\psi(x+x_0,t)|^2 = \mathrm{S}_x\,,$$

which is verified by changing variables under the infinite integration interval.

(ii) translations along p by any p_0

$$\begin{aligned}\psi_{p_0}(x,t) &= \langle x|\,\hat{T}_X(p_0)\,|\psi_t\rangle = \int_{-\infty}^{\infty}\langle x|\,\hat{T}_X(p_0)\,|p\rangle\,\langle p|\psi_t\rangle\,\mathrm{d}p\\ &= \int_{-\infty}^{\infty}\langle x|p+p_0\rangle\,\tilde{\phi}(p,t)\,\mathrm{d}p = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\,x\,(p+p_0)}\tilde{\phi}(p,t)\,\mathrm{d}p\\ &= \psi(x,t)\,\mathrm{e}^{\mathrm{i}\,x\,p_0}\,,\end{aligned}$$

implying $|\psi_{p_0}(x,t)|^2 = |\psi(x,t)|^2$.

Similarly, by applying both translations to $\mathrm{S_p} = -\int_{-\infty}^{\infty}\mathrm{d}p\,|\tilde{\phi}(p,t)|^2\ln|\tilde{\phi}(p,t)|^2$ we conclude that $\mathrm{S_p}$ is invariant under them too. Therefore $\mathrm{S} = \mathrm{S}_x + \mathrm{S_p} - 3\ln\hbar$ is invariant under translations in both x and p. □

CPT Transformations

We will be focusing on fermions, and thus on the Dirac spinors equation, though most of the ideas apply to bosons as well. The QFT Dirac Hamiltonian is

$$\mathcal{H}^{\mathcal{D}} = \int \mathrm{d}^3\mathbf{r}\, \Psi^\dagger(\mathbf{r},t) \left(-\mathrm{i}\hbar\gamma^0\vec{\gamma}\cdot\nabla + mc\gamma^0\right)\Psi(\mathbf{r},t)\,.$$

A QFT solution $\Psi(\mathbf{r},t)$ satisfies $[\mathcal{H}^{\mathcal{D}},\Psi(\mathbf{r},t)] = -\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$ and the C, P, and T symmetries provide new solutions from $\Psi(\mathbf{r},t)$. As usual, $\Psi^{\mathrm{C}}(\mathbf{r},t) = C\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t) = P\Psi(-\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t) = T\Psi^*(\mathbf{r},-t)$, and $\psi^{\mathrm{CPT}}(-\mathbf{r},-t) = CPT\overline{\psi}^{\mathsf{T}}(-\mathbf{r},-t)$, For completeness, we briefly review the three operations, Charge Conjugation, Parity Change, and Time Reversal.

Charge Conjugation transforms particles $\Psi(\mathbf{r},t)$ into antiparticles $\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = (\Psi^\dagger\gamma^0)^{\mathsf{T}}(\mathbf{r},t)$. As $C\gamma^\mu C^{-1} = -\gamma^{\mu\mathsf{T}}$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$ is also a solution for the same Hamiltonian. In the standard representation, $C = \mathrm{i}\gamma^2\gamma^0$ up to a phase. Parity Change $P = \gamma^0$, up to a sign, effects the transformation $\mathbf{r} \mapsto -\mathbf{r}$. Time Reversal effects $t \mapsto -t$ and is carried by the operator $\mathcal{T} = T\hat{K}$, where $\hat{K}$ applies conjugation. In the standard representation $T = \mathrm{i}\gamma^1\gamma^3$, up to a phase.

Theorem 3 (Invariance of the entropy under CPT-transformations)**.** *Given a quantum field $\Psi(\mathbf{r},t)$, its Fourier transform $\Phi(\mathbf{k},t)$, and its entropy S_t, the entropies of $\Psi^*(\mathbf{r},t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t)$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t)$, of and $\Psi^{\mathrm{CPT}}(-\mathbf{r},-t)$, and their corresponding Fourier transforms are all equal to S_t.*

Proof. The probability densities of $\Psi^*(\mathbf{r},t)$, $\Psi^{\mathrm{T}}(\mathbf{r},-t)$, $\Psi^{\mathrm{P}}(-\mathbf{r},t)$, $\Psi^{\mathrm{C}}(\mathbf{r},t)$, and

$\Psi^{\mathrm{CPT}}(-\mathbf{r},-t)$ are

$$
\begin{aligned}
\rho_{\mathrm{r}}^{*}(\mathbf{r},t) &= \Psi^{\mathsf{T}}(\mathbf{r},t)\Psi^{*}(\mathbf{r},t) = \Psi^{\dagger}(\mathbf{r},t)\Psi(\mathbf{r},t) = \rho(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{C}}(\mathbf{r},t) &= \left(\overline{\Psi}^{\mathsf{T}}\right)^{\dagger}(\mathbf{r},t)C^{\dagger}C\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \overline{\Psi}^{*}(\mathbf{r},t)\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{P}}(-\mathbf{r},t) &= \Psi^{\dagger}(\mathbf{r},t)(\gamma^{0})^{\dagger}\gamma^{0}\Psi(\mathbf{r},t) = \Psi^{\dagger}(\mathbf{r},t)\Psi(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{T}}(\mathbf{r},-t) &= \Psi^{\mathsf{T}}(\mathbf{r},t)T^{\dagger}T\Psi^{*}(\mathbf{r},t) = \Psi^{\mathsf{T}}(\mathbf{r},t)\Psi^{*}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,,\\
\rho_{\mathrm{r}}^{\mathrm{CPT}}(-\mathbf{r},-t) &= \left(\overline{\Psi}^{\mathsf{T}}\right)^{\dagger}(\mathbf{r},t)(CPT)^{\dagger}(CPT)\overline{\Psi}^{\mathsf{T}}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,. \qquad (3)
\end{aligned}
$$

As the densities are equal, so are the associated entropies.

Equations (3) also hold for $\Phi(\mathbf{k},t)$ and its density. Thus, both entropies terms in $\mathrm{S}_t = \mathrm{S}_r + \mathrm{S}_k$ are invariant under all CPT transformations. □

Lorentz Transformations

Theorem 4. *The entropy is a relativistic scalar.*

Proof. The probability elements $\mathrm{dP}(\mathbf{r},t) = \rho_{\mathrm{r}}(\mathbf{r},t)\,\mathrm{d}^3\mathbf{r}$ and $\mathrm{dP}(\mathbf{k},t) = \rho_{\mathrm{k}}(\mathbf{k},t)\,\mathrm{d}^3\mathbf{k}$ are invariant under Lorentz transformations since event probabilities do not depend on the frame of reference. Consider a slice of the phase space with frequency $\omega_{\mathrm{k}} = \sqrt{\mathbf{k}^2c^2 + \left(\frac{mc^2}{\hbar}\right)^2}$. The volume elements $\frac{1}{\omega_{\mathrm{k}}}\,\mathrm{d}^3\mathbf{k}$ and $\omega_{\mathrm{k}}\,\mathrm{d}^3\mathbf{r}$, are invariant under the Lorentz group [12], that is, $\frac{1}{\omega_{\mathrm{k}}}\,\mathrm{d}^3\mathbf{k} = \frac{1}{\omega_{\mathrm{k}'}}\,\mathrm{d}^3\mathbf{k}'$ and $\omega_{\mathrm{k}}\,\mathrm{d}^3\mathbf{r} = \omega_{\mathrm{k}'}\,\mathrm{d}^3\mathbf{r}'$, implying $\mathrm{d}V = \mathrm{d}^3\mathbf{k}\,\mathrm{d}^3\mathbf{r} = \mathrm{d}^3\mathbf{k}'\,\mathrm{d}^3\mathbf{r}' = \mathrm{d}V'$, where $\mathbf{r}'$, $\mathbf{k}'$, and $\omega_{\mathrm{k}'}$ result from applying a Lorentz transformation to $\mathbf{r}$, $\mathbf{k}$, and ω_{k}. Thus, from the probability invariant elements we conclude that $\frac{1}{\omega_{\mathrm{k}}}\rho_{\mathrm{r}}(\mathbf{r},t)$ and $\omega_{\mathrm{k}}\rho_{\mathrm{k}}(\mathbf{k},t)$ are also invariant under the group. Thus, the phase space density $\rho_{\mathrm{r}}(\mathbf{r},t)\rho_{\mathrm{k}}(\mathbf{k},t)$ is an invariant under Lorentz transformations. Therefore the entropy is a relativistic scalar. □

Note that in QFT, one scales the operator $\Phi(\mathbf{k},t)$ by $\sqrt{2\omega_{\mathrm{k}}}$, that is, one scales the creation and the annihilation operators $\alpha^\dagger(\mathbf{k}) = \sqrt{\omega_\mathrm{k}}\,\mathbf{a}^\dagger(\mathbf{k})$ and $\alpha(\mathbf{k}) = \sqrt{\omega}\,\mathbf{a}(\mathbf{k})$. In this way, the density operator $\Phi^\dagger(\mathbf{k},t)\Phi(\mathbf{k},t)$ scales with ω_k and becomes a relativistic scalar. Also, with such a scaling, the infinitesimal probability of finding a particle with momentum $\mathbf{p} = \hbar\mathbf{k}$ in the original reference frame is invariant under the Lorentz transformation, though it would be found with momentum $\mathbf{p}' = \hbar\mathbf{k}'$.

QCURVES AND ENTROPY-PARTITION

In [7], we introduced the concept of a *QCurve* to specify a curve (or path) in a Hilbert space parametrized by time. In QM, a QCurve is represented by a triple $\big(\left|\psi_0\right\rangle, U(t), \delta t\big)$, where $\left|\psi_0\right\rangle$ is the initial state, $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$ is the evolution operator, and $[0, \delta t]$ is the time interval of the evolution. Alternatively, we can represent the initial state by $(\langle\mathbf{r}|\psi_0\rangle, \langle\mathbf{k}|\psi_0\rangle)$ and in QFT as $(\Psi(\mathbf{r},0)\left|\text{state}\right\rangle, \Phi(\mathbf{k},0)\left|\text{state}\right\rangle)$.

Definition 1 (Partition of $\mathcal{E}$ from [7])**.** Let $\mathcal{E}$ to be the set of all QCurves. We define a partition of $\mathcal{E}$ based on the entropy evolution into four blocks:

- $\mathcal{C}$**:** Set of the QCurves for which the entropy is a constant.
- $\mathcal{I}$**:** Set of the QCurves for which the entropy is increasing, but it is not a constant.
- $\mathcal{D}$**:** Set of the QCurves for which the entropy is decreasing, but it is not a constant.
- $\mathcal{O}$**:** Set of oscillating QCurves, with the entropy strictly increasing in some subinterval of $[0, \delta t]$ and strictly decreasing in another subinterval of $[0, \delta t]$.

Consider stationary states $\left|\psi_t\right\rangle = \left|\psi_E\right\rangle \mathrm{e}^{-\mathrm{i}\omega t}$ with $\omega = E/\hbar$, where E is an energy eigenvalue of the Hamiltonian, and $\left|\psi_E\right\rangle$ is the time-independent eigenstate of the Hamiltonian associated with E.

Theorem 5. *All stationary states are in* $\mathcal{C}$.

Proof. Follows from the time invariance of the probabilities. □

Dirac's free-particle Hamiltonian in QM [5] is

$$H = -\mathrm{i}\hbar\gamma^0\vec{\gamma}\cdot\nabla + mc\gamma^0\,. \tag{4}$$

It can be diagonalized in the spatial Fourier domain $|\mathbf{k}\rangle$ basis to obtain

$$\omega(\mathbf{k}) = \pm c\sqrt{k^2 + \frac{m^2}{\hbar^2}c^2}\,, \tag{5}$$

where $\omega(\mathbf{k})$ is the frequency component of the Hamiltonian. We focus on the positive energy solutions and so the group velocity becomes

$$\mathbf{v_g}(\mathbf{k}) = \nabla_{\mathrm{k}}\omega(\mathbf{k}) = \frac{\hbar}{m}\frac{\mathbf{k}}{\sqrt{1+(\frac{\hbar k}{mc})^2}}\,. \tag{6}$$

In (9) we will use the Taylor expansion of (5) up to the second order, thus requiring the Hessian $\mathbf{H}(\mathbf{k})$, with the entries

$$\mathbf{H}_{ij}(\mathbf{k}) = \frac{\partial^2\omega(\mathbf{k})}{\partial k_i\partial k_j} = \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{3}{2}}\left[\delta_{i,j}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right) - \left(\frac{\hbar k_i}{mc}\right)\left(\frac{\hbar k_j}{mc}\right)\right] \tag{7}$$

for the positive energy solution. The three (positive) eigenvalues of $\mathbf{H}(\mathbf{k})$ are

$$\begin{aligned}
\lambda_1 &= \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{3}{2}} = \hbar\frac{m^2}{\left(m^2+\mu^2(k)\right)^{\frac{3}{2}}}\,,\\
\lambda_{2,3} &= \frac{\hbar}{m}\left(1+\left(\frac{\hbar k}{mc}\right)^2\right)^{-\frac{1}{2}} = \hbar\frac{1}{\left(m^2+\mu^2(k)\right)^{\frac{1}{2}}}\,,
\end{aligned}$$

where $\mu(k) = \hbar k/c$ is the kinetic energy in mass units. The Hessian is positive definite for positive energy, and so it gives a measure of the dispersion of the wave.

We now consider initial solutions that are localized in space, $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0) = \psi_0(\mathbf{r}-\mathbf{r}_0)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}$, where $\mathbf{r}_0$ is the mean value of $\mathbf{r}$. Assume that the variance, $\int \mathrm{d}^3\mathbf{r}\,(\mathbf{r}-\mathbf{r}_0)^2\rho_{\mathrm{r}}(\mathbf{r})$, is finite, where $\rho_{\mathrm{r}}(\mathbf{r}) = |\psi_0(\mathbf{r})|^2$. In a Cartesian representation, we can write the initial state in the spatial frequency domain as $\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0) = \phi_0(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}_0}$, where $\phi_0(\mathbf{k})$ is the Fourier transform of $\psi_0(\mathbf{r})$, and so the variance of $\rho_{\mathrm{k}}(\mathbf{k}) = |\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)|^2$ is also finite, with the mean in the spatial frequency center $\mathbf{k}_0$.

The time evolution of $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0)$ according a Hamiltonian with a dispersion relation $\omega(\mathbf{k})$, and written via the inverse Fourier transform, is

$$\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0, t) = \frac{1}{(\sqrt{2\pi})^3}\int \Phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}\omega(\mathbf{k})t}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{k}\,. \tag{8}$$

As $\phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0)$ fades away exponentially from $\mathbf{k} = \mathbf{k}_0$, we expand (5) in a Taylor series and approximate it as

$$\omega(\mathbf{k}) \approx \mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0 + \mathbf{v_g}(\mathbf{k}_0)\cdot(\mathbf{k}-\mathbf{k}_0) + \frac{1}{2}(\mathbf{k}-\mathbf{k}_0)^{\mathsf{T}}\mathbf{H}(\mathbf{k}_0)\,(\mathbf{k}-\mathbf{k}_0)\,, \tag{9}$$

where $\mathbf{v_p}(\mathbf{k}_0)$, $\mathbf{v_g}(\mathbf{k}_0)$, and $\mathbf{H}(\mathbf{k}_0)$ are the phase velocity $\frac{\omega(\mathbf{k}_0)}{|\mathbf{k}_0|}\hat{\mathbf{k}}_0$, the group velocity (6), and the Hessian (7) of the dispersion relation $\omega(\mathbf{k})$, respectively. Then after inserting (9) into (8), we obtain the quantum dispersion transform

$$\begin{aligned}\phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0, t) &\approx \frac{1}{Z_k}\,\mathrm{e}^{-\mathrm{i}t\mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0}\,\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0)\,\mathcal{N}\left(\mathbf{k}\mid\mathbf{k}_0, -\mathrm{i}t^{-1}\mathbf{H}^{-1}(\mathbf{k}_0)\right),\\ \psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}, t) &\approx \frac{1}{Z_{\mathrm{r}}}\mathrm{e}^{-\mathrm{i}t\mathbf{v_p}(\mathbf{k}_0)\cdot\mathbf{k}_0}\,\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}) * \mathcal{N}\left(\mathbf{r}\mid\mathbf{r}^t_{\mathbf{k}_0}, \mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right),\end{aligned} \tag{10}$$

where $\mathbf{r}^t_{\mathbf{k}_0} = \mathbf{r}_0 + \mathbf{v_g}(\mathbf{k}_0)t$, $\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0) = \phi_0(\mathbf{k}-\mathbf{k}_0)\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}^t_{\mathbf{k}_0}}$, with Fourier transform $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0})$; $*$ denotes a convolution, Z_r and Z_k normalize the amplitudes, and $\mathcal{N}$ is a normal distribution. Consequently, $\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}, t)$ is the spatial Fourier transform of $\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0, t)$.

The probability densities associated with the probability amplitudes in (10) are

$$\begin{aligned}\rho_{\mathrm{r}}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}, t) &= \frac{1}{Z_{\mathrm{r}}^2}|\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}^t_{\mathbf{k}_0}) * \mathcal{N}\left(\mathbf{r} \mid \mathbf{r}^t_{\mathbf{k}_0}, \mathrm{i}\, t\, \mathbf{H}(\mathbf{k}_0)\right)|^2 \,,\\ \rho_{\mathrm{k}}(\mathbf{k}-\mathbf{k}_0, t) &= \frac{1}{Z_k^2}|\Phi_{\mathbf{r}^t_{\mathbf{k}_0}}(\mathbf{k}-\mathbf{k}_0)|^2 \,.\end{aligned} \tag{11}$$

Lemma 1 (Dispersion Transform and Reference Frames)**.** *The entropy associated with* (11) *is equal to the entropy associated with the simplified probability densities*

$$\begin{aligned}\rho^{\mathrm{S}}_{\mathrm{r}}(\mathbf{r}, t) &= \frac{1}{Z^2}|\psi_0(\mathbf{r}) * \mathcal{N}\left(\mathbf{r} \mid 0, \mathrm{i}\, t\, \mathbf{H}(\mathbf{k}_0)\right)|^2 \,,\\ \rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t) &= \frac{1}{Z_k^2}|\Phi_0(\mathbf{k})|^2 = \rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t=0) \,.\end{aligned} \tag{12}$$

Proof. Consider (11). If the frame of reference is translating the position by $\mathbf{r}^t_{\mathbf{k}_0} = \mathbf{r}_0 + \mathbf{v_g}(\mathbf{k}_0)t$ and the momentum by $\hbar\mathbf{k}_0$, we get the simplified density functions (12).

Theorem 2 shows that the entropy in position and momentum is invariant under translations of the position $\mathbf{r}$ and the spatial frequency $\mathbf{k}$, and that completes the proof. □

The time invariance of the density $\rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k}, t)$, and therefore of S_{k}, reflects the conservation law of momentum for free particles.

We now focus on the case of coherent states, represented by $|\alpha\rangle$, eigenstates of the annihilator operator. In 1D position space they are represented as $\psi_\alpha(x) = \langle x|\alpha\rangle = \frac{\mathrm{e}^{-\frac{p_0^2}{2}}}{\pi^{\frac{1}{4}}}\mathrm{e}^{-\frac{1}{2}\left(x-\sqrt{2}\alpha\right)^2}$, where $\alpha = \frac{1}{\sqrt{2}}(x_0 + \mathrm{i}p_0)$. Squeeze states extend to all eigenstate solutions of the annihilator operator by allowing different variances for the Gaussian solution, and together their representation in 3D position and momentum space are

$$\begin{aligned}
\psi_{\mathrm{k}_0}(\mathbf{r}-\mathbf{r}_0) &= \langle \mathbf{r}|\alpha\rangle = \frac{1}{2^3\pi^{\frac{3}{2}}(\det\boldsymbol{\Sigma})^{\frac{1}{2}}}\,\mathcal{N}\left(\mathbf{r}\mid\mathbf{r}_0,\boldsymbol{\Sigma}\right)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}\,,\\
\Phi_{\mathrm{r}_0}(\mathbf{k}-\mathbf{k}_0) &= \langle \mathbf{k}|\alpha\rangle = \frac{1}{2^3\pi^{\frac{3}{2}}(\det\boldsymbol{\Sigma}^{-1})^{\frac{1}{2}}}\,\mathcal{N}\left(\mathbf{k}\mid\mathbf{k}_0,\boldsymbol{\Sigma}^{-1}\right)\,\mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}_0}\,,
\end{aligned}\tag{13}$$

where $\boldsymbol{\Sigma}$ is the spatial covariance matrix.

Theorem 6. *A QCurve with an initial coherent state* (13) *and evolving according to* (4) *is in* $\mathcal{I}$.

Proof. To describe the evolution of the initial states (13), we apply (10). Then, after applying Lemma 1,

$$\begin{aligned}
\rho^{\mathrm{S}}_{\mathrm{r}}(\mathbf{r},t) &= \frac{1}{Z_2^2}\mathcal{N}\left(\mathbf{r}\mid 0,\boldsymbol{\Sigma}+\mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right)\mathcal{N}\left(\mathbf{r}\mid 0,\boldsymbol{\Sigma}-\mathrm{i}t\mathbf{H}(\mathbf{k}_0)\right) = \mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\boldsymbol{\Sigma}(t)\right)\,,\\
\rho^{\mathrm{S}}_{\mathrm{k}}(\mathbf{k},t) &= \mathcal{N}\left(\mathbf{k}\mid 0,\boldsymbol{\Sigma}^{-1}\right)\,,
\end{aligned}$$

where $\boldsymbol{\Sigma}(t)=\boldsymbol{\Sigma}+t^2\mathbf{H}(\mathbf{k}_0)\boldsymbol{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0)$. Then

$$\begin{aligned}
\mathrm{S} &= \mathrm{S}_{\mathrm{r}}+\mathrm{S}_{\mathrm{k}}\\
&= -\int\mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\boldsymbol{\Sigma}(t)\right)\ln\mathcal{N}\left(\mathbf{r}\mid 0,\frac{1}{2}\boldsymbol{\Sigma}(t)\right)\mathrm{d}^3\mathbf{r}\\
&\quad-\int\mathcal{N}\left(\mathbf{k}\mid 0,\boldsymbol{\Sigma}^{-1}\right)\ln\mathcal{N}\left(\mathbf{k}\mid 0,2\boldsymbol{\Sigma}^{-1}\right)\mathrm{d}^3\mathbf{k}\\
&= 3(1+\ln\pi)+\frac{1}{2}\ln\det\left(\mathbf{I}+t^2(\boldsymbol{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0))^2\right)\,.
\end{aligned}$$

As $\det\left(\mathbf{I}+t^2(\boldsymbol{\Sigma}^{-1}\mathbf{H}(\mathbf{k}_0))^2\right)>0$, the entropy increases over time. □

The theorem suggests that quantum physics has an inherent mechanism to increase entropy for free particles, due to the spatial dispersion property of the Hamiltonian. Note that at $t=0$ a coherent state (13) reaches the minimum possible coordinate-entropy value, while the spin-entropy remains constant.

Time Reflection

We consider a time-independent Hamiltonian, investigate the discrete symmetries C and P, and propose that Time Reversal be augmented with Time Translation, say by δt. We refer to the mapping $t \mapsto -t + \delta t$ as Time Reflection, because as t varies from 0 to δt, $t'(t) = -t + \delta t$ varies as a reflection from δt to 0. We define the Time Reflection quantum field

$$\Psi^{\mathrm{T}_\delta}(\mathbf{r}, -t + \delta t) = \mathcal{T}\Psi(\mathbf{r}, t) = T\Psi^*(\mathbf{r}, t) .$$

Note that in contrast to the case of Time Reversal, $\Psi^{\mathrm{T}_\delta}(\mathbf{r}, t') = \mathcal{T}\Psi(\mathbf{r}, -t' + \delta t)$, and the entropies associated with $\Psi(\mathbf{r}, t)$ and $\Psi^{\mathrm{T}_\delta}(\mathbf{r}, t)$ are generally not equal. Thus, an instantaneous Time Reflection transformation will cause entropy changes.

We next consider a composition of the three transformation, Charge Conjugation, Parity Change, and Time Reflection.

Definition 2 ($\Psi^{\mathrm{CPT}_\delta}$)**.** Let the CPT_δ quantum field be

$$\Psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, -t + \delta t) = \eta_\delta\, CPT\,\overline{\Psi}^{\mathsf{T}}(\mathbf{r}, t) = \eta\gamma^5\,(\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, t)\,, \tag{14}$$

where η is the product of the phases of each operation, η_δ is the phase of time translation, and $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

Definition 3 (Q_{CPT_δ})**.** Let Q_{CPT_δ} be $\big(\psi(\mathbf{r}, 0), U(t), [0, \delta t]\big) \mapsto \big(\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0), U(t), [0, \delta t]\big)$.

Using (14) we see that,

$$\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0) = \eta\gamma^5\,(\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, -0 + \delta t) = \eta\,\gamma^5\,(\Psi^\dagger)^{\mathsf{T}}(\mathbf{r}, \delta t)\,. \tag{15}$$

Theorem 7 (Time Reflection)**.** *Consider a* CPT *invariant quantum field theory (QFT) with energy conservation, such as Standard Model or Wightman axiomatic QFT [13]. Let* $e_0 = (\psi(\mathbf{r}, 0), U(t), [0, \delta t])$ *be a QCurve solution to such QFT. Then,* $e_1 = Q_{\mathrm{CPT}_\delta}(e_0)$ *is (i) a solution to such QFT, (ii) if* e_0 *is in* $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$ *then* e_1 *is respectively in* $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$, *making* $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$ *reflections of* $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$, *respectively.*

Proof. Let $t' = -t + \delta t$. The QCurve e_1 describes the evolution $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ during the period $[0, \delta t]$.

Since e_0 is a solution to a QFT that is CPT-invariant and time-translation invariant, e_1 is also a solution to the QFT, proving (i).

The time evolution of $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, 0)$ from 0 to δt is described by $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$, and by (15) $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t') = \eta\,\gamma^5\,(\Psi^\dagger)^\mathsf{T}(\mathbf{r}, -t' + \delta t) = \eta\,\gamma^5\,\Psi^*(\mathbf{r}, \delta t - t')$. Thus, the evolution of $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ as t' evolves from 0 to δt, by Theorem 3, has the same entropies as $\psi(\mathbf{r}, \delta t - t')$. Since $\psi(\mathbf{r}, \delta t - t')$ traverses the same path as $\psi(\mathbf{r}, t')$ but in the opposite time direction, we conclude that e_1 produces the time evolution states $\psi^{\mathrm{CPT}_\delta}(-\mathbf{r}, t')$ in the time interval $[0, \delta t]$ traversing the same path and with the same entropies as $\psi(\mathbf{r}, t')$, but in the opposite time directions.

Applying the above to a QCurve respectively in $\mathcal{I}$, $\mathcal{D}$, $\mathcal{C}$, $\mathcal{O}$, results in a QCurve respectively in $\mathcal{D}$, $\mathcal{I}$, $\mathcal{C}$, $\mathcal{O}$. Thus, we conclude the proof of (ii). □

For a visualization see Figure 1, which we repeat here from [9] for the reader's convenience.

Entropy Oscillations

Theorem 8 (Coefficients for two states)**.** *Consider a particle in an eigenstate* $|\psi_{E_1}\rangle$ *of a Hamiltonian H that has only two eigenstates* $|\psi_{E_1}\rangle$ *and* $|\psi_{E_2}\rangle$ *with eigenvalues* $E_1 = \hbar\omega_1$ *and* $E_2 = \hbar\omega_2$, *respectively. Let this particle interact with an external field (such as the impact of a Gauge Field), requiring an additional Hamiltonian*

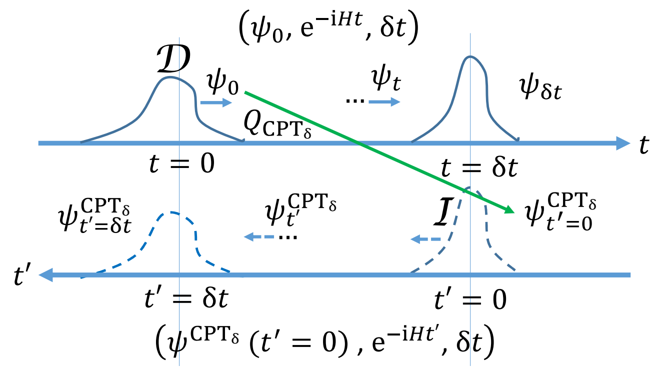

Figure 1. A visualization of the Time Reflection Theorem. (i) Axis t: A QCurve $e_1 = \big(\psi_0(\mathbf{r}), e^{-iHt}, \delta t\big)$. (ii) Axis $t' = \delta t - t$: The antiparticle QCurve is created as $e_2 = Q_{CPT_\delta}(e_1) = \big(\psi^{CPT_\delta}(-\mathbf{r}, t' = 0), e^{-iHt'}, \delta t\big)$. Axis t' shows the evolution as going forward in time t'. The evolution of $\psi^{CPT_\delta}(-\mathbf{r}, t') = \eta\gamma^5 (\Psi^\dagger)^\mathsf{T}(\mathbf{r}, \delta t - t')$ is mirroring the evolution of $\psi(\mathbf{r}, t)$, with $t = t'$ evolving from 0 to δt. If $e_1 \in \mathcal{D}$, then $e_2 \in \mathcal{I}$.

term H^{I} to describe the evolution of this system.

Let $\omega^{\mathrm{I}}_{i,j} = \frac{1}{\hbar}\langle\psi_{E_i}|\, H^{\mathrm{I}}\,|\psi_{E_j}\rangle$, $\omega^{\text{total}}_1 = \omega_1 + \omega^{\mathrm{I}}_{11}$, $\omega^{\text{total}}_2 = \omega_2 + \omega^{\mathrm{I}}_{22}$, $\eta = \sqrt{\left(\omega^{\text{total}}_1 - \omega^{\text{total}}_2\right)^2 + 4(\omega^{\mathrm{I}}_{12})^2}$, *and* $\lambda_\pm = \frac{\omega^{\text{total}}_1 + \omega^{\text{total}}_2 \pm \eta}{2}$. *The probability of the particle to be in state* $|\psi_{E_2}\rangle$ *at time t is*

$$\frac{4(\omega^{\mathrm{I}}_{12})^2}{\eta^2}\,\sin^2\frac{(\lambda_+ - \lambda_-)t}{2}\,.$$

Proof. The Hamiltonians in the basis $|\psi_{E_1}\rangle, |\psi_{E_2}\rangle$ are

$$H = \hbar\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix} \qquad \text{and} \qquad H^{\mathrm{I}} = \hbar\begin{pmatrix}\omega^{\mathrm{I}}_{11} & \omega^{\mathrm{I}}_{12}\\ \omega^{\mathrm{I}}_{12} & \omega^{\mathrm{I}}_{22}\end{pmatrix},$$

where the real values satisfy $\omega^{\mathrm{I}}_{21} = \omega^{\mathrm{I}}_{12}$ as H^{I} is Hermitian. The eigenvalues of the symmetric matrix $H' = H + H^{\mathrm{I}}$ are $\hbar\lambda_\pm$, and so we can decompose it as

$$H' = \hbar \begin{pmatrix} \omega_1^{\text{total}} & \omega_{12}^{\mathrm{I}} \\ \omega_{12}^{\mathrm{I}} & \omega_2^{\text{total}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hbar\lambda_+ & 0 \\ 0 & \hbar\lambda_- \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{16}$$

where

$$\theta = \frac{1}{2} \arcsin \frac{2\omega_{12}^{\mathrm{I}}}{\eta}. \tag{17}$$

The time evolution of $\left|\psi_{E_1}\right\rangle$ is $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}\frac{(H+H^{\mathrm{I}})}{\hbar}t}\left|\psi_{E_1}\right\rangle = \sum_{k=1}^{2} \alpha_k(t)\left|\psi_{E_k}\right\rangle$, and projecting on $\left\langle\psi_{E_j}\right|$, we get $\alpha_j(t) = \left\langle\psi_{E_j}\right| \mathrm{e}^{-\mathrm{i}\frac{(H+H^{\mathrm{I}})}{\hbar}t}\left|\psi_{E_1}\right\rangle$. From (16),

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\frac{H'}{\hbar}t} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda_+ t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\lambda_- t} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda_+ t}\cos^2\theta + \mathrm{e}^{-\mathrm{i}\lambda_- t}\sin^2\theta & \frac{\mathrm{e}^{-\mathrm{i}\lambda_+ t} - \mathrm{e}^{-\mathrm{i}\lambda_- t}}{2}\sin 2\theta \\ \frac{\mathrm{e}^{-\mathrm{i}\lambda_+ t} - \mathrm{e}^{-\mathrm{i}\lambda_- t}}{2}\sin 2\theta & \mathrm{e}^{-\mathrm{i}\lambda_+ t}\sin^2\theta + \mathrm{e}^{-\mathrm{i}\lambda_- t}\cos^2\theta \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\frac{H'}{\hbar}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta\, \mathrm{e}^{-\mathrm{i}\lambda_+ t} + \sin^2\theta\, \mathrm{e}^{-\mathrm{i}\lambda_- t} \\ \sin 2\theta \left(\frac{\mathrm{e}^{-\mathrm{i}\lambda_+ t} - \mathrm{e}^{-\mathrm{i}\lambda_- t}}{2}\right) \end{pmatrix},$$

and so

$$\begin{pmatrix} |\alpha_1(t)|^2 \\ |\alpha_2(t)|^2 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\sin^2 2\theta\,(1 - \cos(\lambda_- - \lambda_+)t) \\ \frac{1}{2}\sin^2 2\theta\,(1 - \cos(\lambda_- - \lambda_+)t) \end{pmatrix}.$$

As $1 - \cos(\lambda_- - \lambda_+)t = 2\sin^2\frac{(\lambda_+ - \lambda_-)t}{2}$, the probability of being in state $\left|\psi_{E_2}\right\rangle$ at time t is $|\alpha_2(t)|^2 = \sin^2 2\theta\, \sin^2\frac{(\lambda_+ - \lambda_-)t}{2}$. Using (17), completes the proof. □

If $\omega_1 \gg \omega_{11}^{\mathrm{I}}$, $\omega_2 \gg \omega_{22}^{\mathrm{I}}$, and $|\omega_1 - \omega_2| \gg \omega_{12}^{\mathrm{I}}$, then $\lambda_{+,-} \approx \omega_{1,2}$, and the coefficient of transition becomes $|\alpha_2(t)|^2 \approx \frac{4(\omega_{12}^{\mathrm{I}})^2}{(\omega_1 - \omega_2)^2}\sin^2\frac{(\omega_2 - \omega_1)t}{2}$, which is Fermi's golden rule [4, 6].

In [7] we showed that when the probability of two states oscillates as above, the entropy oscillates too.

The derivation of $\alpha_2(t)$ can be extended to multiple states. However, for multiple states, the sum over all the frequencies $\lambda_k - \lambda_i$ may cancel the oscillations unless some frequencies dominate the sum, such as when the transition to the ground state dominates other transitions. Thus, to obtain the entropy oscillation in the presence of multiple transitions may require approximations similar to the ones that are usually used in derivations of Fermi's golden rule.

An Entropy Law and a Time Arrow

In classical statistical mechanics, the entropy provides a time arrow through the second law of thermodynamics [2]. We have shown that due to the dispersion property of the fermions Hamiltonian some states in quantum mechanics, such as coherent states, already obey such a law. However, current quantum physics is described as time reversible. In [7] we conjectured the following

Law (The Entropy Law)**.** *The entropy of a quantum system is an increasing function of time.*

The law may help explain why particles are created and/or annihilated in scenarios such as high-speed collision $e^+ + e^- \to 2\gamma$, kaons decay into mesons, and photon creation and emission when the electron in the hydrogen atom transitions from an excited state to the ground state. In those scenarios, while such final states are reachable in a unitary evolution of the initial state, it seems that only those evolutions in which entropy increase are realized. According to the S-matrix formulation [12], similar to Fermi's golden rule in QM, these final states are among the possible transition states. We note that similarly to Fermi's golden rule, these are also entropy oscillation scenarios in which the evolution is blocked from entering a time interval of decreasing entropy. The creation and/or annihilation of a particles seem to occur when the entropy of the evolution from the initial to the final state is oscillating, and but for such events the entropy would decrease, which the conjectured law forbids.

Furthermore, the spin-entropy evolution of system of particles or fields is also subject to this law, which may have implications in all physical scenarios including quantum information and quantum computing.

CONCLUSIONS

The concepts of entropy in quantum phase spaces in [7] were further developed here. We extended the coordinate-entropy in QM to multiple particles. We proved that the coordinate-entropy is invariant under coordinate transformations, Lorentz transformations, and CPT transformations. We analyzed the entropy evolution of coherent states, showing that the Dirac's Hamiltonian has a mechanism to disperse the information and to increase entropy. We proved that Time Reflection transforms QCurves in $\mathcal{C}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{D}$ into QCurves in $\mathcal{C}$, $\mathcal{D}$, $\mathcal{O}$, $\mathcal{I}$, respectively. We proved that for a two-state Hamiltonian, the addition of a Hamiltonian term not only causes a state oscillation (as suggested by Fermi's golden rule when the appropriate approximations hold) but also causes entropy oscillation. In light of the technical advancements here, we reviewed the conjectured entropy law [7]. According to that law, not only a time arrow would emerge, but should the formation of new particles be triggered by the entropy law, the history of the universe would have to be revised through such a lens. Perhaps, the collapse of a wave function occurs not due to measurements, but instead due to the restrictions posed by the entropy law.

ACKNOWLEDGEMENT

This paper is partially based upon work supported by both the National Science Foundation under Grant No. DMS-1439786 and the Simons Foundation Institute Grant Award ID 507536 while the first author was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the spring 2019 semester "Computer Vision" program.

[1] Y. Aharonov and T. Kaufherr. Quantum frames of reference. *Phys. Rev. D*, 30(2):368, 1984.

[2] R. Clausius. *The mechanical theory of heat: With its applications to the steam-engine and to the physical properties of bodies*. J. van Voorst, 1867.

[3] B. S. DeWitt. Point transformations in quantum mechanics. *Phys. Rev.*, 85(4):653, 1952.

[4] P. A. M. Dirac. The quantum theory of the emission and absorption of radiation. *Proc. R. Soc. Lond. A*, 114:243–265, 1927.

[5] P. A. M. Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 126(801):360–365, 1930.

[6] E. Fermi. *Nuclear physics: A course given by Enrico Fermi at the University of Chicago*. University of Chicago Press, 1950.

[7] D. Geiger and Z. Kedem. Quantum entropy. *arXiv preprint arXiv:2106.15375*, 2021.

[8] D. Geiger and Z. Kedem. Spin entropy. *arXiv preprint arXiv:2111.11605*, 2021.

[9] D. Geiger and Z. M. Kedem. Quantum information physics I. New York University, Dept. of Computer Science,TR 2021-996, 2021.

[10] J. von Neumann. *Mathematische grundlagen der quantenmechanik*. Springer-Verlag, 2013.

[11] A. Wehrl. General properties of entropy. *Reviews of Modern Physics*, 50(2):221, 1978.

[12] S. Weinberg. *The quantum theory of fields: Volume 1, (Foundations)*. Cambridge University Press, 1995.

[13] A. S. Wightman. Hilbert's sixth problem: Mathematical treatment of the axioms of physics. In *Proceedings of Symposia in Pure Mathematics*, volume 28, pages 147–240. AMS, 1976.